An Algebraic Structure of Music Theory

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Abstract We may define a binary relation. Then a nonempty finite set equipped with the binary relation is called a circle set. And we define a bijective mapping of the circle set, and the mapping is called a shift. We may construct a pitch structure over a circle set. And we may define a tonic and step of a pitch structure. Then the ordered pair of the tonic and step is called the key of the pitch structure. Then we define a key transpose along a shift. And a key transpose is said to be regular if it consists of stretches, shrinks and a shift. A key transpose is regular if and only if it satisfies some hypotheses.

Contents

1. Introduction

In definition 3.1, we define a binary relation '$\olessthan$'. Then a non-vacuous finite set P equipped with the binary relation $\olessthan$ is called a circle set, and we define a bijective mapping δ that is called shift if the mapping is compatible with $\olessthan$, see definition 3.2.

A circle set has no heads, but we may select a member as a head. Hence we define a tonic(cf. [4])τ of P in definition 3.3.

Let $\mathbb{S}$:= {▬, –, $\otimes$} be a set. The members of $\mathbb{S}$ is called scales(cf. [4]), and we define a function $\lambda \colon P \times P \longrightarrow \mathbb{S}$ given by assigning to an ordered pair of P a scale, see definition 3.4 for more details.

Two unary relations '$\sharp$' and $\flat$' on a circle set P are defined in definitions 3.5 and 3.6, respectively.

Let $\mathfrak{L} := \{\lambda, \tau, \mathbb{S}, \olessthan\}$ be a language. Then we may construct a partial structure $\boldsymbol{M}$ of the language $\mathfrak{L}$ over a circle set P, and the partial structure $\boldsymbol{M}$ is called the pitch structure, see definition 3.7 for more details.

Then we obtain a sequence of the scales, the sequence is called the step of the pitch structure $\boldsymbol{M}$, and denoted by $SS_{\tau_M}(\boldsymbol{M})$, see definition 3.8 for more details. The ordered pair $\langle \tau_M, SS_{\tau_M}(\boldsymbol{M}) \rangle$ is called the key(cf. [4]) of $\boldsymbol{M}$, see definition 3.9.

Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are two pitch structures over a circle set P. Then a bijective mapping $\kappa \colon SS(\boldsymbol{M}) \rightsquigarrow SS(\boldsymbol{N})$ is called a key transpose(cf. [4]) along a shift δ if the mapping κ satisfies the hypotheses of definition 3.10.

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We say that a key transpose κ is regular if κ consists of stretches, shrinks and a shift, see definitions 3.11 and 3.12 for details. And some members of P, that is invariant under κ, are called κ-invariant, see definition 3.13. A key transpose κ is regular if and only if lemma 3.1 and lemma 3.2 holds, see proposition 3.2 for more details.

2. Preliminaries

We recall some definitions in universal algebra.

Definition 2.1 ([2, 3])**.** An ordered pair $\langle L, \sigma \rangle$ is said to be a (first-order) **language** provided that

- L is a nonempty set,
- $\sigma\colon L \to \mathbb{Z}$ is a mapping.

A language $\langle L, \sigma \rangle$ is denoted by $\mathfrak{L}$. If $f \in \mathfrak{L}$ and $\sigma(f) \geq 0$ then f is called an **operation symbol**, and $\sigma(f)$ is called the **arity** of f. If $r \in \mathfrak{L}$ and $\sigma(r) < 0$, then r is called a **relation symbol**, and $-\sigma(r)$ is called the **arity** of r. A language is said to be **algebraic** if it has no relation symbols.

Definition 2.2 ([2])**.** Let X be a nonempty class and n a nonnegative integer. Then an n-ary **partial operation** on X is a mapping from a subclass of X^n to X. If the domain of the mapping is X^n, then it is called an n-ary **operation**. And an n-ary **relation** is a subclass of X^n where $n > 0$. An operation(relation) is said to be **unary**, **binary** or **ternary** if the arity of the operation(relation) is 1, 2 or 3, respectively. And an operation is called **nullary** if the arity is 0.

Definition 2.3 ([2])**.** An ordered pair $\boldsymbol{A} := \langle A, \mathfrak{L} \rangle$ is said to be a **structure** of a language $\mathfrak{L}$ if A is a nonempty class and there exists a mapping which assigns to every n-ary operation symbol $f \in \mathfrak{L}$ an n-ary operation f^A on $\boldsymbol{A}$ and assigns to every n-ary relation symbol $r \in \mathfrak{L}$ an n-ary relation r^A on $\boldsymbol{A}$. If all operation on $\boldsymbol{A}$ are partial operations, then $\boldsymbol{A}$ is called a **partial structure**. A (partial)structure $\boldsymbol{A}$ is said to be a (**partial**)**algebra** if the language $\mathfrak{L}$ is algebraic.

Definition 2.4 ([2, 3])**.** Let $\boldsymbol{A}, \boldsymbol{B}$ be (partial)structures of a language $\mathfrak{L}$. A mapping $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ is said to be a **homomorphism** provided that

$$\varphi(f^A(a_1, \dots, a_n)) = f^B(\varphi(a_1), \dots, \varphi(a_n)) \text{ for every } n\text{-ary operation } f;$$
$$r^A(a_1, \dots, a_n) \Longrightarrow r^B(\varphi(a_1), \dots, \varphi(a_n)) \text{ for every } n\text{-ary relation } r.$$

A homomorphism φ is called an **isomorphism** if φ is bijective.

3. An Algebraic Structure of Music Theory

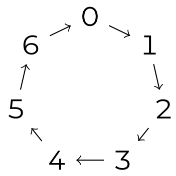
Definition 3.1. Suppose that P is a nonempty finite set. We may define a binary relation '$\lessdot$' on P as follows. For every $s \in P$,

- there is exactly one $u \in P$ such that $u \lessdot s$, and
- there is exactly one $v \in P$ such that $s \lessdot v$.

Remark. The binary relation '$\lessdot$' is *not* an order relation.

Definition 3.2. A **circle set** is a nonempty finite set equipped with the binary relation '$\lessdot$' defined in definition 3.1. Let P be a circle set. Then a bijective mapping $\delta\colon P \to P$ is said to be a **shift** if δ preserves the order of P, i.e., $\delta(p_i) \lessdot \delta(p_j)$ if and only if $p_i \lessdot p_j$.

Example 3.1. The set $X := \{x \in \mathbb{N} \mid x \bmod 7\}$ can be regarded as a circle set.



And it is a shift that a mapping is defined by $i \mapsto \big((i+1) \bmod 7\big)$ for $i \in X$.

Example 3.2. Let A be a non-vacuous finite ordered set, B a non-vacuous countable ordered set. Suppose that $(a_0, b_0), (a_1, b_1) \in A \times B$. If we define

$$(a_0, b_0) \leq (a_1, b_1) \text{ if } \begin{cases} a_0 \leq a_1 & \text{for } b_0 = b_1; \\ b_0 \leq b_1 & \text{for } b_0 \neq b_1, \end{cases} \tag{3.1}$$

then $A \times B$ is an ordered set. Now, let $(a_0, b_0) \sim (a_1, b_1)$ if $a_0 = a_1$. It is clear that $\sim$ is an equivalence relation. Then the quotient[1] set $(A \times B)/\sim$ can be regarded as a circle set.

A circle set P has no head members. But we may select a member τ as a head.

Definition 3.3. Suppose that P is a circle set. Let $\tau := p$ for an arbitrary $p \in P$. We call τ a **tonic** of P.

And we have the following important definitions.

Definition 3.4. Suppose that P is a circle set. Let $\mathbb{S}$ be the set $\{\text{▬}, \text{–}, \otimes\}$. We may define a function $\lambda \colon P \times P \to \mathbb{S}$ given by

$$\lambda(p, p') = \begin{cases} \text{▬ or –} & \text{if } p \oslash p', \\ \otimes & \text{otherwise.} \end{cases} \tag{3.2}$$

And the elements of the set $\mathbb{S}$ is called **scales**.

Recall the definition of unary relations which is defined in definition 2.2. And we have the following definitions.

Definition 3.5. Suppose that P is a circle set. Let $\sharp$ be a unary relation on P such that

(1) $$\lambda(\sharp(s), \sharp(p)) = \lambda(s, p);$$

(2) $$\lambda(s, \sharp(p)) = \begin{cases} \text{▬} & \text{if } \lambda(s,p) = \text{–}, \\ \otimes & \text{if } \lambda(s,p) = \text{▬}; \end{cases}$$

(3) $$\lambda(\sharp(s), p) = \begin{cases} \text{–} & \text{if } \lambda(s,p) = \text{▬}, \\ \otimes & \text{if } \lambda(s,p) = \text{–} \end{cases}$$

for every $s, p \in P$ with $s \oslash p$.

Definition 3.6. Suppose that P is a circle set. Let $\flat$ be a unary relation on P such that

(1) $$\lambda(\flat(s), \flat(p)) = \lambda(s, p);$$

(2) $$\lambda(s, \flat(p)) = \begin{cases} \text{–} & \text{if } \lambda(s,p) = \text{▬}, \\ \otimes & \text{if } \lambda(s,p) = \text{–}; \end{cases}$$

$$\lambda(\flat(s),p) = \begin{cases} \text{▬} & \text{if } \lambda(s,p) = \text{–}, \\ \otimes & \text{if } \lambda(s,p) = \text{▬} \end{cases} \tag{3}$$

for every $s, p \in P$ with $s \oslash p$.

Assumption 3.1. *Let P be a circle set. For simplicity, we assume that*

$$\lambda(\flat(p), \sharp(q)) = \otimes;$$
$$\lambda(\sharp(p), \flat(q)) = \otimes,$$

for all $p, q \in P$. Since $\sharp$ and $\flat$ are unary relations, we have that $\sharp(\sharp(p))$, $\sharp(\flat(p))$, $\flat(\sharp(p))$ and $\flat(\flat(p))$ are invalid for all $p \in P$. So we have not 'double sharp' and 'flat flat'.

Remark 3.1. In fact, that $\sharp$ and $\flat$ are *not* real unary relations.

Let $M = P \cup \{\text{▬}, \text{–}, \otimes\}$. By definitions 3.4 to 3.6, we have that λ is a partial binary operation on M, and that ▬, – and $\otimes$ are nullary operations. Hence we may define a partial structure[definition 2.3] of a language[definition 2.1] $\mathfrak{L}$. Then we have the following definitions.

Definition 3.7. A partial structure $\boldsymbol{M} := \langle M, \mathfrak{L}\rangle$ of the language $\mathfrak{L}$ is called a **pitch structure** over a circle set P provided that the underlying set $M = P \cup \mathbb{S}$ where P equipped with $\oslash$ is a circle set[definition 3.2], and the language is defined to be the set $\mathfrak{L} := \{\lambda, \tau, \mathbb{S}, \oslash\}$ where λ is a partial binary operation defined in definition 3.4, $\oslash$ is a binary relation defined in definition 3.1, τ is a nullary operation defined in definition 3.3, and $\mathbb{S} = \{\text{▬}, \text{–}, \otimes\}$ is the set of nullary operations defined in definition 3.4.

Suppose that $\boldsymbol{M}$ is a pitch structure over a circle set P. We may assume that $|P| = n$ and $\tau := m_0$ for $m_0 \in P$. If $m_i \oslash m_{((i+1) \bmod n)} \in P$, then $\{\lambda(m_i, m_{((i+1) \bmod n)})\}$ constitutes a scale sequence, e.g., $\{\text{▬}, \text{▬}, \text{–}, \dots, \text{▬}\}$.

Definition 3.8. Let $\boldsymbol{M}$ be a pitch structure over a circle set P, $|P| = n$, and $\tau := m_0$ for $m_0 \in P$. Then we define $SS_{\tau_M}(\boldsymbol{M})$ to be the following sequence

$$\{\lambda(m_0, m_1), \lambda(m_1, m_2), \dots, \lambda(m_{n-2}, m_{n-1}), \lambda(m_{n-1}, m_0)\}, \tag{3.3}$$

if we have $m_0 \oslash m_1 \oslash m_2 \oslash \cdots \oslash m_{n-2} \oslash m_{n-1} \oslash m_0 \in P$. And the sequence $SS_{\tau_M}(\boldsymbol{M})$ is called a **step** of the pitch structure $\boldsymbol{M}$ at the tonic m_0.

Remark. For all pitch structure $\boldsymbol{M}$, we have $\otimes \notin SS_\tau(\boldsymbol{M})$.

Proposition 3.1. *Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are two pitch structures. We have that $\boldsymbol{M} \cong \boldsymbol{N}$ implies $SS_{\tau_M}(\boldsymbol{M}) = SS_{\tau_N}(\boldsymbol{N})$.*

Proof. Let $\varphi\colon \boldsymbol{M} \to \boldsymbol{N}$ be an isomorphism. Since the scales in set $\mathbb{S} = \{\text{▬}, \text{–}, \otimes\}$ and τ_M are nullary operations of $\boldsymbol{M}$, we have that $\varphi\restriction\mathbb{S}$ is an identity mapping of $\mathbb{S}$ and $\varphi(\tau_M) = \tau_N$. Observe that λ is a binary operation. By definition 2.4, it is obvious that $SS_{\tau_M}(\boldsymbol{M}) = SS_{\tau_N}(\boldsymbol{N})$. □

Remark 3.2. Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are pitch structures. If there exists a homomorphism $\varphi\colon \boldsymbol{M} \to \boldsymbol{N}$, then φ must be an isomorphism. This is an immediate consequence of definitions 2.4 and 3.1. The isomorphism φ is unique. If we assume that $\boldsymbol{M}, \boldsymbol{N}$ have same underlying set $M = \mathbb{S} \cup P$, then it is clear that $\varphi\restriction P$ is a shift. Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are pitch structures over a circle set P. Let $\tau_M = \tau_N$ and $\boldsymbol{M} \ncong \boldsymbol{N}$. Then it follows $\lambda_M \neq \lambda_N$.

Definition 3.9. Suppose that $\boldsymbol{M}$ is a pitch structure over a circle set P, and the tonic $\tau = m_0$. Then the ordered pair $\langle \tau_M, SS_{\tau_M}(\boldsymbol{M})\rangle$ is called the **key** of $\boldsymbol{M}$.

Definition 3.10. Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are pitch structures over a circle set P, and $\tau_M = m_i$, $\tau_N = m_j$ for m_i, $m_j \in P$. Let δ be a shift[definition 3.2] which assigns m_j to m_i. Then a bijective mapping $\kappa\colon SS_{\tau_M}(\boldsymbol{M}) \rightsquigarrow SS_{\tau_N}(\boldsymbol{N})$ is called a **key transpose** along δ provided that κ assigns $\lambda_N(\delta(m), \delta(m'))$ to $\lambda_M(m, m')$ for every m, $m' \in P$ with $m \olessthan m'$.

Remark. We have that $\boldsymbol{M} \cong \boldsymbol{N}$ implies that κ is an identity mapping.

Example 3.3. Suppose that $P := \{m_0, m_1, m_2, m_3, m_4\}$ is a circle set, $\boldsymbol{M}$ is a pitch structure over P, and $\tau := m_0$. Let $SS(\boldsymbol{M}) = \{\text{▬}, \text{▬}, \text{–}, \text{▬}, \text{–}\}$. If we take $\sharp$, $\flat$ on some members of $\boldsymbol{M}$, e.g., $\sharp(m_0)$ and $\flat(m_2)$, then we obtain a new sequence

$$\begin{aligned}\{\lambda(\sharp(m_0), m_1), \lambda(m_1, \flat(m_2)), \lambda(\flat(m_2), m_3), \lambda(m_3, m_4), \lambda(m_4, \sharp(m_0))\} \\ = \{\text{–}, \text{–}, \text{▬}, \text{▬}, \text{▬}\},\end{aligned}$$

where the unary relations $\sharp$ and $\flat$ are defined in definitions 3.5 and 3.6.

Example 3.4. With the notations of example 3.3, if we change the value of τ, e.g., let $\tau := m_2$, then we also obtain a new sequence

$$\begin{aligned}\{\lambda(m_2, m_3), \lambda(m_3, m_4), \lambda(m_4, m_0), \lambda(m_0, m_1), \lambda(m_1, m_2)\} \\ = \{\text{–}, \text{▬}, \text{–}, \text{▬}, \text{▬}\}.\end{aligned}$$

Definition 3.11. Suppose that $\boldsymbol{M}$ is a pitch structure over a circle set P, and

$$P := \{m_0 \olessthan m_1 \olessthan \cdots \olessthan m_{n-1} \olessthan m_0\}.$$

Let m_i, $m_j \in \boldsymbol{M}$ with $m_i \olessthan m_j$ for $0 \leq i \leq n-1, j = (i+1) \bmod n$. The scale of $\langle m_i, m_j\rangle$ is said to be **shrinkable** if $\lambda(m_i, m_j) = \text{▬}$. By definitions 3.5 and 3.6, we have that both $\lambda(\sharp(m_i), m_j)$ and $\lambda(m_i, \flat(m_j))$ are –. Hence we call $\lambda(\sharp(m_i), m_j)$ and $\lambda(m_i, \flat(m_j))$ a $\sharp$-**shrink** and $\flat$-**shrink**, respectively. The scale of $\langle m_i, m_j\rangle$ is said to be **stretchable** if $\lambda(m_i, m_j) = \text{–}$. And we have that $\lambda(m_i, \sharp(m_j))$ and $\lambda(\flat(m_i, m_j))$ are a $\sharp$-**stretch** and $\flat$-**stretch**, respectively.

Example 3.5. Let the hypotheses be as in example 3.3. We have that the scale of $\langle m_0, m_1\rangle$ is shrinkable, the scale of $\langle m_4, m_0\rangle$ is stretchable. And we have that $\lambda(\sharp(m_0), m_1)$ and $\lambda(m_4, \sharp(m_0))$ are a $\sharp$-shrink and $\sharp$-stretch respectively, and $\lambda(\flat(m_2), m_3)$ and $\lambda(m_1, \flat(m_2))$ are a $\flat$-stretch and $\flat$-shrink respectively.

We may take the two classes of the transposition in examples 3.3 and 3.4 on a pitch structure $\boldsymbol{M}$ simultaneously.

Example 3.6. Let the notations be as in examples 3.3 and 3.4. Suppose that $\boldsymbol{N}$ is a pitch structure over the circle set P, and $\tau_N := m_2$. Let δ be a shift which assigns m_2 to m_0, and $\kappa\colon SS_{\tau_M}(\boldsymbol{M}) \rightsquigarrow SS_{\tau_N}(\boldsymbol{N})$ a key transpose along δ. If we assume that

$$SS_{\tau_N}(\boldsymbol{N}) := \{\text{▬}, \text{▬}, \text{▬}, \text{–}, \text{–}\},$$

then it is clear that κ is equivalent to the process which is defined as follows:

I. Take $\sharp$ and $\flat$ on m_0 and m_2 respectively, as described in example 3.3.
II. Let $\tau_M = m_2$, as described in example 3.4.

Therefore, we may say that the key transpose κ consists of a stretch, shrink and shift. And the order of the process is not important.

Definition 3.12. Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are pitch structures over a circle set P. If $SS(\boldsymbol{M})$ is transposed to $SS(\boldsymbol{N})$ by a key transpose κ in such a way that is described in examples 3.3, 3.4 and 3.6, that is, the key transpose consists of stretches[definition 3.11], shrinks[definition 3.11] and a shift[definition 3.2], then we say that the key transpose κ is **regular**.

Remark. A key transpose may be not regular.

Example 3.7. Suppose that $\boldsymbol{M}$ is a pitch structure over a circle set P, and $|P| = n$. For every $0 \leq i \leq n-1$, there are two **trivial** key transposes. One is

$$\{\sharp(m_i), \sharp(m_{(i+1) \bmod n}), \ldots, \sharp(m_{((i+n-1) \bmod n)}), \sharp(m_i)\},$$

and the other is

$$\{\flat(m_i), \flat(m_{(i+1) \bmod n}), \ldots, \flat(m_{((i+n-1) \bmod n)}), \flat(m_i)\}.$$

They are regular. And there are no changes on all of scales in the case of the trivial key transpose.

Definition 3.13. Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are two pitch structures over a circle set P. Let $\kappa\colon SS_{\tau_M}(\boldsymbol{M}) \rightsquigarrow SS_{\tau_N}(\boldsymbol{N})$ be a nontrivial regular key transpose and $m \in P$. The element m is said to be κ-**invariant** if there are not $\sharp(m)$ and $\flat(m)$ under the key transpose κ.

Definition 3.14. Suppose that $\boldsymbol{M}$ is a pitch structure over a circle set P. Let $P := \{m_0 \circledcirc m_1 \circledcirc \cdots \circledcirc m_{n-1} \circledcirc m_0\}$. Then the directions 3.4 and 3.5 are called **clockwise** and **anticlockwise**, respectively.

$$m_0 \circledcirc m_1 \circledcirc \cdots \circledcirc m_{n-1} \circledcirc m_0$$

$$\longrightarrow \tag{3.4}$$

$$\longleftarrow \tag{3.5}$$

We shall see what properties a key transpose satisfies if it is regular.

Lemma 3.1 ($\sharp$-shrink $\Longleftrightarrow$ $\sharp$-stretch)**.** *Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are pitch structures over a circle set P, and*

$$P := \{p_0 \circledcirc p_1 \circledcirc \cdots \circledcirc p_{n-1} \circledcirc p_0\}.$$

Let $\kappa\colon SS_{\tau_M}(\boldsymbol{M}) \rightsquigarrow SS_{\tau_N}(\boldsymbol{N})$ be a nontrivial key transpose along a shift δ which assigns to τ_M τ_N, and $\lambda_M(p_i, p_j) = \blacksquare$, $\lambda_N(p_i, p_j) = -$ for $p_i \circledcirc p_j \in P$. Then the scale of $\langle p_i, p_j\rangle$ is transformed from $\lambda_M(p_i, p_j)$ to $\lambda_N(p_i, p_j)$ under the key transpose κ via a $\sharp$-shrink, i.e., $\lambda_M(\sharp(p_i), p_j)$ if and only if there exist $p_{i'} \circledcirc p_{j'} \in P$ with $\lambda_M(p_{i'}, p_{j'}) = -$, $\lambda_N(p_{i'}, p_{j'}) = \blacksquare$ such that

① *$j' = (i + d) \bmod n$ with $d \leq 0$, i.e., in the anticlockwise,*
② *the scale of $\langle p_{i'}, p_{j'}\rangle$ is transformed from $\lambda_M(p_{i'}, p_{j'})$ to $\lambda_N(p_{i'}, p_{j'})$ under κ via a $\sharp$-stretch, i.e., $\lambda_M(p_{i'}, \sharp(p_{j'}))$, hence $p_{i'}$ is κ-invariant, and*
③ *κ makes no changes on the scales of the consecutive members pairs in $\{p_{j'} \circledcirc \ldots \circledcirc p_i\}$ if $p_{j'} \neq p_i$.*

Proof. We assume $p_{i'} \circledcirc p_i \circledcirc p_j$. Since $\lambda_M(\sharp(p_i), p_j)$ and assumption 3.1, we have that either

$$\lambda_N(p_{i'}, p_i) = \lambda_M(p_{i'}, p_i), \tag{3.6}$$

or

$$\begin{aligned} \lambda_N(p_{i'}, p_i) &= \blacksquare, \\ \lambda_M(p_{i'}, p_i) &= -. \end{aligned} \tag{3.7}$$

Hence if (3.7) holds, then the proof is complete. Now we assume that equation (3.6) holds, and observe assumption 3.1. Then there exists a $p_{j'} \in P$ such that κ makes no changes on the scales of the consecutive members pairs in $\{p_{j'} \olessthan \ldots \olessthan p_i\}$ by induction. Hence we have that κ takes $\sharp$ on all of elements in $\{p_{j'} \olessthan \ldots \olessthan p_i\}$. It follows that there exists a $p_{i'}$ with $p_{i'} \olessthan p_{j'}$ such that $\lambda_M(p_{i'}, p_{j'}) = \rule{1em}{1.5pt}$, $\lambda_N(p_{i'}, p_{j'}) = \rule{2em}{4pt}$, and the scale of $\langle p_{i'}, p_{j'} \rangle$ is transformed from the former to the latter under κ via a $\sharp$-stretch, i.e., $\lambda_M(p_{i'}, \sharp(p_{j'}))$. Otherwise, the nontrivial key transpose hypotheses would not hold. Hence it is clear that $p_{i'}$ is κ-invariant. On the other hand, we may assume $p_{i'} \olessthan p_{j'} \olessthan p_j$. Then the proof of the converse is similar. This completes the proof. □

Remark 3.3. Let κ be a key transpose along δ. Then we have that κ sends $\lambda_M(p_i, p_j)$ to $\lambda_N(\delta(p_i), \delta(p_j))$ for $p_i \olessthan p_j \in P$, cf. definition 3.10. But in lemmas 3.1 and 3.2, we observe $\lambda_M(p_i, p_j)$ and $\lambda_N(p_i, p_j)$.

We have the following lemma that is similar to lemma 3.1.

Lemma 3.2 ($\flat$-shrink $\iff$ $\flat$-stretch)**.** *Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are pitch structures over a circle set P, and*

$$P := \{p_0 \olessthan p_1 \olessthan \cdots \olessthan p_{n-1} \olessthan p_0\}.$$

Let $\kappa \colon SS_{\tau_M}(\boldsymbol{M}) \rightsquigarrow SS_{\tau_N}(\boldsymbol{N})$ be a nontrivial key transpose along a shift δ which assigns to τ_M τ_N, and $\lambda_M(p_i, p_j) = \rule{2em}{4pt}$, $\lambda_N(p_i, p_j) = \rule{1em}{1.5pt}$ for $p_i \olessthan p_j \in P$. The scale of $\langle p_i, p_j \rangle$ is transformed from $\lambda_M(p_i, p_j)$ to $\lambda_N(p_i, p_j)$ under the key transpose κ via a $\flat$-shrink, i.e., $\lambda_M(p_i, \flat(p_j))$ if and only if there exist $p_{i'} \olessthan p_{j'} \in P$ with $\lambda_M(p_{i'}, p_{j'}) = \rule{1em}{1.5pt}$, $\lambda_N(p_{i'}, p_{j'}) = \rule{2em}{4pt}$ such that

1. *$i' = (j + d) \bmod n$ with $d \geq 0$, i.e., in the clockwise,*
2. *the scale of $\langle p_{i'}, p_{j'} \rangle$ is transformed from $\lambda_M(p_{i'}, p_{j'})$ to $\lambda_N(p_{i'}, p_{j'})$ under κ via a $\flat$-stretch, i.e., $\lambda_M(\flat(p_{i'}), p_{j'})$, hence $p_{j'}$ is κ-invariant, and*
3. *κ makes no changes on the scales of the consecutive members pairs in $\{p_j \olessthan \ldots \olessthan p_{i'}\}$ if $p_j \neq p_{i'}$.*

Proof. This is similar to the proof of lemma 3.1. □

Remark 3.4. Let κ be a nontrivial key transpose. We observe lemmas 3.1 and 3.2. We shall find that a $\sharp$-shrink must be adjoint to a $\sharp$-stretch, and a $\flat$-shrink must be adjoint to a $\flat$-stretch. And we have that κ makes no changes on the scales of the consecutive members pairs between an adjoint pair.

Proposition 3.2. *Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are two pitch structures over a circle set P. Let $\kappa \colon SS(\boldsymbol{M}) \rightsquigarrow SS(\boldsymbol{N})$ be a non-trivial key transpose. Then the key transpose κ is regular if and only if lemma 3.1 and lemma 3.2 hold.*

Proof. Immediate from definitions 3.10 and 3.12 and lemmas 3.1 and 3.2. □

Remark 3.5. Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are pitch structures over a circle set P. Let $\varphi \colon \boldsymbol{M} \to \boldsymbol{N}$ be a homomorphism. Observe remark 3.2. We have that φ is an isomorphism. By proposition 3.1, we have $SS_{\tau_M}(\boldsymbol{M}) = SS_{\tau_N}(\boldsymbol{N})$. And it is clear that $\delta := \varphi \restriction P$ is a shift[definition 3.2] which assigns τ_N to τ_M. If κ is a key transpose along δ then κ is an identity mapping, since we have definition 3.10. And if κ is regular then κ consists of shrinks, stretches and a shift, even if κ is an identity mapping, cf. remark 3.3.

Corollary 3.2.1. *Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are two pitch structures over a circle set P. Let $\kappa \colon SS(\boldsymbol{M}) \rightsquigarrow SS(\boldsymbol{N})$ be a non-trivial key transpose. Then the key transpose κ is regular if and only if the key transpose κ^{-1} is regular.*

Proof. It is clear that lemma 3.1 and lemma 3.2 hold for κ^{-1} if the lemmas hold for κ, and vice versa. □

References

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