Non-Archimedean functional analysis over non-Archimedean field $\widetilde{{}^*\mathbb{R}_c^\#}$. Applications to constructive quantum field theory.

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Abstract. Functional analysis works with TVS (Topological Vector Spaces), classically over archimedean fields like $\mathbb{R}$ and $\mathbb{C}$.Canonical non-Archimedean functional analysis, where alternative but equally valid number systems such as p-adic numbers $\mathbb{Q}_p$ etc. are fundamental, is a fast-growing discipline.

This paper deals with TVS over non-classical non-Archimedean fields ${}^*\mathbb{R}_c^\#$, $\widetilde{{}^*\mathbb{R}_c^\#}$ and ${}^*\mathbb{C}_c^\#$, $\widetilde{{}^*\mathbb{C}_c^\#}$.

Definitions and theorems related to non-Archimedean functional analysis on non-Archemedean field $\widetilde{{}^*\mathbb{R}_c^\#}$ and on complex field $\widetilde{{}^*\mathbb{C}_c^\#} = \widetilde{{}^*\mathbb{R}_c^\#} + i\widetilde{{}^*\mathbb{R}_c^\#}$ are considered.

Applications to constructive quantum field theory also are considered
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[12] https://iopscience.iop.org/article/10.1088/1742-6596/2701/1/012113
[notice in [6] and [12] we abbreviate ${}^*\mathbb{R}_c^\#$ instead $\widetilde{{}^*\mathbb{R}_c^\#}$ for the sake of brevity].
Definitions and theorems appropriate to analysis on non-Archemedean field ${}^*\mathbb{R}_c^\#$ and on complex field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$ are given in [1]-[2].

Content.

Introduction

The incompleteness of set theory ZFC leads one to look for natural nonconservative extensions of ZFC in which one can prove statements independent of ZFC which appear to be “true”. One approach has been to add large cardinal axioms.
Or, one can investigate second-order expansions like Kelley-Morse class theory, KM or Tarski-Grothendieck set theory TG or It is a nonconservative extension of ZFC and is obtained from other axiomatic set theories by the inclusion of Tarski’s axiom which implies the existence of inaccessible cardinals. See also related set theory with a filter quantifier $ZF(aa)$. In this paper we look at a set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$[18], based on bivalent gyper infinitary logic with restricted Modus Ponens Rule [18]. Nonconservative extension namely IST$^{\#}$ of the canonical internal set theory IST was presented in [18].

§1.Bivalent hyper Infinitary first-order logic ${}^{2}L_{\infty^{\#}}^{\#}$ with restricted rules of conclusion.Generalized Deduction Theorem.

Hyper infinitary language $L_{\infty^{\#}}^{\#}$ are defined according to the length of hyper infinitary conjunctions/disjunctions as well as quantification it allows. In that way, assuming a supply of $\kappa < \aleph_0^{\#} = card(\mathbb{N}^{\#})$ variables to be interpreted as ranging over a nonempty domain, one includes in the inductive definition of formulas an infinitary clause for conjunctions and disjunctions, namely, whenever the hypernaturals indexed hyper infinite sequence $\{A_\delta\}_{\delta \in \mathbb{N}^{\#}}$ of formulas has length less than κ, one can form the hyperfinite conjunction/disjunction of them to produce a formula. Analogously, whenever an hypernaturals indexed sequence of variables has length less than λ, one can introduce one of the quantifiers $\forall$ or $\exists$ together with the sequence of variables in front of a formula to produce a new formula. One also stipulates that the length of any well-formed formula is less than $\aleph_0^{\#}$ itself.

The syntax of bivalent hyper infinitary first-order logics ${}^{2}L_{\infty^{\#}}^{\#}$ consists of a (ordered) set of

sorts and a set of function and relation symbols, these latter together with the

corresponding type, which is a subset with less than $\aleph_0^{\#} = card(\mathbb{N}^{\#})$ many sorts. Therefore, we assume that our signature may contain relation and function symbols on $\gamma < \aleph_0^{\#}$ many variables, and we suppose there is a supply of $\kappa < \aleph_0^{\#}$ many fresh variables of each sort. Terms and atomic formulas are defined as usual, and general formulas are defined inductively according to the following rules.

If $\phi, \psi, \{\phi_\alpha : \alpha < \gamma\}$ (for each $\gamma < \kappa$) are formulas of $L^{\#}_{\infty^{\#}}$, the following are also formulas:

(i) $\bigwedge_{\alpha<\gamma} \phi_\alpha, \bigwedge_{\alpha\leq\gamma} \phi_\alpha,$

(ii) $\bigvee_{\alpha<\gamma} \phi_\alpha, \bigvee_{\alpha\leq\gamma} \phi_\alpha,$

(iii) $\phi \to \psi, \phi \wedge \psi, \phi \vee \psi, \neg\phi$

(iv) $\forall_{\alpha<\gamma} x_\alpha \phi$ (also written $\forall \mathbf{x}_\gamma \phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$),

(v) $\exists_{\alpha<\gamma} x_\alpha \phi$ (also written $\exists \mathbf{x}_\gamma \phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$),

(vi) the statement $\bigwedge_{\alpha<\gamma} \phi_\alpha$ holds if and only if for any α such that $\alpha < \gamma$ the statement holds ϕ_α,

(vii) the statement $\bigvee_{\alpha<\gamma} \phi_\alpha$ holds if and only if there exist α such that $\alpha < \gamma$ the statement holds ϕ_α.

Definition 1.1.A valuation of a syntactic system is a function that as signs $\top$ (true) to some of its sentences, and/or $\bot$ (false) to some of its sentences.Precisely, a valuation maps a nonempty subset of the set of sentences into the set $\{\top, \bot\}$. We call a valuation bivalent iff it maps all the sentences into $\{\top, \bot\}$.

Definition 1.2.Let L be a propositional language. L is a classical bivalent propositional language iff its admissible valuations are the functions v such that for all sentences A, B of L the following properties hold

(a) $v(A) \in \{\top, \bot\}$

(b) $v(\neg A) = \top$ iff $v(A) = \bot$

(c) $v(A \wedge B) = \top$ iff $v(A) = v(B) = \top$.

(d) by definition of the classical implication $A \Rightarrow B$ the following truth table holds

	$v(A)$	$v(B)$	$v(A \Rightarrow B)$
(1)	$\top$	$\top$	$\top$
(2)	$\top$	$\bot$	$\bot$
(3)	$\bot$	$\top$	$\top$
(4)	$\bot$	$\bot$	$\top$

Truth table 1.

(e) $v^*(A) \in \{\top, \bot\}$

(f) $v^*(\neg A) = \top$ iff $v^*(A) = \bot$

(g) $v^*(A \wedge B) = \top$ iff $v^*(A) = v^*(B) = \top$.

(h) by definition of the nonclassical implication $A \Rightarrow B$ the following truth table holds

	$v^*(A)$	$v^*(B)$	$v^*(A \Rightarrow B)$
(1)	$\top$	$\top$	$\top$
(2)	$\top$	$\bot$	$\top$
(3)	$\bot$	$\top$	$\top$
(4)	$\bot$	$\bot$	$\top$

Truth table 2.

Remark 1.1.Note that in the case (2) of the truth table 2

$\top = v^*(A \Rightarrow B) \neq v(A \Rightarrow B) = \bot$.

In this case we call implication $A \Rightarrow B$ a weak implication and abbreviate

$$A \Rightarrow_w B \qquad (1.1)$$

We call a statement (1.1) as a weak statement and often abbreviate $v(A \Rightarrow B) = \top_w$ instead (1).

Definition 1.3.[7-8]. A is a valid (logically valid) sentence (in symbols, $\Vdash A$) in L iff every admissible valuation of L satisfies A.

The axioms of hyper infinitary first-order logic ${}^2L^{\#}_{\infty^{\#}}$ consist of the following schemata:

I. **Logical axiom**

A 1. $A \to [B \to A]$

A 2. $[A \to [B \to C] \to [[A \to B] \to [A \to C]]]$

A 3. $[\neg B \to \neg A] \to [A \to B]$

A 4. $[\bigwedge_{i<\alpha}[A \to A_i]] \ \to [A \to \bigwedge_{i<\alpha} A_i], \alpha \in \mathbb{N}^{\#}$

A 5. $[\bigwedge_{i<\alpha} A_i] \to A_j, \alpha \in \mathbb{N}^{\#}$

A 6. $[\forall \mathbf{x}[A \to B] \to [A \to \forall \mathbf{x} B]]$

provided no variable in $\mathbf{x}$ occurs free in A;

A 7. $\forall \mathbf{x} A(\mathbf{x}) \to S_f(A)$,

where $S_f(A)$ is a substitution based on a function f from $\mathbf{x}$ to the terms of the language; in particular:

A 7$'$. $\forall x_i[A(x_i)] \Rightarrow A(\mathbf{t})$ is a wff of ${}^2L^{\#}_{\infty^{\#}}$ and $\mathbf{t}$ is a term of ${}^2L^{\#}_{\infty^{\#}}$ that is free for x_i in $A(x_i)$. Note here that $\mathbf{t}$ may be identical with x_i; so that all wffs $\forall x_i A \Rightarrow A$ are axioms by virtue of axiom (7),see [8].

A 8.Gen (Generalization).

$\forall x_i B$ follows from B.

II.**Restricted rules of conclusion**.

Let $\mathcal{F}_{\mathbf{wff}}$ be a set of the all closed wffs of $L^{\#}_{\infty^{\#}}$.

R1.RMP (Restricted Modus Ponens).

There exist subsets $\Delta_1, \Delta_2 \subset \mathcal{F}_{\mathbf{wff}}$ such that the following rules are satisfied.

From A and $A \Rightarrow B$, conclude B iff $A \notin \Delta_1$ and $(A \Rightarrow B) \notin \Delta_2$,where $\Delta_1, \Delta_2 \subset \mathcal{F}_{\mathbf{wff}}$.

In particular for any $A, B \in \mathcal{F}_{\mathbf{wff}} : A \Rightarrow_w B \in \Delta_2$.

If $A \notin \Delta_1$ and $(A \Rightarrow B) \notin \Delta_2$ we also abbraviate by $A, A \Rightarrow_s B \vdash_{RMP} B$.

R2.RMT (Restricted Modus Tollens)

There exist subsets $\Delta'_1, \Delta'_2 \subset \mathcal{F}_{\mathbf{wff}}$ such that the following rules are satisfied.

$P \Rightarrow Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta'_1$and $(P \Rightarrow Q) \notin \Delta'_2$,where $\Delta'_1, \Delta'_2 \subset \mathcal{F}_{\mathbf{wff}}$.

Remark 1.2.Note that RMP and RMT easily prevent any paradoxes of naive Cantor set theory (NC), see [1],[9].

III.**Additional derived rule of conclusion**.

Particularization rule (RPR)

Remind that canonical unrestricted particularization rule (UPR) reads

UPR: If $\mathbf{t}$ is free for x in $B(x)$, then $\forall x[B(x)] \vdash B(\mathbf{t})$,see [8].

Proof.From $\forall x[B(x)]$ and the instance $\forall x[B(x)] \Rightarrow B(\mathbf{t})$ of axiom (A7), we obtain $B(\mathbf{t})$ by unrestricted modus ponens rule.Since x is free for x in $B(x)$, a special case of unrestricted particularization rule is:$\forall x B \vdash B$.

Definition 1.4.Any formal theory L with a hyper infinitary lenguage $L^{\#}_{\infty^{\#}}$ is defined when the following conditions are satisfied:

1. A hyper infinite set of symbols is given as the symbols of L. A finite or hyperfinite

sequence of symbols of L is called an expression of L.
2. There is a subset of the set of expressions of L called the set of well formed formulas (wffs) of L. There is usually an effective procedure to determine whether a given expression is a wff.
3. There is a set of wfs called the set of axioms of L. Most often, one can effectively decide whether a given wff is an axiom; in such a case, L is called an axiomatic theory.
4. There is a finite set $R_1,\ldots,R_n,$ of relations among wffs, called rules of conclusion. For each $R_i,$ there is a unique positive integer j such that, for every set of j wfs and each wff $B,$ one can effectively decide whether the given j wffs are in the relation R_i to B, and, if so, B is said to follow from or to be a direct consequence of the given wffs by virtue of R_j.
Definition 1.5.A proof in L is a finite or hyperfinite sequence $B_1,\ldots,B_k, k \in \mathbb{N}^{\#}$ of wffs such that for each $i,$either B_i is an axiom of L or B_i is a direct consequence of some of the preceding wffs in the sequence by virtue of one of the rules of inference of L.
Definition 1.6. A theorem of L is a wff B of Y such that B is the last wff of some proof in $L.$ Such a proof is called a proof of B in L.
Definition 1.7. A wff E is said to be a consequence in L of a set of Γ of wffs if and only if there is a finite or hyperfinite sequence $B_1,\ldots,B_k, k \in \mathbb{N}^{\#}$ of wffs such that E is B_k and, for each $i,$either B_i is an axiom or B_i is in Γ, or B_i is a direct consequence by some rule of inference of some of the preceding wffs in the sequence. Such a sequence is colled a proof (or deduction) E from Γ. The members of Γ are called the hypotheses or premisses of the proof.
We use $\Gamma \vdash E$ as an abbreviation for E as a consequence of Γ.
In order to avoid confusion when dealing with more than one theory, we write $\Gamma \vdash_L E,$ adding the subscript L to indicate the theory in question.
If Γ is a finite or hyperfinite set $\{H_i\}_{1 \leq i \leq m}, m \in \mathbb{N}^{\#}$ we write $H_1,\ldots,H_m \vdash E$ instead of $\{H_i\}_{1 \leq i \leq m} \vdash E$.
Lemma 1.1.[18]. $\vdash B \Rightarrow B$ for all wffs B.
Theorem 1.1.(Generalized Deduction Theorem1). If Γ is a set of wffs and B and E are wffs, and $\Gamma, B \vdash E,$ then $\Gamma \vdash B \Rightarrow_s E.$ In pticular, if $B \vdash E$ then $\vdash B \Rightarrow E.$
Proof. Let $E_1,\ldots,E_n, n \in \mathbb{N}^{\#}$ be a proof of E form $\Gamma \cup \{B\}$, where E_n is E.
Let us prove, by hyperfinite induction on $j,$ that $\Gamma \vdash B \Rightarrow_s E_j$ for $1 \leq j \leq n.$
First of all, E_1 must be either in Γ or an axiom of L or B itself.
By axiom schema A1, $E_1 \Rightarrow_s (B \Rightarrow_s E_1)$ is an axiom. Hence, in the first two cases, by MP, $\Gamma \vdash B \Rightarrow_s E_1$ For the third case, when E_1 is B, we have $\vdash B \Rightarrow_s E_1$ by Lemma 1, and, therefore, $\Gamma \vdash B \Rightarrow_s E_1.$ This takes care of the case $j = 1.$
Assume now that: $\vdash B \Rightarrow_s E_k$ for all $k < j, j \in \mathbb{N}^{\#}.$ Either E_j is an axiom, or E_j is in Γ, or E_j is $B,$ or E_j follows by modus ponens from some E_l and E_m where $l < j,$ $m < j,$ and E_m has the form $E_l \Rightarrow_s E_j.$ In the first three cases, $\Gamma \vdash B \Rightarrow_s E_j$ as in the case $j = 1$ above. In the last case, we have, by inductive hypothesis, $\Gamma \vdash B \Rightarrow_s E_l$ and $\Gamma \vdash B \Rightarrow_s (E_l \Rightarrow_s E_j)$ But, by axiom schema (A2),
$\vdash B \Rightarrow_s (E_l \Rightarrow_s E_j) \Rightarrow_s ((B \Rightarrow_s E_l) \Rightarrow_s (B \Rightarrow_s E_j))$
Hence, by MP, $\Gamma \vdash (B \Rightarrow_s E_l) \Rightarrow_s (B \Rightarrow_s E_j)$ and, again by MP, $\Gamma \vdash B \Rightarrow_s E_j.$

Thus, the proof by hyperfinite induction is complete.

The case $j = n \in \mathbb{N}^{\#}$ is the desired result. Notice that, given a deduction of E from Γ and B, the proof just given enables us to construct a deduction of $B \Rightarrow_s E$ from Γ. Also note that axiom schema A3 was not used in proving the generalized deduction theorem.

Remark 1.3.For the remainder of the chapter, unless something is said to the contrary,
we shall omit the subscript L in $\vdash_L$. In addition, we shall use $\Gamma, B \vdash E$ to stand for $\Gamma \cup \{B\} \vdash E$. In general, we let $\Gamma, B_1, \ldots, B_n \vdash E$ stand for $\Gamma \cup \{B_i\}_{1 \leq i \leq n} \vdash E$.

Remark 1.4.We shall use the terminology proof, theorem, consequence, axiomatic, etc. and notation $\Gamma \vdash E$ introduced above.

Proposition 1.1. Every wff B of K that is an instance of a tautology is a theorem of K, and it may be proved using only axioms A1-A3 and MP.

Proposition 1.2.If E does not depend upon B in a deduction showing that $\Gamma, B \vdash E$, then $\Gamma \vdash E$.

Proof.Let $D_1, \ldots, D_n$ be a deduction of E from Γ and B, in which E does not depend upon B. In this deduction, D_n is E. As an inductive hypothesis, let us assume that the proposition is true for all deductions of length less than $n \in \mathbb{N}^{\#}$ If E belongs to Γ or is an axiom, then $\Gamma \vdash E$. If E is a direct consequence of one or two preceding wffs by Gen or MP, then, since E does not depend upon B, neither do these preceding wfs. By the inductive hypothesis, these preceding wfs are deducible from Γ alone. Consequently, so is E .

Theorem 1.2.(Generalized Deduction Theorem 2).Assume that, in some deduction showing that $\Gamma, B \vdash E$, no application of Gen to a wff that depends upon B has as its quantified variable a free variable of B. Then $\Gamma \vdash B \Rightarrow_s E$.

Proof.Let $D_1, \ldots, D_n$ be a deduction of E from Γ and B satisfying the assumption of this theorem. In this deduction, D_n is E. Let us show by hyperfinite induction that $\Gamma \vdash B \Rightarrow_s D_i$ for each $i \leq n \in \mathbb{N}^{\#}$. If D_i is an axiom or belongs to Γ, then $\Gamma \vdash B \Rightarrow_s D_i$, since $D_i \Rightarrow_s (B \Rightarrow_s D_i)$ is an axiom. If D_i is B, then $\Gamma \vdash B \Rightarrow_s D_i$, since, by Proposition 1, $\vdash B \Rightarrow_s B.\cdot$ If there exist j and k less than i such that D_k is $\vdash D_j \Rightarrow_s D_i$, then, by inductive hypothesis, $\Gamma \vdash B \Rightarrow_s D_j$ and $\Gamma \vdash B \Rightarrow_s (D_j \Rightarrow_s D_i)$. Now, by axiom A2,
$\vdash B \Rightarrow_s (D_j \Rightarrow_s D_i) \Rightarrow_s ((B \Rightarrow_s D_j) \Rightarrow_s (B \Rightarrow_s D_i))$. Hence, by MP twice, $\Gamma \vdash B \Rightarrow_s D_i$. Finally, suppose that there is some $j < i$ such that D_i is $\forall x_k D_j$. By the inductive hypothesis, $\Gamma \vdash B \Rightarrow_s D_j$, and, by the hypothesis of the theorem, either D_j does not depend upon B or x_k is not a free variable of B. If D_j does not depend upon B, then, by Proposition 2, $\Gamma \vdash D_j$ and, consequently, by Gen, $\Gamma \vdash \forall x_k D_j$. Thus, $\Gamma \vdash D_i$. Now, by axiom A1, $\vdash D_i \Rightarrow_s (B \Rightarrow_s D_i)$. So, $\Gamma \vdash B \Rightarrow_s D_i$ by MP. If, on the other hand, x_k is not a free variable of B, then, by axiom A5, $\vdash \forall x_k (B \Rightarrow_s D_j) \Rightarrow_s (B \Rightarrow_s \forall x_k D_j)$ Since $\Gamma \vdash B \Rightarrow_s D_j$, we have, by Gen,$\Gamma \vdash \forall x_k (B \Rightarrow_s D_j)$, and so, by MP,$\Gamma \vdash B \Rightarrow_s \forall x_k D_j$ that is, $\Gamma \vdash B \Rightarrow_s D_i$. This completes the induction, and our proposition is just the special case $i = n.$.

§2.Set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$.

Set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ is formulated as a system of axioms based on bivalent hyper infinitary logic ${}^2L^{\#}_{\infty^{\#}}$ with restricted modus ponens rule [1],[18]. The language of set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ is a first-order hyper infinitary language $L^{\#}_{\infty^{\#}}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg\,(x = y)$ and $x \notin y$ for $\neg(x \in y)$. Individual variables $x, y, z, \ldots$, and $x^{\mathbf{CL}}, y^{\mathbf{CL}}, z^{\mathbf{CL}}, \ldots$ of $L^{\#}_{\infty^{\#}}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x)$, $\exists! x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x) \;\&\; \forall y(\varphi(y) \Rightarrow_s x = y)]$. The language $L^{\#}_{\infty^{\#}}$ will also contains the formation of terms of the form $\{x|\varphi(x)\}^{\mathbf{NCL}}$, for any formula $\varphi(x)$ containing the free variable x.

Such terms are called non-classical sets; we shall use upper case letters $A, B, \ldots$, and $A^{\mathbf{NCL}}, B^{\mathbf{NCL}}, \ldots$ for such sets. For each non-classical set $A = \{x|\varphi(x)\}^{\mathbf{NCL}}$ the formulas

$\forall x[x \in A \Leftrightarrow_{s,w} \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow_{s,w} \varphi(x,A)]$ is called the defining axioms for the non-classical set A.

Remark 2.1.Remind that in logic ${}^2L^{\#}_{\infty^{\#}}$ with restricted modus ponens rule the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ does not always guarantee that

$$\alpha, \alpha \Rightarrow \beta \vdash_{\mathbf{RMP}} \beta \tag{2.1}$$

since for some α and β possible

$$\alpha, \alpha \Rightarrow \beta \nvdash_{\mathbf{RMP}} \beta \tag{2.2}$$

even if the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ holds.

Abbreviation 2.1.We shall write for the sake of brevity instead (2.1) by

$$\alpha \Rightarrow_s \beta \tag{2.3}$$

and we shall write instead (2.2) by

$$\alpha \Rightarrow_w \beta. \tag{2.4}$$

Remark 2.2.Let A be an nonclassical set.Note that in set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$the following true formula

$$\exists A \forall x[x \in A \Leftrightarrow \varphi(x,A)] \tag{2.5}$$

does not always guarantee that

$$x \in A, x \in A \Rightarrow \varphi(x,A) \vdash_{RMP} \varphi(x,A) \tag{2.6}$$

even if $x \in A$ holds and (or)

$$\varphi(x,A), \varphi(x,A) \Rightarrow x \in A \vdash_{RMP} x \in A; \tag{2.7}$$

even $\varphi(x,A)$ holds, since for nonclassical set A for some y possible

$$y \in A,\; y \in A \Rightarrow \varphi(y,A) \nvdash_{\mathbf{RMP}} \varphi(y,A) \tag{2.8}$$

and (or)

$$\varphi(y,A),\; \varphi(y,A) \Rightarrow y \in A \nvdash_{\mathbf{RMP}} y \in A. \tag{2.9}$$

Remark 2.3.Note that in this paper the formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x) \wedge x \in u] \tag{2.10}$$

and more general formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x,a) \wedge x \in u] \tag{2.11}$$

is considered as the defining axioms for the classical set a.

Remark 2.4.Let a be a classical set. Note that in $\mathbf{NC}^{\#}_{\infty^{\#}}$: (i) the following true formula

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x,a) \wedge x \in u] \qquad (2.12)$$

always guarantee that

$$x \in a, x \in a \Rightarrow \varphi(x,a) \vdash_{RMP} \varphi(x) \qquad (2.13)$$

if $x \in a$ holds and

$$\varphi(x), \varphi(x) \Rightarrow x \in a \vdash_{RMP} x \in a; \qquad (2.14)$$

if $\varphi(x)$ holds;

In order to emphasize this fact mentioned above in Remark 2.1-2.3,
we rewrite the defining axioms in general case for the nonclassical sets in the following
form

$$\exists A \forall x\{[x \in A \Leftrightarrow_s \varphi(x,A)] \vee [x \in A \Leftrightarrow_w \varphi(x,A)]\} \qquad (2.15)$$

and similarly we rewrite the defining axioms in general case for the classical sets in the
following form

$$\exists a \forall x[x \in a \Leftrightarrow_s \varphi(x) \wedge (x \in u)]. \qquad (2.16)$$

Abbreviation 2.2.We write instead (2.15):

$$\forall x\{[x \in A \Leftrightarrow_{s,w} \varphi(x,A)]\} \qquad (2.17)$$

Definition 2.1. (1) Let A be a nonclassical set defined by formula (2.17).
Assum that: (i) for some y statement $\varphi(y)$ and statement $\varphi(y) \Rightarrow y \in A$ holds and
(ii) $\varphi(y), \varphi(y) \Rightarrow y \in A \nvdash_{RMP} y \in A$, $y \in A, y \in A \Rightarrow \varphi(y) \nvdash_{RMP} \varphi(y)$.
Then we say that y is a weak member of non-classical set A and abbreviate $y \in_w A$.
Abbreviation 2.3. Let A be a nonclassical set defined by formula (2.17) We abbreviate $x \in_{s,w} A$ if the following statement $x \in_s A \vee x \in_w A$ holds, i.e.

$$x \in_{s,w} A \leftrightarrow_{def} (x \in_s A \vee x \in_w A). \qquad (2.18)$$

Definition 2.2.(1) Two nonclassical sets A, B are defined to be equal and we write
$A = B$ if $\forall x[x \in_{s,w} A \Leftrightarrow_s x \in_{s,w} B]$. (2) A is a subset of B, and we often write $A \subset_{s,v} B$, if
$\forall x[x \in_{s,w} A \Rightarrow_s x \in_{s,w} B]$. (3) We also write $\mathbf{CL.Set}(a)$ for the formula
$\exists u \forall x[x \in a \Leftrightarrow x \in u \wedge \varphi(x)]$. (4) We also write $\mathbf{NCL.Set}(A)$ for the formulas
$\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x)]$ and $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x,A)]$.
Remark 2.5.$\mathbf{CL.Set}(u)$ asserts that the set u is a classical set. For any classical set u,
it follows from the defining axiom for the classical set $u = \{x | x \in_s u \wedge \varphi(x)\}$ that
$\mathbf{CL.Set}(\{x | x \in_s u \wedge \varphi(x)\})$.
We shall identify $\{x | x \in_s u\}$ with u, so that sets may be considered as (special sorts of)
nonclassical sets and we may introduce assertions such as $u \subset_s A, u \subseteqq_s A$, etc.
Abbreviation 2.4.Let $\varphi(t)$ be a formula of $\mathbf{NC}^{\#}_{\infty^{\#}}$.
(i) $\forall x \varphi(x)$ and $\forall^{\mathbf{CL}} x \varphi(x)$ abbreviates $\forall x(\mathbf{CL.Set}(x) \Rightarrow \varphi(x))$
(ii) $\exists x \varphi(x)$ and $\exists^{\mathbf{CL}} x \varphi(x)$ abbreviates $\exists x(\mathbf{CL.Set}(x) \Rightarrow \varphi(x))$
(iii) $\forall X \varphi(X)$ and $\forall^{\mathbf{NCL}} X \varphi(X)$ abbreviates $\forall X(\mathbf{NCL.Set}(X) \Rightarrow \varphi(X))$
(iv) $\exists X \varphi(X)$ and $\exists^{\mathbf{NCL}} X \varphi(X)$ abbreviates $\exists X(\mathbf{NCL.Set}(X) \Rightarrow \varphi(X))$
Remark 2.6.If A is a nonclassical set, we write $\exists x \in A\ \varphi(x,A)$ for $\exists x[x \in A \wedge \varphi(x,A)]$

and $\forall x \in A\varphi(x,A)$ for $\forall x[x \in A \Rightarrow \varphi(x,A)]$.
We define now the following sets:
1.$\{u_1,u_2,\ldots,u_n\} = \{x|x = u_1 \vee x = u_2 \vee\ldots\vee x = u_n\}$. 2. $\{A_1,A_2,\ldots,A_n\} =$
$= \{x|x = A_1 \vee x = A_2 \vee\ldots\vee x = A_n\}$. 3.$\cup A = \{x|\exists y[y \in A \wedge x \in y]\}$.
4.$\cap A = \{x|\forall y[y \in A \Rightarrow x \in y]\}$. 5.$A \cup B = \{x|x \in A \vee x \in B\}$.
5.$A \cap B = \{x|x \in A \wedge x \in B\}$. 6.$A - B = \{x|x \in A \wedge x \notin B\}$. 7.$u^+ = u \cup \{u\}$.
8.$\mathbf{P}(A) = \{x|x \subseteq A\}$. 9.$\{x \in A|\varphi(x,A)\} = \{x|x \in A \wedge \varphi(x,A)\}$. 10.$\mathbf{V} = \{x|x = x\}$.
11.$\varnothing = \{x|x \neq x\}$.
The system $\mathbf{NC}^{\#}_{\infty^{\#}}$ of set theory is based on the following axioms:
Extensionality1: $\forall u\forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$
Extensionality2: $\forall A\forall B[\forall x(x \in A \Leftrightarrow_{s,w} x \in B) \Rightarrow A = B]$
Universal Set: $\mathbf{NCL.Set}(\mathbf{V})$
Empty Set: $\mathbf{CL.Set}(\varnothing)$
Pairing1: $\forall u\forall v\ \mathbf{CL.Set}(\{u,v\})$
Pairing2: $\forall A\forall B\ \mathbf{NCL.Set}(\{A,B\})$
Union1: $\forall u\ \mathbf{CL.Set}(\cup u)$
Union2: $\forall A\ \mathbf{NCL.Set}(\cup A)$
Powerset1: $\forall u\ \mathbf{CL.Set}(\mathbf{P}(u))$
Powerset2: $\forall A\ \mathbf{NCL.Set}(\mathbf{P}(A))$
Infinity $\exists a[\varnothing \in a \wedge \forall x \in a(x^+ \in a)]$
Separation1$\forall u_1\forall u_2,\ldots\forall u_n\forall a\exists\mathbf{CL.Set}(\{x \in_s a|\varphi(x,u_1,u_2,\ldots,u_n)\})$
Separation 2 $\forall u_1\forall u_2,\ldots\forall u_n\mathbf{NCl.Set}(\{x \in_{s,w} A|\varphi(x,A;u_1,u_2,\ldots,u_n)\})$
Comprehension1$\forall u_1\forall u_2,\ldots\forall u_n\exists A\forall x[x \in_{s,w} A \Leftrightarrow_{s,w} \varphi(x;u_1,u_2,\ldots,u_n)]$
Comprehension 2 $\forall u_1\forall u_2,\ldots\forall u_n\exists A\forall x[x \in_{s,w} A \Leftrightarrow_{s,w} \varphi(x,A;u_1,u_2,\ldots,u_n)]$
Comprehension 3 $\forall u_1\forall u_2,\ldots\forall u_n\exists a\forall x[x \in_s a \Leftrightarrow_s (a \subset u_1) \wedge \varphi(x,a;u_1,u_2,\ldots,u_n)]$
In particular:
Comprehension 3$'$ $\forall u\exists a\forall x[x \in_s a \Leftrightarrow_s (a \subset u) \wedge \varphi(x,a;u)]$
Hyperinfinity: see subsection 2.1.
Remark 2.7.Note that the axiom of hyper infinity follows from the schemata Comprehension 3.
Definition 2.3. The ordered pair of two sets u,v is defined as usual by

$$\langle u,v\rangle = \{\{u\},\{u,v\}\}. \tag{2.19}$$

Definition 2.4. We define the Cartesian product of two nonclassical sets A and B as usual by

$$A \times_{s,w} B = \{\langle x,y\rangle|x \in_{s,w} A \wedge y \in_{s,w} B\} \tag{2.20}$$

Definition 2.5. A binary relation between two nonclassical sets A,B is a subset $R \subseteq_{s,w} A \times_{s,w} B$. We also write $aR_{s,w}b$ for $< a,b >\in_{s,w} R$. The doman $\mathbf{dom}(R)$ and the range $\mathbf{ran}(R)$ of R are defined by

$$\mathbf{dom}(R) = \{x|\exists y(xR_{s,w}y)\},\mathbf{ran}(R) = \{y : \exists x(xR_{s,w}y)\}. \tag{2.21}$$

Definition 2.6.A relation $F_{s,w}$ is a function, or map, written $\mathbf{Fun}(F_{s,w})$, if for each $a \in_{s,w} \mathbf{dom}(F)$ there is a unique b for which $aF_{s,w}b$. This unique b is written $F(a)$ or Fa. We write $F_{s,w} : A \to B$ for the assertion that $F_{s,w}$ s a function with $\mathbf{dom}(F_{s,w}) = A$ and $\mathbf{ran}(F_{s,w}) = B$. In this case we write $a \mapsto F_{s,w}(a)$ for $F_{s,w}a$.
Definition 2.7.The identity map $\mathbf{1}_A$ on A is the map $A \to A$ given by $a \mapsto a$.

If $X \subseteq_{s,w} A,$ the map $x \mapsto x : X \to A$ is called the insertion map of X into A.

Definition 2.8. If $F_{s,w} : A \to B$ and $X \subseteq_{s,w} A,$ the restriction $F_{s,w}|X$ of $F_{s,w}$ to X is the map $X \to A$ given by $x \mapsto F_{s,w}(x)$. If $Y \subseteq_{s,w} B,$ the inverse image of Y under $F_{s,w}$ is the set

$$F_{s,w}^{-1}[Y] = \{x \in_{s,w} A : F_{s,w}(x) \in_{s,w} Y\}. \tag{2.22}$$

Given two functions $F_{s,w} : A \to B, G_{s,w} : B \to C,$ we define the composite function

$G_{s,w} \circ F_{s,w} : A \to C$ to be the function $a \mapsto G_{s,w}(F_{s,w}(a)).$ If $F_{s,w} : A \to A,$ we write $F_{s,w}^2$ for $F_{s,w} \circ F_{s,w}, F_{s,w}^3$ for $F_{s,w} \circ F_{s,w} \circ F_{s,w}$ etc.

Definition 2.9.A function $F_{s,w} : A \to B$ is said to be monic if for all $x,y \in_{s,w} A, F_{s,w}(x) = F_{s,w}(y)$ implies $x = y,$ epi if for any $b \in_{s,w} B$ there is $a \in_{s,w} A$ for which $b = F_{s,w}(a)$, and bijective, or a bijection, if it is both monic and epi. It is easily shown that

$F_{s,w}$ is bijective if and only if $F_{s,w}$ has an inverse, that is, a map $G_{s,w} : B \to A$ such that $F_{s,w} \circ G_{s,w} = \mathbf{1}_B$ and $G_{s,w} \circ F_{s,w} = \mathbf{1}_A.$

Definition 2.10. Two sets X and Y are said to be equipollent, and we write $X \approx_{s,w} Y,$ if there is a bijection between them.

Definition 2.11. Suppose we are given two sets I, A and an epi map $F_{s,w} : I \to A.$ Then $A = \{F_{s,w}(i)|i \in I\}$ and so, if, for each $i \in_{s,w} I,$ we write a_i for $F_{s,w}(i)$, then A can be presented in the form of an indexed set $\{a_i : i \in_{s,w} I\}$. If A is presented as an indexed set of sets $\{X_i | i \in_{s,w} I\},$ then we write $\bigcup_{i \in I} X_i$ and $\bigcap_{i \in I} X_i$ for $\cup A$ and $\cap A,$ respectively.

Definition 2.12. The projection maps $\pi_1 : A \times_{s,w} B \to A$ and $\pi_2 : A \times_{s,w} B \to B$ are defined to be the maps $< a,b > \mapsto$ a and $< a,b > \mapsto b$ respectively.

Definition 2.13. For sets $A,B,$ the exponential B^A is defined to be the set of all functions from A to B.

Axiom of nonregularity

Remind that a non-empty set u is called regular iff $\forall x[x \neq \varnothing \to (\exists y \in x)(x \cap y = \varnothing)].$ Let's investigate what it says: suppose there were a non-empty x such that $(\forall y \in x)(x \cap y \neq \varnothing).$ For any $z_1 \in x$ we would be able to get $z_2 \in z_1 \cap x.$ Since $z_2 \in x$ we would be able to get $z_3 \in z_2 \cap x.$ The process continues forever: $\ldots \in z_{n+1} \in z_n \ldots \in z_4 \in z_3 \in z_2 \in z_1 \in x.$Thus if we don't wish to rule out such an infinite regress we forced accept the following statement:

$$\exists x[x \neq \varnothing \to (\forall y \in x)(x \cap y \neq \varnothing)]. \tag{2.23}$$

Axiom of hyperinfinity.

Definition 2.14.(i) A non-empty transitive non regular set u is a well formed non regular set iff:

(i) there is unique countable sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$\ldots \in u_{n+1} \in u_n \ldots \in u_4 \in u_3 \in u_2 \in u_1 \in u, \tag{2.24}$$

(ii) for any $n \in \mathbb{N}$ and any $u_{n+1} \in u_n$:

$$u_n = u_{n+1}^+, \tag{2.25}$$

where $a^+ = a \cup \{a\}.$

(ii) we define a function $a^{+[k]}$ inductively by $a^{+[k+1]} = (a^{+[k]})^{+}$.

Definition 2.15. Let u and w are well formed non regular sets. We write $w \prec u$ iff for any $n \in \mathbb{N}$

$$w \in u_n. \tag{2.26}$$

Definition 2.16. We say that an well formed non regular set u is infinite (or hyperfinite) hypernatural number iff:

(I) For any member $w \in u$ one and only one of the following conditions are satified:

(i) $w \in \mathbb{N}$ or

(ii) $w = u_n$ for some $n \in \mathbb{N}$ or

(iii) $w \prec u$.

(II) Let ${}_{\prec}u$ be a set ${}_{\prec}u = \{z | z \prec u\}$, then by relation $(\bullet \prec \bullet)$ a set ${}_{\prec}u$ is densely ordered with no first element.

(III) $\mathbb{N} \subset u$.

Definition 2.17. Assume $u \in \mathbb{N}^{\#}$, then u is infinite (hypernatural) number if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$.

Axiom of hyperinfinity

There exists a set $\mathbb{N}^{\#}$ such that:

(i) $\mathbb{N} \subset \mathbb{N}^{\#}$,

(ii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number v such that $v \prec u$,

(iii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number w such that for any $n \in \mathbb{N} : u^{+[n]} \prec w$,

(iv) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\bullet \prec \bullet)$ with no first and no last element.

Axiom of existence the nonclassical truth predicate

Let A, B be a closed wff's of $\mathbf{NC}^{\#}_{\infty^{\#}}$ ($\mathbf{NC}^{\#}_{\infty^{\#}}$-sentences).There is truth predicate $\mathbf{T}^{\#}[A]$ satisfies the following $\mathbf{T}^{\#}$-schemas:

$$\begin{gathered}
1. \forall x \forall y \{\mathbf{T}^{\#}[x = y] \Leftrightarrow_{s,w} (x = y)\} \\
2. \forall x \forall y \{\mathbf{T}^{\#}[x \in y] \Leftrightarrow_{s,w} (x \in y)\} \\
3. \mathbf{T}^{\#}[\mathbf{T}^{\#}[A]] \Leftrightarrow_{s} \mathbf{T}^{\#}[A] \\
4. \mathbf{T}^{\#}[\neg\mathbf{T}^{\#}[A]] \Leftrightarrow_{s} \mathbf{T}^{\#}[\neg A] \\
5. \mathbf{T}^{\#}[\neg A] \Leftrightarrow_{s} \neg\mathbf{T}^{\#}[A] \\
6. \mathbf{T}^{\#}[\neg\neg A] \Leftrightarrow_{s} \mathbf{T}^{\#}[A] \\
7. \mathbf{T}^{\#}[A \wedge B] \Leftrightarrow_{s} \mathbf{T}^{\#}[A] \wedge \mathbf{T}^{\#}[B] \\
8. \mathbf{T}^{\#}[A \vee B] \Leftrightarrow_{s} \mathbf{T}^{\#}[A] \vee \mathbf{T}^{\#}[B]
\end{gathered} \tag{2.27}$$

and

$$9. \mathbf{T}^{\#}[A] \Leftrightarrow_{s,w} A. \tag{2.28}$$

Definition 2.18.(i) We say that a $\mathbf{NC}^{\#}_{\infty^{\#}}$-sentence is a s-$\mathbf{NC}^{\#}_{\infty^{\#}}$-sentence (strong $\mathbf{NC}^{\#}_{\infty^{\#}}$-sentence relative to $\vdash_{\mathbf{RMP}}$) if

$$\mathbf{T}^{\#}[A] \Leftrightarrow_{s} A. \tag{2.29}$$

(ii) We say that a $\mathbf{NC}^{\#}_{\infty^{\#}}$-sentence is a w-$\mathbf{NC}^{\#}_{\infty^{\#}}$-sentence (weak $\mathbf{NC}^{\#}_{\infty^{\#}}$-sentence relative to $\vdash_{\mathbf{RMP}}$) if

$$\mathbf{T}^{\#}[A] \Leftrightarrow_w A. \tag{2.30}$$

Notations 2.1.(i) We write $x =_s y$ if $\mathbf{T}^{\#}[x = y] \Leftrightarrow_s (x = y)$.
(ii) We write $x =_w y$ and if $\mathbf{T}^{\#}[x = y] \Leftrightarrow_w (x = y)$.
Notations 2.2.(i) We write $x \in_s y$ and will be say that a set if $\mathbf{T}^{\#}[x \in y] \Leftrightarrow_s (x \in y)$.
(ii) We write $x \in_w y$ and will be say that if $\mathbf{T}^{\#}[x \in y] \Leftrightarrow_w (x \in y)$.
Definition 2.19.(i) We will be say that a set y is a s-set if

$$\forall x[x \in y \Leftrightarrow_s x \in_s y] \quad (2.31)$$

(ii) We will be say that a set y is a w-set if

$$\forall x[x \in y \Leftrightarrow_s x \in_s y] \quad (2.32)$$

(iii) We will be say that a set y is a s,w-set if

$$\forall x[x \in y \Leftrightarrow_s x \in_{s,w} y] \quad (2.33)$$

Remark 2.8.For any model M in a first-order language, the definition of the truth predicate of M is the same - we define the elementary diagram of M as the set of all sentences with parameters from M that are true in M, using Tarski's recursive definition of truth, using the T schema. This is the same for a model of ZFC as for any other model in first-order logic. Symbolically

$$\mathbf{T}[A] \Leftrightarrow M \vDash A, \tag{2.34}$$

where $M \vDash A$ stands to A true in model M.
Remark 2.9. Remind that classical truth predicate $\mathbf{T}[A]$ unrestrictedly satisfies the following $\mathbf{T}$-schema [25-27]:

$$\mathbf{T}[A] \Leftrightarrow A, \tag{2.35}$$

i.e., the sentence $A \Leftrightarrow \mathbf{T}[A]$ is true for every sentence A of language L, where $\mathbf{T}[A]$ stands for "the sentence (denoted by) A is true". Unfortunately $\mathbf{T}$-schema incorrect by well known Curry's paradox.
Assume, too, that we have the principle called Assertion (also known as pseudo modus ponens): $(A \wedge (A \Rightarrow B)) \Rightarrow B.$ By diagonalization, self-reference we can get a sentence C such that $C \Leftrightarrow (\mathbf{T}[C] \Rightarrow F)$ where F is anything you like. (For effect, though, make F something obviously false, e.g. $F \equiv \perp \equiv 0 = 1$) By an instance of the $\mathbf{T}$-schema: $\mathbf{T}[C] \Leftrightarrow C$ we immediately get:$\mathbf{T}[C] \Leftrightarrow (\mathbf{T}[C] \Rightarrow F)$. Again, using the same
instance of the $\mathbf{T}$-schema, we can substitute $C[\mathbf{T},F]$ for $\mathbf{T}[C]$ in the above to get (1).
(1) $\vdash C[\mathbf{T},F] \Leftrightarrow (C[\mathbf{T},F] \Rightarrow F)$ [by $\mathbf{T}$-schema and substitution]
(2) $\vdash (C[\mathbf{T},F] \wedge (C[\mathbf{T},F] \Rightarrow F)) \Rightarrow F$ [by assertion]
(3) $\vdash (C[\mathbf{T},F] \wedge C[\mathbf{T},F]) \Rightarrow F$ [by substitution, from (2)]
(4) $\vdash C[\mathbf{T},F] \Rightarrow F$ [by equivalence of C and $C \wedge C$, from (3)]
(5) $\vdash C[\mathbf{T},F]$ [by unrestricted Modus Ponens, from (1) and (4)]
(6) $\vdash F$ [by unrestricted Modus Ponens, from (4) and (5)]
Letting F be anything entailing triviality Curry's paradox quickly 'shows' that the world is trivial.
Remark 2.10.Curry's paradox easily avoided by restricted MP such that:
1. $C[\mathbf{T},F] \Rightarrow F$, $(C[\mathbf{T},F] \Rightarrow F) \Rightarrow C[\mathbf{T},F] \nvdash_{\mathbf{RMP}} C[\mathbf{T},F]$ and
2. $C[\mathbf{T},F], C[\mathbf{T},F] \Rightarrow F \nvdash_{\mathbf{RMP}} F$,
Remark 2.11.The set of all T-sentences $T[\phi] \Leftrightarrow \phi$, where ϕ is any sentence of the

language L_T, that is, where ϕ may contain T, is inconsistent with PA (or any theory that proves the diagonal lemma) because of the Liar paradox [28].
In formal languages, self-reference is also very easy to come by. Any language capable of expressing some basic syntax can generate self-referential sentences via so-called diagonalization (or more properly, any language together with an appropriate theory of syntax or arithmetic). A language containing a truth predicate and this basic syntax will thus have a sentence L such that

$$L \Leftrightarrow \neg\mathbf{Tr}[L] \tag{2.36}$$

This is a 'fixed point' of (the compound predicate) $\neg Tr$, and is, in effect, our simple-untruth Liar.
Other conspicuous ingredients in common Liar paradoxes concern logical behavior of basic connectives or features of implication. A few of the relevant principles are:
Modus ponens (MP): $A, A \Rightarrow B \vdash B$
Excluded middle (LEM): $\vdash A \vee \neg A$
Explosion (EFQ): $A, \neg A \vdash B$
Disjunction principle (DP): If $A \vdash C$ and $B \vdash C$ then $A \vee B \vdash C$
Adjunction: If $A \vdash B$ and $A \vdash C$ then $A \vdash B \wedge C$.
An argument that Liar sentence L implies a contradiction runs as follows.
1. $\mathbf{Tr}[L] \vee \neg\mathbf{Tr}[L]$ [LEM]
2.Case One:
a $\mathbf{Tr}[L]$
b L [2a: release by MP from $\mathbf{T}$ schema (2.35)]
c $\neg\mathbf{Tr}[L]$ [2b: definition of L]
d $\neg\mathbf{Tr}[L] \wedge \mathbf{Tr}[L]$ [2a, 2c: adjunction]
Case Two:
a $\neg\mathbf{Tr}[L]$
b L [3a: definition of L by MP]
c $\mathbf{Tr}[L]$ [3b: by MP from $\mathbf{T}$ schema (2.35)]
d $\neg\mathbf{Tr}[L] \wedge \mathbf{Tr}[L]$ [3a, 3c: adjunction]
4. $\neg\mathbf{Tr}[L] \wedge \mathbf{Tr}[L]$ [1–3: DP]
Remark 2.12. Liar easily avoided by restricted MP such that:
1. $\mathbf{Tr}[L] \nvdash_{\mathbf{RMP}} L$
2. $L \nvdash_{\mathbf{RMP}} \mathbf{Tr}[L]$
3. $\neg\mathbf{Tr}[L] \nvdash_{\mathbf{RMP}} L$
4. $L \nvdash_{\mathbf{RMP}} \neg\mathbf{Tr}[L]$

§3.Nonconservative extension of the model theoretical NSA based on bivalent hyper Infinitary first-order logic ${}^2L^{\#}_{\infty^{\#}}$ with restricted canonical rules of conclusion.

Extending the classical real numbers $\mathbb{R}$ to include infinite and infinitesimal quantities originally enabled D. Laugwitz [1] to view the delta distribution $\delta(x)$ as a nonstandard point function. Independently A. Robinson [2] demonstrated that distributions could be viewed as generalized polynomials. Luxemburg [3] and Sloan [4] presented an alternate representative of distributions as internal functions within the context of canonical

Robinson's theory of nonstandard analysis. For further information on classical nonstandard real analysis, we refer to [8]-[11].

Abbreviation 3.1.In this paper we adopt the following notations. For a standard set E we often write $E_{\mathbf{st}}$. For a set $E_{\mathbf{st}}$ let ${}^{\sigma}E_{\mathbf{st}}$ be a set ${}^{\sigma}E_{\mathbf{st}} = \{{}^{*}x|x \in E_{\mathbf{st}}\}$. We identify z with ${}^{\sigma}z$ i.e., $z \equiv^{\sigma} z$ for all $z \in \mathbb{C}$. Hence, ${}^{\sigma}E_{\mathbf{st}} = E_{\mathbf{st}}$ if $E \subseteq \mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C} = \mathbb{C}$, ${}^{\sigma}\mathbb{R} = \mathbb{R}$, ${}^{\sigma}P = P$, ${}^{\sigma}L_{\uparrow}^{+} = L_{\uparrow}^{+}$, etc. Let ${}^{*}\mathbb{R}_{\approx}$, ${}^{*}\mathbb{R}_{\approx,+}$, ${}^{*}\mathbb{R}_{\mathbf{fin}}$,${}^{*}\mathbb{R}_{\infty}$, and ${}^{*}\mathbb{N}_{\infty}$ denote the sets of infinitesimal hyper-real numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers and infinite hyper natural numbers, respectively. Note that ${}^{*}\mathbb{R}_{\mathbf{fin}} = {}^{*}\mathbb{R}/{}^{*}\mathbb{R}_{\infty}$, ${}^{*}\mathbb{C} = {}^{*}\mathbb{R} + i{}^{*}\mathbb{R}$, ${}^{*}\mathbb{C}_{\mathbf{fin}} = {}^{*}\mathbb{R}_{\mathbf{fin}} + i{}^{*}\mathbb{R}_{\mathbf{fin}}$.

Remind that Robinson nonstandard analysis (RNA) many developed using set-theoretical objects called superstructures [8]-[11]. A superstructure $\mathbf{V}(S)$ over a set S is defined in the following way

$$\mathbf{V}_0(S) = S, \mathbf{V}_{n+1}(S) = \mathbf{V}_n(S) \cup (P(\mathbf{V}_n(S)), \mathbf{V}(S) = \bigcup_{n \in \mathbb{N}} \mathbf{V}_n(S). \tag{3.1}$$

Superstructures of the empty set consist of sets of infinite rank in the cumulative hierarchy and therefore do not satisfy the in…nity axiom. Making $S = \mathbb{R}$ will suffice for virtually any construction necessary in analysis.

Bounded formulas are formulas where all quantifiers occur in the form

$$\forall x(x \in y \Rightarrow \cdots), \exists x(x \in y \Rightarrow \cdots). \tag{3.2}$$

A nonstandard embedding is a mapping

$$*\colon \mathbf{V}(X) \to \mathbf{V}(Y)$$

from a superstructure $\mathbf{V}(X)$ called the standard universum, into another superstructure

$\mathbf{V}(Y)$, called nonstandard universum, satisfying the following postulates:

1. $Y = {}^{*}X$

2.**Transfer Principle**.For every bounded formula $\Phi(x_1,\ldots,x_n)$ and elements $a_1,\ldots,a_n \in \mathbf{V}(X)$, the property Φ is true for $a_1,\ldots,a_n$ in the standard universum if and only if it is true for ${}^{*}a_1,\ldots,{}^{*}a_n$ in the nonstandard universum:

$$\langle \mathbf{V}(X), \in \rangle \models \Phi(a_1,\ldots,a_n) \Leftrightarrow \langle \mathbf{V}(Y), \in \rangle \models \Phi({}^{*}a_1,\ldots,{}^{*}a_n).$$

3.**Non-triviality**.For every infinite set A in the standard universum, the set $\{{}^{*}a|a \in A\}$ is a proper subset of ${}^{*}A$.

Definition 3.1.[10].A set x is <u>internal</u> if and only if x is an element of ${}^{*}A$ for some element A of $\mathbf{V}(\mathbb{R})$. Let X be a set with $A = \{A_i\}_{i \in I}$ a family of subsets of X. Then the collection A has the infinite intersection property, if any infinite subcollection $J \subset I$ has non-empty intersection. Nonstandard universum is κ-saturated if whenever $\{A_i\}_{i \in I}$is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to κ, $\bigcap_{i \in I} A_i \neq \varnothing$.

Remark 3.1.Remind that: (i) for each standard universum $U = \mathbf{V}(X)$ there exists canonical language $\mathcal{L} = \mathcal{L}_U$,(ii) for each nonstandard universum $W = \mathbf{V}(Y)$ there exists corresponding canonical nonstandard language ${}^{*}\mathcal{L} = \mathcal{L}_W$ [10].

3*.**The restricted rules of conclusion**.

If $W \models A$ then $\neg A \nvdash B$,where $B \in \mathcal{L} \wedge B \in {}^{*}\mathcal{L}$.

Thus if A holds in W we cannot obtain from $\neg A$ any formula B whatsoever.

Remark 3.2. We write $* \models A$ instead $W \models A$.

In this paper we apply the following hyper inductive definitions of a sets [18]

$$\exists S \forall \beta(\beta \in {}^*\mathbb{N}) \left[\beta \in S \Leftrightarrow_s \bigwedge_{0 \leq \alpha < \beta} (\alpha \in S \Rightarrow_s \alpha^+ \in S) \right].$$

Definition 3.2.[18].A set $S \subset {}^*\mathbb{N}$is a <u>hyper inductive</u> if the following statement holds

$$\bigwedge_{\alpha \in {}^*\mathbb{N}} (\alpha \in S \Rightarrow_s \alpha^+ \in S), \tag{3.3}$$

where $\alpha^+ \triangleq \alpha + 1$.Obviously a set ${}^*\mathbb{N}$ is a hyper inductive.As we see later there is just one hyper inductive subset of ${}^*\mathbb{N}$,namely ${}^*\mathbb{N}$itself.

We extend up Robinson nonstandard analysis (**RNA**) by adding the following postulate:

4.**Any hyper inductive set** S **is internal**.

Remark 3.3.The statement **4** is not provable in ZFC but provable in set theory $\mathbf{NC}^{\#}_{\infty}$, see [2]-[3].Thus postulates 1-4 gives an nonconservative extension of RNA and we denote such extension by **NERNA**.

Remark 3.4.Note that NERNA of course based on the same gyper infinitary logic with Restricted Modus Ponens Rule as set theory $\mathbf{NC}^{\#}_{\infty}$ [1]-[3].

Remind that in RNA the following induction principle holds.

Theorem 3.1.[6]. Assume that $S \subset {}^*\mathbb{N}$ is internal set, then

$$(1 \in S) \wedge \forall x[x \in S \Rightarrow x + 1] \Rightarrow S = {}^*\mathbb{N}. \tag{3.4}$$

In NERNA Theorem 1.1also holds.

Remark 3.5.It follows from postulate 4 and Theorem 1.1 that any hyper inductive set S is equivalent to ${}^*\mathbb{N}$: $S \equiv {}^*\mathbb{N}$.

Remark 3.6. Note that the following statements are provable in $\mathbf{NC}^{\#}_{\infty}$ [2]-[3]:

5 Axiom of ω-induction

$$\forall S(S \subset_s \mathbb{N}) \left\{ \forall \beta(\beta \in_s \mathbb{N}) \left[\bigwedge_{0 \leq \alpha < \beta} (\alpha \in_s S \Rightarrow_s \alpha^+ \in_s S) \right] \Rightarrow_s S = \mathbb{N} \right\}. \tag{3.5}$$

6 Axiom of hyper infinite induction

$$\forall S(S \subset {}^*\mathbb{N}) \left\{ \forall \beta(\beta \in {}^*\mathbb{N}) \left[\bigwedge_{0 \leq \alpha < \beta} (\alpha \in S \Rightarrow_s \alpha^+ \in S) \right] \Rightarrow_s S = {}^*\mathbb{N} \right\}. \tag{3.6}$$

Thus postulate **5** of the theory NERNA is provable in $\mathbf{NC}^{\#}_{\infty^{\#}}$.

Rules of conclusion

(1) Restricted Modus Ponens Rule (denoted by $\vdash_{\mathbf{RMP}}$) the same as in set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$.

(2) Restricted Modus Tollens Rule (denoted by $\vdash_{\mathbf{RMT}}$) the same as in set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$.

(3) MRR1 (1.Main Restricted rule of conclusion)

Let $\varphi(x)$ be a wff with one free variable x and such that $\exists \bar{n}(\bar{n} \in {}^*\mathbb{N}\backslash\mathbb{N}) \wedge \mathbf{V}(Y) \models \varphi(\bar{n})$, then for all $n \geq \bar{n}$: $\neg\varphi(n) \nvdash_{\mathbf{RMP}} B$,i.e., if statement $\varphi(\bar{n})$ holds in $\mathbf{V}(Y)$ we cannot obtain from $\neg\varphi(n)$,with $n \geq \bar{n}$ any formula B whatsoever.

(4) MRR2 (2.Main Restricted rule of conclusion)

Let $\varphi(x)$ be a wff with one free variable x and such that $\exists \bar{n}(\bar{n} \in {}^*\mathbb{N}) \wedge \mathbf{V}(Y) \models \varphi(\bar{n})$, then for all $n \geq \bar{n}$: $\neg\varphi(n) \nvdash_{\mathbf{RMP}} B$,i.e., if statement $\varphi(\bar{n})$ holds in $\mathbf{V}(Y)$ we cannot obtain from $\neg\varphi(n)$,with $n \geq \bar{n}$ any formula B whatsoever.

Remark 3.5.The MRR1,2 is necessarily in natural way, since by assumption $\neg\varphi(n)$ one obtains directly the apparent contradiction $\varphi(n) \wedge \neg\varphi(n)$ from which by unrestricted modus ponens rule (UMPR) one obtains $\varphi(n) \wedge \neg\varphi(n) \vdash_{\mathbf{UMPR}} B$.

Example 3.1. Remind the proof of the following statement:

Theorem 3.2. The structure $(\mathbb{N},<)$ is a well-ordered set.

Proof.Let X be a nonempty subset of $\mathbb{N}$. Suppose X does not have a $<$ -least element. Then consider the set $\mathbb{N}\backslash X$.

Case (1) $\mathbb{N}\backslash X = \varnothing$. Then $X = \mathbb{N}$ and so 0 is a $<$ -least element. Contradiction.

Case (2) $\mathbb{N}\backslash X \neq \varnothing$.Then $1 \in \mathbb{N}\backslash X$ otherwise 1 is a $<$ -least element. Contradiction.

Case (3) $\mathbb{N}\backslash X \neq \varnothing$. Assume now that there exists an $n \in \mathbb{N}\backslash X$ such that $n \neq 1$.

Since we have supposed that X does not have a least element, thus $n+1 \notin X$.

Thus we see that for all $n : n \in \mathbb{N}\backslash X$ implies that $n+1 \in \mathbb{N}\backslash X$. We can conclude by induction that $n \in \mathbb{N}\backslash X$ for all $n \in \mathbb{N}$. Thus $\mathbb{N}\backslash X = \mathbb{N}$ implies $X = \varnothing$.

This is a contradiction to X being a nonempty subset of $\mathbb{N}$.

Remark 3.6.(i) The proof of the Theorem 3.2 is an example proof by a contradiction. Remind that a mathematical proof employing proof by contradiction usually proceeds as follows:

1.The proposition to be proved is P.

2.We assume P to be false, i.e., we assume $\models \neg P$.

3.It is then shown that $\neg P$ implies falsehood. This is typically accomplished by deriving two mutually contradictory assertions, Q and $\neg Q$, and appealing to the law of noncontradiction.

4.Since assuming P to be false leads to a contradiction, it is concluded that P is in fact true.

(ii) The statement of the Theorem 3.2 obviously is unprovable by a contradiction under MRR2. Note that in the Case (3) there is an $\bar{n} \neq 1, \bar{n} \in X$ and $\bar{n} \notin \mathbb{N}\backslash X$.Thus induction hypothesis $\models \bar{n} \in \mathbb{N}\backslash X$ is not holds since $\bar{n} \notin \mathbb{N}\backslash X \wedge \bar{n} \in \mathbb{N}\backslash X$ is a contradiction and by MRR2

$$\bar{n} \in \mathbb{N}\backslash X \nvdash_{\mathbf{RMP}} \bar{n}+1 \in \mathbb{N}\backslash X.$$

(iii) Note that proof of the Theorem 3.2 mentioned above completely abnormal in fact even in point view of classical proof theory, since basic assuption $\bar{n} \notin \mathbb{N}\backslash X$ which is employed in proof by contradiction, contradicts with induction hypothesis $\models \bar{n} \in \mathbb{N}\backslash X$.

Example 3.2. (i) We set now $X_1 = {}^*\mathbb{N}\backslash\mathbb{N}$, thus ${}^*\mathbb{N}\backslash X_1 = \mathbb{N}$.In contrast with a set X mentioned in Example 3.1, the assumption $n \in {}^*\mathbb{N}\backslash X_1$ implies that $n+1 \in {}^*\mathbb{N}\backslash X_1$ if and only if n is finite, since for any infinite $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ the assumption $n \in {}^*\mathbb{N}\backslash X_1$ contradicts with a true statement $\mathbf{V}(Y) \models n \notin {}^*\mathbb{N}\backslash X_1 = \mathbb{N}$ and therefore in accordance with MRR we cannot obtain for any infinite n from formula $n \in {}^*\mathbb{N}\backslash X_1$ any formula B whatsoever.

Remark 3.7.Notice in order to prove an statement $G = \forall n(n \in {}^*\mathbb{N})P(n)$ <u>by induction</u> one needs to proof that: $P(n) \vdash_{\mathbf{RMP}} P(n+1)$,i.e. by assuming that $P(n)$ is true and then by RMP proving $P(n+1)$.Thus:

(i) any proof by hyperinfinite induction bused on additional assumption that

$$\nvdash_{\mathbf{RMP}} \exists\bar{n}(\bar{n} \in_s {}^*\mathbb{N})[\neg P(\bar{n})]. \tag{3.7}$$

(ii) any proof by ω-induction bused on additional assumption that

$$\nvdash_{\mathbf{RMP}} \exists \bar{n}(\bar{n} \in_s \mathbb{N})[\neg P(\bar{n})]. \tag{3.8}$$

Definition 3.3.κ is a natural number if $\kappa \in_s \mathbf{X}$ for every set $\mathbf{X}$ such that $0 \in_s \mathrm{X}$ and, for any λ, if $\lambda \in_s X$ then $\lambda + 1 \in_s X$, i.e. $\lambda \in_s X \vdash_{\mathbf{RMP}} \lambda + 1 \in_s X$.

We remind now some basic theorem and definitions related to classical naturals.

Definition 3.4. [20]. κ is a natural number if κ belongs to every set X such that $0 \in \mathrm{X}$ and, for any λ, if $\lambda \in X$ then $\lambda + 1 \in X$.

(As usual, $j, k, \ldots, n$ will denote natural numbers.)

Remark 3.8.[20].If the set of all natural numbers exists, we call it $\mathbb{N}$. But it is not necessary for us to assume now that $\mathbb{N}$ exists. The assumption that $\mathbb{N}$ exists is a form of what is called the Axiom of Infinity.

Proposition 3.1.[20] For any κ, $\{\lambda | \lambda \leq \kappa\}$ exists.

Proof. Let $\overline{\overline{\mathrm{A}}} = \kappa$. The desired set is $\left\{\overline{\overline{B}} | B \in_s \mathrm{P}(\mathrm{A})\right\}$, which exists by the Axiom of Replacement.

Theorem 3.3.[20].

(a) 0 is a natural number, $0 \leq k$

(b) If k is a natural number so is $k + 1$, if $k < \lambda$ then $k + 1 \leq \lambda$.

(c) (Induction) Suppose that $P(0)$ ('P holds for 0'); and that, for any natural number n, $P(n) \Rightarrow P(n+1)$ holds. Then for every n, $P(n)$ holds.

Proof [20] (a) and (b) are very easy. For (c), suppose that the whole hypothesis of (c) holds, but that, for some particular $\bar{n}, P(\bar{n})$ fails,i.e. $\neg P(\bar{n})$ holds.Put

$$X = \{m < \bar{n} | P(m)\} \tag{3.9}$$

X exists since $X = \{\lambda | \lambda < \eta$ and λ is a natural number and $P(\lambda)\}$, which exists by Proposition 3.1 and the Separation Axiom. Obviously, $0 \in X$ by Theorem 3.3 (a). It will be enough to show that: $\lambda + 1 \in X$ whenever $\lambda \in X$ - as then X is 'an X' as in Definition 3.3, so, by Definition 3.3, the natural number $\bar{n} \in X$, and so $P(\bar{n})$ holds, a contradiction. Suppose then that $\lambda \in X$, so that $\lambda \leq \bar{n}, \lambda$ is a natural number, and $P(\lambda)$. By our hypothesis (in (c)), $P(\lambda + 1)$. By (b), $\lambda + 1$ is a natural number. Also, $\lambda < \bar{n}$, as $P(\bar{n})$ fails. Hence $\lambda + 1 \leq n$, by Theorem 3.3 (b). So $\lambda + 1 \in X$, as desired.

Remark 3.9.Note that proof of the Theorem 3.3 mentioned above completely abnormal sinse definition (3.9) incorrect. Correct definition reads

$$X = \{m < \bar{n} | \mathbf{T}[P(m)] \wedge P(m)\} \tag{3.10}$$

where $\mathbf{T}[A]$ is a truth predicate such that for any well formed closed formula A of ZFC [24]

$$\mathbf{T}[A] \Leftrightarrow A. \tag{3.11}$$

However as well known such truth predicate is not exists by Curry's paradox. Thus a set X is not exists in general case.

Definition 3.5. An element x is said to be a first element of the linearly ordered set A (with respect to the relation R) if xRy for all $y \in A$. On the other hand, if yRx for all y, then x is said to be a last element of A (with respect to R). Generally speaking, not every set has a first or last element; but if such an element exists, then it is uniquely determined.

Theorem 3.4.[20]-[22]. In a finite non-empty subset X of a linearly ordered set A there is a first element and a last element of X.

Proof. The proof is by induction on the number of elements of X. If X has only one element, then the theorem is obvious. Suppose that the theorem holds for subsets with n elements. Let $X = Y \cup \{a\}$ where $a \notin Y$ and Y has n elements. Let b_1 be the first and b_2 the last element of Y. Since A is linearly ordered, either a precedes b_1 or b_1 precedes a. That element which precedes the other is clearly the first element of Y. Similarly we show that one of the elements a and b_2 is the last element of X.

Corollary 3.1. Every finite subset A of $\mathbb{N}$ has a first element and also a last element.

Proof. From Theorem 3.4 by definitions.

Theorem 3.5.[20]. (a) (**The least element principle**). If for some $\bar{n}, P(\bar{n}),$ then there is a minimal (which is here the same as minimum) n such that $P(n)$.

(b) (**Course-of-values induction**). If, for any n,if $Q(m)$ holds for all $m < n$, then $Q(n);$ then, for all $n, Q(n)$.

Proof of (a). Suppose $P(\bar{n})$. If $(P(m)$ for no $m < \bar{n}$, then $\bar{n}$ is minimal as desired. Otherwise $\{k \in W(\bar{n}) | P(k)\}$ $(W(\bar{n}) = \{m | m < n\}$) is non-empty, and so, being finite, has a least element $m,$ by Corollary 3.1. It is easy to see that m is the least number with the property P, as desired.

Proof of (b). Assume the hypothesis of (b) holds and that, for some $n,$ $Q(n).$ fails. By (a) let k be the least such $n.$ Thus $Q(m)$ holds for all $m < k,$ so by our hypothesis, $Q(k)$ holds, a contradiction.

Theorem 3.6.(s-Induction) Let $P(x)$ be wff of $\mathbf{NC}^{\#}_{\infty}$ with a free variable $x.$ Suppose that

$$\mathbf{T}^{\#}[P(0)] \wedge \mathbf{T}^{\#}[P(0)] \tag{3.12}$$

('P holds for 0');$\mathbf{T}^{\#}[P(0)] \Leftrightarrow_s P(0),$and that, for any natural number n,

$$P(n) \Rightarrow_s P(n+1) \tag{3.13}$$

and for every n,

$$\mathbf{T}^{\#}[P(n)] \Leftrightarrow_s P(n), \tag{3.14}$$

i.e.or every given $n, P(n)$ is s-sentence.Then for every $n \in \mathbb{N},$ $P(n)$ holds, i.e. $\forall n\{\mathbf{T}^{\#}[P(n)] \Leftrightarrow_s P(n)\}.$

Proof. Suppose that the whole hypothesis mentioned above holds, but that, for some particular $\bar{n}, P(\bar{n})$ fails,i.e. $\neg P(\bar{n})$ holds.Put

$$X = \{m < \bar{n} | \mathbf{T}^{\#}[P(m)] \wedge P(m)\} \tag{3.15}$$

X exists since $X = \{\lambda | \lambda < \eta$ and λ is a natural number and $\mathbf{T}^{\#}[P(m)] \wedge P(\lambda)\}$, which exists by Proposition 3.1 and the Separation Axiom. Obviously, $0 \in X$ by Theorem 3.3 (a). It will be enough to show that: $\lambda + 1 \in_s X$ whenever $\lambda \in_s X$ - as then X is 'an X' as in Definition 3.3, so, by Definition 3.3, the natural number $\bar{n} \in_s X,$ and so $P(\bar{n})$ holds, a contradiction. Suppose then that $\lambda \in_s X,$ so that $\lambda \leq \bar{n}, \lambda$ is a natural number, and $P(\lambda)$.By our hypothesis (in (3.13)), $P(\lambda+1)$. By (b), $\lambda + 1$ is a natural number. Also, $\lambda < \bar{n}$, as $P(\bar{n})$ fails. Hence $\lambda + 1 \leq n$, by Theorem 3.3 (b). So $\lambda + 1 \in_s X$, as desired.

Theorem 3.7.[23] Any finite nonempty subset X of $\mathbb{N}$ has minimal and maximal members.

Proof [23].Let X_n consist of $x_1, \ldots, x_n.$ Define $m_1 = x_1$ and m_k as x_k if $x_k < m_{k-1}$ and

m_{k-1} otherwise. Then m_n will be minimal. Similarly, X has a maximal element.
Remark 3.7. This proof in fact based on assumption (the induction hypothesis) that the theorem holds for X_{k-1} consist of $x_1,\ldots,x_{k-1}$,i.e. $m_{k-1} = \min\{x_1,\ldots,x_{k-1}\}$,then it follows $m_{k-1} = \min\{x_1,\ldots,x_{k-1}\} \Rightarrow m_k = \min\{x_1,\ldots,x_k\}$ and by induction we conclude that for all $n \in \mathbb{N}, m_n = \min\{x_1,\ldots,x_n\}$.
Definition 3.6. An element x is said to be a first element of the linearly s-ordered set A (with respect to the s-relation R) if xRy for all $y \in_s A$. On the other hand, if yRx for all $y \in_s A$, then x is said to be a last element of A (with respect to R). Generally speaking, not every set has a first or last element; but if such an element exists, then it is uniquely determined.
Abbreviation 3.2 Let $X_n(A), \overline{\overline{X}}_n(A) = n$ be s-finite non-empty subset of a linearly s-ordered set A suth that there is a first element and a last element of X_n.We shall abbreviated: $[X_n(A), \left(\overline{\overline{X}}_n(A) = n\right)$ is a s-finite non-empty subset of a linearly s-ordered set A suth that there is a first element and a last element of $X_n(A)] \Leftrightarrow \widehat{X}_n(A)$.
Under assumption

$$\nvdash_{\mathbf{RMP}} \exists m(m \in_s \mathbb{N})\exists X_m(A)\left[\neg\widehat{X}_m(A)\right]. \tag{3.16}$$

by axiom of ω-induction we obtain

$$\forall X_n(A)\left[\bigwedge_{n\in\mathbb{N}}\left(\widehat{X}_n(A) \Rightarrow_s \widehat{X}_{n+1}(A)\right)\right] \Rightarrow_s \forall n\forall X_n(A)\left[\widehat{X}_n(A)\right]. \tag{3.17}$$

In particular for $A = \mathbb{N}$ under assumption

$$\nvdash_{\mathbf{RMP}} \exists m(m \in_s \mathbb{N})\exists X_m(\mathbb{N})\left[\neg\widehat{X}_m(\mathbb{N})\right]. \tag{3.18}$$

by axiom of ω-induction we obtain

$$\forall X_n(\mathbb{N})\left[\bigwedge_{n\in\mathbb{N}}\left(\widehat{X}_n(\mathbb{N}) \Rightarrow_s \widehat{X}_{n+1}(\mathbb{N})\right)\right] \Rightarrow_s \forall n\forall X_n(A)\left[\widehat{X}_n(\mathbb{N})\right]. \tag{3.19}$$

7 Axiom of existence non well-ordered s-finite subset of $\mathbb{N}$.

$$\exists m(m \in_s \mathbb{N})\exists X_m(\mathbb{N})\left[\neg\widehat{X}_m(\mathbb{N})\right]. \tag{3.20}$$

§4.Internal Set Theory IST.

The axiomatics IST (Internal Set Theory) was presented in 1977 [19] and in a sense formulates within first-order language the behaviour of standard and internal sets of a nonstandard model of ZFC. This were done by adding the unary standardness predicate "st" to the language of ZFC as well as adding to the axioms of ZFC three new axiom schemes involving the predicate "st": **Idealization**, **Standardization** and **Transfer**.
Remark 4.1. Formulas which do not use the predicate **st** are called internal formulas (or $\in$-formulas) and formulas that use this new predicate are called external formulas (or st-$\in$-formulas).A formula φ is standard if only standard constants occur in φ.
Abbreviaion 4.1. We denote a set of the all naturals by $\mathbb{N}^{\#}$and a set of the all finite naturals by $\mathbb{N}$.
Abbreviaion 4.2. We write $\mathbf{fin}(x)$ meaning 'x is finite'. Let $\varphi(x)$ be a **st**- $\in$ -formula:
1.$\forall^{\mathbf{st}} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow \varphi(x))$. 2.$\exists^{\mathbf{st}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \varphi(x))$.
3.$\forall^{\mathbf{fin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{fin}(x)) \Rightarrow \varphi(x))$. 4.$\exists^{\mathbf{fin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{fin}(x) \wedge \varphi(x))$.

5.$\forall^{\mathbf{stfin}}\, x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \wedge \mathbf{fin}(x)) \;\Rightarrow \varphi(x))$.
6.$\exists^{\mathbf{stfin}}\, x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \mathbf{fin}(x) \wedge \varphi(x))$.
The fundamental axioms of $\mathbf{IST}$:
(**I**) **Idealization**

$$\forall^{\mathbf{stfin}} F\exists y\forall x \in F[R(x,y) \Leftrightarrow \exists b\forall^{\mathbf{st}} xR(x,b)] \tag{4.1}$$

for any internal relation R.
Remark 4.2.The idealization axiom obviously states that saying that for any fixed finite set F there is a y such that $R(x,y)$ holds for all $x \in F$ is the same as saying that there is a b such that for all fixed x the relation $R(x,b)$ holds.
(**II**) **Standardization**

$$\forall^{\mathbf{st}} A\exists^{\mathbf{st}} B\forall^{\mathbf{st}} x(x \in B \Leftrightarrow x \in A \wedge \varphi(x)) \tag{4.2}$$

for every st-$\in$-formula φ with arbitrary (internal) parameters.
(**III**) **Transfer**

$$\forall^{\mathbf{st}} y_1,\ldots,y_n\forall^{\mathbf{st}} x[\varphi(x,y_1,\ldots,y_n)] \Rightarrow \forall x\varphi(x,y_1,\ldots,y_n) \tag{4.3}$$

for all internal $\varphi(x,y_1,\ldots,y_n)$.
Remark 5.3. An importent consequence of (**I**) is the principle of **External Induction**, which states that for any (external or internal) formula φ,one has

$$\varphi(0) \wedge [\forall^{\mathbf{st}} n(\varphi(n) \Rightarrow \varphi(n+1))] \Rightarrow \forall^{\mathbf{st}} n\varphi(n). \tag{4.4}$$

Boundedness

$$\forall x\exists^{\mathbf{st}} y(x \in y) \tag{4.5}$$

and since (2.5) contradicts idealization the following (bounded) form is taken instead:
(**IV**) **Bounded Idealization**
For every $\in$-formula R :

$$\forall^{\mathbf{st}} Y[\forall^{\mathbf{stfin}} F\exists y \in Y(\forall x \in FR(x,y) \Leftrightarrow \exists b(b \in Y)\forall^{\mathbf{st}} xR(x,b))]. \tag{4.6}$$

This gives a subsystem BST, which corresponds to the bounded sets of IST.

§5.Internal Set Theory IST$^{\#}$

The axiomatics $\mathbf{IST}^{\#}$ formulates within infinitary first-order language the behaviour of standard and internal sets of a nonstandard model of $\mathbf{NC}^{\#}_{\infty^{\#}}$. This done by adding the unary standardness predicate "st" to the language of $\mathbf{NC}^{\#}_{\infty^{\#}}$ as well as adding to the axioms of $\mathbf{NC}^{\#}_{\infty^{\#}}$ three new axiom schemes involving the predicate "st": **Idealization**, **Standardization**,**Transfer** and **Axiom of internal hyper infinite induction**.
Remark 5.1.Formulas which do not use the predicate $\mathbf{st}$ are called internal formulas (or $\in_{sw}$ -formulas) and formulas that use this new predicate are called external formulas (or st-$\in_{sw}$ -formulas).A formula φ is standard if only standard constants occur in φ.
Abbreviaion 5.1.We write $\mathbf{fin}(x)$ meaning 'x is finite'. Let $\varphi(x)$ be a $\mathbf{st}$- $\in_{sw}$ -formula:
1.$\forall^{\mathbf{st}}_{s}\, x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow_s \varphi(x))$.
2.$\forall^{\mathbf{st}}_{s,w}\, x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow_{s,w} \varphi(x))$.
3.$\exists^{\mathbf{st}}\, x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \varphi(x))$.
4.$\forall^{\mathbf{fin}}_{s} x\varphi(x)$ abbreviates $\forall x(\mathbf{fin}(x)) \;\Rightarrow_s \varphi(x))$.

5.$\forall_{s,w}^{\mathbf{fin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{fin}(x)) \Rightarrow_{s,w} \varphi(x))$.
6.$\exists^{\mathbf{fin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{fin}(x) \wedge \varphi(x))$.
7.$\forall_{s}^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \wedge \mathbf{fin}(x)) \Rightarrow_{s} \varphi(x))$.
8.$\forall_{s,w}^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \wedge \mathbf{fin}(x)) \Rightarrow_{s,w} \varphi(x))$.
9.$\exists^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \mathbf{fin}(x) \wedge \varphi(x))$.
The fundamental axioms of $\mathbf{IST}^{\#}$:
(I) Idealization for classical sets

$$\forall_{s}^{\mathbf{stfin}} F^{\mathbf{CL}} \exists y^{\mathbf{CL}} \forall x^{\mathbf{CL}} \in_{s} F[R^{\mathbf{CL}}(x,y) \Leftrightarrow_{s} \exists b^{\mathbf{CL}} \forall_{s}^{\mathbf{st}} x R^{\mathbf{CL}}(x,b)] \tag{5.1}$$

for any internal classical relation $R^{\mathbf{CL}}(x,y)$.
Remark 5.2.The idealization axiom obviously states that saying that for any fixed classical finite set F there is a classical y such that $R^{\mathbf{CL}}(x,y)$ holds for all classical $x \in_{s} F$ is the same as saying that there is a classical b such that for all fixed classical x the classical relation $R^{\mathbf{CL}}(x,b)$ holds.
(II) Standardization for classical sets

$$\forall^{\mathbf{st}} A^{\mathbf{CL}} \exists^{\mathbf{st}} B^{\mathbf{CL}} \forall^{\mathbf{st}} x^{\mathbf{CL}} (x \in B \Leftrightarrow_{s} x \in A \wedge \varphi(x)) \tag{5.2}$$

for every st-$\in$-formula φ with arbitrary (internal) parameters.
(III) Transfer for classical sets

$$\forall^{\mathbf{st}} y_1^{\mathbf{CL}},\ldots,y_n^{\mathbf{CL}} \forall^{\mathbf{st}} x^{\mathbf{CL}} [\varphi(x,y_1,\ldots,y_n)] \Rightarrow_{s} \forall x^{\mathbf{CL}} \varphi(x,y_1,\ldots,y_n) \tag{5.3}$$

for all internal $\varphi(x,y_1,\ldots,y_n)$.
Boundedness

$$\forall x^{\mathbf{CL}} \exists^{\mathbf{st}} y^{\mathbf{CL}} (x \in_{s} y) \tag{5.4}$$

and since (5.4) contradicts idealization the following (bounded) form is taken instead:
(IV) Bounded Idealization for classical sets
For every $\in$-formula R :

$$\forall^{\mathbf{st}} Y^{\mathbf{CL}} [\forall^{\mathbf{stfin}} F^{\mathbf{CL}} \exists y^{\mathbf{CL}} \in Y (\forall x^{\mathbf{CL}} (x \in F) R(x,y) \Leftrightarrow_{s} \exists b^{\mathbf{CL}} (b \in Y) \forall^{\mathbf{st}} x R(x,b))]. \tag{5.5}$$

(V) Idealization for nonclassical sets

$$\forall_{s,w}^{\mathbf{stfin}} F^{\mathbf{NCL}} \exists y^{\mathbf{NCL}} \forall x^{\mathbf{NCL}} \in_{s,w} F[R^{\mathbf{NCL}}(x,y) \Leftrightarrow_{s,w} \exists b^{\mathbf{NCL}} \forall_{s,w}^{\mathbf{st}} x R^{\mathbf{NCL}}(x,b)] \tag{5.6}$$

for any internal nonclassical relation $R^{\mathbf{NCL}}(x,y)$.
Remark 5.3.The idealization axiom obviously states that saying that for any fixed nonclassical finite set F there is a classical y such that $R^{\mathbf{NCL}}(x,y)$ holds for all classical $x \in_{s} F$ is the same as saying that there is a classical b such that for all fixed classical x the nonclassical relation $R^{\mathbf{NCL}}(x,b)$ holds.
(VI) Standardization for nonclassical sets

$$\forall_{s,w}^{\mathbf{st}} A^{\mathbf{NCL}} \exists^{\mathbf{st}} B^{\mathbf{NCL}} \forall_{s,w}^{\mathbf{st}} x^{\mathbf{NCL}} (x \in_{s,w} B \Leftrightarrow_{s,w} x \in_{s,w} A \wedge \varphi(x)) \tag{5.7}$$

for every st-$\in_{s,w}$ -formula φ with arbitrary (internal) parameters.
(VII) Transfer for nonclassical sets

$$\forall_{s,w}^{\mathbf{st}} y_1^{\mathbf{NCL}},\ldots,y_n^{\mathbf{NCL}} \forall^{\mathbf{st}} x^{\mathbf{NCL}} [\varphi(x,y_1,\ldots,y_n)] \Rightarrow_{s,w} \forall_{s,w} x^{\mathbf{NCL}} \varphi(x,y_1,\ldots,y_n) \tag{5.8}$$

for all internal $\varphi(x,y_1,\ldots,y_n)$.
Boundedness for nonclassical sets

$$\forall_{s,w} x^{\mathbf{NCL}} \exists^{\mathbf{st}} y^{\mathbf{NCL}} (x \in_{s,w} y) \tag{5.9}$$

and since (5.9) contradicts idealization the following (bounded) form is taken instead:

(**VIII**) **Bounded Idealization for nonclassical sets**
For every $\in_{s,w}$ -formula R :

$$\forall^{\mathbf{st}}_{s,w} Y^{\mathbf{NCL}}\Big[\forall^{\mathbf{stfin}}_{s,w} F^{\mathbf{NCL}} \exists y^{\mathbf{NCL}} \in_{s,w} Y(\forall_{s,w} x^{\mathbf{NCL}}(x \in F)R(x,y) \Leftrightarrow_{s,w} \exists b^{\mathbf{NCL}}(b \in Y)\forall^{\mathbf{st}}_{s,w} xR(x,b))\Big]. \quad (5.10)$$

(**IX**) **Internal Hyper Infinite Induction**

$$\forall S(S \subset_s \mathbb{N}^{\#})\left\{\forall \beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0 \leq \alpha < \beta}(\alpha \in_s S \Rightarrow_s \alpha^{+} \in_s S)\right] \Rightarrow_s S =_s \mathbb{N}^{\#}\right\}. \quad (5.11)$$

The main restricted rules of conclusion.
If $\mathbf{IST}^{\#} \vdash A$ then $\neg A \nvdash B,$ where $B \in \mathcal{L}^{\#}$.
Thus if statement A holds in $\mathbf{IST}^{\#}$ we cannot obtain from $\neg A$ any formula B whatsoever.

Abbreviation 5.2 Let $X_n(A), \overline{\overline{X}}_n(A) = n$ be a s-finite non-empty subset of a linearly s-ordered set A suth that there is a first element and a last element of X_n. We shall abbreviated: $[X_n(A), \left(\overline{\overline{X}}_n(A) = n\right)$ is a s-finite non-empty subset of a linearly s-ordered set A suth that there is a first element and a last element of $X_n(A)] \Leftrightarrow \widehat{X}_n(A)$.

(**X**) **Axiom of existence non well-ordered s-finite subset of** $\mathbb{N}$.

$$\exists m(m \in_s \mathbb{N})\exists X_m(\mathbb{N})\left[\neg \widehat{X}_m(\mathbb{N})\right]. \quad (5.12)$$

§6. Hypernaturals $\mathbb{N}^{\#}$. Axiom of hyperinfinity

Definition 6.1.(i) A non-empty transitive non regular set u is a well formed non regular set iff:
(i) there is unique countable sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$\ldots \in u_{n+1} \in u_n \ldots \in u_4 \in u_3 \in u_2 \in u_1 \in u, \quad (6.1)$$

(ii) for any $n \in \mathbb{N}$ and any $u_{n+1} \in u_n$:

$$u_n = u_{n+1}^{+}, \quad (6.2)$$

where $a^{+} = a \cup \{a\}$.
(ii) we define a function $a^{+[k]}$ inductively by $a^{+[k+1]} = (a^{+[k]})^{+}$.

Definition 6.2. Let u and w are well formed non regular sets. We write $w \prec u$ iff for any $n \in \mathbb{N}$

$$w \in u_n. \quad (6.3)$$

Definition 6.3. We say that an well formed non regular set u is infinite (or hyperfinite) hypernatural number iff:
(I) For any member $w \in u$ one and only one of the following conditions are satified:
(i) $w \in \mathbb{N}$ or
(ii) $w = u_n$ for some $n \in \mathbb{N}$ or
(iii) $w \prec u$.
(II) Let ${}_{\prec}u$ be a set ${}_{\prec}u = \{z | z \prec u\}$, then by relation $(\cdot \prec \cdot)$ a set ${}_{\prec}u$ is densely ordered with no first element.

(III) $\mathbb{N} \subset u$.

Definition 6.4. Assume $u \in \mathbb{N}^{\#}$, then u is infinite (hypernatural) number if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$.

Axiom of hyperinfinity

There exists a set $\mathbb{N}^{\#}$ such that:

(i) $\mathbb{N} \subset \mathbb{N}^{\#}$,

(ii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number v such that $v \prec u$,

(iii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number w such that for any $n \in \mathbb{N} : u^{+[n]} \prec w$,

(iv) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\bullet \prec \bullet)$ with no first and no last element.

§7.Axioms of the nonstandard arithmetic $\mathbf{A}^{\#}$.

Axioms of the nonstandard arithmetic $\mathbf{A}^{\#}$ are:

Axiom of hyperinfinity

There exists a set $\mathbb{N}^{\#}$ such that:

(i) $\mathbb{N} \subset \mathbb{N}^{\#}$

(ii) if u is infinite (hypernatural) number then there exists infinite (hypernatural) number v such that $v \prec u$

(iii) if u is infinite hypernatural number then there exists infinite (hypernatural) number w such that $u \prec w$

(iv) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\bullet \prec \bullet)$ with no first and no last element.

Axioms of infite ω-induction

(i)

$$\forall S(S \subset \mathbb{N})\left\{\left[\bigwedge_{n \in \omega}(n \in S \Rightarrow_s n^{+} \in S)\right] \Rightarrow_s S = \mathbb{N}\right\}. \tag{7.1}$$

(ii) Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$, then

$$\left[\bigwedge_{n \in \omega}(F(n) \Rightarrow_s F(n^{+}))\right] \Rightarrow_s \forall n(n \in \omega)F(n). \tag{7.2}$$

Definition 7.1.(i) Let β be a hypernatural such that $\beta \in \mathbb{N}^{\#} \backslash \mathbb{N}$. Let $[0,\beta] \subset \mathbb{N}^{\#}$ be a set such that $\forall x[x \in [0,\beta] \Leftrightarrow 0 \leq x \leq \beta]$ and let $[0,\beta)$ be a set $[0,\beta) = [0,\beta] \backslash \{\beta\}$.

(ii) Let $\beta \in \mathbb{N}^{\#} \backslash \mathbb{N}$ and let $\beta_{\infty} \subset \mathbb{N}^{\#}$be a set such that

$$\forall x\{x \in \beta_{\infty} \Leftrightarrow \exists k(k \geq 0)[0 \leq x \leq \beta^{+[k]}]\}. \tag{7.3}$$

Definition 7.2.Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is restricted on a classical set S such that $S \subsetneqq_s \mathbb{N}^{\#}$ iff the following condition is satisfied

$$\forall \alpha[\alpha \in \mathbb{N}^{\#} \backslash S \Rightarrow_s \neg F(\alpha)]. \tag{7.4}$$

Definition 7.3.Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is <u>strictly</u> restricted on a set S such that $S \subsetneqq_s \mathbb{N}^{\#}$ iff there is no proper subset $S' \subset S$ such that a wff $F(x)$ is restricted on a set S'.

Example 7.1.(i)Let $\mathbf{fin}(\alpha), \alpha \in \mathbb{N}^{\#}$be a wff formula such that $\mathbf{fin}(\alpha) \Leftrightarrow_s \alpha \in \mathbb{N}$. Obviously wff $\mathbf{fin}(\alpha)$ is strictly restricted on a set $\mathbb{N}$ since $\forall \alpha[\alpha \in \mathbb{N}^{\#} \backslash \mathbb{N} \Rightarrow_s \neg\mathbf{fin}(\alpha)]$.

Let $\mathbf{hfin}(\alpha), \alpha \in \mathbb{N}^{\#}$be a wff formula such that $\mathbf{hfin}(\alpha) \Leftrightarrow_s \alpha \in \mathbb{N}^{\#}\backslash\mathbb{N}$ since $\forall\alpha[\alpha \in \mathbb{N} \Rightarrow_s \neg\mathbf{hfin}(\alpha)]$.

Definition 7.4. Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is unrestricted if wff $F(x)$ is not restricted on any set S such that $S \subsetneqq \mathbb{N}^{\#}$.

Axiom of hyperfinite induction 1

$$\forall S(S \subseteq_s [0,\beta])\forall\beta(\beta \in_s \mathbb{N}^{\#}) \searrow$$
$$\left\{\forall\alpha(\alpha \in_s [0,\beta))\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in_s S \Rightarrow \alpha^{+} \in_s S)\right] \Rightarrow_s S = [0,\beta]\right\}. \tag{7.5}$$

Axiom of hyperfinite induction 1$'$

$$\forall S(S \subseteq_s [0,\beta_\infty])\forall\beta(\beta \in \mathbb{N}^{\#}) \searrow$$
$$\left\{\forall\alpha(\alpha \in [0,\beta_\infty])\left[\bigwedge_{0\leq\alpha<\beta_\infty}(\alpha \in S \Rightarrow \alpha^{+} \in S)\right] \Rightarrow S = [0,\beta_\infty]\right\}. \tag{7.6}$$

Axiom of hyper infinite induction 1

$$\forall S(S \subset_s \mathbb{N}^{\#})\left\{\forall\beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in_s S \Rightarrow \alpha^{+} \in_s S)\right] \Rightarrow_s S =_s \mathbb{N}^{\#}\right\}. \tag{7.7}$$

Definition 7.5.A set $S \subset_s \mathbb{N}^{\#}$is a hyper inductive if the following statement holds

$$\bigwedge_{\alpha\in\mathbb{N}^{\#}}(\alpha \in_s S \Rightarrow_s \alpha^{+} \in_s S). \tag{7.8}$$

Obviously a set $\mathbb{N}^{\#}$ is a hyper inductive. Thus axiom of hyper infinite induction 1 asserts that a set $\mathbb{N}^{\#}$ this is the smallest hyper inductive set.

Axioms of hyperfinite induction 2

Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta]$ then

$$\left[\forall\beta(\beta \in [0,\beta])\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha) \Rightarrow_s F(\alpha^{+}))\right]\right] \Rightarrow_s \forall\alpha(\alpha \in [0,\beta])F(\alpha). \tag{7.9}$$

Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta_\infty]$ then

$$\left[\forall\beta(\beta \in [0,\beta_\infty])\left[\bigwedge_{0\leq\alpha<\beta_\infty}(F(\alpha) \Rightarrow_s F(\alpha^{+}))\right]\right] \Rightarrow_s \forall\alpha(\alpha \in [0,\beta_\infty])F(\alpha). \tag{7.10}$$

Axiom of hyper infinite induction 2

Let $F(x)$ be anrestricted wff of the set theory $\mathrm{NC}^{\#}_{\infty^{\#}}$ then

$$\left[\forall\beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha) \Rightarrow_s F(\alpha^{+}))\right]\right] \Rightarrow_s \forall\beta(\beta \in \mathbb{N}^{\#})F(\beta). \tag{7.11}$$

The main restricted rules of conclusion.

If $\mathbf{A}^{\#} \vdash A$ then $\neg A \nvdash B$, where $B \in \mathcal{L}^{\#}$.

Thus if statement A holds in $\mathbf{A}^{\#}$ we cannot obtain from $\neg A$ any formula B whatsoever.

§8.The Generalized Recursion Theorem.

Theorem 1. Let S be a set, $c \in S$ and $G : S \to S$ is any function with $\mathbf{dom}(G) = S$ and $\mathbf{range}(G) \subseteq S$. Let $W[G] \in \mathbb{N}^{\#} \times S$ be a binary relation such that:

(a) $(1,c) \in W[G]$ and

(b) if $(x,y) \in W[G]$ then $(\mathbf{Sc}(x), G(y)) \in W[G]$.
Then there exists a function $\mathcal{F} : \mathbb{N}^{\#} \to S$ such that:
(i) $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$and $\mathbf{range}(\mathcal{F}) \subseteqq S$;
(ii) $\mathcal{F}(1) = c$;
(iii) for all $x \in \mathbb{N}^{\#}, \mathcal{F}(\mathbf{Sc}(x)) = G(\mathcal{F}(x))$.
1.The desired function $\mathcal{F}$ is a certain hyper inductive relation $\mathbf{W} \subseteqq \mathbb{N}^{\#} \times S$. It is to have the properties:
(ii′) $(1,c) \in \mathbf{W}$;
(iii′) if $(x,y) \in \mathbf{W}$ then $(\mathbf{Sc}(x), G(y)) \in \mathbf{W}$.
Remark 1. The latter is just another way of expressing (iii), that if

$$\mathcal{F}(x) = y \tag{1}$$

then

$$\mathcal{F}(\mathbf{Sc}(x)) = G(y). \tag{2}$$

Remark 2.Note that any relation $\mathbf{W}$ mentioned above is hyper inductive relation since the hyper inductivity conditions (ii′)-(iii′) are satisfied.
However there are many hyper inductive relations which satisfy the conditions (ii′)-(iii′); on such is $\mathbb{N}^{\#} \times S$. What distinguishes the desired function from all these other relations is that we want (a,b) to be on it only as required by (ii′) and (iii′). In other words, it is to be the smallest relation satisfying (ii′)-(iii′). This can be expressed precisely as follows:
(1) Let $\mathbf{M}$ be a set of the relations $\mathbf{W}$ satisfying the conditions (ii′) and (iii′);
then we define

$$\mathcal{F} = \bigcap_{\mathbf{W} \in \mathbf{M}} \mathbf{W}.$$

Hence
(2) whenever $\mathbf{W} \in \mathbf{M}$ then $\mathcal{F} \subseteq \mathbf{W}$.
We shall now show that we can derived from (1) that $\mathcal{F}$ is also one relation in $\mathbf{M}$.
(3) $(1,c) \in \mathcal{F}$.
This follows immediately from the definition of $\bigcap_{\mathbf{W} \in \mathbf{M}}$ and the fact that $(1,c) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$.
(4) If $(x,y) \in \mathcal{F}$ then $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$.
For if $(x,y) \in \mathcal{F}$ then $(x,y) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$; hence by (iii′) $(\mathbf{Sc}(x), G(y)) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$ so that $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$ by (1).
We must now verify that $\mathcal{F}$ ís actually a function, i,e., we wish to show that for any $x, z_1, z_2 \in \mathbb{N}^{\#}$, if $(x,z_1) \in \mathcal{F}$ and $(x,z_2) \in \mathcal{F}$, then $z_1 = z_2$.
We shall prove this by hyper infinite induction on x. Let
(5) $A = \{x | x \in \mathbb{N}^{\#}$ and for all $z_1, z_2 \in \mathbb{N}^{\#}$, if $(x,z_1) \in \mathcal{F}$ and $(x,z_2) \in \mathcal{F}$ then $z_1 = z_2\}$.
We shall show $A = \mathbb{N}^{\#}$ by applying hyper infinite induction. First we have
(6) $1 \in A$.
To prove (6), it suffices to show that for any z, if $(1,z) \in \mathcal{F}$ then $z = c$.
We prove this by contradiction; in other words, suppose to tbe contrary that there is some z with $(1,z) \in \mathcal{F}$ but $z \neq c$. Consider the relation $W = \mathcal{F} \setminus \{(1,z)\}$. Since $(1,c) \in \mathcal{F}$ and $(1,c) \neq (1,z)$, it follows that $(1,c) \in W$. Moreover, whenever $(u,y) \in W$

then $(u,y) \in \mathcal{F}$ and hence $(\mathbf{Sc}(u), G(y)) \in \mathcal{F}$ but $\mathbf{Sc}(u) \neq 1$, so $(\mathbf{Sc}(u), G(y)) \neq (1,z)$, and hence $(\mathbf{Sc}(u), G(y)) \in W$. Thus W satisfies both conditions (ii′) and (iii′); in other words, $\mathbf{W} \in \mathbf{M}$. But then it follows from (2) that $\mathcal{F} \subseteq \mathbf{W}$ however this is elearly false sinee $(1,z) \in \mathcal{F}$ and $(1,z) \notin \mathbf{W}$. Tbus our hypothesis has led us to a contradiction, and henee (6) is proved. Next we show that

(7) for any $x \in \mathbb{N}^{\#}$ if $x \in A$ then $\mathbf{Sc}(x) \in A$.

Suppose that $x \in A$, so that whenever $(x,z_1) \in \mathcal{F}$ and $(x,z_2) \in \mathcal{F}$ then $z_1 = z_2$. We must show that whenever $(\mathbf{Sc}(x), w_1) \in \mathcal{F}$and $(\mathbf{Sc}(x), w_2) \in \mathcal{F}$ then $w_1 = w_2$. To prove this, it suffices to show that

(8) whenever $(\mathbf{Sc}(x), w) \in \mathcal{F}$ then there exists a z with $w = G(z)$ and $(x,z) \in \mathcal{F}$.

For if (8) ia true, we would have for the given w_1, w_2 some $z_1 = z_2$ with $w_1 = G(z_1),\ w_2 = G(z_2),\ (x,z_1) \in \mathcal{F}$ and $(x,z_2) \in \mathcal{F}$. Then, since $x \in A, z_1 = z_2$ and henee $G(z_1) = G(z_2)$, that is, $w_1 = w_2$.

Now to prove (8) suppose, to the contrary, that it is not true; in other words, suppose that we have some w with $(\mathbf{Sc}(x), w) \in \mathcal{F}$ but such that for all z which $(x,z) \in \mathcal{F}$ we have $w \neq G(z)$. Consider the relation $\mathbf{W} = \mathcal{F} \backslash \{(\mathbf{Sc}(x), w)\}$.
We shall show that $\mathbf{W} \in \mathbf{M}$. First of all $(1,c) \in \mathcal{F}$and $(1,c) \neq (\mathbf{Sc}(x), w)$; hence $(1,c) \in \mathbf{W}$. Suppose tbat $(u,y) \in \mathbf{W}$; then $(u,y) \in \mathcal{F}$ and $(\mathbf{Sc}(u), G(y)) \in \mathcal{F}$.
Clearly if $u \neq x$ then $(\mathbf{Sc}(u), G(y)) \neq (\mathbf{Sc}(x), w)$,so that in this case $(Sc(u), G(y)) \in \mathbf{W}$.
On the other hand, if $u = x$ and $(\mathbf{Sc}(u), G(y)) = (\mathbf{Sc}(x), w)$, then $w = G(y)$, where $(x,y) \in \mathcal{F}$, contrary to the choice of w henee $(\mathbf{Sc}(u), G(y)) \neq (\mathbf{Sc}(x), w))$, so again $(\mathbf{Sc}(u), G(y)) \in \mathbf{W}$. Thus whenever $(u,y) \in \mathbf{W}$, also $(\mathbf{Sc}(u), G(y)) \in \mathbf{W}$. Now that we have shown $\mathbf{W} \in \mathbf{M}$ we see by (2) that $\mathcal{F} \subseteq \mathbf{W}$ but this is false since $(\mathbf{Sc}(x), w) \in \mathcal{F}$ and $(\mathbf{Sc}(x), w) \notin \mathbf{W}$. Thus our hypothesis that (8) is incorrect has led to a contradiction, and now (8) is proved. Sinee (7) follows from (8), we have by hyper infinite induction from (6) that $A = \mathbb{N}^{\#}$. Hence

(9) $\mathcal{F}$ is a function.

We have still to prove that $\mathcal{F}$ satisfies,condition (i); we must show that for each $x \in \mathbb{N}^{\#}$ there· is a y with $(x,y) \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathbb{N}^{\#} \times S$, it will then follow that $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$ and $\mathbf{range}(\mathcal{F}) \subseteq S$. Let $B = \mathbf{dom}(\mathcal{F})$, that is,

(10) $B = \{x | x \in \mathbb{N}^{\#}$ and for some $y, (x,y) \in \mathcal{F}\}$.

We prove now by hyper infinite induction that $B = \mathbb{N}^{\#}$. First, $1 \in B$, síPnce $(1,c) \in \mathcal{F}$ by (3). Next, if $x \in B$, pick some y with $(x,y) \in \mathcal{F}$; then by (4), $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$, and henee $\mathbf{Sc}(x) \in B$.

Thus concludes the first part of the proof, that there is at least one function $\mathcal{F}$ satisfying conditions (i)-(iii).

Part 2. We prove that there cannot be more than one such function.
Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ both satisfy the conditions (i)-(iii) we wish to show $\mathcal{F}_1 = \mathcal{F}_2$, i.e., that for all $x \in \mathbb{N}^{\#}, \mathcal{F}_1(x) = \mathcal{F}_2(x)$. Thus is proved by hyper infinite induction on X. By (ii), $\mathcal{F}_1(1) = c$ and $\mathcal{F}_2(1) = c$, so $\mathcal{F}_1(1) = \mathcal{F}_2(1)$. Suppose that $\mathcal{F}_1(x) = \mathcal{F}_2(x)$; then $\mathcal{F}_1(\mathbf{Sc}(x)) = G(\mathcal{F}_1(x))$ and $\mathcal{F}_2(\mathbf{Sc}(x)) = G(\mathcal{F}_2(x))$, so $\mathcal{F}_1(\mathbf{Sc}(x)) = \mathcal{F}_2(\mathbf{Sc}(x))$.

Theorem 2. Let S be a set, $c \in S$ and $G : S \times \mathbb{N}^{\#} \to S$ is a binary function with $\mathbf{dom}(G) = S \times \mathbb{N}^{\#}$ and $\mathbf{range}(G) \subseteqq S$.
Then there exists a function $\mathcal{F} : \mathbb{N}^{\#} \to S$ such that:
(i) $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$and $\mathbf{range}(\mathcal{F}) \subseteqq S$;

(ii) $\mathcal{F}(1) = c$;
(iii) for all $x \in \mathbb{N}^{\#}, \mathcal{F}(\mathbf{Sc}(x)) = G(\mathcal{F}(x), x)$.
We omit the proof of the Theorem 3.4.2 since it can be given by simple modification of the proof to Theorem 3.4.1.

§9.General associatíve and commutative laws.

Definition C.1. Suppose that S is a set on which a binary operation $+$ is defined and under which S is closed. Let $\{x_k\}_{k\in\mathbb{N}^{\#}}$ be any hyper infinite sequence of terms of S. For every $n \in \mathbb{N}^{\#}$ we denote by $Ext\text{-}\sum_{k=1}^{n} x_k$ the element of S uniquely determined by the following conditions:

(i) $Ext\text{-}\sum_{k=1}^{1} x_k = x_1$; (ii) $Ext\text{-}\sum_{k=1}^{n+1} x_k = Ext\text{-}\sum_{k=1}^{n} x_k + x_{n+1}$ for all $n \in \mathbb{N}^{\#}$.

Remark 9.1.This definition is justified on the following grounds. The sequence $\{x_k\}_{k\in\mathbb{N}^{\#}}$is a given external function H with domain $\mathbb{N}^{\#}, H(x_k) = x_k$ for every k. We seek a function F with domain $\mathbb{N}^{\#}$ whose value $F(n)$ is to be $Ext\text{-}\sum_{k=1}^{n} x_k$. Then the conditions (i), (ii) above correspond to the following conditions on F :
(i′) $F(1) = H(1)$; (ii′) $F(n+1) = F(n) + H(n+1)$, for all $n \in \mathbb{N}^{\#}$.
Let (1) $c = H(1)$; (2) $G(n,z) = z + H(n+1)$.
Thus the conditions (i′) and (ii′) are equivalent to
(i″) $F(1) = c$;
(ii″) $F(n+1) = G(n, F(n))$ for all $n \in \mathbb{N}^{\#}$.
Given the function H, the element c of S and the function G are well-defined by (1)-(2).
Then by Theorem B.1 we see that there is a unique function F satisfying (1)-(2) with $\mathbf{dom}(F) = \mathbb{N}^{\#}$ and $\mathbf{range}(F) \subseteq S$. Thus (i′)-(ii′) is just another form of recursive definition.

(Hence it should be expected that various properties of $Ext\text{-}\sum_{k=1}^{n} x_k$ will have to be verified
by hyper infinite induction on $n \in \mathbb{N}^{\#}$.)

Definition 9.2. Suppose that S is a set on which a binary operation $\times$ is defined and under which S is dosed. Let $\{x_k\}_{k\in\mathbb{N}^{\#}}$ be an hyper infinite sequence of terms of S. For every $n \in \mathbb{N}^{\#}$ we denote by $Ext\text{-}\prod_{k=1}^{n} x_k$ the element of S uniquely determined by the following conditions:

(i) $Ext\text{-}\prod_{k=1}^{n} x_k = x_1$; (ii) $Ext\text{-}\prod_{k=1}^{n+1} x_k = \left(Ext\text{-}\prod_{k=1}^{n} x_k\right) \times x_{n+1}$ for all $n \in \mathbb{N}^{\#}$.

Theorem 1.(1) Suppose that S is a set closed under a binary operation $+$ and that $+$ is associative on S, i.e.,for all $x,y,z \in S, x+(y+z) = (x+y)+z$. Let $\{x_k\}_{k\in\mathbb{N}^{\#}}$ be any hyper infinite sequence of terms in S. Then for any $n,m \in \mathbb{N}^{\#}$. we have

$$Ext\text{-}\sum_{k=1}^{n+m} x_k = \left(Ext\text{-}\sum_{k=1}^{n} x_k\right) + \left(Ext\text{-}\sum_{k=1}^{m} x_{n+k}\right). \tag{9.1}$$

(2) Suppose that S is a set closed under a binary operation $\times$ and that $\times$ is associative

on S, i.e.,for all $x,y,z \in S, x\times(y\times z) = (x\times y)\times z$. Let $\{x_k\}_{k\in\mathbb{N}^{\#}}$ be any hyper infinite sequence of terms in S. Then for any $n,m \in \mathbb{N}^{\#}$. we have

$$Ext\text{-}\prod_{k=1}^{n+m} x_k = \left(Ext\text{-}\prod_{k=1}^{n} x_k\right)\times Ext\text{-}\prod_{k=1}^{m} x_{n+k}. \tag{9.2}$$

Proof. We prove (3.5.1); the proof of (2) is completely similar. Let n be fixed; we proceed by hyper infinite induction on m. For $m = 1$ from Eq.(3.8.1) we get

$$Ext\text{-}\sum_{k=1}^{n+1} x_k = \left(Ext\text{-}\sum_{k=1}^{n} x_k\right) + \left(Ext\text{-}\sum_{k=1}^{1} x_{n+k}\right). \tag{9.3}$$

By Definition 3.8.1(i) we obtain

$$Ext\text{-}\sum_{k=1}^{1} x_{n+k} = x_{n+1}. \tag{9.4}$$

Suppose Eq.(3.8.1) is true for $m \in \mathbb{N}^{\#}$. We show that is true for $m+1$, i.e.,that

$$Ext\text{-}\sum_{k=1}^{n+(m+1)} x_k = \left(Ext\text{-}\sum_{k=1}^{n} x_k\right) + \left(Ext\text{-}\sum_{k=1}^{m+1} x_{n+k}\right). \tag{9.4$'$}$$

By associativity $+$ on $\mathbb{N}^{\#}$ we get

$$Ext\text{-}\sum_{k=1}^{n+(m+1)} x_k = Ext\text{-}\sum_{k=1}^{(n+m)+1} x_k. \tag{9.6}$$

From Eq.(3.8.6) by Definition 3.8.1(ii) we obtain

$$Ext\text{-}\sum_{k=1}^{(n+m)+1} x_k = Ext\text{-}\sum_{k=1}^{n+m} x_k + x_{(n+m)+1} = Ext\text{-}\sum_{k=1}^{n+m} x_k + x_{n+(m+1)}. \tag{9.7}$$

From Eq.(3.8.7) by induction hypothesis we obtain

$$Ext\text{-}\sum_{k=1}^{n+m} x_k + x_{n+(m+1)} = \left(Ext\text{-}\sum_{k=1}^{n} x_k + Ext\text{-}\sum_{k=n}^{m} x_k\right) + x_{n+(m+1)}. \tag{9.8}$$

From Eq.(3.8.8) by associativity $+$ on S we get

$$\left(Ext\text{-}\sum_{k=1}^{n} x_k + Ext\text{-}\sum_{k=n}^{m} x_k\right) + x_{n+(m+1)} = Ext\text{-}\sum_{k=1}^{n} x_k + \left(Ext\text{-}\sum_{k=n}^{m} x_k + x_{n+(m+1)}\right). \tag{9.9}$$

From Eq.(3.8.9) by Definition 3.8.1(ii) we obtain

$$Ext\text{-}\sum_{k=1}^{n} x_k + \left(Ext\text{-}\sum_{k=n}^{m} x_k + x_{n+(m+1)}\right) = Ext\text{-}\sum_{k=1}^{n} x_k + Ext\text{-}\sum_{k=n}^{m+1} x_k. \tag{9.10}$$

This equality completes the inductive step and hence the proof of the theorem.

Definition 9.3. Let $\langle x_1,\ldots,x_n\rangle, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^{\#}$. Then $Ext\text{-}\sum_{k=m}^{n} x_k$ and $Ext\text{-}\prod_{k=m}^{n} x_k$ are defined for any $n,m \in \mathbb{N}^{\#}$ by the recursions

(i) $Ext\text{-}\sum_{k=m}^{n} x_k = 0$ and $Ext\text{-}\prod_{k=m}^{n} x_k = 1$ if $n < m$;

(ii) $Ext\text{-}\sum_{k=m}^{n} x_k = \left(Ext\text{-}\sum_{k=m}^{n-1} x_k\right) + x_n$ and

(iii) $Ext\text{-}\prod_{k=m}^{n} x_k = x_n \times \left(Ext\text{-}\prod_{k=m}^{n-1} x_k\right)$ if $m < n$.

The condition (ii) of the above definitíon is justified by recursive definition, see

Appendix B.

Definition 9.4. Let $\langle x_1,\ldots,x_j,\ldots\rangle, j \in \mathbb{N}$ be a countable sequence of elements of $\mathbb{R}_c^\#$. Then ω-sum $Ext\text{-}\sum_{j=m}^{\omega} x_k$ and ω-product $Ext\text{-}\prod_{j=m}^{\omega} x_k$ are defined for any $m \in \mathbb{N}$ by

(iv) $Ext\text{-}\sum_{j=m}^{\omega} x_j \triangleq Ext\text{-}\sum_{j=m}^{n} y_j$, where $\langle y_1,\ldots,y_j,\ldots,y_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ is a hyperfinite sequence such that $x_j = y_j$ for all $j \in \mathbb{N}$ and $y_j = 0$ for all $j \in \mathbb{N}^\#\backslash\mathbb{N}$;

(v) $Ext\text{-}\prod_{j=m}^{\omega} x_j \triangleq Ext\text{-}\prod_{j=m}^{\omega} y_j$, where $\langle y_1,\ldots,y_j,\ldots,y_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ is a hyperfinite sequence such that $x_j = y_j$ for all $j \in \mathbb{N}$ and $y_j = 1$ for all $j \in \mathbb{N}^\#\backslash\mathbb{N}$.

Theorem 9.2.Let $\langle x_1,\ldots,x_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^\#$. Then we have

$$Ext\text{-}\sum_{k=m}^{n} x_k = Ext\text{-}\sum_{k=m}^{n-m+q} x_{k+m-q} \tag{9.11}$$

and

$$z \times \left(Ext\text{-}\sum_{k=m}^{n} x_k \right) = Ext\text{-}\sum_{k=m}^{n} z \times x_k, \tag{9.12}$$

$z \in \mathbb{R}_c^\#$.

Proof.Let $\langle x_1,\ldots,x_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^\#$. Consider now any hyperfinite nonnegative integers

$n_1, n_2, .., n_i, \ldots, n_{\mathbf{t}}, n_i \in \mathbb{N}^\#\backslash\mathbb{N}, 1 \leq i \leq \mathbf{t}$,

and set

$$n = n_1 + n_2 + \ldots + n_{\mathbf{t}}. \tag{9.13}$$

Given $x_1,\ldots,x_n$, we can group these as:

$$x_1,\ldots,x_{n_1};\ \ x_{n_1+1},\ldots,x_{n_1+n_2};\ x_{n_1+n_2+1},\ldots,x_{n_1+n_2+n_3};\ldots x_{n_1+n_2+\ldots n_i+1},\ldots,x_{n_1+n_2+\ldots n_{i+1}};\ldots \tag{9.14}$$

Here, if $n_i = 0$, the corresponding subsequence is regarded as being empty.

Theorem 9.3. Let $\langle x_1,\ldots,x_k,\ldots\rangle$ be an hyper infinite sequence of elements of $\mathbb{R}_c^\#$. Let $\langle n_1,\ldots,n_{\mathbf{t}}\rangle$ be a sequence of nonnegalive integers. For each $i = 1,\ldots,\mathbf{t} \in \mathbb{N}^\#$, let $m_i = \sum_{j=1}^{i-1} n_j$ and let $n = m_t + n_t$. Then

$$Ext\text{-}\sum_{k=1}^{n} x_k = \sum_{i=1}^{\mathbf{t}} \left(Ext\text{-}\sum_{k=1}^{n_i} x_{m_i+k} \right) \tag{9.15}$$

and

$$Ext\text{-}\prod_{k=1}^{n} x_k = \prod_{i=1}^{\mathbf{t}} \left(Ext\text{-}\prod_{k=1}^{n_i} x_{m_i+k} \right). \tag{9.16}$$

Proof. By hyper infinite induction.

Definition 9.5. A function F is said to be a permutation of a set S if it is one-to-one and $\mathbf{dom}(F) = \mathbf{range}(F) = S$.

Definition 9.6. Let $[1,n]$ a set $\{k | k \in \mathbb{N}^\# \wedge (1 \leq k \leq n)\}$

Theorem 9.4.Let $\langle x_1,\ldots,x_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ be an hyperfinite <u>external</u> sequence of elements of $\mathbb{R}_c^\#$. Then for any $n \in \mathbb{N}^\#$ and any permutalion $\mathbf{F}$ of $[1,n]$ following holds

$$Ext\text{-}\sum_{k=1}^{n} x_k = Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)}. \qquad (9.17)$$

The same holds if we replace $Ext\text{-}\sum$ by $Ext\text{-}\prod$.

Proof. The proof is by hyper infinite induction on $n \in \mathbb{N}^{\#}$. For $n = 1$ it is trivial. Suppose that it is true for n. Let $\mathbf{G}$ be a permutation of $[1, n+1]$. Then $G(m) = n+1$ for a unique m, such that $1 \leq m \leq n+1$. Then by Eq.(3.5.15)

$$Ext\text{-}\sum_{k=1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + x_{n+1} + Ext\text{-}\sum_{k=m+1}^{n+1} x_{\mathbf{G}(k)} \qquad (9.18)$$

and by Eq.(3.8.18)

$$Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + x_{n+1} + Ext\text{-}\sum_{k=m+1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{G}(k+1)} + x_{n+1}. \qquad (9.19)$$

Thus by Eq.(3.8.11) we obtain

$$Ext\text{-}\sum_{k=1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{G}(k+1)} + x_{n+1}. \qquad (9.20)$$

To reduce this to the inductive hypothesis, we wish to rewrite the external sum of the first

two terms as $Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)}$ for suitable $\mathbf{F}$. Define $\mathbf{F}$ by

$$\mathbf{F}(k) = \begin{cases} \mathbf{G}(k) & \text{if} \quad 1 \leq k < m \\ \\ \mathbf{G}(k+1) & \text{if} \quad m \leq k \leq n \end{cases} \qquad (9.21)$$

Since all valucs of $\mathbf{G}(k)$ for $k \neq m$, we have for all $k \leq n$

$$1 \leq \mathbf{F}(k) \leq n \qquad (9.22)$$

Now we claim that

$$\mathbf{F} \text{ is a permutation of } [1, n]. \qquad (9.23)$$

By (3.8.21) and (3.8.22) we need only check that $\mathbf{F}$ is one-to one. Suppose that $\mathbf{F}(k_1) = \mathbf{F}(k_2)$.
If both k_1, k_2 are $< m$ or both are $\geq m$, it lollows from (3.8.21) and the fact that $\mathbf{G}$ is a permutation that $k_1 = k_2$. If, say, $k_1 < m \leq k_2$, we have $\mathbf{G}(k_1) = \mathbf{G}(k_2 + 1)$, hence $k_1 = k_2 + 1$, which contradicts our assumption. Thus neither this case· nor, by symmetry, the case $k_2 < m \leq k_1$ can occur. We have from (3.8.20) and (3.8.21) that

$$Ext\text{-}\sum_{k=1}^{m+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{F}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{F}(k)} + x_{n+1} = Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)} + x_{n+1} \qquad (9.24)$$

by (3.8.23) and inductive hypothesis

$$Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)} + x_{n+1} = Ext\text{-}\sum_{k=1}^{n} x_k + x_{n+1} = Ext\text{-}\sum_{k=1}^{n+1} x_k \qquad (9.25)$$

This equality completes the inductive step and hence the proof of the theorem.

§10.Hyperrationals $\mathbb{Q}^{\#}$.

Now that we have the hypernatural numbers $\mathbb{N}^{\#}$, defining hyperintegers and hyperrational numbers is well within reach [2].

Definition 10.1. Let $Z^{\#'} = \mathbb{N}^{\#} \times \mathbb{N}^{\#}$. We can define an equivalence relation $\approx$ on $Z^{\#'}$ by $(a,b) \approx (c,d)$ if and only if $a+d = b+c$. Then we denote the set of all hyperintegers by $\mathbb{Z}^{\#} = Z^{\#'}/\approx$ (The set of all equivalence classes of $Z^{\#'}$ modulo $\approx$).

Definition 10.2. Let $Q^{\#'} = \mathbb{Z}^{\#} \times (\mathbb{Z}^{\#} - \{0\}) \ = \{(a,b) \in \mathbb{Z}^{\#} \times \mathbb{Z}^{\#} | b \neq 0\}$. We can define an equivalence relation $\approx$ on $Q^{\#'}$ by $(a,b) \approx (c,d)$ if and only if $a \times d = b \times c$. Then we denote the set of all hyperrational numbers by $\mathbb{Q}^{\#} = Q'/\approx$ (The set of all equivalence classes of Q' modulo $\approx$).

Definition 10.3. A linearly ordered set $(P,<)$ is called dense if for any $a,b \in P$ such that $a < b$, there exists $z \in P$ such that $a < z < b$.

Lemma 10.1. $(\mathbb{Q}^{\#},<)$ is dense.

Proof. Let $x = (a,b), y = (c,d) \in \mathbb{Q}^{\#}$ be such that $x < y$. Consider $z = (ad+bc, 2bd) \in \mathbb{Q}^{\#}$.
It is easily shown that $x < z < y$.

11.External Cauchy hyperreals $\mathbb{R}^{\#}_{c}$ via Cauchy completion.

Definition 11.1. A hyper infinite sequence of hyperrational numbers (or for the sake of brevity simply hyperrational sequence) is a function from the hypernatural numbers $\mathbb{N}^{\#}$ into the hyperrational numbers $\mathbb{Q}^{\#}$. We usually denote such a function by $n \mapsto a_n$, or by $a : n \to a_n$, so the terms in the sequence are written $\{a_1, a_2, a_3, \ldots, a_n \ldots\}$. To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{\infty^{\#}}$, or $\{a_n\}_{n \in \mathbb{N}^{\#}}$, or for the sake of brevity simply $\{a_n\}$.

Definition 11.2. Let $\{a_n\}$ be a hyperrational sequence. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N \in \mathbb{N}^{\#}\backslash\mathbb{N}$, $N = N(\varepsilon)$ such that, after N (i.e.for all $n > N$), $|a_n| \leq \varepsilon$. We often denote this symbolically by $a_n \to_{\#} 0$.
We can also, at this point, define what it means for a hyperrational sequence #-tends to any given number $q \in \mathbb{Q}^{\#}$: $\{a_n\}$ #-tends to q if the hyperrational sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_{\#} 0$.

Definition 11.3. Let $\{a_n\}$ be a hyper infinite hyperrational sequence. We call $\{a_n\}$ a Cauchy hyperrational sequence if the difference between its terms #-tends to 0.
To be precise: given any hyperrational number $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m,n > N, |a_n - a_m| < \varepsilon$.

Theorem 11.1. If $\{a_n\}$ is a #-convergent hyperrational sequence (that is, $a_n \to_{\#} q$ for some hyperrational number $q \in \mathbb{Q}^{\#}$), then $\{a_n\}$ is a Cauchy hyperrational sequence.

Proof.We know that $a_n \to_{\#} q$. Here is a ubiquitous trick: instead of using ε in the definition, start with an arbitrary small $\varepsilon > 0, \varepsilon \approx 0$ and then choose $N \in \mathbb{N}^{\#}/\mathbb{N}$ so that $|a_n - q| < \varepsilon/2$ when $n > N$. Then if $m,n > N$, we have
$|a_n - a_m| = |(a_n - q) - (a_m - q)| \leq |a_n - q| + |a_m - q| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
This shows that $\{a_n\}$ is a Cauchy hyper infinite sequence.

Theorem 11.2. If $\{a_n\}$ is a Cauchy hyperrational sequence, then it is bounded or hyper bounded; that is, there is some $M \in \mathbb{Q}^{\#}$ finite or hyperfinite such that $|a_n| \leq M$ for all $n \in \mathbb{N}^{\#}$.

Proof.Since $\{a_n\}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some $N \in \mathbb{N}^{\#}/\mathbb{N}$ such that $|a_m - a_n| < 1$ whenever $m,n > N$. Thus, $|a_{N+1} - a_n| < 1$ for $n > N$. We can rewrite this as $a_{N+1} - 1 < a_n < a_{N+1} + 1$.This means that $|a_n|$ is less than the maximum of $|a_{N+1} - 1|$ and $|a_{N+1} + 1|$. So, set M equal to the maximum number in the following list: $\{|a_0|, |a_1|, \ldots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\}$.Then for any term a_n, if $n \leq N$, then $|a_n|$ appears in the list and so $|a_n| \leq M$; if $n > N$, then (as shown above) $|a_n|$ is less than at least one of the last two entries in the list, and so $|a_n| \leq M$.Hence, M is a bound for the sequence.

Definition 11.4. Let S be a set . A relation $x \sim y$ among pairs of elements of S is said to be an equivalence relation if the following three properties hold:

Reflexivity: for any $s \in S, s \sim s$.

Symmetry: for any $s,t \in S$, if $s \sim t$ then $t \sim s$.

Transitivity: for any $s,t,r \in S$, if $s \sim t$ and $t \sim r$, then $s \sim r$.

Theorem 11.3. Let S be a set, with an equivalence relation $(\bullet\sim\bullet)$ on pairs of elements. For $s \in S$,denote by $\mathbf{cl}[s]$ the set of all elements in S that are related to s. Then for any $s,t \in S$,either $\mathbf{cl}[s] = \mathbf{cl}[t]$ or $\mathbf{cl}[s]$ and $\mathbf{cl}[t]$ are disjoint.

The hyperreal numbers $\mathbb{R}_{\mathbf{c}}^{\#}$ will be constructed as equivalence classes of Cauchy hyperrational sequences. Let $\mathcal{F}_{\mathbb{Q}^{\#}}$ denote the set of all Cauchy hyperrational sequences of hyperrational numbers. We define the equivalence relation on $\mathcal{F}_{\mathbb{Q}^{\#}}$.

Definition 11.5. Let $\{a_n\}$ and $\{b_n\}$ be in $\mathcal{F}_{\mathbb{Q}^{\#}}$. Say they are #-equivalent if $a_n - b_n \to_{\#} 0$ i.e., if and only if the hyperrational sequence $\{a_n - b_n\}$ tends to 0.

Theorem 11.4.Definition 11.4 yields an equivalence relation on $\mathcal{F}_{\mathbb{Q}^{\#}}$.

Proof. We need to show that this relation is reflexive, symmetric, and transitive.

Reflexive: $a_n - a_n = 0$, and the sequence all of whose terms are 0 clearly #-converges to 0. So $\{a_n\}$ is related to $\{a_n\}$.

Symmetric: Suppose $\{a_n\}$ is related to $\{b_n\}$, so $a_n - b_n \to_{\#} 0$.
But $b_n - a_n = -(a_n - b_n)$,and since only the absolute value $|a_n - b_n| = |b_n - a_n|$ comes into play in Definition 11.2, it follows that $b_n - a_n \to_{\#} 0$ as well. Hence, $\{b_n\}$ is related to $\{a_n\}$.

Transitive: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 11.1. Suppose $\{a_n\}$ is related to $\{b_n\}$, and $\{b_n\}$ is related to $\{c_n\}$. This means that $a_n - b_n \to_{\#} 0$ and $b_n - c_n \to_{\#} 0$.To be fully precise, let us fix $\varepsilon > 0, \varepsilon \approx 0$; then there exists an $N \in \mathbb{N}^{\#}$ such that for all $n > N, |a_n - b_n| < \varepsilon/2$; also, there exists an $M \in \mathbb{N}^{\#}$ such that for all $n > M, |b_n - c_n| < \varepsilon/2$. Well, then, as long as n is bigger than both N and M, we have that $|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
So, choosing L equal to the max of N,M, we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |a_n - c_n| < \varepsilon$. This means that $a_n - c_n \to_{\#} 0$ – i.e. $\{a_n\}$ is related to $\{c_n\}$.

Definition 11.6. The hyperreal numbers $\mathbb{R}_c^{\#}$ are the equivalence classes $\mathbf{cl}[\{a_n\}]$ of Cauchy sequences of hyperrational numbers, as per Definition 11.5. That is, each such equivalence class is a hyperreal number.

Definition 11.7. Given any hyperrational number $q \in \mathbb{Q}^{\#}$, define a hyperreal number $q^{\#}$ to be the equivalence class of the sequence $q^{\#} = (q,q,q,q,\ldots)$ consisting entirely of q. So we view $\mathbb{Q}^{\#}$ as being inside $\mathbb{R}_c^{\#}$ by thinking of each hyperrational number $q \in \mathbb{Q}^{\#}$ as its associated equivalence class $q^{\#}$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

Definition 11.8. Let $s,t \in \mathbb{R}_c^{\#}$, so there are Cauchy sequences $\{a_n\},\{b_n\}$ of

hyperrational numbers with $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$.
(a) Define $s+t$ to be the equivalence class of the sequence $\{a_n + b_n\}$.
(b) Define $s \times t$ to be the equivalence class of the sequence $\{a_n \times b_n\}$.
Theorem 11.5.The operations $+, \times$ in Definition 8.8 (a),(b) are well-defined.
Proof. Suppose that $\mathbf{cl}[\{a_n\}] = \mathbf{cl}[\{c_n\}]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$. Thus means that $a_n - c_n \to_{\#} 0$ and $b_n - d_n \to_{\#} 0$. Then $(a_n + b_n) - (c_n + d_n) = (a_n - c_n) + (b_n - d_n)$. Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to 0, and so $\mathbf{cl}[\{a_n + b_n\}] = \mathbf{cl}[\{c_n + d_n\}]$.
Multiplication is a little trickier; this is where we will use Theorem 11.2. We will also use another ubiquitous technique: adding 0 in the form of $s - s$. Again, suppose that $\mathbf{cl}[(a_n) = \mathbf{cl}[(c_n)]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$; we wish to show that $\mathbf{cl}[\{a_n \times b_n\}] = \mathbf{cl}[\{c_n \times d_n\}]$, or, in other words, that $a_n \times b_n - c_n \cdot d_n \to_{\#} 0$.Well, we add and subtract one of the other cross terms, say

$$b_n \times c_n : a_n \times b_n - c_n \times d_n = a_n \times b_n + (b_n \times c_n - b_n \times c_n) - c_n \times d_n =$$
$$= (a_n \times b_n - b_n \times c_n) + (b_n \times c_n - c_n \times d_n) = b_n \times (a_n - c_n) + c_n \times (b_n - d_n).$$

Hence, we have $|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n|$. Now, from Theorem 11.2, there are numbers M and L such that $|b_n| \leq M$ and $|c_n| \leq L$ for all $n \in \mathbb{N}^{\#}$. Taking some number K which is bigger than both, we have

$$|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n| \leq K(|a_n - c_n| + |b_n - d_n|).$$

Now, noting that both $a_n - c_n$ and $b_n - d_n$ tend to 0 and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2K$), we see that $a_n \times b_n - c_n \times d_n \to_{\#} 0$.
Theorem 11.6. Given any hyperreal number $s \neq 0$, there is a hyperreal number t such that $s \times t = 1$.
Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $\{0,0,0,0,\ldots\}$. In other words, $s = \mathbf{cl}[\{a_n\}]$ where $a_n - 0$ does not #-converge to 0. From this, we are to deduce the existence of a hyperreal number $t = \mathbf{cl}[\{b_n\}]$ such that $s \times t = \mathbf{cl}[\{a_n \times b_n\}]$ is the same equivalence class as $\mathbf{cl}[\{1,1,1,1,\ldots\}]$. Doing so is actually an easy consequence of the fact that nonzero rational numbers have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{a_n\}$ does not tend to 0), there's no reason any number of the terms in $\{a_n\}$ can't equal 0. However, it turns out that eventually, $a_n \neq 0$.
That is,
Lemma 11.1. If $\{a_n\}$ is a Cauchy hyper infinite sequence which does not #-tend to 0, then there is an $N \in \mathbb{N}^{\#}/\mathbb{N}$ such that, for $n > N, a_n \neq 0$.
We will now use Lemma 11.1 to complete the proof of Theorem 11.7.
Let N be such that $a_n \neq 0$ for $n > N$. Define a hyper infinite sequence b_n of hyperrational numbers as follows:
for $n \leq N, b_n = 0$, and for $n > N, b_n = 1/a_n; \{b_n\} = (0,0,\ldots,0,1/a_{N+1},1/a_{N+2},\ldots)$.
This makes sense since, for $n > N$, an is a nonzero hyperrational number, so $1/a_n$ exists. Then $a_n \cdot b_n$ is equal to $a_n \cdot 0 = 0$ for $n \leq N$, and equals $a_n \cdot b_n = a_n \cdot 1/a_n = 1$ for $n > N$. Well, then, if we
look at the hyper infinite sequence $(1,1,1,1,\ldots)$, we have $(1,1,1,1,\ldots) - (a_n \cdot b_n)$ is the

hyper infinite sequence which is $1-0=1$ for $n \leq N$ and equals $1-1=0$ for $n > N$. Since this sequence is eventually equal to 0, it #-converges to 0, and so $\mathbf{cl}[\{a_n \cdot b_n\}] = \mathbf{cl}[(1,1,1,1,\ldots)] = 1 \in \mathbb{R}_c^{\#}$. This shows that $t = \mathbf{cl}[\{b_n\}]$ is a multiplicative inverse to $s = \mathbf{cl}[\{a_n\}]$.

Definition 11.9. Let $s \in \mathbb{R}_c^{\#}$. Say that s is positive if $s \neq 0$, and if $s = \mathbf{cl}[\{a_n\}]$ for some Cauchy sequence of hyperrational numbers such that for some $N \in \mathbb{N}^{\#}, a_n > 0$ for all $n > N$. Given two hyperreal numbers s, t, say that $s > t$ if $s - t$ is positive.

Theorem 11.7. Let s,t be hyperreal numbers such that $s > t$, and let $r \in \mathbb{R}_c^{\#}$. Then $s + r > t + r$.

Proof. Let $s = \mathbf{cl}[\{a_n\}], t = \mathbf{cl}[\{b_n\}]$, and $r = \mathbf{cl}[\{c_n\}]$. Since $s > t$ i.e., $s - t > 0$, we know that there is an $N \in \mathbb{N}^{\#}$ such that, for $n > N$, $a_n - b_n > 0$. So $a_n > b_n$ for $n > N$. Now, adding c_n to both sides of this inequality (as we know we can do for hyperrational numbers), we have $a_n + c_n > b_n + c_n$ for $n > N$, or $(a_n + c_n) - (b_n + c_n) > 0$ for $n > N$. Note also that $(a_n + c_n) - (b_n + c_n) = a_n - b_n$ does not #-converge to 0, by the assumption that $s - t > 0$. Thus, by Definition 11.8, this means that $s + r = \mathbf{cl}[\{a_n + c_n\}] > \mathbf{cl}[\{b_n + c_n\}] = t + r$.

Theorem 11.8. (Generalized Archimedean property)Let $s,t > 0$ be hyperreal numbers. Then there is $m \in \mathbb{N}^{\#}$ such that $m \times s > t$.

Proof. Let $s,t > 0$ be hyperreal numbers. We need to find a hypernatural number m so that $m \times s > t$. First, recall that, by m in this context, we mean $\mathbf{cl}[\{m,m,m,m,\ldots\}]$. So, letting $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$, what we need to show is that there exists $m \in \mathbb{N}^{\#}$ with

$\mathbf{cl}[\{m,m,m,m,\ldots\}] \times \mathbf{cl}[\{a_1,a_2,a_3,a_4,\ldots\}] = \mathbf{cl}[\{m \times a_1, m \times a_2, m \times a_3, m \times a_4,\ldots\}] >$
$> \mathbf{cl}[\{b_1,b_2,b_3,b_4,\ldots\}]$.

Now, to say that $\mathbf{cl}[\{m \times a_n\}] > \mathbf{cl}[\{b_n\}]$, or $\mathbf{cl}[\{m \times a_n - b_n\}]$ is positive, is, by Definition 11.9, just to say that there is $N \in \mathbb{N}^{\#}$ such that $m \times a_n - b_n > 0$ for all $n > N$, while $m \times a_n - b_n \nrightarrow_{\#} 0$. To be precise, the first statement is:

There exist $m,N \in \mathbb{N}^{\#}$ so that $m \times a_n > b_n$ for all $n > N$.

To produce a contradiction, we assume this is not the case; assume that

(#) for every m and N, there exists an $n > N$ so that $m \times a_n \leq b_n$.

Now, since $\{b_n\}$ is a Cauchy sequence, by Theorem 11.2 it is hyperbounded – there is a hyperrational number $M \in \mathbb{Q}^{\#}$ such that $b_n \leq M$ for all n. Now, by the properties for the hyperrational numbers $\mathbb{Q}^{\#}$, given any hyperrational number $\varepsilon > 0, \varepsilon \approx 0$, there is an $m \in \mathbb{N}^{\#}$ such that $M/m < \varepsilon/2$. Fix such an m. Then if $m \times a_n \leq b_n$, we have $a_n \leq b_n/m \leq M/m < \varepsilon/2$.

Now, $\{a_n\}$ is a Cauchy sequence, and so there exists N so that for $n,k > N, |a_n - a_k| < \varepsilon/2$.

By Asumption (#), we also have an $n > N$ such that $m \times a_n \leq b_n$, which means that $a_n < \varepsilon/2$. But then for every $k > N$, we have that $a_k - a_n < \varepsilon/2$, so $a_k < a_n + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, $a_k < \varepsilon$ for all $k > N$. This proves that $a_k \to_{\#} 0$, which by Definition 11.9 contradicts the fact that $\mathbf{cl}[\{a_n\}] = s > 0$.

Thus, there is indeed some $m \in N$ so that $m \times a_n - b_n > 0$ for all sufficiently infinite large $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. To conclude the proof, we must also show that $m \times a_n - b_n \nrightarrow 0$. Actually, it is possible that $m \times a_n - b_n \to 0$ (for example if $\{a_n\} = \{1,1,1,\ldots\}$ and $\{b_n\} = \{m,m,m,\ldots\}$). But that's okay: then we can simply choose a larger m. That is:

let m be a hypernatural number constructed as above, so that $m \times a_n - b_n > 0$ for all sufficiently large $\in \mathbb{N}^{\#} \backslash \mathbb{N}$. If it happens to be true that $m \times a_n - b_n \nrightarrow 0,$ then the proof is complete.

If, on the other hand, it turned out that $m \times a_n - b_n \to 0,$ then take instead the integer $m+1.$ Since $s = \mathbf{cl}[\{a_n\}] > 0,$ we have a $n > 0$ for all infinite large $n,$ so $(m+1) \times a_n - b_n = m \times a_n - b_n + a_n > a_n > 0$ for all infinite large n, so $m+1$ works just as well as m did in this regard; and since $m \times a_n - b_n \to 0,$ we have $(m+1) \times a_n - b_n = (m \times a_n - b_n) + a_n \nrightarrow 0$ since $s = \mathbf{cl}[\{a_n\}] > 0$ (so $a_n \nrightarrow 0$).

It will be handy to have one more Theorem about how the hyperrationals $\mathbb{Q}^{\#}$ and hyperreals $\mathbb{R}_c^{\#}$ compare before we proceed. This theorem is known as the density of $\mathbb{Q}^{\#}$ in $\mathbb{R}_c^{\#}$, and it follows almost immediately from the construction of the $\mathbb{R}_c^{\#}$ from $\mathbb{Q}^{\#}$.

Theorem 11.9. Given any hyperreal number $r \in \mathbb{R}_c^{\#}$, and any hyperrational number $\varepsilon > 0, \quad \varepsilon \approx 0,$ there is a hyperrational number $q \in \mathbb{Q}^{\#}$ such that $|r - q| < \varepsilon.$

Proof. The hyperreal number r is represented by a Cauchy hyperrational sequence $\{a_n\}.$

Since this sequence is Cauchy, given $\varepsilon > 0, \varepsilon \approx 0,$ there is $N \in \mathbb{N}^{\#}$ so that for all $m, n > N,$

$|a_n - a_m| < \varepsilon.$ Picking some fixed $l > N,$ we can take the hyperrational number q given by $q = \mathbf{cl}[\{a_l, a_l, a_l, \ldots\}].$ Then we have $r - q = \mathbf{cl}[\{a_n - a_l\}_{n \in \mathbb{N}^{\#}}]$, and $q - r = \mathbf{cl}[\{a_l - a_n\}_{n \in \mathbb{N}^{\#}}].$

Now, since $l > N,$ we see that for $n > N, a_n - a_l < \varepsilon$ and $a_l - a_n < \varepsilon,$ which means by Definition 11.9 that $r - q < \varepsilon$ and $q - r < \varepsilon;$ hence, $|r - q| < \varepsilon.$

Definition 11.10.Let $S \subsetneqq \mathbb{R}_c^{\#}$ be a non-empty set of hyperreal numbers.

A hyperreal number $x \in \mathbb{R}_c^{\#}$ is called an upper bound for S if $x \geq s$ for all $s \in S.$

A hyperreal number x is the least upper bound (or supremum $\sup S$) for S if x is an upper bound for S and $x \leq y$ for every upper bound y of $S.$

Remark 11.1.The order $\leq$ given by Definition 11.9 obviously is $\leq$-incomplete.

Definition 11.11. Let $S \subsetneqq \mathbb{R}_c^{\#}$ be a nonempty subset of $\mathbb{R}_c^{\#}.$We we will say that:

(1) S is $\leq$-admissible above if the following conditions are satisfied:

(i) S bounded above;

(ii) let $A(S)$ be a set $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0.$

(2) S is $\leq$-admissible belov if the following condition are satisfied:

(i) S bounded belov;

(ii) let $L(S)$ be a set $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0.$

Theorem 11.10. (i) Any $\leq$-admissible above subset $S \subset \mathbb{R}_c^{\#}$ has the least upper bound property.(ii) Any $\leq$-admissible below subset $S \subset \mathbb{R}_c^{\#}$ has the greatest lower bound property.

Proof. Let $S \subset \mathbb{R}_c^{\#}$ be a nonempty subset, and let M be an upper bound for $S.$ We are going to construct two sequences of hyperreal numbers, $\{u_n\}$ and $\{l_n\}$. First, since S is nonempty, there is some element $s_0 \in S.$ Now, we go through the following hyperinductive procedure to produce numbers $u_0, u_1, u_2, \ldots, u_n, \ldots$ and $l_1, l_2, l_3, \ldots, l_n, \ldots$

(i) Set $u_0 = M$ and $l_0 = s.$

(ii) Suppose that we have already defined u_n and l_n. Consider the number $m_n = (u_n + l_n)/2,$ the average between u_n and l_n.

(1) If m_n is an upper bound for $S,$ define $u_{n+1} = m_n$ and $l_{n+1} = l_n$.

(2) If m_n is not an upper bound for $S,$ define $u_{n+1} = u_n$ and $l_{n+1} = m_n$.

Since $s < M,$ it is easy to prove by hyper infinite induction that (i) $\{u_n\}$ is a non-increasing sequence: $u_{n+1} \leq u_n, n \in \mathbb{N}^{\#}$(ii) $\{l_n\}$ is a non-decreasing sequence $l_{n+1} \geq l_n, n \in \mathbb{N}^{\#}$ and (iii) $u_n - l_n = 2^{-n}(M - s)$.

This gives us the following lemma.

Lemma 11.2. $\{u_n\}$ and $\{l_n\}$ are Cauchy sequences of hyperreal numbers.

Proof. Note that each $l_n \leq M$ for all $n \in \mathbb{N}^{\#}$. Since $\{l_n\}$ is non-decreasing and $u_n - l_n = 2^{-n}(M - s).$, it follows directly that $\{l_n\}$ is Cauchy.

For $\{u_n\}$, we have $u_n \geq s_0$ for all $n \in \mathbb{N}^{\#}$, and so $-u_n \leq -s_0$. Since $\{u_n\}$ is non-increasing, $\{-u_n\}$ is non-decreasing, and so as above, $\{-u_n\}$ is Cauchy. It is easy to verify that, therefore, $\{u_n\}$ is Cauchy.

The following Lemma shows that $\{u_n\}$ does tend to a hyperreal number.

Lemma 11.3. There is a hyperreal number u such that $u_n \to_{\#} u$.

Proof. Fix a term u_n in the sequence $\{u_n\}$. By Theorem 11.9, there is a hyperrational number q_n such that $|u_n - q_n| < 1/n.$ Consider the sequence $\{q_1, q_2, q_3, \ldots, q_n, \ldots\}$ of hyperrational numbers. We will show this hypersequence is Cauchy. Fix $\varepsilon > 0, \varepsilon \approx 0.$ By the Theorem 11.8, we can choose $N \in \mathbb{N}^{\#}$ so that $1/N < \varepsilon/3$. We know, since $\{u_n\}$ is Cauchy, that there is an $M \in \mathbb{N}^{\#}$ such that for $n, m > M, |u_n - u_m| < \varepsilon/3.$ Then, so long as $n, m > max\{N, M\},$ we have

$$|q_n - q_m| = |(q_n - u_n) + (u_n - u_m) + (u_m - q_m)| \leq$$
$$\leq |q_n - u_n| + |u_n - u_m| + |u_m - q_m| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, $\{q_n\}$ is a Cauchy sequence of hyperrational numbers, and so it represents a hyperreal number $u = \mathbf{cl}[\{q_n\}].$ We must show that $u_n - u \to_{\#} 0,$ but this is practically built into the definition of $u.$ To be precise, letting q_n^* be the hyperreal number $\mathbf{cl}[\{q_n, q_n, q_n, \ldots\}],$ we see immediately that $q_n^* - u \to_{\#} 0$ (this is precisely equivalent to the statement that $\{q_n\}$ is Cauchy). But $u_n - q_n^* < 1/n$ by construction; it is easily verify that the assertion that if a sequence $q_n^* \to_{\#} u$ and $u_n - q_n^* \to_{\#} 0,$ then $u_n \to_{\#} u.$ So $\{u_n\}$, a non-increasing sequence of upper bounds for $S,$ tends to a hyperreal number $u.$ As you've guessed, u is the least upper bound of our set S. To prove this, we need one more lemma.

Lemma 11.4. $l_n \to_{\#} u.$

Proof. First, note in the first case above, we have that

$$u_{n+1} - l_{n+1} = m_n - l_n = \frac{u_n + l_n}{2} - l_n = \frac{u_n - l_n}{2}.$$

In the second case, we also have

$$u_{n+1} - l_{n+1} = u_n - m_n = u_n - \frac{u_n + l_n}{2} = \frac{u_n - l_n}{2}.$$

Now, this means that $u_1 - l_1 = \frac{1}{2}(M - s),$ and so $u_2 - l_2 = \frac{1}{2}(u_1 - l_1) = \frac{1}{2^2}(M - s),$ and in general by hyperinfinite induction, $u_n - l_n = 2^{-n}(M - s).$ Since $M > s$ so $L - s > 0,$ and since $2^{-n} < 1/n,$ by the Theorem 11.8, we have for any $\varepsilon > 0$ that $2^{-n}(L - s) < \varepsilon$ for all sufficiently large $n \in \mathbb{N}^{\#}/\mathbb{N}$. Thus, $u_n - l_n = 2^{-n}(L - s) < \varepsilon$ as well, and so $u_n - l_n \to_{\#} 0$. Again, it is easily verify that, since $u_n \to_{\#} u$, we have $l_n \to_{\#} u$ as well.

Proof of Theorem 11.10. First, we show that u is an upper bound. Well, suppose it is not, so that $u < s$ for some $s \in S$. Then $\varepsilon = s - u$ is > 0, and since $u_n \to u$ and is non-increasing, there must be an n so that $u_n - u < \varepsilon$, meaning that $u_n < u + \varepsilon = u + (s - u) = s$. Since u_n is an upper bound for S, however, this is a contradiction. Hence, u is an upper bound for S.

Now, we also know that, for each n, l_n is not an upper bound, meaning that for each n, there is an $s_n \in S$ so that $l_n \leq s_n$. Lemma 11.4 tells us that $l_n \to_{\#} u$, and since the sequence $\{l_n\}$ is non-decreasing, this means that for each $\varepsilon > 0$, there is an $N \in \mathbb{N}^{\#}/\mathbb{N}$ so that for $n > N, l_n > u - \varepsilon$. Hence, for $n > N, s_n \geq l_n > u - \varepsilon$ as well. In particular, for each $\varepsilon > 0$, there is an element $s \in S$ such that $s > u - \varepsilon$. This means that no number smaller than u can be an upper bound for S. Hence, u is the least upper bound for S.

Remark 11.2.Note that assumption in Theorem 11.10 that S is $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$ is necessarily, othervice Theorem 11.10 is not holds. For example let $\Delta = \{\varepsilon | \varepsilon \geq 0 \wedge \varepsilon \approx 0\}$. Obviously a set Δ is not $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$. It is clear that Theorem 11.10 is not holds for a set Δ.

Theorem 11.11.(**Generalized Nested Intervals Theorem**)

Let $\{I_n\}_{n \in \mathbb{N}^{\#}} = \{[a_n, b_n]\}_{n \in \mathbb{N}^{\#}}, [a_n, b_n] \subset \mathbb{R}_c^{\#}$ be a hyper infinite sequence of closed intervals satisfying each of the following conditions:

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots$,

(ii) $b_n - a_n \to_{\#} 0$ as $n \to \infty^{\#}$.

Then $\bigcap_{n=1}^{\infty^{\#}} I_n$ consists of exactly one hyperreal number $x \in \mathbb{R}_c^{\#}$. Moreover both sequences $\{a_n\}$ and $\{b_n\}$ #-converge to x.

Proof.Note that: (a) the set $A = \{a_n | n \in \mathbb{N}^{\#}\}$ is bounded or hyperbouded above by b_1 and (b) the set $A = \{a_n | n \in \mathbb{N}^{\#}\}$ is $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$.

By Theorem 11.10 there exists $\sup A$. Let $\xi = \sup A$.

Since I_n are nested,for any positive hyperintegers m and n we have $a_m \leq a_{m+n} \leq b_{m+n} \leq b_n$, so that $\xi \leq b_n$ for each $n \in \mathbb{N}^{\#}$. Since we obviously have $a_n \leq \xi$ for each $n \in \mathbb{N}^{\#}$, we have $a_n \leq \xi \leq b_n$ for all $n \in \mathbb{N}^{\#}$, which implies $\xi \in \bigcap_{n=1}^{\infty^{\#}} I_n$. Finally, if $\xi, \eta \in \bigcap_{n=1}^{\infty^{\#}} I_n$, with $\xi \leq \eta$, then we get $0 \leq \eta - \xi \leq b_n - a_n$, for all $n \in \mathbb{N}^{\#}$, so that $0 \leq \eta - \xi \leq \inf_{n \in \mathbb{N}^{\#}} |b_n - a_n| = 0$.

Theorem 11.12.(**Generalized Squeeze Theorem**)

Let $\{a_n\}, \{c_n\}$ be two hyper infinite sequences #-converging to L, and $\{b_n\}$ a hyper infinite sequence. If $\forall n \geq K, K \in \mathbb{N}^{\#}$ we have $a_n \leq b_n \leq c_n$, then $\{b_n\}$ also #-converges to L.

Proof. Choose an $\varepsilon > 0, \varepsilon \approx 0$. By definition of the #-limit,there is an $N_1 \in \mathbb{N}^{\#}$ such that for all $n > N_1$ we have $|a_n - L| < \varepsilon$, in other words $L - \varepsilon < a_n < L + \varepsilon$.Similarly, there is an $N_2 \in \mathbb{N}^{\#}$ such that for all $n > N_2$ we have $L - \varepsilon < c_n < L + \varepsilon$. Denote $N = \max(N_1, N_2, K)$. Then for $n > N, L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$, in other words $|b_n - L| < \varepsilon$. Since $\varepsilon > 0, \varepsilon \approx 0$ was arbitrary, by definition of the #-limit this says that $\#\text{-}\lim_{n \to \infty^{\#}} b_n = L$.

Theorem 11.13.(**Corollary of the Generalized Squeeze Theorem**).

If $\#\text{-}\lim_{n \to \infty^{\#}} |a_n| = 0$ then $\#\text{-}\lim_{n \to \infty^{\#}} a_n = 0$.

Proof.We know that $-|a_n| \leq a_n \leq |a_n|$.We want to apply the Generalized Squeeze Theorem.We are given that $\#\text{-}\lim_{n \to \infty^{\#}} |a_n| = 0$.This also implies that

$\#\text{-}\lim_{n\to\infty^\#}(-|a_n|) = 0$. So by the Generalized Squeeze Theorem, $\#\text{-}\lim_{n\to\infty^\#} a_n = 0$.

Theorem 11.14. (**Generalized Bolzano-Weierstrass Theorem**)

Every hyperbounded hyperinfinite sequence has a #-convergent hyper infinite subsequence.

Proof. Let $\{w_n\}_{n\in\mathbb{N}^\#}$ be a hyperbounded hyper infinite sequence. Then, there exists an interval $[a_1,b_1]$ such that $a_1 \leq w_n \leq b_1$ for all $n \in \mathbb{N}^\#$.

Either $\left[a_1, \frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2}, b_1\right]$ contains hyper infinitely many terms of $\{w_n\}$. That is, there exists hyperinfinitely many n in $\mathbb{N}^\#$ such that a_n is in $\left[a_1, \frac{a_1+b_1}{2}\right]$ or there exists hyper infinitely many n in $\mathbb{N}^\#$ such that a_n is in $\left[\frac{a_1+b_1}{2}, b_1\right]$. If $\left[a_1, \frac{a_1+b_1}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_2,b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$. Otherwise, let $[a_2,b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$. Either $\left[a_2, \frac{a_2+b_2}{2}\right]$ or $\left[\frac{a_2+b_2}{2}, b_2\right]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$. If $\left[a_2, \frac{a_2+b_2}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_3,b_3] = \left[a_2, \frac{a_2+b_2}{2}\right]$. Otherwise, let $[a_3,b_3] = \left[\frac{a_2+b_2}{2}, b_2\right]$. By hyper infinite induction, we can continue this construction and obtain hyper infinite sequence of intervals $\{[a_n,b_n]\}_{n\in\mathbb{N}^\#}$ such that:

(i) for each $n \in \mathbb{N}^\#, [a_n,b_n]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$,

(ii) for each $n \in \mathbb{N}^\#, [a_{n+1},b_{n+1}] \subseteq [a_n,b_n]$ and

(iii) for each $n \in \mathbb{N}^\#, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.

Then generalized nested intervals theorem implies that the intersection of all of the intervals $[a_n,b_n]$ is a single point w. We will now construct a hyper infinite subsequence of $\{w_n\}_{n\in\mathbb{N}^\#}$ which will #-converge to w.

Since $[a_1,b_1]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$, there exists $k_1 \in \mathbb{N}^\#$ such that w_{k_1} is in $[a_1,b_1]$. Since $[a_2,b_2]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$, there exists $k_2 \in \mathbb{N}^\#, k_2 > k_1,$ such that w_{k_2} is in $[a_2,b_2]$. Since $[a_3,b_3]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$, there exists $k_3 \in \mathbb{N}^\#, k_3 > k_2,$ such that w_{k_3} is in $[a_3,b_3]$. Continuing this process by hyper infinite induction, we obtain hyper infinite sequence $\{w_{k_n}\}_{n\in\mathbb{N}^\#}$ such that $w_{k_n} \in [a_n,b_n]$ for each $n \in \mathbb{N}^\#$. The sequence $\{w_{k_n}\}_{n\in\mathbb{N}^\#}$ is a subsequence of $\{w_n\}_{n\in\mathbb{N}^\#}$ since $k_{n+1} > k_n$ for each $n \in \mathbb{N}^\#$. Since $a_n \to_\# w,$ and $a_n \leq w_n \leq b_n$ for each $n \in \mathbb{N}^\#$, the squeeze theorem implies that $w_{k_n} \to_\# w$.

Definition 11.12. Let $\{a_n\}$ be a $\mathbb{R}_c^\#$-valued hyper infinite sequence i.e., $a_n \in \mathbb{R}_c^\#, n \in \mathbb{N}^\#$. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N \in \mathbb{N}^\#\backslash\mathbb{N}$, $N = N(\varepsilon)$ such that, for all $n > N$, $|a_n| \leq \varepsilon$. We often denote this symbolically by $a_n \to_\# 0$.

We can also, at this point, define what it means for a hyperreal sequence #-tends to a given number $q \in \mathbb{R}_c^\#$: $\{a_n\}$ #-tends to q if the hyperreal sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_\# 0$.

Definition 11.13. Let $\{a_n\}, n \in \mathbb{N}^\#$ be a hyperreal sequence. We call $\{a_n\}$ a Cauchy hyperreal sequence if the difference between its terms #-tends to 0. To be precise: given any hyperreal number $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N = N(\varepsilon)$ such that for any $m,n > N, |a_n - a_m| < \varepsilon$.

Theorem 11.15. If $\{a_n\}$ is a #-convergent hyperreal sequence (that is, $a_n \to_\# b$ for some hyperreal number $b \in \mathbb{R}_c^\#$), then $\{a_n\}$ is a Cauchy hyperreal sequence.

Theorem 11.16. If $\{a_n\}$ is a Cauchy hyperreal sequence, then it is bounded or hyper bounded; that is, there is some $M \in \mathbb{R}_c^\#$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}^\#$.

Theorem 11.17. Any Cauchy hyperreal sequence $\{a_n\}$ has a #-limit in $\mathbb{R}_c^\#$ i.e.,there exists $b \in \mathbb{R}_c^\#$ such that $a_n \to_\# b$.

Proof.By Definition 11.13 given $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N = N(\varepsilon)$ such that for any $n, n' > N$,

$$|a_n - a_{n'}| < \varepsilon. \tag{11.1}$$

From (11.1) for any $n, n' > N$ we get

$$a_{n'} - \varepsilon < a_n < a_n + \varepsilon. \tag{11.2}$$

The generalized Bolzano-Weierstrass theorem implies there is a #-convergent hyper infinite subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to_\# b$ for some hyperreal number $b \in \mathbb{R}_c^\#$. Let us show that the sequence $\{a_n\}$ also #-convergent to this $b \in \mathbb{R}_c^\#$. We can choose $k \in \mathbb{N}^\#$ so large that $n_k > N$ and

$$|a_{n_k} - b| < \varepsilon. \tag{11.3}$$

We choose now in (11.1) $n' = n_k$ and therefore

$$|a_n - a_{n_k}| < \varepsilon. \tag{11.4}$$

From (11.3) and (11.4) for any $n > N$ we get

$$|(a_{n_k} - b) + (a_n - a_{n_k})| = |a_n - b| < 2\varepsilon. \tag{11.5}$$

Thus $a_n \to_\# b$ as well. .

12.The Extended Hyperreal Number System $\hat{\mathbb{R}}_c^\#$

Definition 12.1.(a) A set $S \subset \mathbb{N}^\#$ is hyperfinite if $card(S) = card(\{x | 0 \leq x \leq n\})$, where $n \in \mathbb{N}^\# \backslash \mathbb{N}$. (b) A set $S \subseteq \mathbb{N}^\#$ is hyper infinite if $card(S) = card(\mathbb{N}^\#)$

Notation 12.1. If F is an arbitrary collection of sets, then $\cup\{S | S \in F\}$is the set of all elements that are members of at least one of the sets in F , and $\cap\{S | S \in F\}$ is the set of all elements that are members of every set in F. The union and intersection of finitely or hyper finitely many sets $S_k, 0 \leq k \leq n \in \mathbb{N}^\#$ are also written as $\cup_{k=0}^n S_k$ and $\cap_{k=0}^n S_k$. The union and intersection of an hyperinfinite sequence $S_k, k \in \mathbb{N}^\#$ of sets are written as $\cup_{k=0}^{\infty^\#} S$ or $\cup_{n \in \mathbb{N}^\#} S$ and $\cap_{k=0}^{\infty^\#} S$ or $\cap_{n \in \mathbb{N}^\#} S$ correspondingly.

A nonempty set S of hyperreal numbers $\mathbb{R}_c^\#$ is unbounded above if it has no hyperfinite upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to the hyperreal number system two points, $+\infty^\#$ (which we also write more simply as $\infty^\#$) and $-\infty^\#$,and to define the order relationships between them and any hyperreal number $x \in \mathbb{R}_c^\#$ by $-\infty^\# < x < \infty^\#$.

We call $-\infty^\#$ and $\infty^\#$ points at hyperinfinity. If S is a nonempty set of hyperreals, we write $\sup S = \infty^\#$ to indicate that S is hyper unbounded above, and $\inf S = -\infty^\#$ to indicate that S is hyper unbounded below. .

12.1.#-Open and #-Closed Sets on $\hat{\mathbb{R}}_c^\#$.

Definition 12.2.If a and b are in the extended hyperreals and $a < b$, then the #-open interval (a,b) is defined by $(a,b) \triangleq \{x | a < x < b\}$. :

The #-open intervals $(a, \infty^\#)$ and $(-\infty^\#, b)$ are semi-hyper infinite if a and b are finite or hyperfinite, and $(-\infty^\#, \infty^\#)$ is the entire hyperreal line.

If $-\infty^\# < a < b < \infty^\#$, the set $[a,b] \triangleq \{x | a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#}, a)$ and $(b, \infty^{\#})$. We say that $[a,b]$ is a #-closed interval. Semi-hyper infinite #-closed intervals are sets of the form $[a,\infty) = \{x|a \leq x\}$ and $(-\infty^{\#}, a] = \{x|x \leq a\}$, where a is finite or hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#}, a)$ and $(a, \infty^{\#})$, respectively.

Definition 12.3.If $x_0 \in \mathbb{R}_c^{\#}$ is a hyperreal number and $\varepsilon > 0, \varepsilon \approx 0$ then the #-open interval

$(x_0 - \varepsilon, x_0 + \varepsilon)$ is an #-neighborhood of x_0. If a set $S \subset \mathbb{R}_c^{\#}$ contains an #-neighborhood of x_0, then S is a #-neighborhood of x_0, and x_0 is an #-interior point of S. The set of #-interior points of S is the #-interior of S, denoted by #-$\mathbf{Int}(S)$.

(i) If every point of S is an #-interior point (that is, $S =$ #-$\mathbf{Int}(S)$), then S is #-open.

(ii) A set S is #-closed if $S^c = \mathbb{R}_c^{\#} \backslash S$ is #-open.

Example 12.1. An open interval (a,b) is an #-open set, because if $x_0 \in (a,b)$ and $\varepsilon \leq \min\{x_0 - a; b - x_0\}$, then $(x_0 - \varepsilon, x_0 + \varepsilon) \subset (a,b)$.

Remark 12.1.The entire hyperline $\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#}, \infty^{\#})$ is #-open, and therefore $\varnothing$ is #-closed.However, $\varnothing$ is also #-open, for to deny this is to say that $\varnothing$ contains a point that is not an #-interior point, which is absurd because $\varnothing$ contains no points. Since $\varnothing$ is #-open, $\hat{\mathbb{R}}_c^{\#}$ is #-closed. Thus, $\hat{\mathbb{R}}_c^{\#}$ and $\varnothing$ are both #-open and #-closed.

Remark 12.2.They are not the only subsets of $\hat{\mathbb{R}}_c^{\#}$ with this property mentioned above.

Definition 12.4.A deleted #-neighborhood of a point x_0 is a set that contains every point of some #-neighborhood of x_0 except for x_0 itself. For example, $S = \{x|0 < |x - x_0| < \varepsilon\}$, where $\varepsilon \approx 0$, is a deleted #-neighborhood of x_0. We also say that it is a deleted ε-#-neighborhood of x_0.

Theorem 12.1.(a) The union of #-open sets is #-open:

(b) The #-intersection of #-closed sets is #-closed:

These statements apply to arbitrary collections, hyperfinite or hyperinfinite, of #-open and #-closed sets.

Proof (a) Let L be a collection of #-open sets and $S = \cup\{G|G \in L\}$.

If $x_0 \in S$, then $x_0 \in G_0$ for some G_0 in L, and since G_0 is #-open, it contains some ε-#-neighborhood of x_0. Since $G_0 \subset S$, this ε-#–neighborhood is in S, which is consequently a #-neighborhood of x_0.Thus, S is a #-neighborhood of each of its points, and therefore #-open, by definition.

(b) Let F be a collection of #-closed sets and $T = \cap\{H|H \in F\}$. Then $T^c = \cup\{H^c|H \in F\}$ and, since each H^c is #-open, T^c is #-open, from (a). Therefore, T is #-closed, by definition.

Example 12.2. If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] = \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#} a)$ and $(b, \infty^{\#})$. We say that $[a,b]$ is a #-closed interval. The set $[a,b) = \{x|a \leq x < b\}$ is a half-#-closed or half-#-open interval if $-\infty^{\#} < a < b < \infty^{\#}$, as is $(a,b] = \{x|a < x \leq b\}$ however, neither of these sets is #-open or #-closed. Semi-infinite #-closed intervals are sets of the form $[a, \infty^{\#}) = \{x|a \leq x\}$ and $(-\infty^{\#}, a] = \{x|x \leq a\}$, where a is hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#}, a)$ and $(a, \infty^{\#})$,respectively.

Definition 12.5. Let S be a subset of $\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#}, \infty^{\#})$. Then

(a) x_0 is a #-limit point of S if every deleted #-neighborhood of x_0 contains a point of S.

(b) x_0 is a boundary point of S if every #-neighborhood of x_0 contains at least one point in S and one not in S. The set of #-boundary points of S is the #-boundary of S, denoted

by #-∂S. The #-closure of S, denoted by #-$\bar{S}$, is $S \cup$ #-∂S.

(c) x_0 is an #-isolated point of S if $x_0 \in S$ and there is a #-neighborhood of x_0 that contains no other point of S.

(d) x_0 is #-exterior to S if x_0 is in the #-interior of S^c. The collection of such points is the #-exterior of S.

Theorem 12.2. A set S is #-closed if and only if no point of S^c is a #-limit point of S.

Proof. Suppose that S is #-closed and $x_0 \in S^c$. Since S^c is #-open, there is a #-neighborhood of x_0 that is contained in S^c and therefore contains no points of S. Hence, x_0 cannot be a #-limit point of S. For the converse, if no point of S^c is a #-limit point of S then every point in S^c must have a #-neighborhood contained in S^c. Therefore, S^c is #-open and S is #-closed.

Corollary 12.1.A set S is #-closed if and only if it contains all its #-limit points.

If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S.

Proposition 12.1. If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S. .

12.2. #-Open Coverings

Definition 12.6.A collection H of #-open sets of $\mathbb{R}_c^{\#}$ is an #-open covering of a set S if every point in S is contained in a set H belonging to H; that is, if $S \subset \cup\{F|F \in H\}$.

Definition 12.7.A set $S \subset \mathbb{R}_c^{\#}$ is called #-compact (or hyper compact) if each of its #-open covers has a finite or hyperfinite subcover. .

Theorem 12.3.(**Generalized Heine–Borel Theorem**) If H is an #-open covering of a #-closed and hyper bounded subset S of the hyperreal line $\mathbb{R}_c^{\#}$ (or of the $\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$) then S has an #-open covering $\widetilde{H}$ consisting of hyper finite many #-open sets belonging to H.

Proof. If a set S in $\mathbb{R}_c^{\#n}$ is hyper bounded, then it can be enclosed within an n-box $T_0 = [-a,a]^n$ where $a > 0$. By the property above, it is enough to show that T_0 is #-compact.

Assume, by way of contradiction, that T_0 is not #-compact. Then there exists an hyper infinite open cover $C_{\infty^{\#}}$ of T_0 that does not admit any hyperfinite subcover. Through bisection of each of the sides of T_0, the box T_0 can be broken up into $2n$ sub n-boxes, each of which has diameter equal to half the diameter of T_0. Then at least one of the $2n$ sections of T_0 must require an hyper infinite subcover of $C_{\infty^{\#}}$, otherwise $C_{\infty^{\#}}$ itself would have a hyperfinite subcover, by uniting together the hyperfinite covers of the sections. Call this section T_1. Likewise, the sides of T_1 can be bisected, yielding 2^n sections of T_1, at least one of which must require an hyper infinite subcover of $C_{\infty^{\#}}$. Continuing in like manner yields a decreasing hyper infinite sequence of nested n-boxes:

$T_0 \supset T_1 \supset T_2 \supset \ldots \supset T_k \supset \ldots, k \in \mathbb{N}^{\#}$, where the side length of T_k is $(2a)/2^k$, which #-converges to 0 as k tends to hyper infinity, $k \to \infty^{\#}$. Let us define a hyper infinite sequence $\{x_k\}_{k \in \mathbb{N}^{\#}}$ such that each $x_k : x_k \in T_k$. This hyper infinite sequence is Cauchy, so it must #-converge to some #-limit L. Since each $T_k$is #-closed, and for each k the sequence $\{x_k\}_{k \in \mathbb{N}^{\#}}$ is eventually always inside T_k, we see that $L \in T_k$ for each $k \in \mathbb{N}^{\#}$. Since $C_{\infty^{\#}}$ covers T_0, then it has some member $U \in C_{\infty^{\#}}$ such that $L \in U$.

Since U is open, there is an n-ball $B(L) \subseteq U$. For large enough k, one has $T_k \subseteq B(L) \subseteq U$, but then the infinite number of members of $C_{\infty^{\#}}$ needed to cover T_k can be replaced by just one: U, a contradiction.Thus, T_0 is #-compact. Since S is #-closed and a subset of the #-compact set T_0, then S is also #-compact.
As an application of the Generalized Heine–Borel theorem, we give a short proof of the Generalized Bolzano–Weierstrass Theorem.
Theorem 12.4.(Generalized Bolzano–Weierstrass Theorem) Every hyper bounded hyper infinite set $S \subset \mathbb{R}_c^{\#}$ has at least one #-limit point.
Proof. We will show that a hyper bounded nonempty set without a #-limit point can contain only finite or a hyper finite number of points. If S has no #-limit points, then S is #-closed and every point $x \in S$ has an w-#-open neighborhood N_x that contains no point of S other than x. The collection $H = \{N_x | x \in S\}$ is an w-#-open covering for S. Since S is also hyper bounded, Theorem 12.3 implies that S can be covered by finite or a hyper finite collection of sets from H, say $N_{x_1},\ldots,N_{x_n}, n \in \mathbb{N}^{\#}$. Since these sets contain only $x_1,\ldots,x_n$ from S, it follows that $S = \{x_k\}_{1 \leq k \leq n}, n \in \mathbb{N}^{\#}$.

13.External non-Archimedean field ${}^*\mathbb{R}_c^{\#}$. via Cauchy completion of internal non-Archimedean field ${}^*\mathbb{R}$.

Definition 13.1. A hyper infinite sequence of hyperreal numbers from ${}^*\mathbb{R}$ is a function $a : \mathbb{N}^{\#} \to {}^*\mathbb{R}$ from hypernatural numbers $\mathbb{N}^{\#}$ into the hyperreal numbers ${}^*\mathbb{R}$.
We usually denote such a function by $n \mapsto a_n$, or by $a : n \to a_n$, so the terms in the sequence are written $\{a_1, a_2, a_3, \ldots, a_n \ldots\}$. To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{\infty^{\#}}$, or $\{a_n\}_{n \in \mathbb{N}^{\#}}$, or for the sake of brevity simply $\{a_n\}$.
Definition 13.2. Let $\{a_n\}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N \in \mathbb{N}^{\#} \backslash \mathbb{N}$, $N = N(\varepsilon)$ such that, after N (i.e.for all $n > N$), $|a_n| \leq \varepsilon$. We denote this symbolically by $a_n \to_{\#} 0$.
We can also, at this point, define what it means for a hyper infinite ${}^*\mathbb{R}$-valued sequence #-tends to any given number $q \in {}^*\mathbb{R}$: $\{a_n\}$ #-tends to q if the hyper infinite sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_{\#} 0$.
Definition 13.3. Let $\{a_n\}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence. We call $\{a_n\}$ a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence if the difference between its terms #-tends to 0. To be precise: given any hyperreal number such that $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m, n > N, |a_n - a_m| < \varepsilon$.
Theorem 13.1.If $\{a_n\}$ is a #-convergent hyper infinite ${}^*\mathbb{R}$-valued sequence (that is, $a_n \to_{\#} q$ for some hyperreal number $q \in {}^*\mathbb{R}$), then $\{a_n\}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence.
Proof. We know that $a_n \to_{\#} q$. Here is a ubiquitous trick: instead of using ε in the definition Definition 13.3, start with an arbitrary infinite small $\varepsilon > 0, \varepsilon \approx 0$ and then choose $N \in \mathbb{N}^{\#} \backslash \mathbb{N}$ so that $|a_n - q| < \varepsilon/2$ when $n > N$. Then if $m, n > N$, we have $|a_n - a_m| = |(a_n - q) - (a_m - q)| \leq |a_n - q| + |a_m - q| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This shows that $\{a_n\}_{n \in \mathbb{N}^{\#}}$ is a Cauchy sequence.
Theorem 13.2. If $\{a_n\}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence, then it is bounded or hyper bounded; that is, there is some finite or hyperfinite $M \in {}^*\mathbb{R}$ such

that $|a_n| \leq M$ for all $n \in \mathbb{N}^{\#}$.

Proof.Since $\{a_n\}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some $N \in \mathbb{N}^{\#}$ such that $|a_m - a_n| < 1$whenever $m,n > N$. Thus, $|a_{N+1} - a_n| < 1$ for $n > N$. We can rewrite this as $a_{N+1} - 1 < a_n < a_{N+1} + 1$.This means that $|a_n|$ is less than the maximum of $|a_{N+1} - 1|$ and $|a_{N+1} + 1|$. So, set M equal to the maximum number in the following list: $\{|a_0|, |a_1|, \ldots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\}$.Then for any term a_n, if $n \leq N$, then $|a_n|$ appears in the list and so $|a_n| \leq M$; if $n > N$, then
(as shown above) $|a_n|$ is less than at least one of the last two entries in the list, and so $|a_n| \leq M$.Hence, $M \in {}^*\mathbb{R}$ is a bound for the sequence $\{a_n\}$.

Definition 13.4. Let S be a set. A relation $x \sim y$ among pairs of elements of S is said to be an equivalence relation if the following three properties hold:
Reflexivity: for any $s \in S, s \sim s$.
Symmetry: for any $s,t \in S$, if $s \sim t$ then $t \sim s$.
Transitivity: for any $s,t,r \in S$, if $s \sim t$ and $t \sim r$, then $s \sim r$.

Theorem 13.3. Let S be a set, with an equivalence relation $(\bullet \sim \bullet)$ on pairs of elements. For $s \in S$,denote by $\mathbf{cl}[s]$ the set of all elements in S that are related to s. Then for any $s,t \in S$,either $\mathbf{cl}[s] = \mathbf{cl}[t]$ or $\mathbf{cl}[s]$ and $\mathbf{cl}[t]$ are disjoint.

The hyperreal numbers ${}^*\mathbb{R}_c^{\#}$ will be constructed as equivalence classes of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences. Let $\mathcal{F}_{{}^*\mathbb{R}}$ denote the set of all Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers. We define the equivalence relation on $\mathcal{F}_{{}^*\mathbb{R}}$.

Definition 13.5. Let $\{a_n\}$ and $\{b_n\}$ be in $\mathcal{F}_{{}^*\mathbb{R}}$. Say they are #-equivalent if $a_n - b_n \to_{\#} 0$ i.e., if and only if the hyper infinite ${}^*\mathbb{R}$-valued sequence $a_n - b_n$ #-tends to 0.

Theorem 13.4.Definition 13.5 yields an equivalence relation on $\mathcal{F}_{{}^*\mathbb{R}}$.

Proof. We need to show that this relation is reflexive, symmetric, and transitive.
Reflexive: $a_n - a_n = 0$, and the hyper infinite sequence all of whose terms are 0 clearly #-converges to 0. So $\{a_n\}$ is related to $\{a_n\}$.
Symmetric: Suppose $\{a_n\}$ is related to $\{b_n\}$, so $a_n - b_n \to_{\#} 0$.
But $b_n - a_n = -(a_n - b_n)$,and since only the absolute value $|a_n - b_n| = |b_n - a_n|$ comes into play in Definition 13.2, it follows that $b_n - a_n \to_{\#} 0$ as well. Hence, $\{b_n\}$ is related to $\{a_n\}$.
Transitive: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 10.1. Suppose $\{a_n\}$ is related to $\{b_n\}$, and $\{b_n\}$ is related to $\{c_n\}$. This means that $a_n - b_n \to_{\#} 0$ and $b_n - c_n \to_{\#} 0$.To be fully precise, let us fix $\varepsilon > 0, \varepsilon \approx 0$; then there exists an $N \in \mathbb{N}^{\#}$ such that for all $n > N, |a_n - b_n| < \varepsilon/2$; also, there exists an M such that for all $n > M$, $|b_n - c_n| < \varepsilon/2$. Well, then, as long as n is bigger than both N and M, we have that $|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
So, choosing L equal to the max of N,M, we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |a_n - c_n| < \varepsilon$. This means that $a_n - c_n \to_{\#} 0$ – i.e. $\{a_n\}$ is related to $\{c_n\}$.

Definition 13.6. The external hyperreal numbers ${}^*\mathbb{R}_c^{\#}$ are the equivalence classes $\mathbf{cl}[\{a_n\}]$ of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers, as per Definition 13.5. That is, each such equivalence class is an external hyperreal number.

Definition 13.7. Given any hyperreal number $q \in {}^*\mathbb{R}$, define a hyperreal number $q^{\#}$ to be the equivalence class of the hyper infinite ${}^*\mathbb{R}$-valued sequence $q^{\#} = (q,q,q,q,\ldots)$

consisting entirely of q. So we view $^*\mathbb{R}$ as being inside $^*\mathbb{R}_c^\#$ by thinking of each hyperreal number q as its associated equivalence class $q^\#$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

Definition 13.8. Let $s,t \in {}^*\mathbb{R}_c^\#$, so there are Cauchy hyper infinite $^*\mathbb{R}$-valued sequences $\{a_n\},\{b_n\}$ of hyperreal numbers with $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$.

(a) Define $s+t$ to be the equivalence class of the sequence $\{a_n + b_n\}$.

(b) Define $s \times t$ to be the equivalence class of the sequence $\{a_n \times b_n\}$.

Theorem 13.5.The operations $+,\times$ in Definition 13.8 (a),(b) are well-defined.

Proof. Suppose that $\mathbf{cl}[\{a_n\}] = \mathbf{cl}[\{c_n\}]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$. Thus means that $a_n - c_n \to_\# 0$ and $b_n - d_n \to_\# 0$. Then $(a_n + b_n) - (c_n + d_n) = (a_n - c_n) + (b_n - d_n)$. Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to 0, and so $\mathbf{cl}[\{a_n + b_n\}] = \mathbf{cl}[\{c_n + d_n\}]$.

Multiplication is a little trickier; this is where we will use Theorem 13.3. We will also use another ubiquitous technique: adding 0 in the form of $s - s$. Again, suppose that $\mathbf{cl}[(a_n)] = \mathbf{cl}[(c_n)]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$; we wish to show that $\mathbf{cl}[\{a_n \times b_n\}] = \mathbf{cl}[\{c_n \times d_n\}]$, or, in other words, that $a_n \times b_n - c_n \cdot d_n \to_\# 0$. Well, we add and subtract one of the other cross terms, say

$b_n \times c_n$: $a_n \times b_n - c_n \times d_n = a_n \times b_n + (b_n \times c_n - b_n \times c_n) - c_n \times d_n =$

$= (a_n \times b_n - b_n \times c_n) + (b_n \times c_n - c_n \times d_n) = b_n \times (a_n - c_n) + c_n \times (b_n - d_n)$.

Hence, we have $|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n|$. Now, from Theorem 13.2, there are numbers M and L such that $|b_n| \leq M$ and $|c_n| \leq L$ for all $n \in \mathbb{N}^\#$. Taking some number K which is bigger than both, we have

$|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n| \leq K(|a_n - c_n| + |b_n - d_n|)$.

Now, noting that both $a_n - c_n$ and $b_n - d_n$ tend to 0 and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2K$), we see that $a_n \times b_n - c_n \times d_n \to_\# 0$.

Theorem 13.6. Given any hyperreal number $s \in {}^*\mathbb{R}_c^\#$, $s \neq 0$, there is a hyperreal number $t \in {}^*\mathbb{R}_c^\#$ such that $s \times t = 1$.

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $\{0,0,0,0,\ldots\}$. In other words, $s = \mathbf{cl}[\{a_n\}]$ where $a_n - 0$ does not #-converge to 0. From this, we are to deduce the existence of a hyperreal number $t = \mathbf{cl}[\{b_n\}]$ such that $s \times t = \mathbf{cl}[\{a_n \times b_n\}]$ is the same equivalence class as $\mathbf{cl}[\{1,1,1,1,\ldots\}]$. Doing so is actually an easy consequence of the fact that nonzero hyperreal numbers have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{a_n\}$ does not tend to 0), there's no reason any number of the terms in $\{a_n\}$ can't equal 0. However, it turns out that eventually, $a_n \neq 0$.

That is:

Lemma 13.1. If $\{a_n\}$ is a Cauchy sequence which does not #-tend to 0, then there is an $N \in \mathbb{N}^\#$ such that, for $n > N, a_n \neq 0$.

Definition 13.9. Let $s \in {}^*\mathbb{R}_c^\#$. Say that s is positive if $s \neq 0$, and if $s = \mathbf{cl}[\{a_n\}]$ for some Cauchy sequence of hyperreal numbers such that for some $N \in \mathbb{N}^\#, a_n > 0$ for all $n > N$. Given two hyperreal numbers s, t, say that $s > t$ if $s - t$ is positive.

Theorem 13.7. Let $s,t \in {}^*\mathbb{R}_c^\#$ be hyperreal numbers such that $s > t,$ and let $r \in {}^*\mathbb{R}_c^\#$. Then $s+r > t+r.$

Proof. Let $s = \mathbf{cl}[\{a_n\}], t = \mathbf{cl}[\{b_n\}],$ and $r = \mathbf{cl}[\{c_n\}].$ Since $s > t$ i.e., $s-t > 0,$ we know that there is an $N \in \mathbb{N}^\#$ such that, for $n > N,$ $a_n - b_n > 0.$ So $a_n > b_n$ for $n > N.$ Now, adding c_n to both sides of this inequality (as we know we can do for hyperreal numbers ${}^*\mathbb{R}$), we have $a_n + c_n > b_n + c_n$ for $n > N,$ or $(a_n + c_n) - (b_n + c_n) > 0$ for $n > N.$ Note also that $(a_n + c_n) - (b_n + c_n) = a_n - b_n$ does not #-converge to $0,$ by the assumption that $s - t > 0.$ Thus, by Definition 13.8, this means that $s + r = \mathbf{cl}[\{a_n + c_n\}] > \mathbf{cl}[\{b_n + c_n\}] = t + r.$

Theorem 13.8. Let $s,t \in {}^*\mathbb{R}_c^\#$ $s,t > 0$ be hyperreal numbers.Then there is $m \in \mathbb{N}^\#$ such that $m \times s > t.$

Proof. Let $s,t > 0$ be hyperreal numbers. We need to find a natural number m so that $m \times s > t.$ First, recall that, by m in this context, we mean $\mathbf{cl}[\{m,m,m,m,\ldots\}].$ So, letting $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}],$what we need to show is that there exists m with

$$\mathbf{cl}[\{m,m,m,m,\ldots\}] \times \mathbf{cl}[\{a_1,a_2,a_3,a_4,\ldots\}] =$$
$$\mathbf{cl}[\{m \times a_1, m \times a_2, m \times a_3, m \times a_4,\ldots\}] >$$
$$> \mathbf{cl}[\{b_1,b_2,b_3,b_4,\ldots\}].$$

Now, to say that $\mathbf{cl}[\{m \times a_n\}] > \mathbf{cl}[\{b_n\}],$ or $\mathbf{cl}[\{m \times a_n - b_n\}]$ is positive, is, by Definition 13.9, just to say that there is $N \in \mathbb{N}^\#$ such that $m \times a_n - b_n > 0$ for all $n > N,$ while $m \times a_n - b_n \nrightarrow_\# 0.$ To be precise, the first statement is:

There exist $m,N \in \mathbb{N}^\#$ so that $m \times a_n > b_n$ for all $n > N.$

To produce a contradiction, we assume this is not the case; assume that

(#) for every m and $N,$ there exists an $n > N$ so that $m \times a_n \leq b_n.$

Now, since $\{b_n\}$ is a Cauchy sequence, by Theorem 13.2 it is hyperbounded - there is a hyperreal number $M \in {}^*\mathbb{R}$ such that $b_n \leq M$ for all $n \in \mathbb{N}^\#.$ Now, by the properties for the hyperreal numbers ${}^*\mathbb{R},$ given any hyperreal number such that $\varepsilon > 0, \varepsilon \approx 0,$ there is an $m \in \mathbb{N}^\#$ such that $M/m < \varepsilon/2.$ Fix such an $m.$ Then if $m \times a_n \leq b_n,$ we have $a_n \leq b_n/m \leq M/m < \varepsilon/2.$

Now, $\{a_n\}$ is a Cauchy sequence, and so there exists N so that for $n,k > N, |a_n - a_k| < \varepsilon/2.$

By Asumption (#), we also have an $n > N$ such that $m \times a_n \leq b_n,$ which means that $a_n < \varepsilon/2.$ But then for every $k > N,$ we have that $a_k - a_n < \varepsilon/2,$ so $a_k < a_n + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$ Hence, $a_k < \varepsilon$ for all $k > N.$ This proves that $a_k \rightarrow_\# 0,$ which by Definition 13.9 contradicts the fact that $\mathbf{cl}[\{a_n\}] = s > 0.$

Thus, there is indeed some $m \in N$ so that $m \times a_n - b_n > 0$ for all sufficiently infinite large $n \in \mathbb{N}^\# \backslash \mathbb{N}$. To conclude the proof, we must also show that $m \times a_n - b_n \nrightarrow_\# 0.$ Actually, it is possible that $m \times a_n - b_n \rightarrow_\# 0$ (for example if $\{a_n\} = \{1,1,1,\ldots\}$ and $\{b_n\} = \{m,m,m,\ldots\}$). But that's okay: then we can simply choose a larger $m.$ That is: let m be a hypernatural number constructed as above, so that $m \times a_n - b_n > 0$ for all sufficiently large $\in \mathbb{N}^\# \backslash \mathbb{N}$. If it happens to be true that $m \times a_n - b_n \nrightarrow_\# 0,$ then the proof is complete.

If, on the other hand, it turned out that $m \times a_n - b_n \rightarrow_\# 0,$ then take instead the integer $m+1.$Since $s = \mathbf{cl}[\{a_n\}] > 0,$ we have a n > 0 for all infinite large n, so $(m+1) \times a_n - b_n = m \times a_n - b_n + a_n > a_n > 0$ for all infinite large n, so $m+1$ works just as well as m did in this regard; and since $m \times a_n - b_n \rightarrow 0,$ we have

$(m+1) \times a_n - b_n = (m \times a_n - b_n) + a_n \nrightarrow_{\#} 0$ since $s = \mathbf{cl}[\{a_n\}] > 0$ (so $a_n \nrightarrow_{\#} 0$).
It will be handy to have one more Theorem about how the hyperreals ${}^*\mathbb{R}$ and hyperreals ${}^*\mathbb{R}_c^\#$ compare before we proceed. This theorem is known as the density of ${}^*\mathbb{R}$ in ${}^*\mathbb{R}_c^\#$, and it follows almost immediately from the construction of the ${}^*\mathbb{R}_c^\#$ from ${}^*\mathbb{R}$.

Theorem 13.9. Given any hyperreal number $r \in {}^*\mathbb{R}_c^\#$, and any hyperreal number $\varepsilon > 0,\ \varepsilon \approx 0,$ there is a hyperreal number $q \in {}^*\mathbb{R}$ such that $|r - q| < \varepsilon$.

Proof. The hyperreal number r is represented by a Cauchy ${}^*\mathbb{R}$-valued sequence $\{a_n\}$. Since this sequence is Cauchy, given $\varepsilon > 0, \varepsilon \approx 0,$ there is $N \in \mathbb{N}^\#$ so that for all $m,n > N,$
$|a_n - a_m| < \varepsilon$. Picking some fixed $l > N,$ we can take the hyperreal number q given by $q = \mathbf{cl}[\{a_l, a_l, a_l, \ldots\}]$. Then we have $r - q = \mathbf{cl}[\{a_n - a_l\}_{n\in\mathbb{N}^\#}]$, and $q - r = \mathbf{cl}[\{a_l - a_n\}_{n\in\mathbb{N}^\#}]$.
Now, since $l > N,$ we see that for $n > N, a_n - a_l < \varepsilon$ and $a_l - a_n < \varepsilon,$ which means by Definition 13.9 that $r - q < \varepsilon$ and $q - r < \varepsilon;$ hence, $|r - q| < \varepsilon$.

Definition 13.10.Let $S \subsetneqq {}^*\mathbb{R}_c^\#$ be a non-empty set of hyperreal numbers.
A hyperreal number $x \in {}^*\mathbb{R}_c^\#$ is called an upper bound for S if $x \geq s$ for all $s \in S$.
A hyperreal number x is the least upper bound (or supremum $\sup S$) for S if x is an upper
bound for S and $x \leq y$ for every upper bound y of S.

Remark 13.1.The order $\leq$ given by Definition 10.9 obviously is $\leq$ -incomplete.

Definition 13.11. Let $S \subsetneqq {}^*\mathbb{R}_c^\#$ be a nonempty subset of ${}^*\mathbb{R}_c^\#$. We we will say that:
(1) S is $\leq$ -admissible above if the following conditions are satisfied:
(i) S bounded or hyperbounded above;
(ii) let $A(S)$ be a set $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$.
(2) S is $\leq$-admissible belov if the following condition are satisfied:
(i) S bounded belov;
(ii) let $L(S)$ be a set $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.

Theorem 13.10. (i) Any $\leq$-admissible above subset $S \subset {}^*\mathbb{R}_c^\#$ has the least upper bound property.(ii) Any $\leq$-admissible below subset $S \subset {}^*\mathbb{R}_c^\#$ has the greatest lower bound property.

Proof. Let $S \subset {}^*\mathbb{R}_c^\#$ be a nonempty subset, and let M be an upper bound for S. We are going to construct two sequences of hyperreal numbers, $\{u_n\}$ and $\{l_n\}$. First, since S is nonempty, there is some element $s_0 \in S$. Now, we go through the following hyperinductive procedure to produce numbers $u_0, u_1, u_2, \ldots, u_n, \ldots$ and $l_1, l_2, l_3, \ldots, l_n, \ldots$
(i) Set $u_0 = M$ and $l_0 = s$.
(ii) Suppose that we have already defined u_n and l_n. Consider the number $m_n = (u_n + l_n)/2,$ the average between u_n and l_n.
(1) If m_n is an upper bound for $S,$ define $u_{n+1} = m_n$ and $l_{n+1} = l_n$.
(2) If m_n is not an upper bound for $S,$ define $u_{n+1} = u_n$ and $l_{n+1} = l_n$.

Remark 13.1.Since $s < M,$ it is easy to prove by hyper infinite induction that
(i) $\{u_n\}$ is a non-increasing sequence: $u_{n+1} \leq u_n, n \in \mathbb{N}^\#$and $\{l_n\}$ is a non-decreasing sequence $l_{n+1} \geq l_n, n \in \mathbb{N}^\#,$ (ii) u_n is an upper bound for S for all $n \in \mathbb{N}^\#$ and l_n is never an upper bound for S for any $n \in \mathbb{N}^\#$,(iii) $u_n - l_n = 2^{-n}(M - s)$.

This gives us the following lemma.

Lemma 13.2. $\{u_n\}$ and $\{l_n\}$ are Cauchy ${}^*\mathbb{R}$-valued sequences of hyperreal numbers.

Proof. Note that each $l_n \leq M$ for all $n \in \mathbb{N}^\#$. Since $\{l_n\}$ is non-decreasing and $u_n - l_n = 2^{-n}(M-s)$, it follows directly that $\{l_n\}$ is Cauchy.

For $\{u_n\}$, we have $u_n \geq s_0$ for all $n \in \mathbb{N}^\#$, and so $-u_n \leq -s_0$.

Since $\{u_n\}$ is non-increasing, $\{-u_n\}$ is non-decreasing, and so as above, $\{-u_n\}$ is Cauchy. It is easy to verify that, therefore, $\{u_n\}$ is Cauchy.

The following Lemma shows that $\{u_n\}$ does #-tend to a hyperreal number $u \in {}^*\mathbb{R}^\#_c$.

Lemma 13.3. There is a hyperreal number $u \in {}^*\mathbb{R}^\#_c$ such that $u_n \to_\# u$.

Proof. Fix a term u_n in the sequence $\{u_n\}$. By Theorem 13.9, there is a hyperreal number $q_n \in {}^*\mathbb{R}, n \in \mathbb{N}^\#$ such that $|u_n - q_n| < 1/n$. Consider the sequence $\{q_1, q_2, q_3, \ldots, q_n, \ldots\}$ of hyperreal numbers. We will show this sequence is Cauchy. Fix $\varepsilon > 0, \varepsilon \approx 0$. By the Theorem 13.8, we can choose $N \in \mathbb{N}^\#$ so that $1/N < \varepsilon/3$. We know, since $\{u_n\}$ is Cauchy, that there is an $M \in \mathbb{N}^\#$ such that for $n,m > M$, $|u_n - u_m| < \varepsilon/3$. Then, so long as $n,m > max\{N,M\}$, we have

$$|q_n - q_m| = |(q_n - u_n) + (u_n - u_m) + (u_m - q_m)| \leq$$
$$\leq |q_n - u_n| + |u_n - u_m| + |u_m - q_m| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, $\{q_n\}$ is a Cauchy sequence of internal hyperreal numbers, and so it represents the external hyperreal number $u = \mathbf{cl}[\{q_n\}]$. We must show that $u_n - u \to_\# 0$, but this is practically built into the definition of u. To be precise, letting q_n^* be the hyperreal number
$\mathbf{cl}[\{q_n, q_n, q_n, \ldots\}]$, we see immediately that $q_n^* - u \to_\# 0$ (this is precisely equivalent to the statement that $\{q_n\}$ is Cauchy). But $u_n - q_n^* < 1/n$ by construction; it is easily verify that the assertion that if a sequence $q_n^* \to_\# u$ and $u_n - q_n^* \to_\# 0$, then $u_n \to_\# u$. So $\{u_n\}$, a non-increasing sequence of upper bounds for S, tends to a hyperreal
number u. As you've guessed, u is the least upper bound of our set S. To prove this, we
need one more lemma.

Lemma 13.4. $l_n \to_\# u$.

Proof. First, note in the first case above, we have that

$$u_{n+1} - l_{n+1} = m_n - l_n = \frac{u_n + l_n}{2} - l_n = \frac{u_n - l_n}{2}.$$

In the second case, we also have

$$u_{n+1} - l_{n+1} = u_n - m_n = u_n - \frac{u_n + l_n}{2} = \frac{u_n - l_n}{2}.$$

Now, this means that $u_1 - l_1 = \frac{1}{2}(M-s)$, and so $u_2 - l_2 = \frac{1}{2}(u_1 - l_1) = \frac{1}{2^2}(L-s)$, and in general by hyperinfinite induction, $u_n - l_n = 2^{-n}(M-s)$. Since $M > s$ so $M - s > 0$, and since $2^{-n} < 1/n$, by the Theorem 13.8, we have for any $\varepsilon > 0$ that $2^{-n}(M-s) < \varepsilon$ for all sufficiently large $n \in \mathbb{N}^\#$. Thus, $u_n - l_n = 2^{-n}(M-s) < \varepsilon$ as well, and so $u_n - l_n \to_\# 0$. Again, it is easily verify that, since $u_n \to_\# u$, we have $l_n \to_\# u$ as well.

Remark 13.2.Note that assumption in Theorem 13.10 that S is $\leq$-admissible above subset of $\mathbb{R}^\#_c$ is necessarily, othervice Theorem 13.10 is not holds.

Theorem 13.11.(**Generalized Nested Intervals Theorem**)

Let $\{I_n\}_{n\in\mathbb{N}^\#} = \{[a_n,b_n]\}_{n\in\mathbb{N}^\#}, [a_n,b_n] \subset \mathbb{R}_c^\#$ be a hyper infinite sequence of closed intervals satisfying each of the following conditions:
(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq\ldots\supseteq I_n \supseteq\ldots,$
(ii) $b_n - a_n \to_\# 0$ as $n \to \infty^\#$.
Then $\bigcap_{n=1}^{\infty^\#} I_n$ consists of exactly one hyperreal number $x \in \mathbb{R}_c^\#$. Moreover both sequences $\{a_n\}$ and $\{b_n\}$ #-converge to x.
Proof.Note that: (a) the set $A = \{a_n|n \in \mathbb{N}^\#\}$ is hyperbouded above by $b_1$and
(b) the set $A = \{a_n|n \in \mathbb{N}^\#\}$ is $\leq$-admissible above subset of $\mathbb{R}_c^\#$.
By Theorem 13.10 there exists $\sup A$. Let $\xi = \sup A$.
Since I_n are nested,for any positive hyperintegers m and n we have
$a_m \leq a_{m+n} \leq b_{m+n} \leq b_n$,so that $\xi \leq b_n$ for each $n \in \mathbb{N}^\#$. Since we obviously have $a_n \leq \xi$ for each $n \in \mathbb{N}^\#$,we have $a_n \leq \xi \leq b_n$ for all $n \in \mathbb{N}^\#$,which implies $\xi \in \bigcap_{n=1}^{\infty^\#} I_n$. Finally, if $\xi,\eta \in \bigcap_{n=1}^{\infty^\#} I_n$, with $\xi \leq \eta$, then we get $0 \leq \eta - \xi \leq b_n - a_n$,for all $n \in \mathbb{N}^\#$,so that
$0 \leq \eta - \xi \leq \inf_{n\in\mathbb{N}^\#}|b_n - a_n| = 0$.
Theorem 13.12.(**Generalized Squeeze Theorem**)
Let $\{a_n\},\{c_n\}$ be two hyper infinite sequences #-converging to L,and $\{b_n\}$ a hyper infinite sequence. If $\forall n \geq K, K \in \mathbb{N}^\#$ we have $a_n \leq b_n \leq c_n$, then $\{b_n\}$ also #-converges to L.
Proof. Choose an $\varepsilon > 0, \varepsilon \approx 0$. By definition of the #-limit,there is an $N_1 \in \mathbb{N}^\#$ such that for all $n > N_1$ we have $|a_n - L|< \varepsilon$, in other words $L - \varepsilon < a_n < L + \varepsilon$.Similarly, there is an $N_2 \in \mathbb{N}^\#$ such that for all $n > N_2$ we have $L - \varepsilon < c_n < L + \varepsilon$. Denote $N = \max(N_1,N_2,K)$. Then for $n > N, L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$, in other words $|b_n - L|< \varepsilon$. Since $\varepsilon > 0, \varepsilon \approx 0$ was arbitrary, by definition of the #-limit this says that #-$\lim_{n\to\infty^\#} b_n = L$.
Theorem 13.13.(**Corollary of the Generalized Squeeze Theorem**).
If #-$\lim_{n\to\infty^\#}|a_n|= 0$ then #-$\lim_{n\to\infty^\#} a_n = 0$.
Proof.We know that $-|a_n|\leq a_n \leq |a_n|$.We want to apply the Generalized Squeeze Theorem.We are given that #-$\lim_{n\to\infty^\#}|a_n|= 0$.This also implies that #-$\lim_{n\to\infty^\#}(-|a_n|) = 0$. So by the Generalized Squeeze Theorem, #-$\lim_{n\to\infty^\#} a_n = 0$.
Theorem 13.14. (**Generalized Bolzano**-**Weierstrass Theorem**)
Every hyperbounded hyper infinite ${}^*\mathbb{R}_c^\#$-valued sequence has a #-convergent hyper infinite subsequence.
Proof. Let $\{w_n\}_{n\in\mathbb{N}^\#}$ be a hyperbounded hyper infinite sequence. Then, there exists an interval $[a_1,b_1]$ such that $a_1 \leq w_n \leq b_1$ for all $n \in \mathbb{N}^\#$.
Either $\left[a_1,\frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2},b_1\right]$ contains hyper infinitely many terms of $\{w_n\}$. That is, there exists hyper infinitely many n in $\mathbb{N}^\#$ such that a_n is in $\left[a_1,\frac{a_1+b_1}{2}\right]$ or there exists hyper infinitely many n in $\mathbb{N}^\#$ such that a_n is in $\left[\frac{a_1+b_1}{2},b_1\right]$. If $\left[a_1,\frac{a_1+b_1}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_2,b_2] = \left[a_1,\frac{a_1+b_1}{2}\right]$. Otherwise, let $[a_2,b_2] = \left[\frac{a_1+b_1}{2},b_1\right]$.
Either $\left[a_2,\frac{a_2+b_2}{2}\right]$ or $\left[\frac{a_2+b_2}{2},b_2\right]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$. If $\left[a_2,\frac{a_2+b_2}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_3,b_3] = \left[a_2,\frac{a_2+b_2}{2}\right]$.
Otherwise, let $[a_3,b_3] = \left[\frac{a_2+b_2}{2},b_2\right]$. By hyper infinite induction, we can continue this construction and obtain hyper infinite sequence of intervals $\{[a_n,b_n]\}_{n\in\mathbb{N}^\#}$ such that:
(i) for each $n \in \mathbb{N}^\#, [a_n,b_n]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$,

(ii) for each $n \in \mathbb{N}^{\#}, [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and
(iii) for each $n \in \mathbb{N}^{\#}, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.
Then generalized nested intervals theorem implies that the intersection of all of the intervals $[a_n, b_n]$ is a single point w. We will now construct a hyper infinite subsequence of $\{w_n\}_{n \in \mathbb{N}^{\#}}$ which will #-converge to w.
Since $[a_1, b_1]$ contains hyper infinitely many terms of $\{w_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_1 \in \mathbb{N}^{\#}$ such that w_{k_1} is in $[a_1, b_1]$. Since $[a_2, b_2]$ contains hyper infinitely many terms of $\{w_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_2 \in \mathbb{N}^{\#}, k_2 > k_1$, such that w_{k_2} is in $[a_2, b_2]$. Since $[a_3, b_3]$ contains hyper infinitely many terms of $\{w_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_3 \in \mathbb{N}^{\#}, k_3 > k_2$, such that w_{k_3} is in $[a_3, b_3]$. Continuing this process by hyper infinite induction, we obtain hyper infinite sequence $\{w_{k_n}\}_{n \in \mathbb{N}^{\#}}$ such that $w_{k_n} \in [a_n, b_n]$ for each $n \in \mathbb{N}^{\#}$. The sequence $\{w_{k_n}\}_{n \in \mathbb{N}^{\#}}$ is a subsequence of $\{w_n\}_{n \in \mathbb{N}^{\#}}$ since $k_{n+1} > k_n$ for each $n \in \mathbb{N}^{\#}$. Since $a_n \to_{\#} w$, and $a_n \leq w_n \leq b_n$ for each $n \in \mathbb{N}^{\#}$, the squeeze theorem implies that $w_{k_n} \to_{\#} w$.
Definition 13.12. Let $\{a_n\}$ be a hyperreal sequence i.e.,$a_n \in {}^*\mathbb{R}_c^{\#}, n \in \mathbb{N}^{\#}$. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N \in \mathbb{N}^{\#} \backslash \mathbb{N}$, $N = N(\varepsilon)$ such that,for all $n > N$, $|a_n| \leq \varepsilon$. We often denote this symbolically by $a_n \to_{\#} 0$.
We can also, at this point, define what it means for a hyperreal sequence #-tends to a given number $q \in {}^*\mathbb{R}_c^{\#}$: $\{a_n\}$ #-tends to q if the hyperreal sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_{\#} 0$.
Definition 13.13. Let $\{a_n\}, n \in \mathbb{N}^{\#}$ be a hyperreal sequence. We call $\{a_n\}$ a Cauchy hyperreal sequence if the difference between its terms #-tends to 0. To be precise: given any hyperreal number $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m, n > N, |a_n - a_m| < \varepsilon$.
Theorem 13.15. If $\{a_n\}$ is a #-convergent hyperreal sequence (that is, $a_n \to_{\#} b$ for some hyperreal number $b \in \mathbb{R}_c^{\#}$), then $\{a_n\}$ is a Cauchy hyperreal sequence.
Theorem 13.16. If $\{a_n\}$ is a Cauchy hyperreal sequence, then it is hyper bounded; that is, there is some $M \in \mathbb{R}_c^{\#}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}^{\#}$.
Theorem 13.17. Any Cauchy hyperreal sequence $\{a_n\}$ has a #-limit in ${}^*\mathbb{R}_c^{\#}$ i.e., there exists $b \in {}^*\mathbb{R}_c^{\#}$ such that $a_n \to_{\#} b$.
Proof.By Definition 13.13 given $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $n, n' > N$,

$$|a_n - a_{n'}| < \varepsilon. \tag{13.1}$$

From (13.1) for any $n, n' > N$ we get

$$a_{n'} - \varepsilon < a_n < a_n + \varepsilon. \tag{13.2}$$

The generalized Bolzano-Weierstrass theorem implies there is a #-convergent hyper infinite subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to_{\#} b$ for some hyperreal number $b \in {}^*\mathbb{R}_c^{\#}$. Let us show that the sequence $\{a_n\}$ also #-convergent to this $b \in {}^*\mathbb{R}_c^{\#}$.
We can choose $k \in \mathbb{N}^{\#}$ so large that $n_k > N$ and

$$|a_{n_k} - b| < \varepsilon. \tag{13.3}$$

We choose now in (13.1) $n' = n_k$ and therefore

$$|a_n - a_{n_k}| < \varepsilon. \tag{13.4}$$

From (13.3) and (13.4) for any $n > N$ we get

$$|(a_{n_k} - b) + (a_n - a_{n_k})| = |a_n - b| < 2\varepsilon. \tag{13.5}$$

Thus $a_n \to_{\#} b$ as well.

Remark 13.3.Note that there exist canonical natural embedings

$$\mathbb{R} \hookrightarrow {}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R}_c^{\#}. \tag{13.6}$$

13.1.The Extended Hyperreal Number System ${}^*\hat{\mathbb{R}}_c^{\#}$

Definition 13.14.(a) A set $S \subset \mathbb{N}^{\#}$ is hyperfinite if $card(S) = card(\{x|0 \leq x \leq n\})$, $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. (b) A set $S \subseteq \mathbb{N}^{\#}$ is hyper infinite if $card(S) = card(\mathbb{N}^{\#})$.

Notation 13.2. If F is an arbitrary collection of subsets of ${}^*\mathbb{R}_c^{\#}$, then $\cup\{S|S \in F\}$is the set of all elements that are members of at least one of the sets in F , and $\cap\{S|S \in F\}$ is the set of all elements that are members of every set in F. The union and intersection of finitely or hyperfinitely many sets $S_k, 0 \leq k \leq n \in \mathbb{N}^{\#}$ are also written as $\cup_{k=0}^{n} S_k$ and $\cap_{k=0}^{n} S_k$. The union and intersection of an hyperinfinite sequence $S_k, k \in \mathbb{N}^{\#}$ of sets are written as $\cup_{k=0}^{\infty^{\#}} S$ or $\cup_{n\in\mathbb{N}^{\#}} S$ and $\cap_{k=0}^{\infty^{\#}} S$ or $\cap_{n\in\mathbb{N}^{\#}} S$ correspondingly.

A nonempty set S of hyperreal numbers ${}^*\mathbb{R}_c^{\#}$ is unbounded above if it has no hyperfinite upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to the hyperreal number system two points, $+\infty^{\#}$ (which we also write more simply as $\infty^{\#}$) and $-\infty^{\#}$,and to define the order relationships between them and any hyperreal number $x \in {}^*\mathbb{R}_c^{\#}$ by $-\infty^{\#} < x < \infty^{\#}$.

We call $-\infty^{\#}$ and $\infty^{\#}$ points at hyperinfinity. If S is a nonempty set of hyperreals, we write $\sup S = \infty^{\#}$ to indicate that S is unbounded above, and $\inf S = -\infty^{\#}$ to indicate that S is unbounded below.

13.2. #-Open and #-Closed Sets on ${}^*\hat{\mathbb{R}}_c^{\#}$.

Definition 13.15.If a and b are in the extended hyperreals and $a < b$, then the open interval (a,b) is defined by $(a,b) \triangleq \{x|a < x < b\}$. :

The open intervals $(a,+\infty^{\#})$ and $(-\infty^{\#},b)$ are semi-hyperinfinite if a and b are finite or hyperfinite, and $(-\infty^{\#},\infty^{\#})$ is the entire hyperreal line.

If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] \triangleq \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#},a)$ and $(b,\infty^{\#})$. We say that $[a,b]$ is a #-closed interval. Semi-hyper infinite #-closed intervals are sets of the form $[a,\infty) = \{x|a \leq x\}$ and $(-\infty^{\#},a] = \{x|x \leq a\}$,where a is finite or hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#},a)$ and $(a,\infty^{\#})$,respectively.

Definition 13.16.If $x_0 \in \mathbb{R}_c^{\#}$ is a hyperreal number and $\varepsilon > 0, \varepsilon \approx 0$ then the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is an #-neighborhood of x_0. If a set $S \subset {}^*\mathbb{R}_c^{\#}$ contains an #-neighborhood of x_0, then S is a #-neighborhood of x_0, and x_0 is an #-interior point of S.

The set of #-interior points of S is the #-interior of S, denoted by #-$Int(S)$.

(i) If every point of S is an #-interior point (that is, $S =$ #-$Int(S)$), then S is #-open.

(ii) A set S is #-closed if $S^c = {}^*\mathbb{R}_c^{\#}\backslash S$ is #-open.

Example 13.1. An open interval (a,b) is an #-open set, because if $x_0 \in (a,b)$ and $\varepsilon \leq \min\{x_0 - a; b - x_0\}$, then $(x_0 - \varepsilon, x_0 + \varepsilon) \subset (a,b)$

Remark 13.4.The entire hyperline ${}^*\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#},\infty^{\#})$ is #-open, and therefore $\varnothing$ is

#-closed.

However, $\varnothing$ is also #-open, for to deny this is to say that $\varnothing$ contains a point that is not an #-interior point, which is absurd because $\varnothing$ contains no points. Since $\varnothing$ is #-open, ${}^*\hat{\mathbb{R}}_c^\#$ is #-closed. Thus, ${}^*\hat{\mathbb{R}}_c^\#$ and $\varnothing$ are both #-open and #-closed.

Remark 13.5.They are not the only subsets of ${}^*\hat{\mathbb{R}}_c^\#$ with this property.

Definition 13.17.A deleted #-neighborhood of a point x_0 is a set that contains every point of some #-neighborhood of x_0 except for x_0 itself. For example, $S = \{x|0 < |x - x_0| < \varepsilon\}$, where $\varepsilon \approx 0$, is a deleted #-neighborhood of x_0. We also say that it is a deleted ε-#-neighborhood of x_0.

Theorem 13.18.(a) The union of #-open sets is #-open:

(b) The #-intersection of #-closed sets is #-closed:

These statements apply to arbitrary collections, hyperfinite or hyperinfinite, of #-open and #-closed sets.

Proof (a) Let L be a collection of #-open sets and $S = \cup\{G|G \in L\}$.

If $x_0 \in S$, then $x_0 \in G_0$ for some G_0 in L, and since G_0 is #-open, it contains some ε-#-neighborhood of x_0. Since $G_0 \subset S$, this ε-#–neighborhood is in S, which is consequently a #-neighborhood of x_0. Thus, S is a #-neighborhood of each of its points, and therefore #-open, by definition.

(b) Let F be a collection of #-closed sets and $T = \cap\{H|H \in F\}$. Then $T^c = \cup\{H^c|H \in F\}$ and, since each H^c is #-open, T^c is #-open, from (a). Therefore, T is #-closed, by definition.

Example 13.2. If $-\infty^\# < a < b < \infty^\#$, the set $[a,b] = \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^\# a)$ and $(b,\infty^\#)$. We say that $[a,b]$ is a #-closed interval. The set $[a,b) = \{x|a \leq x < b\}$ is a half-#-closed or half-#-open interval if $-\infty^\# < a < b < \infty^\#$, as is $(a,b] = \{x|a < x \leq b\}$ however, neither of these sets is #-open or #-closed. Semi-infinite #-closed intervals are sets of the form $[a,\infty^\#) = \{x|a \leq x\}$ and $(-\infty^\#,a] = \{x|x \leq a\}$,where a is hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^\#,a)$ and $(a,\infty^\#)$,respectively.

Definition 13.18. Let S be a subset of $\hat{\mathbb{R}}_c^\# = (-\infty^\#,\infty^\#)$. Then

(a) x_0 is a #-limit point of S if every deleted #-neighborhood of x_0 contains a point of S.

(b) x_0 is a boundary point of S if every #-neighborhood of x_0 contains at least one point in S and one not in S. The set of #-boundary points of S is the #-boundary of S, denoted by #-∂S. The #-closure of S, denoted by #-$\bar{S}$, is $S \cup$ #-∂S.

(c) x_0 is an #-isolated point of S if $x_0 \in S$ and there is a #-neighborhood of x_0 that contains no other point of S.

(d) x_0 is #-exterior to S if x_0 is in the #-interior of S^c. The collection of such points is the #-exterior of S.

Theorem 13.19. A set S is #-closed if and only if no point of S^c is a #-limit point of S.

Proof. Suppose that S is #-closed and $x_0 \in S^c$. Since S^c is #-open, there is a #-neighborhood of x_0 that is contained in S^c and therefore contains no points of S. Hence, x_0 cannot be a #-limit point of S. For the converse, if no point of S^c is a #-limit point of S then every point in S^c must have a #-neighborhood contained

in S^c. Therefore, S^c is #-open and S is #-closed.
Corollary 13.1.A set S is #-closed if and only if it contains all its #-limit points.
If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S.
Proposition 13.1. If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S.   .

13.3. #-Open Coverings

Definition 13.19.A collection H of #-open sets of $\mathbb{R}_c^\#$ is an #-open covering of a set S if every point in S is contained in a set H belonging to H; that is, if $S \subset \cup\{F|F \in H\}$.
Definition 13.20.A set $S \subset \mathbb{R}_c^\#$ is called #-compact (or hyper compact) if each of its #-open covers has a hyperfinite subcover.   .
Theorem 13.20.(**Generalized Heine–Borel Theorem**) If H is an #-open covering of a #-closed and hyper bounded subset S of the hyperreal line ${}^*\mathbb{R}_c^\#$ (or of the ${}^*\mathbb{R}_c^{\#n}, n \in \mathbb{N}^\#$) then S has an #-open covering $\widetilde{H}$ consisting of hyper finite many #-open sets belonging to H.
Proof. If a set S in ${}^*\mathbb{R}_c^{\#n}$ is hyper bounded, then it can be enclosed within an n-box $T_0 = [-a,a]^n$ where $a > 0$. By the property above, it is enough to show that T_0 is #-compact.
Assume, by way of contradiction, that T_0 is not #-compact. Then there exists an hyper infinite open cover $C_{\infty^\#}$ of T_0 that does not admit any hyperfinite subcover. Through bisection of each of the sides of T_0, the box T_0 can be broken up into $2n$ sub n-boxes, each of which has diameter equal to half the diameter of T_0. Then at least one of the $2n$ sections of T_0 must require an hyper infinite subcover of $C_{\infty^\#}$, otherwise $C_{\infty^\#}$ itself would have a hyperfinite subcover, by uniting together the hyperfinite covers of the sections. Call this section T_1. Likewise, the sides of T_1 can be bisected, yielding 2^n sections of T_1, at least one of which must require an hyper infinite subcover of $C_{\infty^\#}$. Continuing in like manner yields a decreasing hyper infinite sequence of nested n-boxes: $T_0 \supset T_1 \supset T_2 \supset \ldots \supset T_k \supset \ldots, k \in \mathbb{N}^\#$, where the side length of T_k is $(2a)/2^k$, which #-converges to 0 as k tends to hyper infinity, $k \to \infty^\#$. Let us define a hyper infinite sequence $\{x_k\}_{k\in\mathbb{N}^\#}$ such that each $x_k : x_k \in T_k$. This hyper infinite sequence is Cauchy,   so it must #-converge to some #-limit L. Since each $T_k$is #-closed, and for each k the sequence $\{x_k\}_{k\in\mathbb{N}^\#}$ is eventually always inside T_k, we see that $L \in T_k$ for each $k \in \mathbb{N}^\#$. Since $C_{\infty^\#}$ covers T_0, then it has some member $U \in C_{\infty^\#}$ such that $L \in U$. Since U is   open, there is an n-ball $B(L) \subseteq U$. For large enough k, one has $T_k \subseteq B(L) \subseteq U$, but then the hyper infinite number of members of $C_{\infty^\#}$ needed to cover T_k can be replaced by just one: U, a contradiction.Thus, T_0 is #-compact. Since S is #-closed and a subset of the #-compact set T_0, then S is also #-compact.
As an application of the Generalized Heine–Borel theorem, we give a short proof of the Generalized Bolzano–Weierstrass Theorem.
Theorem 13.21.(Generalized Bolzano–Weierstrass Theorem) Every hyper bounded hyper infinite set $S \subset {}^*\mathbb{R}_c^\#$ has at least one #-limit point.
Proof. We will show that a hyper bounded nonempty set without a #-limit point can contain only finite or a hyper finite number of points. If S has no #-limit points, then S is #-closed and every point $x \in S$ has an #-open neighborhood N_x that contains no point of S other than x.The collection $H = \{N_x|x \in S\}$ is an #-open covering for S. Since S is also hyper bounded, Theorem 13.20 implies that S can be

covered by finite or a hyper finite collection of sets from H, say $N_{x_1},\ldots,N_{x_n}, n \in \mathbb{N}^{\#}$. Since these sets contain only $x_1,\ldots,x_n$ from S, it follows that $S = \{x_k\}_{1\leq k\leq n}, n \in \mathbb{N}^{\#}$.

13.External Cauchy hyperreals $\mathbb{R}_c^{\#}$ and ${}^*\mathbb{R}_c^{\#}$ axiomatically.

A model for the Cauchy hyperreal number system consists of a set $\mathbb{R}_c^{\#}$, two distinct elements 0 and 1 of $\mathbb{R}_c^{\#}$, two binary operations $+$ and $\times$ on $\mathbb{R}^{\#}$ (called addition and multiplication, respectively), and a binary relation $\leq$ on $\mathbb{R}^{\#}$, satisfying the following properties.

Axioms:

I.$(\mathbb{R}_c^{\#},+,\times)$ forms a field i.e.,

(i) For all $x,y,$ and z in $\mathbb{R}^{\#}$, $x+(y+z) = (x+y)+z$ and $x\times(y\times z) = (x\times y)\times z$.
(associativity of addition and multiplication)

(ii) For all x and y in $\mathbb{R}^{\#}$, $x+y = y+x$ and $x\times y = y\times x$.
(commutativity of addition and multiplication)

(iii)For all $x,y,$ and z in $\mathbb{R}^{\#}$,$x\times(y+z) = (x\times y)+(x\times z)$.
(distributivity of multiplication over addition)

(iv)For all x in $\mathbb{R}^{\#}$, $x+0 = x$.
(existence of additive identity)

0 is not equal to 1, and for all x in $\mathbb{R}^{\#}$, $x\times 1 = x$.
(existence of multiplicative identity)

(v) For every x in $\mathbb{R}^{\#}$, there exists an element $-x$ in $\mathbb{R}^{\#}$, such that $x+(-x) = 0$.
(existence of additive inverses)

(vi)For every $x \neq 0$ in $\mathbb{R}^{\#}$, there exists an element $x-1$ in $\mathbb{R}^{\#}$, such that $x\times\ x-1 = 1$.
(existence of multiplicative inverses)

II.$(\mathbb{R}^{\#},\leq)$ forms a totally ordered set. In other words,

(i) For all x in $\mathbb{R}^{\#}$, $x \leq x$. (reflexivity)

(ii) For all x and y in $\mathbb{R}^{\#}$, if $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)

(iii)For all $x,y,$ and z in $\mathbb{R}^{\#}$, if $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

(iv)For all x and y in $\mathbb{R}^{\#}$, $x \leq y$ or $y \leq x$. (totality)

The field operations $+$ and $\times$ on $\mathbb{R}^{\#}$ are compatible with the order $\leq$. In other words,

(v)For all x,y and z in $\mathbb{R}^{\#}$, if $x \leq y$, then $x+z \leq y+z$. (preservation of order under addition)

(vi) For all x and y in $\mathbb{R}^{\#}$, if $0 \leq x$ and $0 \leq y$, then $0 \leq x\times\ y$ (preservation of order under multiplication)

III.Non-Archimedean property

$\mathbb{Q}^{\#} \subset \mathbb{R}^{\#}$ i.e.,$\mathbb{R}^{\#}$ is non-Archimedean ordered field.

Remark 13.1.Here a hyperrational is by definition a ratio of two hyperintegers. Consider the ring $\mathbb{Q}^{\#}_{\mathbf{fin}}$ of all limited (i.e. finite) elements in $\mathbb{Q}^{\#}$. Then $\mathbb{Q}^{\#}_{\mathbf{fin}}$ has a unique maximal ideal $\mathbf{I}^{\#}_{\approx}$, the infinitesimals or infinitesimal numbers are quantities that are closer to zero than any real number from the field $\mathbb{R}$, but are not zero.The quotient ring $\mathbb{Q}^{\#}_{\mathbf{fin}}/\mathbf{I}^{\#}_{\approx}$ gives the field $\mathbb{R}$ of real numbers.

Definition 13.1. An element $x \in \mathbb{R}^{\#}$ is called finite if $|x| < r$ for some $r \in \mathbb{Q},\ r > 0$.

As we shall see in a moment in bivalent case,

Theorem 13.1.Every finite $x \in \mathbb{R}^{\#}$ is infinitely close to some (unique) $r \in \mathbb{R}$ in the sense that $|x - r|$ is either 0 or positively infinitesimal in $\mathbb{R}^{\#}$. This unique r is called the standard part of x and is denoted by $st(x)$.

Proof. Let $x \in \mathbb{R}^{\#}$ be finite. Let $D_1,$ be the set of $r \in \mathbb{R}$ such that $r < x$ and D_2 the set of $r' \in \mathbb{R}$ such that $x < r'$. The pair (D_1, D_2) forms a Dedekind cut in $\mathbb{R}$, hence determines a unique $r_0 \in \mathbb{R}$. A simple argument shows that $|x - r_0|$ is infinitesimal, i.e., $st(x) = r_0$.

Notation 13.1.We usually write $x \approx 0$ iff $x \in \mathbf{I}^{\#}_{\approx}$.

Definition 13.2. A hypersequence of hyperreal numbers is any function $a : \mathbb{N}^{\#} \to \mathbb{R}^{\#}$. Often hypersequences such as these are called hyperreal hypersequences, hypersequences of hyperreal numbers or hypersequences in $\mathbb{R}^{\#}$ to make it clear that the elements of the sequence are hyperreal numbers. Analogous definitions can be given for sequences of hypernatural numbers, hyperintegers, etc.

Notation 13.2.However, we usually write a_n for the image of $n \in \mathbb{N}^{\#}$ under $a,$ rather than $a(n)$.The values a_n are often called the elements of the hypersequence $(x_n)_{n \in \mathbb{N}^{\#}}$.

Definition 13.3. We call $x \in \mathbb{R}^{\#}$ the limit of the hypersequence $(x_n)_{n \in \mathbb{N}^{\#}}$ if the following condition holds: for each hyperreal number $\varepsilon \in \mathbb{R}^{\#}$ such that $\varepsilon \approx 0, \varepsilon > 0,$ there exists a hypernatural number $N \in \mathbb{N}^{\#}$ such that, for every hypernatural number $n \geq N,$ we have $|x_n - x| < \varepsilon.$

Definition 13.4.The hypersequence $(x_n)_{n \in \mathbb{N}^{\#}}$ is said to #-converge to the #-limit x, written $x_n \to x, n \to \infty^{\#}$ or $\lim_{n \to \infty^{\#}}(x_n) = x.$ Symbolically, this reads:

$$\forall \varepsilon[(\varepsilon \approx 0) \wedge (\varepsilon > 0)][\exists N \in \mathbb{N}^{\#}(\forall n \in \mathbb{N}^{\#}(n \geq N \Longrightarrow |x_n - x| < \varepsilon))]. \qquad (13.1)$$

If a hypersequence $(x_n)_{n \in \mathbb{N}^{\#}}$ converges to some limit, then it is convergent; otherwise it is #-divergent. A hypersequence that has zero as a #-limit is sometimes called a null hypersequence.

Limits of hypersequences behave well with respect to the usual arithmetic operations. If $a_n \to a, n \to \infty^{\#}$ and $b_n \to b, n \to \infty^{\#}$, then $a_n + b_n \to a + b, n \to \infty^{\#}$ and $a_n \times b_n \to a \times b, n \to \infty^{\#}$ if neither b_n or any b_n is zero, $a_n \times b_n \to a \times b, n \to \infty^{\#}$.

The following properties of limits of real hypersequences provided, in each equation below, that the limits on the right exist.

The limit of a hypersequence is unique.

1.$\#\text{-}\lim_{n \to \infty^{\#}}(a_n \pm b_n) = \#\text{-}\lim_{n \to \infty^{\#}} a_n \pm \#\text{-}\lim_{n \to \infty^{\#}} b_n$

2.$\#\text{-}\lim_{n \to \infty^{\#}}(c \times a_n) = c \times \#\text{-}\lim_{n \to \infty^{\#}} a_n$

3.$\#\text{-}\lim_{n \to \infty^{\#}}(a_n \times b_n) = (\#\text{-}\lim_{n \to \infty^{\#}}) \times (\#\text{-}\lim_{n \to \infty^{\#}} b_n)$

4.$\#\text{-}\lim_{n \to \infty^{\#}}(a_n/b_n) = \#\text{-}\lim_{n \to \infty^{\#}} a_n / \#\text{-}\lim_{n \to \infty^{\#}} b_n$ provided $\#\text{-}\lim_{n \to \infty^{\#}} b_n \neq 0$

5.$\#\text{-}\lim_{n \to \infty^{\#}} a_n^p = [\#\text{-}\lim_{n \to \infty^{\#}} a_n]^p$

6. If $a_n \leq b_n$ where n greater than some $N,$ then $\#\text{-}\lim_{n \to \infty^{\#}} a_n \leq \#\text{-}\lim_{n \to \infty^{\#}} b_n$

7. (Squeeze theorem) If $a_n \leq c_n \leq b_n,$ and $\#\text{-}\lim_{n \to \infty^{\#}} a_n = \#\text{-}\lim_{n \to \infty^{\#}} b_n = L,$ then $\#\text{-}\lim_{n \to \infty^{\#}} c_n = L.$

Definition 13.5.A hyper infinite sequence (x_n) is said to tend to hyperinfinity, written

$x_n \to \infty^\#$ or #-$\lim_{n\to\infty^\#} x_n = \infty^\#$, if for every $K \in \mathbb{R}^\#$, there is an $N \in \mathbb{N}^\#$ such that for every $n \geq N$; that is, the hypersequence terms are eventually larger than any fixed K. Similarly, $x_n \to -\infty^\#$ if for every $K \in \mathbb{R}^\#$, there is an $N \in \mathbb{N}^\#$ such that for every $n \geq N$, $x_n < K$. If a hypersequence tends to infinity or minus infinity, then it is divergent. However, a divergent hypersequence need not tend to plus or minus hyperinfinity

Definition 13.6.A hypersequence $(x_n)_{n\in\mathbb{N}^\#}$of hyperreal numbers is called a Cauchy hypersequence if for every positive hyperreal number ε, there is a positive hyperinteger $N \in \mathbb{N}^\#$ such that for all hypernatural numbers $m,n > N : |x_m - x_n| < \varepsilon$,where the vertical bars denote the absolute value. In a similar way one can define Cauchy hypersequences of hyperrational numbers,etc. Cauchy formulated such a condition by requiring $|x_m - x_n| \approx 0$ i.e., to be infinite small for every pair of infinite large $m,n \in \mathbb{N}^\#$.

Definition 13.7.Let $\mathbb{R}_c^\#$ be the set of Cauchy hypersequences of hyperrational numbers.

That is, hypersequences $(x_n)_{n\in\mathbb{N}^\#}$ of hyperrational numbers such that for every hyperrational $\varepsilon > 0$, there exists an hyperinteger $N \in \mathbb{N}^\# \backslash \mathbb{N}$ such that for all hypernatural numbers $m,n > N, |x_m - x_n| < \varepsilon$. Here the vertical bars as usial denote the absolute value.

Definition 13.8. A standard procedure to force all Cauchy hypersequences in a metric space to converge is adding new points to the metric space in a process called completion. $\mathbb{R}_c^\#$ is defined as the completion of $\mathbb{Q}^\#$ with respect to the metric $|x - y|$, as will be detailed below.

Definition 13.9. Cauchy hypersequences $(x_n)_{n\in\mathbb{N}^\#}$ and $(y_n)_{n\in\mathbb{N}^\#}$ can be added and multiplied as follows:

$$(x_n)_{n\in\mathbb{N}^\#} + (y_n)_{n\in\mathbb{N}^\#} = (x_n + y_n)_{n\in\mathbb{N}^\#}, \tag{13.2}$$

and

$$(x_n)_{n\in\mathbb{N}^\#} \times (y_n)_{n\in\mathbb{N}^\#} = (x_n \times y_n)_{n\in\mathbb{N}^\#}. \tag{13.3}$$

Definition 13.10. Two Cauchy hypersequences are called equivalent if and only if the difference between them tends to zero. This defines an equivalence relation that is compatible with the operations (16.2)-(16.3) defined above, and the set $\mathbb{R}_c^\#$ of all equivalence classes $\mathbf{cl}[(x_n)_{n\in\mathbb{N}^\#}]$ can be shown to satisfy all axioms of the hyperreal numbers.

We can embed $\mathbb{Q}^\#$ into $\mathbb{R}_c^\#$ by identifying the rational number $r \in \mathbb{Q}^\#$ with the equivalence class of the hypersequence $(r_n)_{n\in\mathbb{N}^\#}$ with $r_n = r$ for all $n \in \mathbb{N}^\#$.

Remark 13.2.Comparison between hyperreal numbers is obtained by defining the following comparison between Cauchy hypersequences:

$$(x_n)_{n\in\mathbb{N}^\#} \geq (y_n)_{n\in\mathbb{N}^\#} \tag{13.4}$$

if and only if x is equivalent to y or there exists an hyperinteger $N \in \mathbb{N}^\#$ such that $x_n \geq y_n$ for all $n > N$.

Remark 13.3.By construction, every hyperreal number $x \in \mathbb{R}_c^\#$ is represented by a Cauchy hyper infinite sequence of hyperrational numbers. This representation is far from unique; every hyperrational hypersequence that converges to x is a representation of x. This reflects the observation that one can often use different hypersequences to approximate the same hyperreal number.The equation $0.999\ldots = 1$ states that the hyper infinite sequences $(0, 0.9, 0.99, 0.999, \ldots)$ and $(1, 1, 1, 1, \ldots)$ are equivalent, i.e., their difference #-converges to 0.

IV.The field $\mathbb{R}^{\#}$ is complete in the following sense:

Definition 13.11.Let $S \subsetneqq \mathbb{R}_c^{\#}$ be a non-empty set of hyperreal numbers.

A hyperreal number $x \in \mathbb{R}_c^{\#}$ is called an upper bound for S if $x \geq s$ for all $s \in S$.

A hyperreal number x is the least upper bound (or supremum $\sup S$) for S if x is an upper

bound for S and $x \leq y$ for every upper bound y of S.

Remark 13.4.The order $\leq$ given by Eq.(14.4) obviously is $\leq$-incomplete.

Definition 13.12. Let $S \subsetneqq \mathbb{R}_c^{\#}$ be a nonempty subset of $\mathbb{R}_c^{\#}$.We we will say that:

(1) S is $\leq$-admissible above if the following conditions are satisfied:

(i) S bounded above;

(ii) let $A(S)$ be a set $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$.

(2) S is $\leq$-admissible belov if the following condition are satisfied:

(i) S bounded belov;

(ii) let $L(S)$ be a set $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.

Theorem 13.2.(i) Every $\leq$-admissible above subset $S \subsetneqq \mathbb{R}_c^{\#}$ has a supremum $\sup S$.

(ii) Every $\leq$-admissible belov subset $S \subsetneqq \mathbb{R}_c^{\#}$ has infinum $\inf S$.

Proof.Let $S \subsetneqq \mathbb{R}_c^{\#}$ be a nonempty subset of $\mathbb{R}_c^{\#}$, and let $M \in \mathbb{Q}^{\#}$ be an hyperrational upper bound for S. We are going to construct two hypersequences of hyperrational numbers, $(u_n)_{n \in \mathbb{N}^{\#}}$and $(l_n)_{n \in \mathbb{N}^{\#}}$. First, since S is nonempty, there is some element $s_0 \in S$.

We can choose a hyperrational number $L \in \mathbb{Q}^{\#}$ such that $L < s_0$.Now, we go through the following hyperinductive procedure to produce hyperrational numbers $u_0, u_1, u_2, \ldots$ and $l_0, l_1,\ l_2, l_3, \ldots$.

(i) Set $u_0 = M$ and $l_0 = L$.

(ii) Suppose that we have already defined u_n and $l_n, n \in \mathbb{N}^{\#}$.

Consider the number $m_n = (u_n + l_n)/2$,i.e.,the average between u_n and l_n.

(1) If m_n is an upper bound for S, define $u_{n+1} = m_n$ and $l_{n+1} = l_n$.

(2) If m_n is not an upper bound for S, define $u_{n+1} = u_n$ and $l_{n+1} = m_n$.

Since $l_0 < M$, it is easy to prove by hyperinfinite induction that $(u_n)_{n \in \mathbb{N}^{\#}}$ is a non-increasing hypersequence, i.e.$u_{n+1} \leq u_n$ and $(l_n)_{n \in \mathbb{N}^{\#}}$ is a non-decreasing hypersequence, i.e. $l_{n+1} \geq l_n$.

Remark 13.5. Note that in the first case above, we have that

$$u_{n+1} - l_{n+1} = m_n - l_n = \frac{u_n + l_n}{2} - l_n = \frac{u_n - l_n}{2}. \tag{13.5}$$

In the second case, we also have that

$$u_{n+1} - l_{n+1} = u_n - m_n = u_n - \frac{u_n + l_n}{2} = \frac{u_n - l_n}{2}. \tag{13.6}$$

Now, this means that $u_1 - l_1 = \frac{1}{2}(M - L)$ and so $u_2 - l_2 = \frac{1}{2}(u_1 - l_1) = \frac{1}{2^2}(M - L)$, and in general by hyperinfinite induction one obtains

$$u_n - l_n = 2^{-n}(M - L). \tag{13.7}$$

Since $M > L$ so $M - L > 0$, and since $2^{-n} < n^{-1}$ we have for any $\varepsilon > 0, \varepsilon \approx 0$ that $2^{-n}(M - L) < \varepsilon$ for all sufficiently large $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. Thus, $u_n - l_n < \varepsilon$ as well, and so

$$\#\text{-}\lim_{n \to \infty^{\#}}(u_n - l_n) = 0. \tag{13.8}$$

This defines two hypersequences of hyperrationals, and so we have hyperreal numbers

$l = (l_n)_{n \in \mathbb{N}^{\#}}$ and $u = (u_n)_{n \in \mathbb{N}^{\#}}$. It is easy to prove, by induction on $n \in \mathbb{N}^{\#}$ that:

(i) u_n is an upper bound for S for all $n \in \mathbb{N}^{\#}$and

(ii) l_n is never an upper bound for S for any $n \in \mathbb{N}^{\#}$.

Thus u is an upper bound for S. To see that it is a least upper bound, notice that the #-limit of $(u_n - l_n)_{n \in \mathbb{N}^{\#}}$ is 0, and so $l = u$. Now suppose $b < u = l$ is a smaller upper bound

for S. Since $(l_n)_{n \in \mathbb{N}^{\#}}$ is monotonic increasing it is easy to see that $b < l_n$ for some $n \in \mathbb{N}^{\#}$.

But l_n is not an upper bound for S and so neither is b. Hence u is a least upper bound for S.

14.§14.1.External non-Archimedean field $\widetilde{\mathbb{R}_c^{\#}}$ via special extension of external non-Archimedean field $\mathbb{R}_c^{\#}$.

Notation 14.1.3. Let $\Delta \subset \mathbb{R}_c^{\#}$ and $\Delta \neq \{0\}$. Then we write $\Delta > 0$ iff $a \in \Delta \Rightarrow a > 0$.

Definition 14.1.13. Let $\Delta \subset \mathbb{R}_c^{\#}$ and $\Delta > 0$. Assume that: $a, b \in \Delta \Rightarrow a + b \in \Delta$. Then we say that Δ is a positive idempotent in $\mathbb{R}_c^{\#}$.

Notation 14.1.4. We will denote by $\mathbb{R}^{\#}_{c+,\mathbf{fin}}$ a set of the all positive finite number in $\mathbb{R}_c^{\#}$ except infinitesimals in $\mathbb{R}_c^{\#}$.

Remark 14.1.6.Note that a set $\mathbb{R}^{\#}_{c+,\mathbf{fin}} \backslash \{0_{\mathbb{R}_c^{\#}}\} \subset \mathbb{R}_c^{\#}$ is a positive idempotent in $\mathbb{R}_c^{\#}$.

Proposition 14.1.1. Let $\Delta \subset \mathbb{R}_c^{\#}$ is a positive idempotent in $\mathbb{R}_c^{\#}$. Then the following are equivalent.[In what follows assume $a, b > 0_{\mathbb{R}_c^{\#}}$].

(i) $a \in \Delta \Rightarrow 2a \in \Delta$,

(ii) $a \in \Delta \Rightarrow na \in \Delta$ for all standard integers $n \in \mathbb{N}$,

(iii) $a \in \Delta \Rightarrow ra \in \Delta$ for all finite $r \in \mathbb{R}_c^{\#}$.

Proof. All parts are immediate from the Definition 14.1.13.

Notation 14.1.4. $\Delta_{\approx}^{\#+} \triangleq \{\delta \in \mathbb{R}_c^{\#} | \delta > 0, \delta \approx 0\}$, i.e. $\Delta_{\approx}^{\#+}$ is a set of the all positive infinitesimals in $\mathbb{R}^{\#}_{c+}$; $\Delta_{\approx}^{\#-} \triangleq \{\delta \in \mathbb{R}_c^{\#} | \delta < 0, \delta \approx 0_{\mathbb{R}_c^{\#}}\}$, i.e. $\Delta_{\approx}^{\#+}$ is a set of the all negative infinitesimals in $\mathbb{R}_c^{\#}$. Note that $\Delta_{\approx}^{\#-} = -\Delta_{\approx}^{\#+}$.

Remark 14.1.7.Note that a set $\Delta_{\approx}^{\#+} \subset \mathbb{R}_c^{\#}$ is a positive idempotent in $\mathbb{R}_c^{\#}$ and $\Delta_{\approx}^{\#-}$ is a negative idempotent in $\mathbb{R}_c^{\#}$.

Definition 14.1.14. Let $\{a_n\}_{n=0}^{\infty}$ be $\mathbb{R}^{\#}_{c+,\mathbf{fin}}$- valued countable sequence $a : \mathbb{N} \to \mathbb{R}^{\#}_{c+,\mathbf{fin}}$ such that:

(i) there is $M \in \mathbb{N}$ such that $\{a_n\}_{n=M}^{\infty}$ is monotonically decreasing $\mathbb{R}^{\#}_{c+,\mathbf{fin}}$- valued countable sequence $a : \mathbb{N} \to \mathbb{R}^{\#}_{c+,\mathbf{fin}} \backslash \{0_{\mathbb{R}_c^{\#}}\}$

(ii) there is $M \in \mathbb{N}$ such that for all $n > M, a_n \neq 0_{\mathbb{R}_c^{\#}}$ [it follows from (ii)]

(iii) for all $n \in \mathbb{N}, a_n \not\approx 0_{\mathbb{R}_c^\#}$ and for any $\epsilon > 0, \epsilon \not\approx 0_{\mathbb{R}_c^\#}, \epsilon \in \mathbb{R}^\#_{c+,\mathbf{fin}}$ there is $N \in \mathbb{N}$ such that for all $n > N : a_n < \epsilon$ and we denote a set of the all these sequences by $\Delta_\omega^{+\downarrow 0}$.
We define a set $\Delta_\omega^{-\downarrow 0}$ by $c_n \in \Delta_\omega^{-\downarrow 0} \iff \{-c_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$. Note that $\Delta_\omega^{-\downarrow 0} = -\Delta_\omega^{+\downarrow 0}$.

Remark 14.1.8. Note that a set $\Delta_\omega^{+\downarrow 0}$ is a positive idempotent in $\mathbb{R}_c^\#$ and a set $\Delta_\omega^{-\downarrow 0}$ is a negative idempotent in $\mathbb{R}_c^\#$.

Proposition 14.1.2. (1) Let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ and $\{b_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ then:
(i) $\{a_n\}_{n=0}^\infty + \{b_n\}_{n=0}^\infty \triangleq \{a_n + b_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$
(ii) $\{a_n\}_{n=0}^\infty - \{b_n\}_{n=0}^\infty \triangleq \{a_n - b_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0} \cup \Delta_\omega^{-\downarrow 0} \cup \Delta_\approx^{\#+} \cup \Delta_\approx^{\#-} \cup \{0_{\mathbb{R}_c^\#}\}_{n=0}^\infty$
where $\{0_{\mathbb{R}_c^\#}\}_{n=0}^\infty$ is a countable $0_{\mathbb{R}_c^\#}$- valued sequence.
(iii) $\{a_n\}_{n=0}^\infty \times \{b_n\}_{n=0}^\infty \triangleq \{a_n \times b_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$.
(2) Let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{-\downarrow 0}$ and $\{b_n\}_{n=0}^\infty \in \Delta_\omega^{-\downarrow 0}$ then we define
(i) $\{a_n\}_{n=0}^\infty + \{b_n\}_{n=0}^\infty \triangleq \{a_n + b_n\}_{n=0}^\infty \in \Delta_\omega^{-\downarrow 0}$
(ii) $\{a_n\}_{n=0}^\infty - \{b_n\}_{n=0}^\infty \triangleq \{a_n - b_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0} \cup \Delta_\omega^{-\downarrow 0}$
(iii) $\{a_n\}_{n=0}^\infty \times \{b_n\}_{n=0}^\infty \triangleq \{a_n \times b_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$
(3) Let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0} \cup \Delta_\omega^{-\downarrow 0}$ and $x, y \in \mathbb{R}_c^\#$ then we define
(iv) $x + y\{a_n\}_{n=0}^\infty \triangleq \{x + ya_n\}_{n=0}^\infty$

Proof. Immediately by definitions and by Definition 14.1.14.

Definition 14.1.15. We define the relation $(\bullet \leq \bullet)$ on a set $\Delta_\omega^{+\downarrow 0}$ by:
let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ and $\{b_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ then $\{a_n\}_{n=0}^\infty \leq \{b_n\}_{n=0}^\infty$ iff there is $N \in \mathbb{N}$ such that for all $n > N : a_n \leq b_n$ and similarly we define the relation $(\bullet \leq \bullet)$ on a set $\Delta_\omega^{-\downarrow 0}$ by:
let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{-\downarrow 0}$ and $\{b_n\}_{n=0}^\infty \in \Delta_\omega^{-\downarrow 0}$ then $\{a_n\}_{n=0}^\infty \leq \{b_n\}_{n=0}^\infty$ iff there is $N \in \mathbb{N}$ such that for all $n > N : a_n \leq b_n$

Definition 14.1.16. (1) We define the relation $(\bullet < \bullet)$ on a set $\Delta_\omega^{+\downarrow} \times \mathbb{R}^\#_{c+,\mathbf{fin}}$ by:
let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ and $x \in \mathbb{R}^\#_{c+,\mathbf{fin}}$ then $\{a_n\}_{n=0}^\infty < x$ iff there is $N \in \mathbb{N}$ such that for all $n > N : a_n < x$.
(2) We define the relation $(\bullet < \bullet)$ on a set $\Delta_\approx^{\#+} \times \Delta_\omega^{+\downarrow}$ by: let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ and $x \in \Delta_\approx^{\#+}$ then $x < \{a_n\}_{n=0}^\infty$ iff there is $N \in \mathbb{N}$ such that for all $n > N : x < a_n$.
(3) Let $\{a_n\}_{n=0}^\infty$ be $\Delta_\approx^{\#+}$- valued countable sequence $a : \mathbb{N} \to \Delta_\approx^{\#+}$, and we denote a set of the all these sequences by $\Delta_{\approx,\omega}^{\#+}$.
We define the relation $(\bullet < \bullet)$ on a set $\Delta_{\approx,\omega}^{\#+} \times \Delta_\omega^{+\downarrow}$ by: let $\{a_n\}_{n=0}^\infty \in \Delta_{\approx,\omega}^{\#+}$ and $x \in \Delta_\approx^{\#+}$ then $\{a_n\}_{n=0}^\infty < x$ iff there is $N \in \mathbb{N}$ such that for all $n > N : a_n < x$.

Proposition 14.1.2. Let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ $\{a_n\}_{n=0}^\infty \neq 0_{\mathbb{R}_c^\#}$ then there is $N \in \mathbb{N}$ such that $0_{\mathbb{R}_c^\#} < \Delta_\approx^{\#+} < \{a_n\}_{n=0}^\infty < \mathbb{R}^\#_{c+,\mathbf{fin}} \backslash \{0_{\mathbb{R}_c^\#}\}$.

Proof. Immediately by definitions and by Definition 14.1.15.

Remark 14.1.9. Note that it follows from Proposition 14.1.2 that

$$0_{\mathbb{R}_c^\#} < \Delta_\approx^{\#+} < \Delta_\omega^{+\downarrow 0} < \mathbb{R}^\#_{c+,\mathbf{fin}} \backslash \{0_{\mathbb{R}_c^\#}\}. \qquad (14.1.9)$$

Definition 14.1.17. Let $\{a_n\}_{n=0}^\infty$ be monotonically increasing $\mathbb{R}^\#_{c+,\mathbf{fin}}$- valued countable sequence $a : \mathbb{N} \to \mathbb{R}^\#_{c+,\mathbf{fin}} \backslash \Delta_\approx^+$ such that:
(i) there is $M \in \mathbb{N}$ such that for all $n > M, a_n \neq 0_{\mathbb{R}_c^\#}$
(ii) there is $N \in \mathbb{N}$ such that for all $n > N$ and for any $\xi > 0_{\mathbb{R}_c^\#}, \xi \in \mathbb{R}^\#_{c+,\mathbf{fin}}$ $a_n > \xi$ and we denote a set of the all these sequences by $\Delta_\omega^{+\downarrow\infty}$. We define a set $\Delta_\omega^{-\downarrow\infty}$ by $c_n \in \Delta_\omega^{-\downarrow\infty} \iff \{-c_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow\infty}$. Note that $\Delta_\omega^{-\downarrow\infty} = -\Delta_\omega^{+\downarrow\infty}$.

Proposition 14.1.3.(1) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$ then:
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty} \cup \Delta_{\omega}^{-\downarrow\infty} \cup \Delta_{\approx}^{\#+} \cup \Delta_{\approx}^{\#-} \cup \Delta_{\omega}^{+\downarrow 0} \cup \Delta_{\omega}^{-\downarrow 0}\{0_{\mathbb{R}_c^{\#}}\}_{n=0}^{\infty}$
where $\{0_{\mathbb{R}_c^{\#}}\}_{n=0}^{\infty}$ is a countable $0_{\mathbb{R}_c^{\#}}$- valued sequence.
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$.
(2) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow\infty}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow\infty}$ then we define
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow\infty}$
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty} \cup \Delta_{\omega}^{-\downarrow\infty}$
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$
(3) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$ and $x,y \in \mathbb{R}_c^{\#}$ then we define
(iv) $x_n + y_n\{a_n\}_{n=0}^{\infty} \triangleq \{x_n + y_n a_n\}_{n=0}^{\infty}$ and we denote a set of the all these sequences by $\left\{\Delta_{\omega}^{+\downarrow\infty}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\right\}$.
Proof. Immediately by definitions and by Definition 14.1.16.
Remark 14.1.10.Note that $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty} \iff \{a_n^{-1}\}_{n=N}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$.
Definition 14.1.18.(1) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ and let $\{A_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ be a hyper infinite sequence

$$\{A_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}} = (a_0, a_1, \ldots, a_k, \ldots, \{a_n\}_{n=0}^{\infty}, \ldots), \tag{14.1.10}$$

i.e. for any infinite $m \in \mathbb{N}^{\#}\backslash\mathbb{N}, A_m \equiv \{a_n\}_{n=0}^{\infty}$. We will denote a set of the all these hyper infinite sequences by $\widetilde{\Delta_{\omega}^{+\downarrow 0}}$ and a set of the all hyper infinite sequences $\widetilde{\{-a_n\}_{n=0}^{\infty}}$ by $\widetilde{\Delta_{\omega}^{-\downarrow 0}}$.(2) Let $\{x_n + y_n a_n\}_{n=0}^{\infty} \in \left\{\Delta_{\omega}^{+\downarrow 0}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\right\}$ and let

$$\begin{gathered}\{x_n + y_n A_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{x_n + y_n a_n\}_{n=0}^{\infty}} \\ (x_0 + y_0 a_0, x_1 + y_1 a_1, \ldots, \{x_k + y_k a_k\}, \ldots, \{x_n +_n y a_n\}_{n=0}^{\infty}, \ldots),\end{gathered} \tag{14.1.11}$$

i.e. for any infinite $m \in \mathbb{N}^{\#}\backslash\mathbb{N}, A_m \equiv \{x_n + y_n a_n\}_{n=0}^{\infty}$. We will denote a set of the all these hyper infinite sequences by $\overline{\left\{\Delta_{\omega}^{+\downarrow 0}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\right\}}$.
Definition 14.1.19.Let $\{A_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ and $\{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{b_n\}_{n=0}^{\infty}}$ be in $\widetilde{\Delta_{\omega}^{+\downarrow 0}}$.
Then we define:
(i) $\{A_n\}_{n=0}^{\infty^{\#}} + \{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}} + \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n + b_n\}_{n=0}^{\infty}} = \{A_n + B_n\}_{n=0}^{\infty^{\#}} \in \widetilde{\Delta_{\omega}^{+\downarrow 0}}$
(ii) $\{A_n\}_{n=0}^{\infty^{\#}} - \{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}} - \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n - b_n\}_{n=0}^{\infty}} =$
$= \{A_n - B_n\}_{n=0}^{\infty^{\#}} \in \widetilde{\Delta_{\omega}^{+\downarrow 0}} \cup \widetilde{\Delta_{\omega}^{-\downarrow 0}} \cup \{0_{\mathbb{R}_c^{\#}}\}_{n=0}^{\infty^{\#}}$
(iii) $\{A_n\}_{n=0}^{\infty^{\#}} \times \{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}} \times \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n \times b_n\}_{n=0}^{\infty}} = \{A_n \times B_n\}_{n=0}^{\infty^{\#}} \in \widetilde{\Delta_{\omega}^{+\downarrow 0}}$
Let $\{A_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ and $\{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{b_n\}_{n=0}^{\infty}}$ be in $\left\{\Delta_{\omega}^{+\downarrow 0}, \{x_{1,n}\}_{n=0}^{\infty}, \{y_{1n}\}_{n=0}^{\infty},\right\}$ and $\{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{b_n\}_{n=0}^{\infty}}$ be in $\left\{\Delta_{\omega}^{+\downarrow 0}, \{x_{2,n}\}_{n=0}^{\infty}, \{y_{2,n}\}_{n=0}^{\infty},\right\}$. Then we define:
(iv) $\{A_n\}_{n=0}^{\infty^{\#}} \dotplus \{B_n\}_{n=0}^{\infty^{\#}} = \overline{\{x_{1,n} + y_{1,n} a_n\}_{n=0}^{\infty}} + \overline{\{x_{2,n} + y_{2,n} b_n\}_{n=0}^{\infty}} \triangleq$
$\triangleq \overline{\{x_{1,n} + x_{2,n} \dotplus y_{1,n} a_n \dotplus y_{2,n} b_n\}_{n=0}^{\infty}} = \{x_{1,n} + +x_{2,n} \dotplus y_{1,n} A_n + y_{2,n} B_n\}_{n=0}^{\infty^{\#}}$
Definition 14.1.20.Let $\{\Psi_n\}_{n=0}^{\infty^{\#}}$ be in $\widetilde{\Delta_{\omega}^{+\downarrow 0}}$, i.e. for all $n \in \mathbb{N}^{\#}$, $\Psi_n \in \widetilde{\Delta_{\omega}^{+\downarrow 0}}$. Say $\{\Psi_n\}_{n=0}^{\infty^{\#}}$ #-tends to $0_{\mathbb{R}_c^{\#}}$as $n \to \infty^{\#}$ iff for any given $\varepsilon > 0_{\mathbb{R}_c^{\#}}, \varepsilon \approx 0_{\mathbb{R}_c^{\#}}$ there is a hypernatural number $N \in \mathbb{N}^{\#}\backslash\mathbb{N}, N = N(\varepsilon)$ such that for any $n > N, |\Psi_n| < \varepsilon$.

Definition 14.1.21. Let $\{\Psi_n\}_{n=0}^{\infty^\#}$ be a hyper infinite sequence such that for all $n \in \mathbb{N}^\#, \Psi_n \in \widetilde{\Delta_\omega^{+\downarrow 0}}$. We call $\{\Psi_n\}_{n=0}^{\infty^\#}$ a Cauchy hyper infinite sequence if the difference between its terms #-tends to $0_{\mathbb{R}_c^\#}$. To be precise: given any $\varepsilon > 0_{\mathbb{R}_c^\#}, \varepsilon \approx 0_{\mathbb{R}_c^\#}$ there is a hypernatural number $N \in \mathbb{N}^\# \backslash, N = N(\varepsilon)$ such that for any $m,n > N, |\Psi_n - \Psi_m| < \varepsilon$.

Theorem 14.1.3.Let $\{\Psi_n\}_{n=0}^{\infty^\#}$ be in $\widetilde{\Delta_\omega^{+\downarrow 0}}$. If $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a #-convergent hyper infinite sequence (that is, $\Psi_n \to_\# \Phi$ as $n \to \infty^\#$ for some $\Phi \in \widetilde{\Delta_\omega^{+\downarrow 0}}$), then $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a Cauchy hyper infinite sequence.

Proof.We know that $\Psi_n \to_\# \Phi$. Here is a ubiquitous trick: instead of using ε in the definition, start with an arbitrary infinitesimall $\varepsilon > 0, \varepsilon \approx 0_{\mathbb{R}_c^\#}$ and then choose N so that $|\Psi_n - \Phi| < \varepsilon/2$ when $n > N$. Then if $m,n > N$, we have

$|\Psi_n - \Psi_m| = |(\Psi_n - \Phi) - (\Psi_m - \Phi)| \leq |\Psi_n - \Phi| + |\Psi_m - \Phi| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

This shows that $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a Cauchy hyper infinite sequence.

Theorem 14.1.4.If $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a Cauchy hyper infinite sequence, then it is bounded in $\mathbb{R}_c^\#$; that is, there is some number $M \in \mathbb{R}_c^\#$ such that $|\{\Psi_n\}_{n=0}^{\infty^\#}| \leq M$ for all $n \in \mathbb{N}^\#$.

Proof. Since $\{\Psi_n\}_{n=0}^{\infty^\#}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some N such that $|\Psi_m - \Psi_n| < 1$ whenever $m,n > N$. Thus, $|\Psi_{N+1} - \Psi_n| < 1$ for $n > N$. We can rewrite this as $\Psi_{N+1} - 1 < \Psi_n < \Psi_{N+1} + 1$. This means that $|\Psi_n|$ is less than the maximum of $|\Psi_{N+1} - 1|$ and $|\Psi_{N+1} + 1|$. So, set $M \in \mathbb{R}_c^\#$ larger than any number in the following list: $\{|\Psi_0|, |\Psi_1|, \ldots, |\Psi_N|, |\Psi_{N+1} - 1|, |\Psi_{N+1} + 1|\}$.

Then for any term Ψ_n, if $n \leq N$, then $|\Psi_n|$ appears in the list and so $|\Psi n| \leq M$; if $n > N$, then (as shown above) $|\Psi_n|$ is less than at least one of the last two entries in the list, and so $|\Psi_n| \leq M$. Hence, M is a bound for the sequence.

Let Ξ denote the set of all Cauchy hyper infinite sequences We must define an equivalence relation on Ξ.

Definition 14.1.22. Let $\mathbf{S}$ be a set of objects. A relation among pairs of elements of $\mathbf{S}$ is said to be an equivalence relation if the following three properties hold:

Reflexivity: for any $s \in \mathbf{S}$, s is related to s.

Symmetry: for any $s,t \in \mathbf{S}$, if s is related to t then t is related to s.

Transitivity: for any $s,t,r \in \mathbf{S}$, if s is related to t and t is related to r, then s is related to r.

The following well known proposition goes most of the way to showing that an equivalence relation divides a set into bins.

Theorem 14.1.5. Let $\mathbf{S}$ be a set, with an equivalence relation on pairs of elements. For $s \in \mathbf{S}$, denote by $[s]$ the set of all elements in $\mathbf{S}$ that are related to s. Then for any $s,t \in \mathbf{S}$, either $[s] = [t]$ or $[s]$ and $[t]$ are disjoint.

The sets $[s]$ for $s \in \mathbf{S}$ are called the equivalence classes, and they are the bins.

Corollary 14.1.1. If $\mathbf{S}$ is a set with an equivalence relation on pairs of elements, then the equivalence classes are non-empty disjoint sets whose union is all of $\mathbf{S}$.

Definition 14.1.23.Let $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$ and $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$ be in $\widetilde{\Delta_\omega^{+\downarrow 0}}$. Say they are equivalent (i.e. related) if $|\Psi_{1,n} - \Psi_{2,n}| \to_\# 0_{\mathbb{R}_c^\#}$ as $n \to \infty^\#$, i.e. if the sequence $\{|\Psi_{1,n} - \Psi_{2,n}|\}_{n=0}^{\infty^\#}$ #-tends to $0_{\mathbb{R}_c^\#}$.

Proposition 14.1.4.Definition 4.1.23 yields an equivalence relation on Ξ.
Proof. we need to show that this relation is reflexive, symmetric, and transitive.

• **Reflexive**: $\Psi_n - \Psi_n = 0_{\mathbb{R}_c^\#}$, and the sequence all of whose terms are $0_{\mathbb{R}_c^\#}$ clearly converges to $0_{\mathbb{R}_c^\#}$.So $\{\Psi_n\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_n\}_{n=0}^{\infty^\#}$.

• **Symmetric**: Suppose $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$, so $\Psi_{1,n} - \Psi_{2,n} \to_\# 0_{\mathbb{R}_c^\#}$. But $\Psi_{2,n} - \Psi_{1,n} = -(\Psi_{1,n} - \Psi_{2,n})$, and since only the absolute value $|\Psi_{1,n} - \Psi_{2,n}| = |\Psi_{2,n} - \Psi_{1,n}|$ comes into play in Definition 14.1.23,it follows that $\Psi_{2,n} - \Psi_{1,n} \to_\# 0_{\mathbb{R}_c^\#}$ as well. Hence, $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$.

• **Transitive**: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 14.1.4. Suppose $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$, and $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{3,n}\}_{n=0}^{\infty^\#}$. This means that $\Psi_{1,n} - \Psi_{2,n} \to_\# 0_{\mathbb{R}_c^\#}$ and $\Psi_{2,n} - \Psi_{3,n} \to_\# 0_{\mathbb{R}_c^\#}$.
To be fully precise, let us fix infinite small $\varepsilon > 0_{\mathbb{R}_c^\#}$; then there exists an $N \in \mathbb{N}^\# \backslash \mathbb{N}$ such that for all $n > N, |\Psi_{1,n} - \Psi_{2,n}| < \varepsilon/2$; also, there exists an M such that for all $n > M, |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2$. Well, then, as long as $n > \max(N,M)$, we have that $|\Psi_{2,n} - \Psi_{3,n}| = |(\Psi_{1,n} - \Psi_{2,n}) + (\Psi_{2,n} - \Psi_{3,n})| \leq |\Psi_{1,n} - \Psi_{2,n}| + |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon 2 + \varepsilon 2 = \varepsilon$.
So, choosing L equal to the max of N,M , we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |\Psi_{1,n} - \Psi_{3,n}| < \varepsilon$. This means that $\Psi_{1,n} - \Psi_{3,n} \to_\# 0_{\mathbb{R}_c^\#}$ – i.e. $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{3,n}\}_{n=0}^{\infty^\#}$.
So, we really have equivalence relation, and so by Corollary 14.1.1, the set Ξ is partitioned into disjoint subsets (equivalence classes).

Definition 14.1.24. The hyperreal numbers $\widetilde{\mathbb{R}_c^\#}$ are the equivalence classes $\left[\{\Psi_{1,n}\}_{n=0}^{\infty^\#}\right]$ of Cauchy hyper infinite sequences of, as per Definition 14.1.23.
That is, each such equivalence class is a hyperreal number in $\widetilde{\mathbb{R}_c^\#}$.

Definition 14.1.25.Let $s,t \in \widetilde{\mathbb{R}_c^\#}$, so there are Cauchy hyper infinite sequences $\{\Psi_n\}_{n=0}^{\infty^\#}$ and $\{\Phi_n\}_{n=0}^{\infty^\#}$ with $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]$ and $t = \left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]$.
(a) Define $s+t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n \dotplus \Phi_n\}_{n=0}^{\infty^\#}$.
(b) Define $s \times t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n \times \Phi_n\}_{n=0}^{\infty^\#}$.

Proposition 14.1.5.The operations $+, \cdot$ in Definition 14.1.25 (a),(b) are well-defined.
Proof. Suppose that $\left[\{\Psi_n\}_{n=0}^{\infty^\#}\right] = \left[\{\Psi_{1,n}\}_{n=0}^{\infty^\#}\right]$ and $\left[\{\Phi_n\}_{n=0}^{\infty^\#}\right] = \left[\{\Phi_{1,n}\}_{n=0}^{\infty^\#}\right]$. Thus means that $\Psi_n - \Psi_{1,n} \to_\# 0_{\mathbb{R}_c^\#}$ and $\Phi_n - \Phi_{1,n} \to_\# 0_{\mathbb{R}_c^\#}$. Then $(\Psi_n + \Phi_n) - (\Psi_{1,n} + \Phi_{1,n}) = (\Psi_n - \Psi_{1,n}) + (\Phi_n - \Phi_{1,n})$. Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to $0_{\mathbb{R}_c^\#}$, and so $[(\Psi_n + \Phi_n)] = [(\Psi_{1,n} + \Phi_{1,n})]$.
Multiplication is a little trickier; this is where we will use Theorem 14.1.4. We will also use another ubiquitous technique: adding $0_{\mathbb{R}_c^\#}$ in the form of $s - s$. Again, suppose that
$\left[\{\Psi_n\}_{n=0}^{\infty^\#}\right] = \left[\{\Psi_{1,n}\}_{n=0}^{\infty^\#}\right]$ and $\left[\{\Phi_n\}_{n=0}^{\infty^\#}\right] = \left[\{\Phi_{1,n}\}_{n=0}^{\infty^\#}\right]$; we wish to show that $\left[\{\Psi_n \times \Phi_n\}_{n=0}^{\infty^\#}\right] = \left[\{\Psi_{1,n} \times \Phi_{1,n}\}_{n=0}^{\infty^\#}\right]$, or, in other words, that $\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n} \to_\# 0_{\mathbb{R}_c^\#}$. Well, we add and subtract one of the other cross terms, say $\Phi_n \times \Psi_{1,n}$:

$\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n} = \Psi_n \times \Phi_n + (\Phi_n \times \Psi_{1,n} - \Phi_n \times \Psi_{1,n}) - \Psi_{1,n} \times \Phi_{1,n} =$
$= (\Psi_n \times \Phi_n - \Phi_n \times \Psi_{1,n}) + (\Phi_n \times \Psi_{1,n} - \Psi_{1,n} \times \Phi_{1,n}) =$
$= \Phi_n \times (\Psi_n - \Psi_{1,n}) + \Psi_{1,n} \times (\Phi_n - \Phi_{1,n})$.
Hence, we have $|\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n}| \leq |\Phi_n| \times |\Psi_n - \Psi_{1,n}| + |\Psi_{1,n}| \cdot |\Phi_n - \Phi_{1,n}|$.
Now, from Theorem 14.1.4, there are numbers M and L such that $|\Phi_n| \leq M$ and $|\Psi_{1,n}| \leq L$ for all $n \in \mathbb{N}^\#$. Taking some number R (for example $R = M + L$) which is bigger than both, we have
$|\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n}| \leq |\Phi_n| \times |\Psi_n - \Psi_{1,n}| + |\Psi_{1,n}| \times |\Phi_n - \Phi_{1,n}| \leq$
$\leq R(|\Psi_n - \Psi_{1,n}| + |\Phi_n - \Phi_{1,n}|)$.
Now, noting that both an - cn and $\Phi_n - \Phi_{1,n}$ #-tend to $0_{\mathbb{R}_c^\#}$ and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2R$, we see that
$\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n} \to_\# 0_{\mathbb{R}_c^\#}$

Theorem 14.1.6. Given any hyperreal number

$$s \in \widetilde{\mathbb{R}_c^\#},\ s \neq 0_{\widetilde{\mathbb{R}_c^\#}} = \left[\widetilde{0_{\mathbb{R}_c^\#}}\right] = [(0_{\mathbb{R}_c^\#}, 0_{\mathbb{R}_c^\#}, 0_{\mathbb{R}_c^\#}, 0_{\mathbb{R}_c^\#}, \ldots)],$$

there is a hyperreal number $t \in \widetilde{\mathbb{R}_c^\#}$ such that

$$s \times t = 1_{\widetilde{\mathbb{R}_c^\#}} = \left[\widetilde{1_{\mathbb{R}_c^\#}}\right] = [(1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, \ldots)].$$

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $0_{\widetilde{\mathbb{R}_c^\#}} = (0_{\mathbb{R}_c^\#}, 0_{\mathbb{R}_c^\#}, 0_{\mathbb{R}_c^\#}, 0_{\mathbb{R}_c^\#}, \ldots)$. In other words, $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]$ where $\Psi_n - 0_{\mathbb{R}_c^\#}$ does not #-converge to $0_{\mathbb{R}_c^\#}$as $n \to \infty^\#$. From this, we are to deduce the existence of a hyperreal number
$t = \left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]$ such that $s \times t = \left[\{\Psi_n \times \Phi_n\}_{n=0}^{\infty^\#}\right]$ is the same equivalence class as $1_{\widetilde{\mathbb{R}_c^\#}} = [(1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, \ldots)]$. Doing so is actually an easy consequence of the fact that
nonzero hyperreal numbers from $\mathbb{R}_c^\#$ have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{\Psi_n\}_{n=0}^{\infty^\#}$ does not #-tend to $0_{\mathbb{R}_c^\#}$), there's no reason any number of the terms in $\{\Psi_n\}_{n=0}^{\infty^\#}$ can't equal $0_{\mathbb{R}_c^\#}$. However, it turns out that eventually, $\Psi_n \neq 0_{\mathbb{R}_c^\#}$.
That is,

Lemma 14.1.1. If $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a Cauchy hyper infinite sequence which does not #-tend to $0_{\mathbb{R}_c^\#}$, then there is an $N \in \mathbb{N}^\#$ such that, for $n > N$, $\Psi_n \neq 0_{\mathbb{R}_c^\#}$.

We will now use it to complete the proof of Theorem 14.1.6.
Let $N \in \mathbb{N}^\#$ be such that $\Psi_n \neq 0_{\mathbb{R}_c^\#}$ for $n > N$. Define hyper infinite sequence Φ_n of hyperreal numbers from $\widetilde{\mathbb{R}_c^\#}$ as follows:
for $n \leq N, \Phi_n = 0_{\mathbb{R}_c^\#}$, and for $n > N$, $\Phi_n = 1/\Psi_n$:
$\{\Phi_n\}_{n=0}^{\infty^\#} = (0_{\mathbb{R}_c^\#}, 0_{\mathbb{R}_c^\#}, \ldots, 0_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}/\Psi_{N+1}, 1/\Psi_{N+2}, \ldots)$.
This makes sense since, for $n > N$, an is a nonzero hyperreal number, so $1_{\mathbb{R}_c^\#}/\Psi_n$ exists.
Then $\Psi_n \times \Phi_n$ is equal to $\Psi_n \times 0_{\mathbb{R}_c^\#} = 0_{\mathbb{R}_c^\#}$ for $n \leq N$, and equals
$\Psi_n \times \Phi_n = \Psi_n \times 1_{\mathbb{R}_c^\#}/\Psi_n = 1_{\mathbb{R}_c^\#}$ for $n > N$
Well, then, if we look at the hyper infinite sequence $1_{\widetilde{\mathbb{R}_c^\#}} = (1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, \ldots)$, we have $(1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, \ldots) - (\Psi_n \times \Phi_n)$ is the sequence which is $1_{\widetilde{\mathbb{R}_c^\#}} - 0_{\widetilde{\mathbb{R}_c^\#}} = 1_{\widetilde{\mathbb{R}_c^\#}}$

for $n \leq N$ and equals $1_{\widetilde{\mathbb{R}_c^\#}} - 1_{\widetilde{\mathbb{R}_c^\#}} = 0_{\mathbb{R}_c^\#}$ for $n > N$. Since this hyper infinite sequence is eventually equal to $0_{\mathbb{R}_c^\#}$, it #-converges to $0_{\mathbb{R}_c^\#}$ as $n \to \infty^\#$, and so $\left[\{\Psi_n \times \Phi_n\}_{n=0}^{\infty^\#}\right] = [(1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, \ldots)] = 1_{\widetilde{\mathbb{R}_c^\#}} \in \widetilde{\mathbb{R}_c^\#}$. This shows that $t = \left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]$ is a multiplicative inverse to $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]$.

Definition 14.1.26. Let $s \in \widetilde{\mathbb{R}_c^\#}$. Say that s is positive if $s \neq 0_{\widetilde{\mathbb{R}_c^\#}}$, and if $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]$ for some Cauchy hyper infinite sequence such that for some $N, \Psi_n > 0_{\mathbb{R}_c^\#}$ for all $n > N$. Given two hyperreal numbers s, t, say that $s > t$ if $s - t$ is positive.

Theorem 14.1.7. Let $s, t \in \widetilde{\mathbb{R}_c^\#}$ be hyperreal numbers such that $s > t$, and let $r \in \widetilde{\mathbb{R}_c^\#}$. Then $s + r > t + r$.

Proof. Let $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right], t = \left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]$, and $r = \left[\{\Theta_n\}_{n=0}^{\infty^\#}\right]$. Since $s > t$, i.e. $s - t > 0$, we know that there is an N such that, for $n > N, \Psi_n - \Phi_n > 0$. So $\Psi_n > \Phi_n$ for $n > N$. Now, adding Θ_n to both sides of this inequality , we have $\Psi_n + \Theta_n > \Phi_n + \Theta_n$ for $n > N$, or $(\Psi_n + \Theta_n) - (\Phi_n + \Theta_n) > 0_{\mathbb{R}_c^\#}$ for $n > N$. Note also that $(\Psi_n + \Theta_n) - (\Phi_n + \Theta_n) = \Psi_n - \Phi_n$ does not #-converge to $0_{\mathbb{R}_c^\#}$ as $n \to \infty^\#$, by the assumption that $s - t > 0_{\widetilde{\mathbb{R}_c^\#}}$. Thus, by Definition 14.1.26, this means that:
$s + r = \left[\{\Psi_n + \Theta_n\}_{n=0}^{\infty^\#}\right] > \left[\{\Phi_n + \Theta_n\}_{n=0}^{\infty^\#}\right] = t + r$.

Definition 14.1.27. There is canonical imbeding

$$\mathbb{R}_c^\# \hookrightarrow \widetilde{\mathbb{R}_c^\#} \tag{14.1.14}$$

defined by

$$a \longmapsto [\widetilde{a}] \tag{14.1.15}$$

where $\widetilde{a}$ is hyper infinite sequence $\widetilde{a} = (a, a, \ldots.), a \in \mathbb{R}_c^\#$.

Notation 14.1.5. $\widehat{a} = (a, a, \ldots.) \in \widetilde{\mathbb{R}_c^\#}, a \in \widetilde{\mathbb{R}_c^\#}$.

Remark 14.1.11.If $a \in \widetilde{\mathbb{R}_c^\#}$ we will identify hyperreal a with hyper infinite sequence $\{a_n\}_{n=0}^{\infty^\#} = a_0, a_1, \ldots, a_{N-1}, \widehat{a}_N, N \in \mathbb{N}^\#$since $a = \#\text{-}\lim_{n\to\infty^\#} a_n$.

Definition 14.1.28. (i) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $\widetilde{\mathbb{R}_c^\#}, \{a_n\}_{n=0}^{k} \subset \widetilde{\mathbb{R}_c^\#}$.

We define external hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{k}} \subset \widetilde{\mathbb{R}_c^\#}$ by

$$\begin{gathered}\{A_n; k\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{k}} = \\ = (a_0, a_1, \ldots, a_m, \ldots, a_{k-1}, a_k, \widehat{a}_k).\end{gathered} \tag{14.1.16}$$

(ii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $\widetilde{\mathbb{R}_c^\#} : \{a_n\}_{n=0}^{\infty} \subset \widetilde{\mathbb{R}_c^\#}$.

We define hyper infinite sequence $\{A_n\}_{n=0}^{\infty^\#} = \overbrace{\{a_n\}_{n=0}^{\infty}} \subset \widetilde{\mathbb{R}_c^\#}$ by

$$\begin{gathered}\{A'_n; \infty\}_{n=0}^{\infty^\#} = \overbrace{\{a_n\}_{n=0}^{\infty}} = \\ = \left(a_0, a_1, , \ldots, a_k, \ldots \{a_n\}_{n=0}^{\infty}, \widehat{\{a_n\}_{n=0}^{\infty}}\right) \in [[\{a_n\}_{n=0}^{\infty}]].\end{gathered} \tag{14.1.17}$$

(iii) Let $\{a_n\}_{n=0}^{N}, N \in \mathbb{N}^\# \backslash \mathbb{N}$ be external hyperfinite sequence in $\widetilde{\mathbb{R}_c^\#} : \{a_n\}_{n=0}^{N} \subset \widetilde{\mathbb{R}_c^\#}$.

We define hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{N}} \subset \widetilde{\mathbb{R}_c^\#}$ by

$$\{A_n;N\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{N}} =$$
$$= (a_0,a_1,\ldots,a_m,\ldots,a_{N-1},a_N,\widehat{a}_N). \tag{14.1.18}$$

Definition 14.1.29.(i) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $\widetilde{\mathbb{R}_c^\#}$, $\{a_n\}_{n=0}^{k} \subset \widetilde{\mathbb{R}_c^\#}$. We define external finite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n = \overbrace{\{c_n\}_{n=0}^{k}} = \left(c_0,c_1,\ldots,\ \{c_j\}_{n=0}^{n=j},\ldots,c_{k-1},\ldots,c_k,\widehat{c}_k\right) \in [[c_k]] \tag{14.1.19}$$

where $c_0 = a_0, c_j = Ext\text{-}\sum_{n=0}^{n=j} a_n, 0 \le n \le k.$

(ii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $\widetilde{\mathbb{R}_c^\#} : \{a_n\}_{n=0}^{\infty} \subset \widetilde{\mathbb{R}_c^\#}$. We define external countable sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n = \overbrace{\{c_n\}_{n=0}^{\infty}} =$$
$$= \left(c_0,c_1,\ldots,c_k,\ldots,\{c_n\}_{n=0}^{\infty},\widehat{\{c_n\}_{n=0}^{\infty}}\right) \in [[\{c_n\}_{n=0}^{\infty}]] \tag{14.1.20}$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n, k \in \mathbb{N}.$

(iii) Let $\{a_n\}_{n=0}^{n=N}, N \in \mathbb{N}^\#\backslash\mathbb{N}$ be external hyperfinite sequence in $\widetilde{\mathbb{R}_c^\#} : \{a_n\}_{n=0}^{N} \subset \widetilde{\mathbb{R}_c^\#}$. We define external hyperfinite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = \overbrace{\{c_n\}_{n=0}^{n=N}} = (c_0,c_1,\ldots,c_k,\ldots,c_{N-1},c_N,\widehat{c}_N) \tag{14.1.21}$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n,\ 0 \le k \le N, c_N = Ext\text{-}\sum_{n=0}^{n=N} a_n.$

(iv) Let $\{a_n\}_{n=0}^{n=N}, N \in \mathbb{N}^\#\backslash\mathbb{N}$ be external hyperfinite sequence in $\widetilde{\mathbb{R}_c^\#} : \{a_n\}_{n=0}^{N} \subset \widetilde{\mathbb{R}_c^\#}$ such that $a_n \equiv 0$ for all $n \in \mathbb{N}^\#\backslash\mathbb{N}$. We assume that

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n. \tag{14.1.22}$$

Example 14.1.1.Consider the G.P: $\alpha,\alpha r,\alpha r^2,\ldots.\alpha r^{N-1}, N \in \mathbb{N}^\#, \alpha \in \widetilde{\mathbb{R}_c^\#}$,

$r \in \widetilde{\mathbb{R}_c^\#}$ be the first term and the ratio of the G.P respectively. Then for any

$N \in \mathbb{N}^\#$ by Proposition 14.1.6 and Definition 14.1.29 one obtains that

$$Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1} \alpha r^{n-1} = \alpha \overbrace{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{\widetilde{\mathbb{R}_c^\#}} - r}} =. \alpha \overbrace{\frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}} - r}} - \alpha \overbrace{\frac{r^N}{1_{\widetilde{\mathbb{R}_c^\#}} - r}}. \tag{14.1.23}$$

and

$$Ext\text{-}\widehat{\sum}_{n=1}^{\infty} \alpha r^{n-1} =. \alpha \overbrace{\frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}} - r}} - \alpha \overbrace{\left\{\frac{r^n}{1_{\widetilde{\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}}. \tag{14.1.24}$$

Example 14.1.2.Consider the G.P: $\alpha,\alpha r,\alpha r^2,\ldots.\alpha r^{N-1}, N \in {}^*\mathbb{N}, \alpha \in \widetilde{\mathbb{R}_c^\#}, r \in \widetilde{\mathbb{R}_c^\#}$,

$r > 0, r \neq 1.$ Note that

$$\alpha\widehat{\frac{1_{\widetilde{\mathbb{R}_c^\#}}-r^N}{1_{\mathbb{R}_c^\#}-r}} = Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1}\alpha r^{n-1} =$$

$$= Ext\text{-}\widehat{\sum}_{n=1}^{\infty}\alpha r^{n-1} + Ext\text{-}\widehat{\sum}_{(n\in\mathbb{N}^\#\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha r^{n-1} = \tag{14.1.25}$$

$$= \alpha\widehat{\frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}}-r}} - \alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{\mathbb{R}_c^\#}}-r}\right\}_{n=1}^{\infty}} + Ext\text{-}\widehat{\sum}_{(n\in\mathbb{N}^\#\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha r^{n-1}.$$

From (14.1.25) we obtain

$$Ext\text{-}\widehat{\sum}_{(n\in{}^*\mathbb{N}\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha r^{n-1} = \alpha\widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r^N}{1_{{}^*\mathbb{R}_c^\#}-r}} - \alpha\widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r}} + \alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r}\right\}_{n=1}^{\infty}} =$$

$$\alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r}\right\}_{n=1}^{\infty}} - \alpha\widehat{\frac{r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r}}. \tag{14.1.26}$$

Assume that: (i) $r < 1_{\widetilde{\mathbb{R}_c^\#}}$, then from (14.1.26) we obtain

$$Ext\text{-}\widehat{\sum}_{(n\in\mathbb{N}^\#\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha r^{n-1} > 0_{\widetilde{\mathbb{R}_c^\#}}. \tag{14.1.27}$$

(ii) $r > 1_{\widetilde{\mathbb{R}_c^\#}}$, then from (14.1.26) we obtain

$$Ext\text{-}\widehat{\sum}_{(n\in\mathbb{N}^\#\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha r^{n-1} = \alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{\mathbb{R}_c^\#}}-r}\right\}_{n=1}^{\infty}} \dot{+} \alpha\widehat{\frac{r^N}{r-1_{\widetilde{\mathbb{R}_c^\#}}}} > 0_{\widetilde{\mathbb{R}_c^\#}}. \tag{14.1.28}$$

Proposition 14.1.6.(i) Consider the G.P: $\alpha,\alpha r,\alpha r^2,\ldots.\alpha r^{N-1}, N\in\mathbb{N}^\#$. Let S_N, $\alpha\in\mathbb{R}_c^\#, r\in\mathbb{R}_c^\#$ be the sum of N terms, first term and the ratio of the G.P respectively. Then for any $N\in\mathbb{N}^\#$the statement Φ_N holds

$$\Phi_N \Leftrightarrow_s Ext\text{-}\sum_{n=1}^{n=N-1}\alpha r^{n-1} = \alpha\frac{1_{\mathbb{R}_c^\#}-r^N}{1_{\mathbb{R}_c^\#}-r}. \tag{14.1.29}$$

Proof.(i) Directly by hyperinfinite induction. Note that $\Phi_N \Rightarrow_s \Phi_{N+1}$:

$$S_{N+1} = Ext\text{-}\sum_{n=1}^{n=N}\alpha r^{n-1} = Ext\text{-}\sum_{n=1}^{n=N-1}\alpha r^{n-1} + \alpha r^N = \alpha\frac{1_{\mathbb{R}_c^\#}-r^N}{1_{\mathbb{R}_c^\#}-r} + \alpha r^N =$$

$$= \alpha\frac{1_{\mathbb{R}_c^\#}-r^N}{1_{\mathbb{R}_c^\#}-r} + \alpha\frac{\left(1_{\mathbb{R}_c^\#}-r\right)r^N}{1_{\mathbb{R}_c^\#}-r} = \alpha\frac{1_{\mathbb{R}_c^\#}-r^N+r^N-r^{N+1}}{1_{\mathbb{R}_c^\#}-r} = \tag{14.1.30}$$

$$= \alpha\frac{1_{\mathbb{R}_c^\#}-r^{N+1}}{1_{\mathbb{R}_c^\#}-r}.$$

Thus $S_{N+1} = \alpha\frac{1_{\mathbb{R}_c^\#}+r^{N+1}}{1_{\mathbb{R}_c^\#}-r}$ and therefore Φ_{N+1}holds.

(ii) Consider the G.P: $\alpha,\alpha r,\alpha r^2,\ldots.\alpha r^{N-1}, N\in\mathbb{N}^\#$. Let S_N, $\alpha\in\widetilde{\mathbb{R}_c^\#}, r\in\widetilde{\mathbb{R}_c^\#}$ be the sum of N terms, first term and the ratio of the G.P respectively. Then for any $N\in{}^*\mathbb{N}$the statement $\widetilde{\Phi}_N$ holds

$$\widetilde{\Phi}_N \Leftrightarrow_s Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1} \alpha r^{n-1} = \alpha \frac{1_{\widetilde{\mathbb{R}_c^\#}} - r^N}{1_{\widetilde{\mathbb{R}_c^\#}} - r}. \tag{14.1.31}$$

Notice that (i)⇒(ii) by definitions.

Definition 14.1.30. Let $\{a_n\}_{n=0}^{\infty^\#}, n \in \mathbb{N}^\#$ be external hyperinfinite sequence in $\widetilde{\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{\infty^\#} \subset \widetilde{\mathbb{R}_c^\#}$. We define external hyperinfinite sum $Ext\text{-}\widehat{\sum}_{n=0}^{\infty^\#} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{\infty^\#} a_n = \#\text{-}\lim_{N \to \infty^\#}\left(Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n\right) \tag{14.1.32}$$

if #-limit in (14.1.31) exists.

Example 14.1.3.Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots. \alpha r^{n-1}, n \in \mathbb{N}^\#, \alpha \in \widetilde{\mathbb{R}_c^\#}, r \in \widetilde{\mathbb{R}_c^\#}$.
From (14.1.27) we obtain

$$\begin{gathered} Ext\text{-}\widehat{\sum}_{n=0}^{\infty^\#} \alpha r^{n-1} = \#\text{-}\lim_{N \to \infty^\#}\left(Ext\text{-}\widehat{\sum}_{n=0}^{n=N} \alpha r^{n-1}\right) = \#\text{-}\lim_{N \to \infty^\#} \alpha \frac{1_{\widetilde{\mathbb{R}_c^\#}} - r^N}{1_{\widetilde{\mathbb{R}_c^\#}} - r} = \\ = \alpha \frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}} - r} \end{gathered} \tag{14.1.33}$$

since $\#\text{-}\lim_{N \to \infty^\#} r^N = 0_{\widetilde{\mathbb{R}_c^\#}}$ if $|r| < 1$. From (14.1.33) and (14.1.25) we obtain

$$\begin{gathered} \alpha \frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}} - r} = Ext\text{-}\widehat{\sum}_{n=0}^{{}^*\infty} \alpha r^{n-1} = Ext\text{-}\widehat{\sum}_{n=0}^{\infty} \alpha r^{n-1} + Ext\text{-}\widehat{\sum}_{n \in \mathbb{N}^\#} \alpha r^{n-1} = \\ \widehat{\alpha \frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}} - r}} - \widehat{\alpha \left\{\frac{r^n}{1_{\widetilde{\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} + Ext\text{-}\widehat{\sum}_{n \in \mathbb{N}^\# \setminus \mathbb{N}} \alpha r^{n-1}. \end{gathered} \tag{14.1.34}$$

From (14.1.34) we obtain

$$\begin{gathered} Ext\text{-}\widehat{\sum}_{n \in \mathbb{N}^\# \setminus \mathbb{N}} \alpha r^{n-1} = \widehat{\alpha \frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}} - r}} \alpha - \left(\widehat{\frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}} - r}} - \widehat{\alpha \left\{\frac{r^n}{1_{\widetilde{\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}}\right) = \\ = \widehat{\alpha \left\{\frac{r^n}{1_{\widetilde{\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} > 0_{\widetilde{\mathbb{R}_c^\#}}. \end{gathered} \tag{14.1.35}$$

Definition 14.1.31. Let $\{a_n\}_{n=0}^{\infty}$ be $\mathbb{R}_c^\#$- valued countable sequence
$a : \mathbb{N} \to {}^*\mathbb{R}_c^\#$ such that:
(i) there is $M \in \mathbb{N}$ such that for all $n > M, a_n \neq 0_{{}^*\mathbb{R}_c^\#}$,
we denote a set of the all these sequences by $\Xi_\omega^{\pm,\neq 0}$.
We define a set $-\Xi_\omega^{\pm,\neq 0}$ by $\{c_n\}_{n=0}^{\infty} \in -\Xi_\omega^{\pm,\neq 0} \Leftrightarrow \{-c_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0}$. Note that $\Xi_\omega^{\pm,\neq 0} = -\Xi_\omega^{\pm,\neq 0}$.
(ii) there is countable subsequence $\{a_{n_k}\}_{k=m}^{\infty} \subset \{a_n\}_{n=0}^{\infty}$ such that $a_{n_k} = 0_{\mathbb{R}_c^\#}, k \geq m$ and $a_n \neq 0_{\mathbb{R}_c^\#}$ iff $a_n \notin \{a_{n_k}\}_{k=m}^{\infty}$,
we denote a set of the all these countable sequences by $\Xi_\omega^{\pm,\neq 0 \vee = 0}$.
We define a set $-\Xi_\omega^{\pm,\neq 0 \vee = 0}$ by $\{c_n\}_{n=0}^{\infty} \in -\Xi_\omega^{\pm,\neq 0 \vee = 0} \Leftrightarrow \{-c_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0 \vee = 0}$. Note that $\Xi_\omega^{\pm,\neq 0 \vee = 0} = -\Xi_\omega^{\pm,\neq 0 \vee = 0}$.

Definition 14.1.31.
(1) Let $\{a_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0}$ and $\{b_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0}$ then we define
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0}$
(iv) $\left(\{a_n\}_{n=0}^{\infty}\right)^{-1} \triangleq \{a_n^{-1}\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0}$
(2) Let $\{a_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$ and $\{b_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$ then we define
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$
(iv) $\left(\{a_n\}_{n=0}^{\infty}\right)^{-1_*} \triangleq \{a_n^{1_*}\}_{n=0}^{\infty}$ where

$$a_n^{1_*} = \begin{cases} a_n^{-1} & \text{if} \quad a_n \neq 0_{{}^*\mathbb{R}_c^{\#}} \\ 0_{{}^*\mathbb{R}_c^{\#}} & \text{if} \quad a_n = 0_{{}^*\mathbb{R}_c^{\#}} \end{cases} \tag{14.1.36}$$

Note that
(i) $\left(\left(\{a_n\}_{n=0}^{\infty}\right)^{-1_*}\right)^{-1_*} = \{a_n\}_{n=0}^{\infty}$
(ii) $\{a_n\}_{n=0}^{\infty} \times \left(\{a_n\}_{n=0}^{\infty}\right)^{-1_*} = \breve{1}_{{}^*\mathbb{R}_c^{\#}}$ where $\breve{1}_{{}^*\mathbb{R}_c^{\#}} = \{\alpha_n\}_{n=0}^{\infty}$ is countable sequence such that

$$\alpha_n = \begin{cases} 1_{\mathbb{R}_c^{\#}} & \text{if} \quad a_n \neq 0_{\mathbb{R}_c^{\#}} \\ 0_{\mathbb{R}_c^{\#}} & \text{if} \quad \alpha_n = 0_{\mathbb{R}_c^{\#}} \end{cases} \tag{14.1.37}$$

Definition 14.1.32. We say that
$\left(\{a_n\}_{n=0}^{\infty}\right)^{-1_*} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$ is a quasi inverse of $\{a_n\}_{n=0}^{\infty}$.

Definition 14.1.33.(1) Let $\{a_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$ and let $\{A_n\}_{n=0}^{{}^*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ be a hyper infinite sequence

$$\{A_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}} = \left(a_0, a_1, \ldots, a_k, \ldots, \{a_n\}_{n=0}^{\infty}, \ldots\right) \tag{14.1.38}$$

i.e. for any infinite $m \in \mathbb{N}^{\#} \backslash \mathbb{N}, A_m \equiv \{a_n\}_{n=0}^{\infty}$. We will denote a set of the all these hyper infinite sequences by $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee = 0}$
(2) Let $\{x_n + y_n a_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee = 0}$ and let

$$\begin{gathered} \{x_n + y_n A_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{x_n + y_n a_n\}_{n=0}^{\infty}} = \\ \left(x_0 + y_0 a_0, x_1 + y_1 a_1, \ldots, x_k + y_k a_k, \ldots, \{x_n +_n y a_n\}_{n=0}^{\infty}, \ldots\right), \end{gathered} \tag{14.1.39}$$

i.e. for any infinite $m \in \mathbb{N}^{\#} \backslash \mathbb{N}, A_m \equiv \{x_n + y_n a_n\}_{n=0}^{\infty}$. We will denote a set of the all these hyper infinite sequences by $\overline{\{\Xi_{\omega}^{\pm,\neq 0 \vee = 0}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\}}$.

Definition 14.1.34. Let $\{A_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ and $\{B_n\}_{n=0}^{{}^*\infty} = \widetilde{\{b_n\}_{n=0}^{\infty}}$ be in $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee = 0}$.
Then we define:
(i) $\{A_n\}_{n=0}^{\infty^{\#}} + \{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}} + \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n + b_n\}_{n=0}^{\infty}} = \{A_n + B_n\}_{n=0}^{\infty^{\#}} \in \widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee = 0}$
(ii) $\{A_n\}_{n=0}^{\infty^{\#}} - \{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}} - \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n - b_n\}_{n=0}^{\infty}} =$
$= \{A_n - B_n\}_{n=0}^{\infty^{\#}} \in \widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee = 0}$
(iii) $\{A_n\}_{n=0}^{\infty^{\#}} \times \{B_n\}_{n=0}^{\infty^{\#}} = \widetilde{\{a_n\}_{n=0}^{\infty}} \times \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n \times b_n\}_{n=0}^{\infty}} = \{A_n \times B_n\}_{n=0}^{\infty^{\#}} \in \widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee = 0}$

Definition 14.1.35.Let $\{\Psi_n\}_{n=0}^{\infty^\#}$ be in $\widetilde{\widetilde{\Xi}}_\omega^{\pm,\neq 0\vee=0}$,i.e. for all $n\in\mathbb{N}^\#$, $\Psi_n\in\Xi_\omega^{\pm,\neq 0\vee=0}$. Say $\{\Psi_n\}_{n=0}^{\infty^\#}$ #-tends to $0_{\mathbb{R}_c^\#}$ as $n\to\infty^\#$ iff for any given $\varepsilon>0_{\mathbb{R}_c^\#}$,$\varepsilon\approx 0_{\mathbb{R}_c^\#}$ there is a hypernatural number $N\in\mathbb{N}^\#\backslash\mathbb{N}, N=N(\varepsilon)$ such that for any $n>N, |\Psi_n|<\varepsilon$.

Definition 14.1.36. Let $\{\Psi_n\}_{n=0}^{\infty^\#}$ be a hyper infinite sequence such that for all $n\in\mathbb{N}^\#,\Psi_n\in\widetilde{\widetilde{\Xi}}_\omega^{\pm,\neq 0\vee=0}$.We call $\{\Psi_n\}_{n=0}^{\infty^\#}$ a Cauchy hyper infinite sequence if the difference between its terms #-tends to $0_{\mathbb{R}_c^\#}$. To be precise: given any $\varepsilon>0_{\mathbb{R}_c^\#}$, $\varepsilon\approx 0_{\mathbb{R}_c^\#}$there is a hypernatural number $N\in\mathbb{N}^\#\backslash\mathbb{N}, N=N(\varepsilon)$ such that for any $m,n>N, |\Psi_n-\Psi_m|<\varepsilon$.

Theorem 14.1.8.Let $\{\Psi_n\}_{n=0}^{\infty^\#}$ be in $\widetilde{\widetilde{\Xi}}_\omega^{\pm,\neq 0\vee=0}$.If $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a #-convergent hyper infinite sequence (that is, $\Psi_n\to_\#\ \Phi$ as $n\to\infty^\#$ for some $\Phi\in\widetilde{\widetilde{\Xi}}_\omega^{\pm,\neq 0\vee=0}$), then $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a Cauchy hyper infinite sequence.

Proof.We know that $\Psi_n\to_\#\ \Phi$. Here is a ubiquitous trick: instead of using ε in the definition, start with an arbitrary infinitesimall $\varepsilon>0_{\mathbb{R}_c^\#}, \varepsilon\approx 0_{\mathbb{R}_c^\#}$ and then choose N so that $|\Psi_n-\Phi|<\varepsilon/2$ when $n>N$. Then if $m,n>N$, we have $|\Psi_n-\Psi_m|=|(\Psi_n-\Phi)-(\Psi_m-\Phi)|\leq|\Psi_n-\Phi|+|\Psi_m-\Phi|<\varepsilon/2+\varepsilon/2=\varepsilon$.
This shows that $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a Cauchy hyper infinite sequence.

Theorem 14.1.9.If $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a Cauchy hyper infinite sequence, then it is bounded in $\mathbb{R}_c^\#$; that is, there is some number $M\in\mathbb{R}_c^\#$ such that $|\{\Psi_n\}_{n=0}^{\infty^\#}|\leq M$ for all $n\in\mathbb{N}^\#$.

Proof. Since $\{\Psi_n\}_{n=0}^{*\infty}$ is Cauchy, setting $\varepsilon=1$ we know that there is some N such that $|\Psi_m-\Psi_n|<1$ whenever $m,n>N$. Thus, $|\Psi_{N+1}-\Psi_n|<1$ for $n>N$. We can rewrite this as $\Psi_{N+1}-1<\Psi_n<\Psi_{N+1}+1$.This means that $|\Psi_n|$ is less than the maximum of $|\Psi_{N+1}-1|$ and $|\Psi_{N+1}+1|$. So, set $M\in\mathbb{R}_c^\#$ larger than any number in the following list: $\{|\Psi_0|,|\Psi_1|,\ldots,|\Psi_N|,|\Psi_{N+1}-1|,|\Psi_{N+1}+1|\}$.
Then for any term Ψ_n, if $n\leq N$, then $|\Psi_n|$ appears in the list and so $|\Psi_n|\leq M$;if $n>N$, then (as shown above) $|\Psi_n|$ is less than at least one of the last two entries in the list, and so $|\Psi_n|\leq M$. Hence, M is a bound for the sequence.
Let $\widetilde{\widetilde{\Xi}}$ denote the set of all Cauchy hyper infinite sequences. We must define an equivalence relation on $\widetilde{\widetilde{\Xi}}$.

Definition 14.1.37. Let $\mathbf{S}$ be a set of objects. A relation among pairs of elements of $\mathbf{S}$ is said to be an equivalence relation if the following three properties hold:

Reflexivity: for any $s\in\mathbf{S}$, s is related to s.

Symmetry: for any $s,t\in\mathbf{S}$, if s is related to t then t is related to s.

Transitivity: for any $s,t,r\in\mathbf{S}$, if s is related to t and t is related to r, then s is related to r.

The following well known proposition goes most of the way to showing that an equivalence relation divides a set into bins.

Theorem 14.1.5.10. Let $\mathbf{S}$ be a set, with an equivalence relation on pairs of elements. For $s\in\mathbf{S}$, denote by $[s]$ the set of all elements in $\mathbf{S}$ that are related to s. Then for any $s,t\in\mathbf{S}$, either $[s]=[t]$ or $[s]$ and $[t]$ are disjoint.
The sets $[s]$ for $s\in\mathbf{S}$ are called the equivalence classes, and they are the bins.

Corollary 14.1.2. If $\mathbf{S}$ is a set with an equivalence relation on pairs of elements, then the equivalence classes are non-empty disjoint sets whose union is all of **S**.

Definition 14.1.38.Let $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$ and $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$ be in $\widetilde{\Xi}_{\omega}^{\pm,\neq 0\vee=0}$. Say they are equivalent (i.e. related) if $|\Psi_{1,n} - \Psi_{2,n}| \to_{\#} 0_{\mathbb{R}_c^\#}$ as $n \to \infty^\#$, i.e. if the hyper infinite sequence $\{|\Psi_{1,n} - \Psi_{2,n}|\}_{n=0}^{\infty^\#}$ #-tends to $0_{\mathbb{R}_c^\#}$.

Proposition 14.1.4. Definition 4.1.38 yields an equivalence relation on $\widetilde{\Xi}_{\omega}^{\pm,\neq 0\vee=0}$.

Proof. We need to show that this relation is reflexive, symmetric, and transitive.

• **Reflexive**: $\Psi_n - \Psi_n = 0_{\mathbb{R}_c^\#}$, and the sequence all of whose terms are $0_{\mathbb{R}_c^\#}$ clearly #-converges to $0_{\mathbb{R}_c^\#}$.So $\{\Psi_n\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_n\}_{n=0}^{\infty^\#}$.

• **Symmetric**: Suppose $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$, so $\Psi_{1,n} - \Psi_{2,n} \to_{\#} 0_{\mathbb{R}_c^\#}$. But $\Psi_{2,n} - \Psi_{1,n} = -(\Psi_{1,n} - \Psi_{2,n})$, and since only the absolute value $|\Psi_{1,n} - \Psi_{2,n}| = |\Psi_{2,n} - \Psi_{1,n}|$ comes into play in Definition 14.1.35,it follows that $\Psi_{2,n} - \Psi_{1,n} \to_{\#} 0_{\mathbb{R}_c^\#}$ as well. Hence, $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$.

• **Transitive**: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 14.1.4. Suppose $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$, and $\{\Psi_{2,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{3,n}\}_{n=0}^{\infty^\#}$. This means that $\Psi_{1,n} - \Psi_{2,n} \to_{\#} 0_{\mathbb{R}_c^\#}$ and $\Psi_{2,n} - \Psi_{3,n} \to_{\#} 0_{\mathbb{R}_c^\#}$.
To be fully precise, let us fix infinite small $\varepsilon > 0_{\mathbb{R}_c^\#}$; then there exists an $N \in \mathbb{N}^\#\backslash\mathbb{N}$ such that for all $n > N, |\Psi_{1,n} - \Psi_{2,n}| < \varepsilon/2$; also, there exists an M such that for all $n > M, |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2$. Well, then, as long as $n > \max(N,M)$, we have that $|\Psi_{2,n} - \Psi_{3,n}| = |(\Psi_{1,n} - \Psi_{2,n}) + (\Psi_{2,n} - \Psi_{3,n})| \leq |\Psi_{1,n} - \Psi_{2,n}| + |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So, choosing L equal to the max of N,M , we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |\Psi_{1,n} - \Psi_{3,n}| < \varepsilon$. This means that $\Psi_{1,n} - \Psi_{3,n} \to_{\#} 0_{{}^*\mathbb{R}_c^\#}$, i.e. $\{\Psi_{1,n}\}_{n=0}^{\infty^\#}$ is related to $\{\Psi_{3,n}\}_{n=0}^{\infty^\#}$.

So, we really have equivalence relation, and so by Corollary 14.1.2, the set $\widetilde{\Xi}_{\omega}^{\pm,\neq 0\vee=0}$ is partitioned into disjoint subsets (equivalence classes).

Definition 14.1.39. The hyperreal numbers $\widetilde{\mathbb{R}_c^\#}$ contain: (1) all the equivalence classes $\left[\{\Psi_{1,n}\}_{n=0}^{\infty^\#}\right]$ of Cauchy hyper infinite sequences of, as per Definition 14.1.38 and (2) the all gyperreals $\mathbb{R}_c^\# \subset \widetilde{\mathbb{R}_c^\#}$ by canonical imbedding $\mathbb{R}_c^\# \hookrightarrow \widetilde{\mathbb{R}_c^\#}$ (14.1.42)-(14.1.43).

That is, each such equivalence class is a hyperreal number in $\widetilde{\mathbb{R}_c^\#}$.

Definition 14.1.40. Let $s,t \in \widetilde{\mathbb{R}_c^\#}$, so there are Cauchy hyper infinite sequences $\{\Psi_n\}_{n=0}^{\infty^\#}$ and $\{\Phi_n\}_{n=0}^{\infty^\#}$ with $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]$ and $t = \left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]$.

(a) Define $s + t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n \dot{+} \Phi_n\}_{n=0}^{\infty^\#}$.

(b) Define $s \times t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n \times \Phi_n\}_{n=0}^{\infty^\#}$.

Proposition 14.1.5.The operations $+,\times$ in Definition 14.1.25 (a),(b) are well-defined.

Proof. Suppose that $\left[\{\Psi_n\}_{n=0}^{\infty^\#}\right] = \left[\{\Psi_{1,n}\}_{n=0}^{\infty^\#}\right]$ and $\left[\{\Phi_n\}_{n=0}^{\infty^\#}\right] = \left[\{\Phi_{1,n}\}_{n=0}^{\infty^\#}\right]$.
Thus means that $\Psi_n - \Psi_{1,n} \to_{\#} 0_{\mathbb{R}_c^\#}$ and $\Phi_n - \Phi_{1,n} \to_{\#} 0_{\mathbb{R}_c^\#}$. Then $(\Psi_n + \Phi_n) - (\Psi_{1,n} + \Phi_{1,n}) = (\Psi_n - \Psi_{1,n}) + (\Phi_n - \Phi_{1,n})$.Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to $0_{\mathbb{R}_c^\#}$, and so $[(\Psi_n + \Phi_n)] = [(\Psi_{1,n} + \Phi_{1,n})]$.

Multiplication is a little trickier; this is where we will use Theorem 14.1.10. We will also use another ubiquitous technique: adding $0_{\mathbb{R}_c^\#}$ in the form of $s-s$. Again, suppose that

$\left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]=\left[\{\Psi_{1,n}\}_{n=0}^{\infty^\#}\right]$ and $\left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]=\left[\{\Phi_{1,n}\}_{n=0}^{\infty^\#}\right]$; we wish to show that $\left[\{\Psi_n\times\Phi_n\}_{n=0}^{\infty^\#}\right]=\left[\{\Psi_{1,n}\times\Phi_{1,n}\}_{n=0}^{\infty^\#}\right]$, or, in other words, that $\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}\to_\# 0_{\mathbb{R}_c^\#}$. Well, we add and subtract one of the other cross terms, say $\Phi_n\times\Psi_{1,n}$:

$\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}=\Psi_n\times\Phi_n+(\Phi_n\times\ \Psi_{1,n}-\Phi_n\times\ \Psi_{1,n})-\Psi_{1,n}\times\Phi_{1,n}=$

$=(\Psi_n\times\ \Phi_n-\Phi_n\times\ \Psi_{1,n})+(\Phi_n\times\ \Psi_{1,n}-\Psi_{1,n}\times\ \Phi_{1,n})=$

$=\Phi_n\times\ (\Psi_n-\Psi_{1,n})+\Psi_{1,n}\times\ (\Phi_n-\Phi_{1,n})$.

Hence, we have $|\Psi_n\times\ \Phi_n-\Psi_{1,n}\times\ \Phi_{1,n}|\leq|\Phi_n|\times|\Psi_n-\Psi_{1,n}|+|\Psi_{1,n}|\cdot|\Phi_n-\Phi_{1,n}|$.

Now, from Theorem 14.1.9, there are numbers M and L such that $|\Phi_n|\leq M$ and $|\Psi_{1,n}|\leq L$ for all $n\in\mathbb{N}^\#$. Taking some number R (for example $R=M+L$) which is bigger than both, we have

$|\Psi_n\times\ \Phi_n-\Psi_{1,n}\times\ \Phi_{1,n}|\leq|\Phi_n|\times|\Psi_n-\Psi_{1,n}|+|\Psi_{1,n}|\times|\Phi_n-\Phi_{1,n}|\leq$

$\leq R(|\Psi_n-\Psi_{1,n}|+|\Phi_n-\Phi_{1,n}|)$.

Now, noting that both $\Psi_n-\Psi_{1,n}$ and $\Phi_n-\Phi_{1,n}$ #-tend to $0_{\mathbb{R}_c^\#}$ and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2R$, we see that

$\Psi_n\times\Phi_n-\Psi_{1,n}\times\ \Phi_{1,n}\to_\#\ 0_{\mathbb{R}_c^\#}$

Theorem 14.2.11. Given any hyperreal number $s\in\widetilde{\mathbb{R}_c^\#}$, $s\neq 0_{\widetilde{\mathbb{R}_c^\#}}$, there is a hyperreal number $t\in\widetilde{\mathbb{R}_c^\#}$ such that $s\times\ t=1_{\widetilde{\mathbb{R}_c^\#}}$ or $s\times\ t=\breve{1}_{\widetilde{\mathbb{R}_c^\#}}$.

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of

$$0_{\widetilde{\mathbb{R}_c^\#}}=(0_{\mathbb{R}_c^\#},0_{\mathbb{R}_c^\#},0_{\mathbb{R}_c^\#},0_{\mathbb{R}_c^\#},\ldots). \tag{14.1.40}$$

In other words, $s=\left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]$ where $\Psi_n-0_{\widetilde{\mathbb{R}_c^\#}}$ does not #-converge to $0_{\mathbb{R}_c^\#}$.

From this, we are to deduce the existence of a hyperreal number $t=\left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]$ such that $s\times\ t=\left[\{\Psi_n\times\Phi_n\}_{n=0}^{\infty^\#}\right]$ is the same equivalence class as $1_{\widetilde{\mathbb{R}_c^\#}}=[(1_{\mathbb{R}_c^\#},1_{\mathbb{R}_c^\#},1_{\mathbb{R}_c^\#},1_{\mathbb{R}_c^\#},\ldots)]$ or as some $\breve{1}_{\widetilde{\mathbb{R}_c^\#}}$. Doing so is actually an easy consequence of the fact that nonzero hyperreal numbers from $\mathbb{R}_c^\#$ have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{\Psi_n\}_{n=0}^{\infty^\#}$ does not #-tend to $0_{\mathbb{R}_c^\#}$ as $n\to\infty^\#$), there's no reason any number of the terms in $\{\Psi_n\}_{n=0}^{\infty^\#}$ can't equal $0_{\widetilde{\mathbb{R}_c^\#}}$. However, it turns out that eventually, $\Psi_n\neq 0_{\mathbb{R}_c^\#}$.

That is,

Lemma 14.1.2. If $\{\Psi_n\}_{n=0}^{\infty^\#}$ is a Cauchy hyper infinite sequence which does not #-tends to $0_{\mathbb{R}_c^\#}$, then there is an $N\in\mathbb{N}^\#$ such that, for $n>N$, $\Psi_n\neq 0_{\mathbb{R}_c^\#}$.

We will now use it to complete the proof of Theorem 14.2.11.

Let $N\in\mathbb{N}^\#$ be such that $\Psi_n\neq 0_{\mathbb{R}_c^\#}$ for $n>N$. Define hyper infinite sequence Φ_n of hyperreal numbers from $\widetilde{\mathbb{R}_c^\#}$ as follows:

for $n\leq N,\Phi_n=0_{\mathbb{R}_c^\#}$, and for $n>N$, $\Phi_n=1_{\mathbb{R}_c^\#}/\Psi_n$:

$\{\Phi_n\}_{n=0}^{\infty^\#}=(0_{\mathbb{R}_c^\#},0_{\mathbb{R}_c^\#},\ldots,0_{\mathbb{R}_c^\#},1_{\mathbb{R}_c^\#}/\Psi_{N+1},1_{\mathbb{R}_c^\#}/\Psi_{N+2},\ldots)$.

This makes sense since, for $n > N$, an is a nonzero hyperreal number, so $1_{\mathbb{R}_c^\#}/\Psi_n$ exists.
Then $\Psi_n \times \Phi_n$ is equal to $\Psi_n \times 0_{\mathbb{R}_c^\#} = 0_{\mathbb{R}_c^\#}$ for $n \leq N$, and equals $\Psi_n \times \Phi_n = \Psi_n \times \ 1_{\mathbb{R}_c^\#}/\Psi_n = 1_{\mathbb{R}_c^\#}$ for $n > N$
Well, then, if we look at the hyper infinite sequence

$$1_{\widetilde{\mathbb{R}_c^\#}} = (1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, \ldots), \tag{14.1.41}$$

we have $(1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, \ldots) - (\Psi_n \times \ \Phi_n)$ is the hyper infinite sequence which is $1_{\widetilde{\mathbb{R}_c^\#}} - 0_{\widetilde{\mathbb{R}_c^\#}} = 1_{\widetilde{\mathbb{R}_c^\#}}$ for $n \leq N$ and equals $1_{\widetilde{\mathbb{R}_c^\#}} - 1_{\widetilde{\mathbb{R}_c^\#}} = 0_{\widetilde{\mathbb{R}_c^\#}}$ for $n > N$. Since this hyper infinite sequence is eventually equal to $0_{\mathbb{R}_c^\#}$, it #-converges to $0_{\mathbb{R}_c^\#}$ as $n \to \infty^\#$, and so $\left[\{\Psi_n \times \Phi_n\}_{n=0}^{\infty^\#}\right] = [(1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, 1_{\mathbb{R}_c^\#}, \ldots)] = 1_{\widetilde{\mathbb{R}_c^\#}} \in \widetilde{\mathbb{R}_c^\#}$ and similarly $\left[\{\Psi_n \times \Phi_n\}_{n=0}^{\infty^\#}\right] = \breve{1}_{\widetilde{\mathbb{R}_c^\#}} \in \widetilde{\mathbb{R}_c^\#}$. This shows that $t = \left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]$ is a multiplicative inverse (and similarly quasi inverse) to $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]$.

Definition 14.2.41. Let $s \in \widetilde{\mathbb{R}_c^\#}$. Say that s is positive if $s \neq 0_{\widetilde{\mathbb{R}_c^\#}}$, and if $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right]$ for some Cauchy hyper infinite sequence such that for some N, $\Psi_n > 0_{\mathbb{R}_c^\#}$ for all $n > N$. Given two hyperreal numbers $s, t \in \widetilde{\mathbb{R}_c^\#}$, say that $s > t$ if $s - t$ is positive.

Theorem 14.1.7. Let $s, t \in \widetilde{\mathbb{R}_c^\#}$ be hyperreal numbers such that $s > t$, and let $r \in \widetilde{{}^*\mathbb{R}_c^\#}$. Then $s + r > t + r$.

Proof. Let $s = \left[\{\Psi_n\}_{n=0}^{\infty^\#}\right], t = \left[\{\Phi_n\}_{n=0}^{\infty^\#}\right]$, and $r = \left[\{\Theta_n\}_{n=0}^{\infty^\#}\right]$. Since $s > t$, i.e. $s - t > 0$, we know that there is an N such that, for $n > N, \Psi_n - \Phi_n > 0$. So $\Psi_n > \Phi_n$ for $n > N$. Now, adding Θ_n to both sides of this inequality , we have $\Psi_n + \Theta_n > \Phi_n + \Theta_n$ for $n > N$, or $(\Psi_n + \Theta_n) - (\Phi_n + \Theta_n) > 0_{\mathbb{R}_c^\#}$ for $n > N$. Note also that $(\Psi_n + \Theta_n) - (\Phi_n + \Theta_n) = \Psi_n - \Phi_n$ does not #-converge to $0_{{}^*\mathbb{R}_c^\#}$ as $n \to {}^*\infty$, by the assumption that $s - t > 0_{\widetilde{\mathbb{R}_c^\#}}$. Thus, by Definition 14.2.41, this means that: $s + r = \left[\{\Psi_n + \Theta_n\}_{n=0}^{\infty^\#}\right] > \left[\{\Phi_n + \Theta_n\}_{n=0}^{\infty^\#}\right] = t + r$.

Definition 14.1.42. There is canonical imbeding

$$\mathbb{R}_c^\# \hookrightarrow \widetilde{\mathbb{R}_c^\#} \tag{14.1.42}$$

defined by

$$a \mapsto \widetilde{a} \tag{14.1.43}$$

where $\widetilde{a}$ is hyper infinite sequence $\widetilde{a} = (a, a, \ldots.) \in \widetilde{\mathbb{R}_c^\#}, a \in \mathbb{R}_c^\#$.

Notation 14.1.5. $\widehat{a} = (a, a, \ldots.) \in \widetilde{\mathbb{R}_c^\#}, a \in \widetilde{\mathbb{R}_c^\#}$.

Definition 14.1.43. (i) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $\widetilde{\mathbb{R}_c^\#}, \{a_n\}_{n=0}^{k} \subset \widetilde{\mathbb{R}_c^\#}$.
We define external hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{k}} \subset \widetilde{\mathbb{R}_c^\#}$ by

$$\begin{gathered}\{A_n; k\}_{n=0}^{\infty^\#} = \overbrace{\{a_n\}_{n=0}^{k}} = \\ = \left(a_0, a_1, \ldots, a_m, \ldots, a_{k-1}, a_k, \widehat{a_k}\right).\end{gathered} \tag{14.1.44}$$

(ii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $\widetilde{\mathbb{R}_c^\#} : \{a_n\}_{n=0}^{\infty} \subset \widetilde{\mathbb{R}_c^\#}$.

We define hyper infinite sequence $\{A_n\}_{n=0}^{\infty^\#} = \overbrace{\{a_n\}_{n=0}^{\infty}} \subset \widetilde{\mathbb{R}_c^\#}$ by

$$\{A'_n;\infty\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{\infty}} = \\ = \left(\widehat{a}_0, a_1, \ldots, a_k, \ldots \{a_n\}_{n=0}^{\infty}, \widehat{\{a_n\}_{n=0}^{\infty}}\right). \tag{14.1.45}$$

(iii) Let $\{a_n\}_{n=0}^{N}, N \in \mathbb{N}^\# \backslash \mathbb{N}$ be external hyperfinite sequence in $\widetilde{\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{N} \subset \widetilde{\mathbb{R}_c^\#}$.
We define hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{N}} \subset \widetilde{\mathbb{R}_c^\#}$ by

$$\{A_n;N\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{N}} = \\ = \left(a_0, a_1, , \ldots, a_m, \ldots, a_{N-1} a_N, \widehat{a_N}\right). \tag{14.1.46}$$

Definition 14.1.44.(i) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $\widetilde{\mathbb{R}_c^\#}, \{a_n\}_{n=0}^{N} \subset \widetilde{\mathbb{R}_c^\#}$.
We define external finite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n = \overbrace{\{c_n\}_{n=0}^{k}} = (c_0, c_1, \ldots, c_k, \widehat{c}_k) \tag{14.1.47}$$

where $c_0 = a_0, c_j = Ext\text{-}\sum_{n=0}^{n=j} a_n, 0 \leq j \leq k.$
(ii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $\widetilde{\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{\infty} \subset \widetilde{\mathbb{R}_c^\#}$. We define external countable sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n = \overbrace{\{c_n\}_{n=0}^{\infty}} = \\ = \left(c_0, c_1, \ldots, c_k, \ldots \{c_n\}_{n=0}^{\infty}, \widehat{\{c_n\}_{n=0}^{\infty}}\right) \in \left[\widehat{\{c_n\}_{n=0}^{\infty}}\right] \tag{14.1.48}$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n, k \in \mathbb{N}.$
(iii) Let $\{a_n\}_{n=0}^{n=N}, N \in {}^*\mathbb{N} \backslash \mathbb{N}$ be external hyperfinite sequence in $\widetilde{{}^*\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^\#}$.
We define external hyperfinite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = \overbrace{\{c_n\}_{n=0}^{n=N}} = (c_0, c_1, \ldots, c_k, \ldots, c_N, \widehat{c}_N) \tag{14.1.49}$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n,\ 0 \leq k \leq N, c_N = Ext\text{-}\sum_{n=0}^{n=N} a_n.$
(iv) Let $\{a_n\}_{n=0}^{n=N}, N \in \mathbb{N}^\#$ be external hyperfinite sequence in $\widetilde{\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{N} \subset \widetilde{\mathbb{R}_c^\#}$ such that $a_n \equiv 0$ for all $n \in \mathbb{N}^\# \backslash \mathbb{N}$. We assume that

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n. \tag{14.1.50}$$

Example 14.1.3. Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots . \alpha r^{N-1}, N \in \mathbb{N}^\#, \alpha \in \widetilde{\mathbb{R}_c^\#}$,
$r \in \widetilde{\mathbb{R}_c^\#}$ be the first term and the ratio of the G.P respectively. Then for any $N \in \mathbb{N}^\#$ by Proposition 14.1.6 and Definition 14.1.44 one obtains that

$$Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1} \alpha r^{n-1} = \alpha \widehat{\frac{1_{\widetilde{\mathbb{R}_c^\#}} - r^N}{1_{\widetilde{\mathbb{R}_c^\#}} - r}} =. \alpha \widehat{\frac{1_{\widetilde{\mathbb{R}_c^\#}}}{1_{\widetilde{\mathbb{R}_c^\#}} - r}} - \alpha \widehat{\frac{r^N}{1_{\widetilde{\mathbb{R}_c^\#}} - r}}. \tag{14.1.51}$$

and

$$Ext\text{-}\widehat{\sum}_{n=1}^{\infty}\alpha r^{n-1} =. \alpha\widehat{\frac{1_{\widetilde{\mathbb{R}^{\#}_c}}}{1_{\widetilde{\mathbb{R}^{\#}_c}}-r}} - \alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{\mathbb{R}^{\#}_c}}-r}\right\}_{n=1}^{\infty}}. \qquad (14.1.52)$$

Example 14.1.4.Consider the G.P: $\alpha,\alpha r,\alpha r^2,\ldots.\alpha r^{N-1}, N \in \mathbb{N}^{\#}, \alpha \in \widetilde{\mathbb{R}^{\#}_c}, r \in \widetilde{\mathbb{R}^{\#}_c}$, $r < 0_{\widetilde{\mathbb{R}^{\#}_c}}$, $|r| < 1$. Note that

$$\begin{gathered}
\alpha\widehat{\frac{1_{\widetilde{\mathbb{R}^{\#}_c}}-r^N}{1_{\mathbb{R}^{\#}_c}-r}} = Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1}\alpha r^{n-1} = \\
= Ext\text{-}\widehat{\sum}_{n=1}^{\infty}\alpha r^{n-1} + Ext\text{-}\widehat{\sum}_{(n\in\mathbb{N}^{\#}\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha r^{n-1} = \qquad (14.1.53)\\
= \alpha\widehat{\frac{1_{\widetilde{\mathbb{R}^{\#}_c}}}{1_{\widetilde{\mathbb{R}^{\#}_c}}-r}} - \alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{\mathbb{R}^{\#}_c}}-r}\right\}_{n=1}^{\infty}} + Ext\text{-}\widehat{\sum}_{(n\in\mathbb{N}^{\#}\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha r^{n-1}.
\end{gathered}$$

From (14.1.53) we obtain

$$\begin{gathered}
Ext\text{-}\widehat{\sum}_{(n\in\mathbb{N}^{\#}\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha r^{n-1} = \alpha\widehat{\frac{1-_{\widetilde{\mathbb{R}^{\#}_c}} r^N}{1_{\mathbb{R}^{\#}_c}-r}} - \alpha\widehat{\frac{1_{\widetilde{\mathbb{R}^{\#}_c}}}{1_{\widetilde{\mathbb{R}^{\#}_c}}-r}} + \alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{\mathbb{R}^{\#}_c}}-r}\right\}_{n=1}^{\infty}} = \\
\alpha\widehat{\left\{\frac{\left(-1_{\widetilde{{}^*\mathbb{R}^{\#}_c}}\right)^n|r|^n}{1_{\widetilde{\mathbb{R}^{\#}_c}}-r}\right\}_{n=1}^{\infty}} - \alpha\widehat{\frac{r^N}{1_{\widetilde{\mathbb{R}^{\#}_c}}-r}}.
\end{gathered} \qquad (14.1.54)$$

Assume that: (i) $r < 0_{\widetilde{\mathbb{R}^{\#}_c}}$, $|r| < 1$ then from (14.1.54) we obtain

$$Ext\text{-}\widehat{\sum}_{(n\in\mathbb{N}^{\#}\setminus\mathbb{N})\wedge(n\leq N-1)}\alpha\left(-1_{\widetilde{\mathbb{R}^{\#}_c}}\right)^{n-1}|r|^{n-1} \neq 0_{\widetilde{\mathbb{R}^{\#}_c}}. \qquad (14.1.55)$$

§14.2.External non-Archimedean field $\widetilde{{}^*\mathbb{R}^{\#}_c}$ via special extension of non-Archimedean field ${}^*\mathbb{R}^{\#}_c$

Notation 14.2.3. Let $\Delta \subset {}^*\mathbb{R}^{\#}_c$ and $\Delta \neq \{0\}$. Then we write $\Delta > 0$ iff $a \in \Delta \Rightarrow a > 0$.
Definition 14.2.13. Let $\Delta \subset {}^*\mathbb{R}^{\#}_c$ and $\Delta > 0$. Assume that: $a,b \in \Delta \Rightarrow a+b \in \Delta$. Then we say that Δ is a positive idempotent in ${}^*\mathbb{R}^{\#}_c$.
Notation 14.2.4. We will denote by ${}^*\mathbb{R}^{\#}_{c+,\mathbf{fin}}$ a set of the all positive finite number in $\mathbb{R}^{\#}_c$ except infinitesimals in ${}^*\mathbb{R}^{\#}_c$.
Remark 14.2.6.Note that a set ${}^*\mathbb{R}^{\#}_{c+,\mathbf{fin}}\setminus\{0\} \subset {}^*\mathbb{R}^{\#}_c$ is a positive idempotent in ${}^*\mathbb{R}^{\#}_c$.
Proposition 14.2.1.Let $\Delta \subset {}^*\mathbb{R}^{\#}_c$ is a positive idempotent in ${}^*\mathbb{R}^{\#}_c$. Then the following are equivalent.[In what follows assume $a,b > 0$].
(i) $a \in \Delta \Rightarrow 2a \in \Delta$,
(ii) $a \in \Delta \Rightarrow na \in \Delta$ for all standard integers $n \in \mathbb{N}$,
(iii) $a \in \Delta \Rightarrow ra \in \Delta$ for all finite $r \in {}^*\mathbb{R}^{\#}_{c+}$.
Proof. All parts are immediate from the Definition 14.2.13.
Notation 14.2.4. $\Delta^{\#+}_{\approx} \triangleq \{\delta \in {}^*\mathbb{R}^{\#}_{c+} | \delta > 0, \delta \approx 0\}$, i.e. $\Delta^{\#+}_{\approx}$ is a set of the all positive infinitesimals in ${}^*\mathbb{R}^{\#}_{c+}$; $\Delta^{\#-}_{\approx} \triangleq \{\delta \in {}^*\mathbb{R}^{\#}_{c+} | \delta < 0, \delta \approx 0_{\mathbb{R}^{\#}_c}\}$, i.e. $\Delta^{\#-}_{\approx} = \Delta^{\#+}_{\approx}$ is a set of the all negative infinitesimals in ${}^*\mathbb{R}^{\#}_c$. Note that $\Delta^{\#-}_{\approx} = -\Delta^{\#+}_{\approx}$.

Remark 14.2.7.Note that a set $\Delta_{\approx}^{\#+} \subset {}^*\mathbb{R}_c^\#$ is a positive idempotent in ${}^*\mathbb{R}_c^\#$ and $\Delta_{\approx}^{\#-}$ is a negative idempotent in ${}^*\mathbb{R}_c^\#$.

Definition 14.2.14. Let $\{a_n\}_{n=0}^{\infty}$ be ${}^*\mathbb{R}_{c+,\mathbf{fin}}^\#$- valued countable sequence $a : \mathbb{N} \to {}^*\mathbb{R}_{c+,\mathbf{fin}}^\#$ such that:

(i) there is $M \in \mathbb{N}$ such that $\{a_n\}_{n=M}^{\infty}$ is monotonically decreasing ${}^*\mathbb{R}_{c+,\mathbf{fin}}^\#$- valued countable sequence $a : \mathbb{N} \to {}^*\mathbb{R}_{c+,\mathbf{fin}}^\# \backslash \{0_{\mathbb{R}_c^\#}\}$

(ii) there is $M \in \mathbb{N}$ such that for all $n > M, a_n \neq 0_{{}^*\mathbb{R}_c^\#}$ [it follows from (ii)]

(iii) for all $n \in \mathbb{N}, a_n \not\approx 0_{{}^*\mathbb{R}_c^\#}$ and for any $\epsilon > 0, \epsilon \not\approx 0_{\mathbb{R}_c^\#}, \epsilon \in {}^*\mathbb{R}_{c+,\mathbf{fin}}^\#$ there is $N \in \mathbb{N}$ such that for all $n > N : a_n < \epsilon$ and we denote a set of the all these sequences by $\Delta_{\omega}^{+\downarrow 0}$.

We define a set $\Delta_{\omega}^{-\downarrow 0}$ by $c_n \in \Delta_{\omega}^{-\downarrow 0} \Leftrightarrow \{-c_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$.Note that $\Delta_{\omega}^{-\downarrow 0} = -\Delta_{\omega}^{+\downarrow 0}$.

Remark 14.2.8.Note that a set $\Delta_{\omega}^{+\downarrow 0}$ is a positive idempotent in ${}^*\mathbb{R}_c^\#$ and a set $\Delta_{\omega}^{-\downarrow 0}$ is a negative idempotent in ${}^*\mathbb{R}_c^\#$.

Proposition 14.2.2.(1) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ then:

(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$

(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0} \cup \Delta_{\omega}^{-\downarrow 0} \cup \Delta_{\approx}^{\#+} \cup \Delta_{\approx}^{\#-} \cup \{0_{{}^*\mathbb{R}_c^\#}\}_{n=0}^{\infty}$

where $\{0_{{}^*\mathbb{R}_c^\#}\}_{n=0}^{\infty}$ is a countable $0_{{}^*\mathbb{R}_c^\#}$- valued sequence.

(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$.

(2) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow 0}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow 0}$ then we define

(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow 0}$

(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0} \cup \Delta_{\omega}^{-\downarrow 0}$

(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$

(3) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0} \cup \Delta_{\omega}^{-\downarrow 0}$ and $x, y \in \mathbb{R}_c^\#$ then we define

(iv) $x + y\{a_n\}_{n=0}^{\infty} \triangleq \{x + ya_n\}_{n=0}^{\infty}$

Proof. Immediately by definitions and by Definition 14.2.14.

Definition 14.2.15. We define the relation $(\cdot \leq \cdot)$ on a set $\Delta_{\omega}^{+\downarrow 0}$ by:

let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ then $\{a_n\}_{n=0}^{\infty} \leq \{b_n\}_{n=0}^{\infty}$ iff there is $N \in \mathbb{N}$ such that for all $n > N : a_n \leq b_n$ and similarly we define the relation $(\cdot \leq \cdot)$ on a set $\Delta_{\omega}^{-\downarrow 0}$ by: let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow 0}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow 0}$ then $\{a_n\}_{n=0}^{\infty} \leq \{b_n\}_{n=0}^{\infty}$ iff there is $N \in \mathbb{N}$ such that for all $n > N : a_n \leq b_n$

Definition 14.2.16. (1) We define the relation $(\cdot < \cdot)$ on a set $\Delta_{\omega}^{+\downarrow} \times {}^*\mathbb{R}_{c+,\mathbf{fin}}^\#$ by:

let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ and $x \in {}^*\mathbb{R}_{c+,\mathbf{fin}}^\#$ then $\{a_n\}_{n=0}^{\infty} < x$ iff there is $N \in \mathbb{N}$ such that for all $n > N : a_n < x$.

(2) We define the relation $(\cdot < \cdot)$ on a set $\Delta_{\approx}^{\#+} \times \Delta_{\omega}^{+\downarrow}$ by: let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ and $x \in \Delta_{\approx}^{\#+}$ then $x < \{a_n\}_{n=0}^{\infty}$ iff there is $N \in \mathbb{N}$ such that for all $n > N : x < a_n$.

(3) Let $\{a_n\}_{n=0}^{\infty}$ be $\Delta_{\approx}^{\#+}$- valued countable sequence $a : \mathbb{N} \to \Delta_{\approx}^{\#+}$, and we denote a set of the all these sequences by $\Delta_{\approx,\omega}^{\#+}$.

We define the relation $(\cdot < \cdot)$ on a set $\Delta_{\approx,\omega}^{\#+} \times \Delta_{\omega}^{+\downarrow}$ by: let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\approx,\omega}^{\#+}$ and $x \in \Delta_{\approx}^{\#+}$ then $\{a_n\}_{n=0}^{\infty} < x$ iff there is $N \in \mathbb{N}$ such that for all $n > N : a_n < x$.

Proposition 14.2.2.Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ $\{a_n\}_{n=0}^{\infty} \neq 0_{{}^*\mathbb{R}_c^\#}$ then there is $N \in \mathbb{N}$ such that $0_{{}^*\mathbb{R}_c^\#} < \Delta_{\approx}^{\#+} < \{a_n\}_{n=0}^{\infty} < {}^*\mathbb{R}_{c+,\mathbf{fin}}^\# \backslash \{0_{{}^*\mathbb{R}_c^\#}\}$.

Proof. Immediately by definitions and by Definition 14.2.16.

Remark 14.2.9.Note that it follows from Proposition 14.2.2 that

$$0_{{}^*\mathbb{R}_c^\#} < \Delta_{\approx}^{\#\downarrow} < \Delta_{\omega}^{+\downarrow 0} < {}^*\mathbb{R}_{c+,\mathbf{fin}}^\# \backslash \{0_{{}^*\mathbb{R}_c^\#}\}. \quad (14.2.9)$$

Definition 14.2.17. Let $\{a_n\}_{n=0}^{\infty}$ be monotonically increasing ${}^*\mathbb{R}_{c+,\mathbf{fin}}^\#$- valued countable sequence $a : \mathbb{N} \to {}^*\mathbb{R}_{c+,\mathbf{fin}}^\# \backslash \Delta_{\approx}^{\downarrow}$ such that:
(i) there is $M \in \mathbb{N}$ such that for all $n > M, a_n \neq 0_{{}^*\mathbb{R}_c^\#}$
(ii) there is $N \in \mathbb{N}$ such that for all $n > N$ and for any $\xi > 0_{{}^*\mathbb{R}_c^\#}, \xi \in {}^*\mathbb{R}_{c+,\mathbf{fin}}^\#$, $a_n > \xi$
and we denote a set of the all these sequences by $\Delta_{\omega}^{+\downarrow\infty}$. We define a set $\Delta_{\omega}^{-\downarrow\infty}$ by $c_n \in \Delta_{\omega}^{-\downarrow\infty} \Leftrightarrow \{-c_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$. Note that $\Delta_{\omega}^{-\downarrow\infty} = -\Delta_{\omega}^{+\downarrow\infty}$.

Proposition 14.2.3.(1) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$ then we define
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty} \cup \Delta_{\omega}^{-\downarrow\infty} \cup \Delta_{\approx}^{\#\downarrow} \cup \Delta_{\approx}^{\#-} \cup \Delta_{\omega}^{+\downarrow 0} \cup$
$\cup \Delta_{\omega}^{-\downarrow 0} \cup \{0_{{}^*\mathbb{R}_c^\#}\}_{n=0}^{\infty}$ where $\{0_{\mathbb{R}_c^\#}\}_{n=0}^{\infty}$ is a countable $0_{\mathbb{R}_c^\#}$- valued sequence.
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$.
(2) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow\infty}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow\infty}$ then we define
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{-\downarrow\infty}$
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty} \cup \Delta_{\omega}^{-\downarrow\infty}$
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$
(3) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty}$ and $x, y \in \mathbb{R}_c^\#$ then we define
(iv) $x_n + y_n\{a_n\}_{n=0}^{\infty} \triangleq \{x_n + y_n a_n\}_{n=0}^{\infty}$ and we denote a set of the all these sequences by $\left\{\Delta_{\omega}^{+\downarrow\infty}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\right\}$.

Proof. Immediately by definitions and by Definition 14.2.16.

Remark 14.2.10.Note that $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow\infty} \Leftrightarrow \{a_n^{-1}\}_{n=N}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$.

Definition 14.2.18.(1) Let $\{a_n\}_{n=0}^{\infty} \in \Delta_{\omega}^{+\downarrow 0}$ and let $\{A_n\}_{n=0}^{{}^*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ be a hyper infinite sequence

$$\{A_n\}_{n=0}^{{}^*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}} = (a_0, a_1, \ldots, a_k, \ldots, \{a_n\}_{n=0}^{\infty}, \ldots) \quad (14.2.10)$$

i.e. for any infinite $m \in {}^*\mathbb{N}\backslash\mathbb{N}, A_m \equiv \{a_n\}_{n=0}^{\infty}$. We will denote a set of the all these hyper infinite sequences by $\widetilde{\Delta_{\omega}^{+\downarrow 0}}$ and a set of the all hyper infinite sequences $\widetilde{\{-a_n\}_{n=0}^{\infty}}$ by $\widetilde{\Delta_{\omega}^{-\downarrow 0}}$. (2) Let $\{x_n + y_n a_n\}_{n=0}^{\infty} \in \left\{\Delta_{\omega}^{+\downarrow 0}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\right\}$ and let

$$\begin{gathered}\{x_n + y_n A_n\}_{n=0}^{{}^*\infty} = \widetilde{\{x_n + y_n a_n\}_{n=0}^{\infty}} \\ (x_0 + y_0 a_0, x_1 + y_1 a_1, \ldots, x_k + y_k a_k, \ldots, \{x_n + y_n a_n\}_{n=0}^{\infty}, \ldots),\end{gathered} \quad (14.2.11)$$

i.e. for any infinite $m \in {}^*\mathbb{N}\backslash\mathbb{N}, A_m \equiv \{x_n + y_n a_n\}_{n=0}^{\infty}$. We will denote a set of the all these hyper infinite sequences by $\widetilde{\left\{\Delta_{\omega}^{+\downarrow 0}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\right\}}$.

Definition 14.2.19.Let $\{A_n\}_{n=0}^{{}^*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ and $\{B_n\}_{n=0}^{{}^*\infty} = \widetilde{\{b_n\}_{n=0}^{\infty}}$ be in $\widetilde{\Delta_{\omega}^{+\downarrow 0}}$.
Then we define:
(i) $\{A_n\}_{n=0}^{{}^*\infty} + \{B_n\}_{n=0}^{{}^*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}} + \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n + b_n\}_{n=0}^{\infty}} = \{A_n + B_n\}_{n=0}^{{}^*\infty} \in \widetilde{\Delta_{\omega}^{+\downarrow 0}}$
(ii) $\{A_n\}_{n=0}^{{}^*\infty} - \{B_n\}_{n=0}^{{}^*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}} - \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n - b_n\}_{n=0}^{\infty}} =$
$= \{A_n - B_n\}_{n=0}^{{}^*\infty} \in \widetilde{\Delta_{\omega}^{+\downarrow 0}} \cup \widetilde{\Delta_{\omega}^{-\downarrow 0}} \cup \{0_{\mathbb{R}_c^\#}\}_{n=0}^{{}^*\infty}$
(iii) $\{A_n\}_{n=0}^{{}^*\infty} \times \{B_n\}_{n=0}^{{}^*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}} \times \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n \times b_n\}_{n=0}^{\infty}} = \{A_n \times B_n\}_{n=0}^{\infty^\#} \in \widetilde{\Delta_{\omega}^{+\downarrow 0}}$

Let $\{A_n\}_{n=0}^{*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ and $\{B_n\}_{n=0}^{*\infty} = \widetilde{\{b_n\}_{n=0}^{\infty}}$ be in $\left\{\Delta_{\omega}^{+\downarrow 0}, \{x_{1,n}\}_{n=0}^{\infty}, \{y_{1n}\}_{n=0}^{\infty},\right\}$ and $\{B_n\}_{n=0}^{*\infty} = \widetilde{\{b_n\}_{n=0}^{\infty}}$ be in $\left\{\Delta_{\omega}^{+\downarrow 0}, \{x_{2,n}\}_{n=0}^{\infty}, \{y_{2,n}\}_{n=0}^{\infty},\right\}$. Then we define:

(iv) $\{A_n\}_{n=0}^{\infty^{\#}} \dotplus \{B_n\}_{n=0}^{*\infty} = \widetilde{\{x_{1,n} + y_{1,n}a_n\}_{n=0}^{\infty}} + \widetilde{\{x_{2,n} + y_{2,n}b_n\}_{n=0}^{\infty}} \triangleq$

$\triangleq \widetilde{\{x_{1,n} + x_{2,n} \dotplus y_{1,n}a_n \dotplus y_{2,n}b_n\}_{n=0}^{\infty}} = \{x_{1,n} + +x_{2,n} \dotplus y_{1,n}A_n + y_{2,n}B_n\}_{n=0}^{*\infty}$

Definition 14.2.20.Let $\{\Psi_n\}_{n=0}^{*\infty}$ be in $\widetilde{\Delta_{\omega}^{+\downarrow 0}}$, i.e. for all $n \in {}^*\mathbb{N}$, $\Psi_n \in \Delta_{\omega}^{+\downarrow 0}$. Say $\{\Psi_n\}_{n=0}^{*\infty}$ #-tends to $0_{{}^*\mathbb{R}_c^{\#}}$ as $n \to {}^*\infty$ iff for any given $\varepsilon > 0_{\mathbb{R}_c^{\#}}, \varepsilon \approx 0_{{}^*\mathbb{R}_c^{\#}}$ there is a hypernatural number $N \in {}^*\mathbb{N}\backslash\mathbb{N}, N = N(\varepsilon)$ such that for any $n > N, |\Psi_n| < \varepsilon$.

Definition 14.2.21. Let $\{\Psi_n\}_{n=0}^{*\infty}$ be a hyper infinite sequence such that for all $n \in {}^*\mathbb{N}, \Psi_n \in \widetilde{\Delta_{\omega}^{+\downarrow 0}}$. We call $\{\Psi_n\}_{n=0}^{*\infty}$ a Cauchy hyper infinite sequence if the difference between its terms #-tends to $0_{{}^*\mathbb{R}_c^{\#}}$. To be precise: given any $\varepsilon > 0_{{}^*\mathbb{R}_c^{\#}}$, $\varepsilon \approx 0_{{}^*\mathbb{R}_c^{\#}}$ there is a hypernatural number $N \in {}^*\mathbb{N}\backslash\mathbb{N}, N = N(\varepsilon)$ such that for any $m,n > N, |\Psi_n - \Psi_m| < \varepsilon$.

Theorem 14.2.3.Let $\{\Psi_n\}_{n=0}^{*\infty}$ be in $\widetilde{\Delta_{\omega}^{+\downarrow 0}}$. If $\{\Psi_n\}_{n=0}^{*\infty}$ is a #-convergent hyper infinite sequence (that is, $\Psi_n \to_{\#} \Phi$ as $n \to {}^*\infty$ for some $\Phi \in \widetilde{\Delta_{\omega}^{+\downarrow 0}}$), then $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence.

Proof.cWe know that $\Psi_n \to_{\#} \Phi$. Here is a ubiquitous trick: instead of using ε in the definition, start with an arbitrary infinitesimall $\varepsilon > 0_{{}^*\mathbb{R}_c^{\#}}, \varepsilon \approx 0_{{}^*\mathbb{R}_c^{\#}}$ and then choose N so that $|\Psi_n - \Phi| < \varepsilon/2$ when $n > N$. Then if $m,n > N$, we have
$|\Psi_n - \Psi_m| = |(\Psi_n - \Phi) - (\Psi_m - \Phi)| \leq |\Psi_n - \Phi| + |\Psi_m - \Phi| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
This shows that $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence.

Theorem 14.2.4.If $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence, then it is bounded in ${}^*\mathbb{R}_c^{\#}$; that is, there is some number $M \in {}^*\mathbb{R}_c^{\#}$ such that $|\{\Psi_n\}_{n=0}^{*\infty}| \leq M$ for all $n \in {}^*\mathbb{N}$.

Proof. Since $\{\Psi_n\}_{n=0}^{*\infty}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some N such that $|\Psi_m - \Psi_n| < 1$ whenever $m,n > N$. Thus, $|\Psi_{N+1} - \Psi_n| < 1$ for $n > N$. We can rewrite this as $\Psi_{N+1} - 1 < \Psi_n < \Psi_{N+1} + 1$. This means that $|\Psi_n|$ is less than the maximum of $|\Psi_{N+1} - 1|$ and $|\Psi_{N+1} + 1|$. So, set $M \in {}^*\mathbb{R}_c^{\#}$ larger than any number in the following list: $\{|\Psi_0|, |\Psi_1|, \ldots, |\Psi_N|, |\Psi_{N+1} - 1|, |\Psi_{N+1} + 1|\}$.
Then for any term Ψ_n, if $n \leq N$, then $|\Psi_n|$ appears in the list and so $|\Psi_n| \leq M$; if $n > N$, then (as shown above) $|\Psi_n|$ is less than at least one of the last two entries in the list, and so $|\Psi_n| \leq M$. Hence, M is a bound for the sequence.
Let Ξ denote the set of all Cauchy hyper infinite sequences We must define an equivalence relation on Ξ.

Definition 14.2.22. Let $\mathbf{S}$ be a set of objects. A relation among pairs of elements of $\mathbf{S}$ is said to be an equivalence relation if the following three properties hold:

Reflexivity: for any $s \in \mathbf{S}$, s is related to s.

Symmetry: for any $s,t \in \mathbf{S}$, if s is related to t then t is related to s.

Transitivity: for any $s,t,r \in \mathbf{S}$, if s is related to t and t is related to r, then s is related to r.

The following well known proposition goes most of the way to showing that an equivalence relation divides a set into bins.

Theorem 14.2.5. Let $\mathbf{S}$ be a set, with an equivalence relation on pairs of elements. For $s \in \mathbf{S}$, denote by $[s]$ the set of all elements in $\mathbf{S}$ that are related to s. Then for any $s,t \in \mathbf{S}$, either $[s] = [t]$ or $[s]$ and $[t]$ are disjoint.
The sets $[s]$ for $s \in \mathbf{S}$ are called the equivalence classes, and they are the bins.
Corollary 14.2.1. If $\mathbf{S}$ is a set with an equivalence relation on pairs of elements, then the equivalence classes are non-empty disjoint sets whose union is all of $\mathbf{S}$.

Definition 14.2.23.Let $\{\Psi_{1,n}\}_{n=0}^{*\infty}$ and $\{\Psi_{2,n}\}_{n=0}^{*\infty}$ be in $\widetilde{\Delta_\omega^{+\downarrow 0}}$. Say they are equivalent (i.e. related) if $|\Psi_{1,n} - \Psi_{2,n}| \to_\# 0_{^*\mathbb{R}_c^\#}$ as $n \to {}^*\infty$, i.e. if the hyper infinite sequence $\{|\Psi_{1,n} - \Psi_{2,n}|\}_{n=0}^{*\infty}$ #-tends to $0_{^*\mathbb{R}_c^\#}$.

Proposition 14.2.4.Definition 4.2.23 yields an equivalence relation on

$$\Xi = \overline{\left\{\Delta_\omega^{+\downarrow 0}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\right\}}.$$

Proof. we need to show that this relation is reflexive, symmetric, and transitive.
• **Reflexive**: $\Psi_n - \Psi_n = 0_{^*\mathbb{R}_c^\#}$, and the sequence all of whose terms are $0_{^*\mathbb{R}_c^\#}$ clearly converges to $0_{\mathbb{R}_c^\#}$.So $\{\Psi_n\}_{n=0}^{*\infty}$ is related to $\{\Psi_n\}_{n=0}^{*\infty}$.
• **Symmetric**: Suppose $\{\Psi_{1,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{2,n}\}_{n=0}^{*\infty}$, so $\Psi_{1,n} - \Psi_{2,n} \to_\# 0_{^*\mathbb{R}_c^\#}$.
But $\Psi_{2,n} - \Psi_{1,n} = -(\Psi_{1,n} - \Psi_{2,n})$,and since only the absolute value $|\Psi_{1,n} - \Psi_{2,n}| =$
$= |\Psi_{2,n} - \Psi_{1,n}|$ comes into play in Definition 14.2.20,it follows that $\Psi_{2,n} - \Psi_{1,n} \to_\# 0_{^*\mathbb{R}_c^\#}$ as well. Hence, $\{\Psi_{2,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{1,n}\}_{n=0}^{*\infty}$.
• **Transitive**: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 14.1.4.
Suppose $\{\Psi_{1,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{2,n}\}_{n=0}^{*\infty}$, and $\{\Psi_{2,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{3,n}\}_{n=0}^{*\infty}$.
This means that $\Psi_{1,n} - \Psi_{2,n} \to_\# 0_{\mathbb{R}_c^\#}$ and $\Psi_{2,n} - \Psi_{3,n} \to_\# 0_{\mathbb{R}_c^\#}$.
To be fully precise, let us fix infinite small $\varepsilon > 0_{^*\mathbb{R}_c^\#}$; then there exists an $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that for all $n > N, |\Psi_{1,n} - \Psi_{2,n}| < \varepsilon/2$; also, there exists an M such that for all $n > M, |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2$. Well, then, as long as $n > \max(N,M)$, we have that
$|\Psi_{2,n} - \Psi_{3,n}| = |(\Psi_{1,n} - \Psi_{2,n}) + (\Psi_{2,n} - \Psi_{3,n})| \leq |\Psi_{1,n} - \Psi_{2,n}| + |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
So, choosing L equal to the max of N,M , we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |\Psi_{1,n} - \Psi_{3,n}| < \varepsilon$. This means that $\Psi_{1,n} - \Psi_{3,n} \to_\# 0_{^*\mathbb{R}_c^\#}$, i.e. $\{\Psi_{1,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{3,n}\}_{n=0}^{*\infty}$.
So, we really have equivalence relation, and so by Corollary 14.2.1, the set Ξ is partitioned into disjoint subsets (equivalence classes).

Definition 14.2.24. The hyperreal numbers $\widetilde{^*\mathbb{R}_c^\#}$ are the equivalence classes $\left[\{\Psi_{1,n}\}_{n=0}^{*\infty}\right]$ of Cauchy hyper infinite sequences of, as per Definition 14.2.23.
That is, each such equivalence class is a hyperreal number in $\widetilde{^*\mathbb{R}_c^\#}$.

Definition 14.2.25.Let $s,t \in \widetilde{^*\mathbb{R}_c^\#}$, so there are Cauchy hyper infinite sequences $\{\Psi_n\}_{n=0}^{*\infty}$ and $\{\Phi_n\}_{n=0}^{*\infty}$ with $s = \left[\{\Psi_n\}_{n=0}^{*\infty}\right]$ and $t = \left[\{\Phi_n\}_{n=0}^{*\infty}\right]$.
(a) Define $s + t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n \dot{+} \Phi_n\}_{n=0}^{*\infty}$.
(b) Define $s \times t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n \times \Phi_n\}_{n=0}^{*\infty}$.
Proposition 14.2.5.The operations $+,\times$ in Definition 14.2.25 (a),(b) are well-defined.
Proof. Suppose that $\left[\{\Psi_n\}_{n=0}^{*\infty}\right] = \left[\{\Psi_{1,n}\}_{n=0}^{*\infty}\right]$ and $\left[\{\Phi_n\}_{n=0}^{*\infty}\right] = \left[\{\Phi_{1,n}\}_{n=0}^{*\infty}\right]$.
Thus means that $\Psi_n - \Psi_{1,n} \to_\# 0_{^*\mathbb{R}_c^\#}$ and $\Phi_n - \Phi_{1,n} \to_\# 0_{^*\mathbb{R}_c^\#}$. Then

$(\Psi_n+\Phi_n)-(\Psi_{1,n}+\Phi_{1,n}) = (\Psi_n-\Psi_{1,n})+(\Phi_n-\Phi_{1,n})$. Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to $0_{{}^*\mathbb{R}_c^\#}$, and so $[(\Psi_n+\Phi_n)] = [(\Psi_{1,n}+\Phi_{1,n})]$.

Multiplication is a little trickier; this is where we will use Theorem 14.2.4. We will also use another ubiquitous technique: adding $0_{{}^*\mathbb{R}_c^\#}$ in the form of $s-s$. Again, suppose that

$\left[\{\Psi_n\}_{n=0}^{*\infty}\right] = \left[\{\Psi_{1,n}\}_{n=0}^{*\infty}\right]$ and $\left[\{\Phi_n\}_{n=0}^{*\infty}\right] = \left[\{\Phi_{1,n}\}_{n=0}^{*\infty}\right]$; we wish to show that $\left[\{\Psi_n\times\Phi_n\}_{n=0}^{*\infty}\right] = \left[\{\Psi_{1,n}\times\Phi_{1,n}\}_{n=0}^{*\infty}\right]$, or, in other words, that $\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}\to_\# 0_{{}^*\mathbb{R}_c^\#}$. Well, we add and subtract one of the other cross terms, say $\Phi_n\times\Psi_{1,n}$:

$$\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n} = \Psi_n\times\Phi_n+(\Phi_n\times\Psi_{1,n}-\Phi_n\times\Psi_{1,n})-\Psi_{1,n}\times\Phi_{1,n} =$$
$$= (\Psi_n\times\Phi_n-\Phi_n\times\Psi_{1,n})+(\Phi_n\times\Psi_{1,n}-\Psi_{1,n}\times\Phi_{1,n}) =$$
$$= \Phi_n\times(\Psi_n-\Psi_{1,n})+\Psi_{1,n}\times(\Phi_n-\Phi_{1,n}).$$

Hence, we have $|\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}|\leq|\Phi_n|\times|\Psi_n-\Psi_{1,n}|+|\Psi_{1,n}|\cdot|\Phi_n-\Phi_{1,n}|$.

Now, from Theorem 14.2.4, there are numbers M and L such that $|\Phi_n|\leq M$ and $|\Psi_{1,n}|\leq L$ for all $n\in{}^*\mathbb{N}$. Taking some number R (for example $R=M+L$) which is bigger than both, we have

$$|\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}|\leq|\Phi_n|\times|\Psi_n-\Psi_{1,n}|+|\Psi_{1,n}|\times|\Phi_n-\Phi_{1,n}|\leq$$
$$\leq R(|\Psi_n-\Psi_{1,n}|+|\Phi_n-\Phi_{1,n}|).$$

Now, noting that both $\Psi_n-\Psi_{1,n}$ and $\Phi_n-\Phi_{1,n}$ #-tend to $0_{{}^*\mathbb{R}_c^\#}$ and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2R$, we see that

$$\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}\to_\# 0_{{}^*\mathbb{R}_c^\#}$$

Theorem 14.2.6. Given any hyperreal number $s\in\widetilde{{}^*\mathbb{R}_c^\#}$, $s\neq 0_{\widetilde{{}^*\mathbb{R}_c^\#}}$, there is a hyperreal number $t\in\widetilde{{}^*\mathbb{R}_c^\#}$ such that $s\times t=1_{\widetilde{{}^*\mathbb{R}_c^\#}}$.

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of

$$0_{\widetilde{{}^*\mathbb{R}_c^\#}} = (0_{{}^*\mathbb{R}_c^\#},0_{{}^*\mathbb{R}_c^\#},0_{{}^*\mathbb{R}_c^\#},0_{{}^*\mathbb{R}_c^\#},\ldots). \tag{14.2.12}$$

In other words, $s=\left[\{\Psi_n\}_{n=0}^{*\infty}\right]$ where $\Psi_n-0_{\widetilde{{}^*\mathbb{R}_c^\#}}$ does not #-converge to $0_{{}^*\mathbb{R}_c^\#}$.

From this, we are to deduce the existence of a hyperreal number $t=\left[\{\Phi_n\}_{n=0}^{*\infty}\right]$ such that $s\times t=\left[\{\Psi_n\times\Phi_n\}_{n=0}^{*\infty}\right]$ is the same equivalence class as $1_{\widetilde{{}^*\mathbb{R}_c^\#}}=[(1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},\ldots)]$. Doing so is actually an easy consequence of the fact that nonzero hyperreal numbers from ${}^*\mathbb{R}_c^\#$ have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{\Psi_n\}_{n=0}^{\infty^\#}$ does not #-tend to $0_{{}^*\mathbb{R}_c^\#}$ as $n\to{}^*\infty$), there's no reason any number of the terms in $\{\Psi_n\}_{n=0}^{*\infty}$ can't equal $0_{\widetilde{{}^*\mathbb{R}_c^\#}}$. However, it turns out that eventually, $\Psi_n\neq 0_{{}^*\mathbb{R}_c^\#}$.

That is,

Lemma 14.2.1. If $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence which does not #-tends to $0_{\mathbb{R}_c^\#}$, then there is an $N\in{}^*\mathbb{N}$ such that, for $n>N$, $\Psi_n\neq 0_{\mathbb{R}_c^\#}$.

We will now use it to complete the proof of Theorem 14.2.6.

Let $N\in\mathbb{N}^\#$ be such that $\Psi_n\neq 0_{\mathbb{R}_c^\#}$ for $n>N$. Define hyper infinite sequence Φ_n of hyperreal numbers from $\widetilde{\mathbb{R}_c^\#}$ as follows:

for $n \leq N, \Phi_n = 0_{{}^*\mathbb{R}_c^\#}$, and for $n > N$, $\Phi_n = 1/\Psi_n$:
$\{\Phi_n\}_{n=0}^{{}^*\infty} = (0_{{}^*\mathbb{R}_c^\#}, 0_{{}^*\mathbb{R}_c^\#}, \dots, 0_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}/\Psi_{N+1}, 1/\Psi_{N+2}, \dots)$.
This makes sense since, for $n > N$, an is a nonzero hyperreal number, so $1_{\mathbb{R}_c^\#}/\Psi_n$ exists.
Then $\Psi_n \times \Phi_n$ is equal to $\Psi_n \times 0_{{}^*\mathbb{R}_c^\#} = 0_{{}^*\mathbb{R}_c^\#}$ for $n \leq N$, and equals $\Psi_n \times \Phi_n = \Psi_n \times\ 1_{{}^*\mathbb{R}_c^\#}/\Psi_n = 1_{{}^*\mathbb{R}_c^\#}$ for $n > N$
Well, then, if we look at the hyper infinite sequence

$$1_{\widetilde{{}^*\mathbb{R}_c^\#}} = (1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, \dots), \tag{14.2.13}$$

we have $(1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, \dots) - (\Psi_n \times\ \Phi_n)$ is the sequence which is $1_{\widetilde{{}^*\mathbb{R}_c^\#}} - 0_{\widetilde{{}^*\mathbb{R}_c^\#}} = 1_{\widetilde{{}^*\mathbb{R}_c^\#}}$ for $n \leq N$ and equals $1_{\widetilde{{}^*\mathbb{R}_c^\#}} - 1_{\widetilde{{}^*\mathbb{R}_c^\#}} = 0_{\widetilde{{}^*\mathbb{R}_c^\#}}$ for $n > N$. Since this hyper infinite sequence is eventually equal to $0_{\mathbb{R}_c^\#}$, it #-converges to $0_{\mathbb{R}_c^\#}$ as $n \to {}^*\infty$, and so $\left[\{\Psi_n \times \Phi_n\}_{n=0}^{{}^*\infty}\right] = [(1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, 1_{{}^*\mathbb{R}_c^\#}, \dots)] = 1_{\widetilde{{}^*\mathbb{R}_c^\#}} \in \widetilde{{}^*\mathbb{R}_c^\#}$. This shows that $t = \left[\{\Phi_n\}_{n=0}^{{}^*\infty}\right]$ is a multiplicative inverse to $s = \left[\{\Psi_n\}_{n=0}^{{}^*\infty}\right]$.

Definition 14.2.26. Let $s \in \widetilde{{}^*\mathbb{R}_c^\#}$. Say that s is positive if $s \neq 0_{\widetilde{{}^*\mathbb{R}_c^\#}}$, and if $s = \left[\{\Psi_n\}_{n=0}^{{}^*\infty}\right]$ for some Cauchy hyper infinite sequence such that for some N, $\Psi_n > 0_{{}^*\mathbb{R}_c^\#}$ for all $n > N$. Given two hyperreal numbers $s, t \in \widetilde{{}^*\mathbb{R}_c^\#}$, say that $s > t$ if $s - t$ is positive.

Theorem 14.2.7. Let $s, t \in \widetilde{{}^*\mathbb{R}_c^\#}$ be hyperreal numbers such that $s > t$, and let $r \in \widetilde{{}^*\mathbb{R}_c^\#}$. Then $s + r > t + r$.

Proof. Let $s = \left[\{\Psi_n\}_{n=0}^{{}^*\infty}\right], t = \left[\{\Phi_n\}_{n=0}^{{}^*\infty}\right]$, and $r = \left[\{\Theta_n\}_{n=0}^{{}^*\infty}\right]$. Since $s > t$, i.e. $s - t > 0$, we know that there is an N such that, for $n > N, \Psi_n - \Phi_n > 0$. So $\Psi_n > \Phi_n$ for $n > N$. Now, adding Θ_n to both sides of this inequality , we have $\Psi_n + \Theta_n > \Phi_n + \Theta_n$ for $n > N$, or $(\Psi_n + \Theta_n) - (\Phi_n + \Theta_n) > 0_{\mathbb{R}_c^\#}$ for $n > N$. Note also that $(\Psi_n + \Theta_n) - (\Phi_n + \Theta_n) = \Psi_n - \Phi_n$ does not #-converge to $0_{{}^*\mathbb{R}_c^\#}$ as $n \to {}^*\infty$, by the assumption that $s - t > 0_{\widetilde{\mathbb{R}_c^\#}}$. Thus, by Definition 14.2.26, this means that:
$s + r = \left[\{\Psi_n + \Theta_n\}_{n=0}^{{}^*\infty}\right] > \left[\{\Phi_n + \Theta_n\}_{n=0}^{{}^*\infty}\right] = t + r$.

Definition 14.2.27. There is canonical imbeding

$$ {}^*\mathbb{R}_c^\# \hookrightarrow \widetilde{{}^*\mathbb{R}_c^\#} \tag{14.2.14}$$

defined by

$$a \mapsto \widetilde{a} \tag{14.2.15}$$

where $\widetilde{a}$ is hyper infinite sequence $\widetilde{a} = (a, a, \dots.) \in \widetilde{{}^*\mathbb{R}_c^\#}, a \in {}^*\mathbb{R}_c^\# \cup \Delta_\omega^{+\downarrow 0}$.

Notation 14.2.5. $\widehat{a} = (a, a, \dots.) \in \widetilde{{}^*\mathbb{R}_c^\#}, a \in \widetilde{{}^*\mathbb{R}_c^\#}$.

Remark14.2.11.Let $a \in \widetilde{{}^*\mathbb{R}_c^\#}$. We will be identity $a \in \widetilde{{}^*\mathbb{R}_c^\#}$ with any $\{a_n\}_{n=0}^{{}^*\infty} \subset \widetilde{{}^*\mathbb{R}_c^\#}$ such that $\#\text{-}\lim_{n\to{}^*\infty} a_n = a$ and we denote by $[[a]]$ the equivalence class corresponding to $a \in \widetilde{{}^*\mathbb{R}_c^\#}$.

Definition 14.2.28. (i) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $\widetilde{{}^*\mathbb{R}_c^\#}, \{a_n\}_{n=0}^{k} \subset \widetilde{{}^*\mathbb{R}_c^\#}$.
We define external hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{k}} \subset \widetilde{{}^*\mathbb{R}_c^\#}$by

$$\{A_n;k\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{k}} = \\ = \left(a_0,a_1,\ldots,a_m,\ldots,a_{k-1},\widehat{a_k}\right) \in [[a_k]]. \qquad (14.2.16)$$

(ii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $\widetilde{{}^*\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{\infty} \subset \widetilde{{}^*\mathbb{R}_c^\#}$.
We define hyper infinite sequence $\{A_n\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{\infty}} \subset \widetilde{{}^*\mathbb{R}_c^\#}$ by

$$\{A_n';\infty\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{\infty}} = \\ = \left(a_0,a_1,\ldots,a_k,\ldots,\widehat{\{a_n\}_{n=0}^{\infty}}\right) \in [[\{a_n\}_{n=0}^{\infty}]]. \qquad (14.2.17)$$

(iii) Let $\{a_n\}_{n=0}^{N}, N \in {}^*\mathbb{N}\backslash\mathbb{N}$ be external hyperfinite sequence in $\widetilde{{}^*\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^\#}$.
We define hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{N}} \subset \widetilde{{}^*\mathbb{R}_c^\#}$ by

$$\{A_n;N\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{N}} = \\ = \left(a_0,a_1,,\ldots,a_n,\ldots,a_{N-1},\widehat{a_N}\right) \in [[a_N]]. \qquad (14.2.18)$$

Definition 14.2.29.(i) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $\widetilde{{}^*\mathbb{R}_c^\#}, \{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^\#}$.
We define external finite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n = \overbrace{\{c_n\}_{n=0}^{k}} = (c_0,c_1,\ldots,c_k,\ldots,\widehat{c}_k) \in [[c_k]] \qquad (14.2.19)$$

where $c_0 = a_0, c_j = Ext\text{-}\sum_{n=0}^{n=j} a_n, 0 \leq j \leq k$.
(ii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $\widetilde{{}^*\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{\infty} \subset \widetilde{{}^*\mathbb{R}_c^\#}$. We define external countable sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n = \overbrace{\{c_n\}_{n=0}^{\infty}} = \\ = \left(c_0,c_1,\ldots,c_k,\ldots\{c_n\}_{n=0}^{\infty},\widehat{\{c_n\}_{n=0}^{\infty}}\right) \in [[\{c_n\}_{n=0}^{\infty}]] \qquad (14.2.20)$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n, k \in \mathbb{N}$.
(iii) Let $\{a_n\}_{n=0}^{n=N}, N \in {}^*\mathbb{N}\backslash\mathbb{N}$ be external hyperfinite sequence in $\widetilde{{}^*\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^\#}$.
We define external hyperfinite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = \overbrace{\{c_n\}_{n=0}^{n=N}} = (c_0,c_1,\ldots,c_k,\ldots,c_N,\widehat{c}_N) \qquad (14.2.21)$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n,\ 0 \leq k \leq N, c_N = Ext\text{-}\sum_{n=0}^{n=N} a_n$.
(iv) Let $\{a_n\}_{n=0}^{n=N}, N \in {}^*\mathbb{N}$ be external hyperfinite sequence in $\widetilde{{}^*\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^\#}$ such that $a_n \equiv 0$ for all $n \in {}^*\mathbb{N}\backslash\mathbb{N}$. We assume that

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n, \qquad (14.2.22)$$

Example 14.2.1.Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots. \alpha r^{N-1}, N \in {}^*\mathbb{N}, \alpha \in \widetilde{{}^*\mathbb{R}_c^\#}$,
$r \in \widetilde{{}^*\mathbb{R}_c^\#}$ be the first term and the ratio of the G.P respectively. Then for any

$N \in {}^*\mathbb{N}$ by Proposition 14.2.6 and Definition 14.2.29 one obtains that

$$Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1} \alpha r^{n-1} = \alpha \widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} =_{.} \alpha \widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} - \alpha \widehat{\frac{r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}}. \tag{14.2.23}$$

and

$$Ext\text{-}\widehat{\sum}_{n=1}^{\infty} \alpha r^{n-1} =_{.} \alpha \widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} - \alpha \widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}}. \tag{14.2.24}$$

Example 14.2.2.Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots. \alpha r^{N-1}, N \in {}^*\mathbb{N}, \alpha \in \widetilde{{}^*\mathbb{R}_c^\#}, r \in \widetilde{{}^*\mathbb{R}_c^\#}$, $r > 0, r \neq 1$. Note that

$$\begin{aligned} \alpha \widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{{}^*\mathbb{R}_c^\#} - r}} &= Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1} \alpha r^{n-1} = \\ &= Ext\text{-}\widehat{\sum}_{n=1}^{\infty} \alpha r^{n-1} + Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\setminus\mathbb{N}) \wedge (n \leq N-1)} \alpha r^{n-1} = \\ &= \alpha \widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} - \alpha \widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} + Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\setminus\mathbb{N}) \wedge (n \leq N-1)} \alpha r^{n-1}. \end{aligned} \tag{14.2.25}$$

From (14.2.25) we obtain

$$\begin{aligned} Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\setminus\mathbb{N}) \wedge (n \leq N-1)} \alpha r^{n-1} &= \alpha \widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{{}^*\mathbb{R}_c^\#} - r}} - \alpha \widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} + \alpha \widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} = \\ & \alpha \widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} - \alpha \widehat{\frac{r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}}. \end{aligned} \tag{14.2.26}$$

Assume that: (i) $r < 1_{\widetilde{{}^*\mathbb{R}_c^\#}}$,then from (14.2.26) we obtain

$$Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\setminus\mathbb{N}) \wedge (n \leq N-1)} \alpha r^{n-1} > 0_{\widetilde{{}^*\mathbb{R}_c^\#}}. \tag{14.2.27}$$

(ii) $r > 1_{\widetilde{{}^*\mathbb{R}_c^\#}}$,then from (14.2.26) we obtain

$$Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\setminus\mathbb{N}) \wedge (n \leq N-1)} \alpha r^{n-1} = \alpha \widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} \dot{+} \alpha \widehat{\frac{r^N}{r - 1_{\widetilde{{}^*\mathbb{R}_c^\#}}}} > 0_{\widetilde{{}^*\mathbb{R}_c^\#}}. \tag{14.2.28}$$

Proposition 14.2.6.(i) Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots. \alpha r^{N-1}, N \in {}^*\mathbb{N}$. Let S_N, $\alpha \in {}^*\mathbb{R}_c^\#, r \in \widetilde{{}^*\mathbb{R}_c^\#}$ be the sum of N terms, first term and the ratio of the G.P respectively. Then for any $N \in {}^*\mathbb{N}$the statement Φ_N holds

$$\Phi_N \Longleftrightarrow_s Ext\text{-}\sum_{n=1}^{n=N-1} \alpha r^{n-1} = \alpha \frac{1_{{}^*\mathbb{R}_c^\#} - r^N}{1_{{}^*\mathbb{R}_c^\#} - r}. \tag{14.2.29}$$

Proof.(i) Directly by hyperinfinite induction. Note that $\Phi_N \Rightarrow_s \Phi_{N+1}$:

$$S_{N+1} = Ext\text{-}\sum_{n=1}^{n=N} \alpha r^{n-1} = Ext\text{-}\sum_{n=1}^{n=N-1} \alpha r^{n-1} + \alpha r^N = \alpha \frac{1_{{}^*\mathbb{R}_c^\#} - r^N}{1_{{}^*\mathbb{R}_c^\#} - r} + \alpha r^N =$$
$$= \alpha \frac{1_{{}^*\mathbb{R}_c^\#} - r^N}{1_{{}^*\mathbb{R}_c^\#} - r} + \alpha \frac{(1_{{}^*\mathbb{R}_c^\#} - r) r^N}{1_{{}^*\mathbb{R}_c^\#} - r} = \alpha \frac{1_{{}^*\mathbb{R}_c^\#} - r^N + r^N - r^{N+1}}{1_{{}^*\mathbb{R}_c^\#} - r} = \tag{14.2.30}$$
$$= \alpha \frac{1_{{}^*\mathbb{R}_c^\#} - r^{N+1}}{1_{{}^*\mathbb{R}_c^\#} - r}.$$

Thus $S_{N+1} = \alpha \frac{1_{{}^*\mathbb{R}_c^\#} + r^{N+1}}{1_{{}^*\mathbb{R}_c^\#} - r}$ and therefore Φ_{N+1}holds.

(ii) Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots. \alpha r^{N-1}, N \in {}^*\mathbb{N}$. Let S_N, $\alpha \in \widetilde{{}^*\mathbb{R}_c^\#}, r \in \widetilde{{}^*\mathbb{R}_c^\#}$ be the sum of N terms, first term and the ratio of the G.P respectively. Then for any $N \in {}^*\mathbb{N}$the statement $\widetilde{\Phi}_N$ holds

$$\widetilde{\Phi}_N \Leftrightarrow_s Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1} \alpha r^{n-1} = \alpha \frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}. \tag{14.2.31}$$

Notice that (i)⇒(ii) by definitions.

Definition 14.2.30. Let $\{a_n\}_{n=0}^{{}^*\infty}, n \in {}^*\mathbb{N}$ be external hyperinfinite sequence in $\widetilde{{}^*\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{{}^*\infty} \subset \widetilde{{}^*\mathbb{R}_c^\#}$. We define external hyperinfinite sum $Ext\text{-}\widehat{\sum}_{n=0}^{{}^*\infty} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{{}^*\infty} a_n = \#\text{-}\lim_{N \to {}^*\infty} \left(Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n \right) \tag{14.2.32}$$

if #-limit in (14.2.31) exists.

Example 14.2.3.Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots. \alpha r^{n-1}, n \in {}^*\mathbb{N}^\#, \alpha \in \widetilde{{}^*\mathbb{R}_c^\#}, r \in \widetilde{{}^*\mathbb{R}_c^\#}$.
From (14.2.27) we obtain

$$Ext\text{-}\widehat{\sum}_{n=0}^{{}^*\infty} \alpha r^{n-1} = \#\text{-}\lim_{N \to {}^*\infty} \left(Ext\text{-}\widehat{\sum}_{n=0}^{n=N} \alpha r^{n-1} \right) = \#\text{-}\lim_{N \to \infty^\#} \alpha \frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r} =$$
$$= \alpha \frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r} \tag{14.2.33}$$

since $\#\text{-}\lim_{N \to {}^*\infty} r^N = 0_{\widetilde{{}^*\mathbb{R}_c^\#}}$ if $|r| < 1$. From (14.2.33) and (14.2.25) we obtain

$$\alpha \frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r} = Ext\text{-}\widehat{\sum}_{n=0}^{{}^*\infty} \alpha r^{n-1} = Ext\text{-}\widehat{\sum}_{n=0}^{\infty} \alpha r^{n-1} + Ext\text{-}\widehat{\sum}_{n \in {}^*\mathbb{N} \setminus \mathbb{N}} \alpha r^{n-1} =$$
$$\widehat{\alpha \frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} - \widehat{\alpha \left\{ \frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r} \right\}_{n=1}^{\infty}} + Ext\text{-}\widehat{\sum}_{n \in {}^*\mathbb{N} \setminus \mathbb{N}} \alpha r^{n-1}. \tag{14.2.34}$$

From (14.2.34) we obtain

$$Ext\text{-}\widehat{\sum}_{n\in{}^*\mathbb{N}\mathbb{N}}\alpha r^{n-1} = \alpha\widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r}}\alpha - \left(\widehat{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r}} - \alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r}\right\}_{n=1}^{\infty}}\right) = \tag{14.2.35}$$

$$= \alpha\widehat{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}}-r}\right\}_{n=1}^{\infty}} > 0.$$

Definition 14.2.31. Let $\{a_n\}_{n=0}^{\infty}$ be ${}^*\mathbb{R}_c^\#$- valued countable sequence
$a : \mathbb{N} \to {}^*\mathbb{R}_c^\#$ such that:
(i) there is $M \in \mathbb{N}$ such that for all $n > M, a_n \neq 0_{{}^*\mathbb{R}_c^\#}$,
we denote a set of the all these sequences by $\Xi_\omega^{\pm,\neq 0}$.
We define a set $-\Xi_\omega^{\pm,\neq 0}$ by $\{c_n\}_{n=0}^{\infty} \in -\Xi_\omega^{\pm,\neq 0} \iff \{-c_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0}$. Note that
$\Xi_\omega^{\pm,\neq 0} = -\Xi_\omega^{\pm,\neq 0}$.
(ii) there is countable subsequence $\{a_{n_k}\}_{k=m}^{\infty} \subset \{a_n\}_{n=0}^{\infty}$ such that $a_{n_k} = 0_{{}^*\mathbb{R}_c^\#}, k \geq m$
and $a_n \neq 0_{{}^*\mathbb{R}_c^\#}$ iff $a_n \notin \{a_{n_k}\}_{k=m}^{\infty}$,
we denote a set of the all these countable sequences by $\Xi_\omega^{\pm,\neq 0\vee=0}$.
We define a set $-\Xi_\omega^{\pm,\neq 0\vee=0}$ by $\{c_n\}_{n=0}^{\infty} \in -\Xi_\omega^{\pm,\neq 0\vee=0} \iff \{-c_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$. Note that
$\Xi_\omega^{\pm,\neq 0\vee=0} = -\Xi_\omega^{\pm,\neq 0\vee=0}$.

Definition 14.2.31.
(1) Let $\{a_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0}$ and $\{b_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0}$ then we define
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0}$
(iv) $(\{a_n\}_{n=0}^{\infty})^{-1} \triangleq \{a_n^{-1}\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0}$
(2) Let $\{a_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$ and $\{b_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$ then we define
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$
(iv) $(\{a_n\}_{n=0}^{\infty})^{-1_*} \triangleq \{a_n^{1_*}\}_{n=0}^{\infty}$ where

$$a_n^{1_*} = \begin{cases} a_n^{-1} & \text{if } \ a_n \neq 0_{{}^*\mathbb{R}_c^\#} \\ 0_{{}^*\mathbb{R}_c^\#} & \text{if } \ a_n = 0_{{}^*\mathbb{R}_c^\#} \end{cases} \tag{14.2.36}$$

Note that
(i) $\left((\{a_n\}_{n=0}^{\infty})^{-1_*}\right)^{-1_*} = \{a_n\}_{n=0}^{\infty}$
(ii) $\{a_n\}_{n=0}^{\infty} \times (\{a_n\}_{n=0}^{\infty})^{-1_*} = \breve{1}_{{}^*\mathbb{R}_c^\#}$ where $\breve{1}_{{}^*\mathbb{R}_c^\#} = \{\alpha_n\}_{n=0}^{\infty}$ is countable sequence
such that

$$\alpha_n = \begin{cases} 1_{{}^*\mathbb{R}_c^\#} & \text{if } \ a_n \neq 0_{{}^*\mathbb{R}_c^\#} \\ 0_{{}^*\mathbb{R}_c^\#} & \text{if } \ \alpha_n = 0_{{}^*\mathbb{R}_c^\#} \end{cases} \tag{14.2.37}$$

Definition 14.2.32. We say that
$(\{a_n\}_{n=0}^{\infty})^{-1_*} \in \Xi_\omega^{\pm,\neq 0\vee=0}$ is a quasi inverse of $\{a_n\}_{n=0}^{\infty}$.

Definition 14.2.33.(1) Let $\{a_n\}_{n=0}^{\infty} \in \Xi_\omega^{\pm,\neq 0\vee=0}$ and let $\{A_n\}_{n=0}^{*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ be a hyper infinite sequence

$$\{A_n\}_{n=0}^{*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}} = (a_0, a_1, \ldots, a_k, \ldots, \{a_n\}_{n=0}^{\infty}, \ldots, \{a_n\}_{n=0}^{\infty}, \ldots) \qquad (14.2.38)$$

i.e. for any infinite $m \in {}^*\mathbb{N}\backslash\mathbb{N}, A_m \equiv \{a_n\}_{n=0}^{\infty}$. We will denote a set of the all these hyper infinite sequences by $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$

(2) Let $\{x_n + y_n a_n\}_{n=0}^{\infty} \in \Xi_{\omega}^{\pm,\neq 0 \vee =0}$ and let

$$\begin{gathered}\{x_n + y_n A_n\}_{n=0}^{*\infty} = \overline{\{x_n + y_n a_n\}_{n=0}^{\infty}} = \\ (x_0 + y_0 a_0, x_1 + y_1 a_1, \ldots, x_k + y_k a_k, \ldots, \{x_n +_n y a_n\}_{n=0}^{\infty}, \ldots),\end{gathered} \qquad (14.2.39)$$

i.e. for any infinite $m \in {}^*\mathbb{N}\backslash\mathbb{N}, A_m \equiv \{x_n + y_n a_n\}_{n=0}^{\infty}$. We will denote a set of the all these hyper infinite sequences by $\overline{\{\Xi_{\omega}^{\pm,\neq 0 \vee =0}, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\}}$.

Definition 14.2.34.Let $\{A_n\}_{n=0}^{*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}}$ and $\{B_n\}_{n=0}^{*\infty} = \widetilde{\{b_n\}_{n=0}^{\infty}}$ be in $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$.
Then we define:

(i) $\{A_n\}_{n=0}^{*\infty} + \{B_n\}_{n=0}^{*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}} + \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n + b_n\}_{n=0}^{\infty}} = \{A_n + B_n\}_{n=0}^{*\infty} \in \widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$

(ii) $\{A_n\}_{n=0}^{*\infty} - \{B_n\}_{n=0}^{*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}} - \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n - b_n\}_{n=0}^{\infty}} =$
$= \{A_n - B_n\}_{n=0}^{*\infty} \in \widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$

(iii) $\{A_n\}_{n=0}^{*\infty} \times \{B_n\}_{n=0}^{*\infty} = \widetilde{\{a_n\}_{n=0}^{\infty}} \times \widetilde{\{b_n\}_{n=0}^{\infty}} \triangleq \widetilde{\{a_n \times b_n\}_{n=0}^{\infty}} = \{A_n \times B_n\}_{n=0}^{\infty^{\#}} \in \widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$

Definition 14.2.35.Let $\{\Psi_n\}_{n=0}^{*\infty}$ be in $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$,i.e. for all $n \in {}^*\mathbb{N}$, $\Psi_n \in \Xi_{\omega}^{\pm,\neq 0 \vee =0}$.
Say $\{\Psi_n\}_{n=0}^{*\infty}$ #-tends to $0_{^*\mathbb{R}_c^{\#}}$ as $n \to {}^*\infty$ iff for any given $\varepsilon > 0_{\mathbb{R}_c^{\#}}, \varepsilon \approx 0_{^*\mathbb{R}_c^{\#}}$ there is a hypernatural number $N \in {}^*\mathbb{N}\backslash\mathbb{N}, N = N(\varepsilon)$ such that for any $n > N, |\Psi_n| < \varepsilon$.

Definition 14.2.36. Let $\{\Psi_n\}_{n=0}^{*\infty}$ be a hyper infinite sequence such that for all $n \in {}^*\mathbb{N}, \Psi_n \in \widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$.We call $\{\Psi_n\}_{n=0}^{*\infty}$ a Cauchy hyper infinite sequence if the difference between its terms #-tends to $0_{^*\mathbb{R}_c^{\#}}$. To be precise: given any $\varepsilon > 0_{^*\mathbb{R}_c^{\#}}$, $\varepsilon \approx 0_{^*\mathbb{R}_c^{\#}}$there is a hypernatural number $N \in {}^*\mathbb{N}\backslash\mathbb{N}, N = N(\varepsilon)$ such that for any $m, n > N, |\Psi_n - \Psi_m| < \varepsilon$.

Theorem 14.2.8.Let $\{\Psi_n\}_{n=0}^{*\infty}$ be in $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$.If $\{\Psi_n\}_{n=0}^{*\infty}$ is a #-convergent hyper infinite sequence (that is, $\Psi_n \to_{\#} \Phi$ as $n \to {}^*\infty$ for some $\Phi \in \widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$), then $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence.

Proof.We know that $\Psi_n \to_{\#} \Phi$. Here is a ubiquitous trick: instead of using ε in the definition, start with an arbitrary infinitesimall $\varepsilon > 0_{^*\mathbb{R}_c^{\#}}, \varepsilon \approx 0_{^*\mathbb{R}_c^{\#}}$ and then choose N so that $|\Psi_n - \Phi| < \varepsilon/2$ when $n > N$. Then if $m, n > N$, we have
$|\Psi_n - \Psi_m| = |(\Psi_n - \Phi) - (\Psi_m - \Phi)| \leq |\Psi_n - \Phi| + |\Psi_m - \Phi| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
This shows that $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence.

Theorem 14.2.9.If $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence, then it is bounded in ${}^*\mathbb{R}_c^{\#}$; that is, there is some number $M \in {}^*\mathbb{R}_c^{\#}$ such that $|\{\Psi_n\}_{n=0}^{*\infty}| \leq M$ for all $n \in {}^*\mathbb{N}$.

Proof. Since $\{\Psi_n\}_{n=0}^{*\infty}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some N such that $|\Psi_m - \Psi_n| < 1$ whenever $m, n > N$. Thus, $|\Psi_{N+1} - \Psi_n| < 1$ for $n > N$. We can rewrite this as $\Psi_{N+1} - 1 < \Psi_n < \Psi_{N+1} + 1$.This means that $|\Psi_n|$ is less than the maximum of $|\Psi_{N+1} - 1|$ and $|\Psi_{N+1} + 1|$. So, set $M \in {}^*\mathbb{R}_c^{\#}$ larger than any number in the following list: $\{|\Psi_0|, |\Psi_1|, \ldots, |\Psi_N|, |\Psi_{N+1} - 1|, |\Psi_{N+1} + 1|\}$.
Then for any term Ψ_n, if $n \leq N$, then $|\Psi_n|$ appears in the list and so $|\Psi_n| \leq M$;if $n > N$,

then (as shown above) $|\Psi_n|$ is less than at least one of the last two entries in the list, and so $|\Psi_n| \leq M$. Hence, M is a bound for the sequence.

Let Ξ denote the set of all Cauchy hyper infinite sequences We must define an equivalence relation on Ξ.

Definition 14.2.37. Let $\mathbf{S}$ be a set of objects. A relation among pairs of elements of $\mathbf{S}$ is said to be an equivalence relation if the following three properties hold:

Reflexivity: for any $s \in \mathbf{S}$, s is related to s.

Symmetry: for any $s,t \in \mathbf{S}$, if s is related to t then t is related to s.

Transitivity: for any $s,t,r \in \mathbf{S}$, if s is related to t and t is related to r, then s is related to r.

The following well known proposition goes most of the way to showing that an equivalence relation divides a set into bins.

Theorem 14.2.10. Let $\mathbf{S}$ be a set, with an equivalence relation on pairs of elements. For $s \in \mathbf{S}$, denote by $[s]$ the set of all elements in $\mathbf{S}$ that are related to s. Then for any $s,t \in \mathbf{S}$, either $[s] = [t]$ or $[s]$ and $[t]$ are disjoint.

The sets $[s]$ for $s \in \mathbf{S}$ are called the equivalence classes, and they are the bins.

Corollary 14.2.2. If $\mathbf{S}$ is a set with an equivalence relation on pairs of elements, then the equivalence classes are non-empty disjoint sets whose union is all of $\mathbf{S}$.

Definition 14.2.38.Let $\{\Psi_{1,n}\}_{n=0}^{*\infty}$ and $\{\Psi_{2,n}\}_{n=0}^{*\infty}$ be in $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$. Say they are equivalent (i.e. related) if $|\Psi_{1,n} - \Psi_{2,n}| \to_{\#} 0_{^*\mathbb{R}_c^\#}$ as $n \to {}^*\infty$, i.e. if the hyper infinite sequence $\{|\Psi_{1,n} - \Psi_{2,n}|\}_{n=0}^{*\infty}$ #-tends to $0_{^*\mathbb{R}_c^\#}$.

Proposition 14.2.4. Definition 4.2.38 yields an equivalence relation on $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$.

Proof. we need to show that this relation is reflexive, symmetric, and transitive.

• **Reflexive**: $\Psi_n - \Psi_n = 0_{^*\mathbb{R}_c^\#}$, and the sequence all of whose terms are $0_{^*\mathbb{R}_c^\#}$ clearly converges to $0_{\mathbb{R}_c^\#}$.So $\{\Psi_n\}_{n=0}^{*\infty}$ is related to $\{\Psi_n\}_{n=0}^{*\infty}$.

• **Symmetric**: Suppose $\{\Psi_{1,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{2,n}\}_{n=0}^{*\infty}$, so $\Psi_{1,n} - \Psi_{2,n} \to_{\#} 0_{^*\mathbb{R}_c^\#}$. But $\Psi_{2,n} - \Psi_{1,n} = -(\Psi_{1,n} - \Psi_{2,n})$,and since only the absolute value $|\Psi_{1,n} - \Psi_{2,n}| =$ $= |\Psi_{2,n} - \Psi_{1,n}|$ comes into play in Definition 14.2.35,it follows that $\Psi_{2,n} - \Psi_{1,n} \to_{\#} 0_{^*\mathbb{R}_c^\#}$ as well. Hence, $\{\Psi_{2,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{1,n}\}_{n=0}^{*\infty}$.

• **Transitive**: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 14.2.4. Suppose $\{\Psi_{1,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{2,n}\}_{n=0}^{*\infty}$, and $\{\Psi_{2,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{3,n}\}_{n=0}^{*\infty}$. This means that $\Psi_{1,n} - \Psi_{2,n} \to_{\#} 0_{\mathbb{R}_c^\#}$ and $\Psi_{2,n} - \Psi_{3,n} \to_{\#} 0_{\mathbb{R}_c^\#}$.

To be fully precise, let us fix infinite small $\varepsilon > 0_{^*\mathbb{R}_c^\#}$; then there exists an $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that for all $n > N, |\Psi_{1,n} - \Psi_{2,n}| < \varepsilon/2$; also, there exists an M such that for all $n > M, |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2$. Well, then, as long as $n > \max(N,M)$, we have that $|\Psi_{2,n} - \Psi_{3,n}| = |(\Psi_{1,n} - \Psi_{2,n}) + (\Psi_{2,n} - \Psi_{3,n})| \leq |\Psi_{1,n} - \Psi_{2,n}| + |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So, choosing L equal to the max of N,M , we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |\Psi_{1,n} - \Psi_{3,n}| < \varepsilon$. This means that $\Psi_{1,n} - \Psi_{3,n} \to_{\#} 0_{^*\mathbb{R}_c^\#}$, i.e. $\{\Psi_{1,n}\}_{n=0}^{*\infty}$ is related to $\{\Psi_{3,n}\}_{n=0}^{*\infty}$.

So, we really have equivalence relation, and so by Corollary 14.2.2, the set $\widetilde{\Xi}_{\omega}^{\pm,\neq 0 \vee =0}$ is partitioned into disjoint subsets (equivalence classes).

Definition 14.2.39. (1) The hyperreal numbers $\widetilde{^*\mathbb{R}_c^\#}$ are the equivalence classes

$\left[\{\Psi_{1,n}\}_{n=0}^{*\infty}\right]$ of Cauchy hyper infinite sequences of, as per Definition 14.2.38 and (2) the all gyperreals ${}^*\mathbb{R}_c^\# \subset \widetilde{{}^*\mathbb{R}_c^\#}$ by the canonical imbedding ${}^*\mathbb{R}_c^\# \hookrightarrow \widetilde{{}^*\mathbb{R}_c^\#}$ (14.1.42)-(14.1.43).

That is, each such equivalence class is a hyperreal number in $\widetilde{{}^*\mathbb{R}_c^\#}$.

Definition 14.2.40. Let $s,t \in \widetilde{{}^*\mathbb{R}_c^\#}$, so there are Cauchy hyper infinite sequences $\{\Psi_n\}_{n=0}^{*\infty}$ and $\{\Phi_n\}_{n=0}^{*\infty}$ with $s = \left[\{\Psi_n\}_{n=0}^{*\infty}\right]$ and $t = \left[\{\Phi_n\}_{n=0}^{*\infty}\right]$.

(a) Define $s+t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n \dotplus \Phi_n\}_{n=0}^{*\infty}$.

(b) Define $s \times t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n \times \Phi_n\}_{n=0}^{*\infty}$.

Proposition 14.2.5.The operations $+,\times$ in Definition 14.2.25 (a),(b) are well-defined.

Proof. Suppose that $\left[\{\Psi_n\}_{n=0}^{*\infty}\right] = \left[\{\Psi_{1,n}\}_{n=0}^{*\infty}\right]$ and $\left[\{\Phi_n\}_{n=0}^{*\infty}\right] = \left[\{\Phi_{1,n}\}_{n=0}^{*\infty}\right]$.

Thus means that $\Psi_n - \Psi_{1,n} \to_\# 0_{{}^*\mathbb{R}_c^\#}$ and $\Phi_n - \Phi_{1,n} \to_\# 0_{{}^*\mathbb{R}_c^\#}$. Then $(\Psi_n + \Phi_n) - (\Psi_{1,n} + \Phi_{1,n}) = (\Psi_n - \Psi_{1,n}) + (\Phi_n - \Phi_{1,n})$. Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to $0_{{}^*\mathbb{R}_c^\#}$, and so $[(\Psi_n + \Phi_n)] = [(\Psi_{1,n} + \Phi_{1,n})]$.

Multiplication is a little trickier; this is where we will use Theorem 14.2.10. We will also use another ubiquitous technique: adding $0_{{}^*\mathbb{R}_c^\#}$ in the form of $s-s$. Again, suppose that

$\left[\{\Psi_n\}_{n=0}^{*\infty}\right] = \left[\{\Psi_{1,n}\}_{n=0}^{*\infty}\right]$ and $\left[\{\Phi_n\}_{n=0}^{*\infty}\right] = \left[\{\Phi_{1,n}\}_{n=0}^{*\infty}\right]$; we wish to show that $\left[\{\Psi_n \times \Phi_n\}_{n=0}^{*\infty}\right] = \left[\{\Psi_{1,n} \times \Phi_{1,n}\}_{n=0}^{*\infty}\right]$, or, in other words, that

$\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n} \to_\# 0_{{}^*\mathbb{R}_c^\#}$. Well, we add and subtract one of the other cross terms, say $\Phi_n \times \Psi_{1,n}$:

$\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n} = \Psi_n \times \Phi_n + (\Phi_n \times \Psi_{1,n} - \Phi_n \times \Psi_{1,n}) - \Psi_{1,n} \times \Phi_{1,n} =$

$= (\Psi_n \times \Phi_n - \Phi_n \times \Psi_{1,n}) + (\Phi_n \times \Psi_{1,n} - \Psi_{1,n} \times \Phi_{1,n}) =$

$= \Phi_n \times (\Psi_n - \Psi_{1,n}) + \Psi_{1,n} \times (\Phi_n - \Phi_{1,n})$.

Hence, we have $|\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n}| \leq |\Phi_n| \times |\Psi_n - \Psi_{1,n}| + |\Psi_{1,n}| \cdot |\Phi_n - \Phi_{1,n}|$.

Now, from Theorem 14.2.9, there are numbers M and L such that $|\Phi_n| \leq M$ and $|\Psi_{1,n}| \leq L$ for all $n \in {}^*\mathbb{N}$. Taking some number R (for example $R = M+L$) which is bigger than both, we have

$|\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n}| \leq |\Phi_n| \times |\Psi_n - \Psi_{1,n}| + |\Psi_{1,n}| \times |\Phi_n - \Phi_{1,n}| \leq$

$\leq R(|\Psi_n - \Psi_{1,n}| + |\Phi_n - \Phi_{1,n}|)$.

Now, noting that both $\Psi_n - \Psi_{1,n}$ and $\Phi_n - \Phi_{1,n}$ #-tend to $0_{{}^*\mathbb{R}_c^\#}$ and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2R$, we see that

$\Psi_n \times \Phi_n - \Psi_{1,n} \times \Phi_{1,n} \to_\# 0_{{}^*\mathbb{R}_c^\#}$

Theorem 14.2.11. Given any hyperreal number $s \in \widetilde{{}^*\mathbb{R}_c^\#}$, $s \neq 0_{\widetilde{{}^*\mathbb{R}_c^\#}}$, there is a hyperreal number $t \in \widetilde{{}^*\mathbb{R}_c^\#}$ such that $s \times t = 1_{\widetilde{{}^*\mathbb{R}_c^\#}}$ or $s \times t = \breve{1}_{\widetilde{{}^*\mathbb{R}_c^\#}}$.

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of

$$0_{\widetilde{{}^*\mathbb{R}_c^\#}} = (0_{{}^*\mathbb{R}_c^\#}, 0_{{}^*\mathbb{R}_c^\#}, 0_{{}^*\mathbb{R}_c^\#}, 0_{{}^*\mathbb{R}_c^\#}, \ldots). \tag{14.2.40}$$

In other words, $s = \left[\{\Psi_n\}_{n=0}^{*\infty}\right]$ where $\Psi_n - 0_{\widetilde{{}^*\mathbb{R}_c^\#}}$ does not #-converge to $0_{{}^*\mathbb{R}_c^\#}$.

From this, we are to deduce the existence of a hyperreal number $t = \left[\{\Phi_n\}_{n=0}^{*\infty}\right]$

such that $s\times\ t=\left[\{\Psi_n\times\Phi_n\}_{n=0}^{*\infty}\right]$ is the same equivalence class as $1_{\widetilde{{}^*\mathbb{R}_c^\#}}=[(1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},\ldots)]$ or as some $\breve{1}_{\widetilde{{}^*\mathbb{R}_c^\#}}$. Doing so is actually an easy consequence of the fact that nonzero hyperreal numbers from ${}^*\mathbb{R}_c^\#$ have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{\Psi_n\}_{n=0}^{\infty^\#}$ does not #-tend to $0_{{}^*\mathbb{R}_c^\#}$ as $n\to{}^*\infty$), there's no reason any number of the terms in $\{\Psi_n\}_{n=0}^{*\infty}$ can't equal $0_{\widetilde{{}^*\mathbb{R}_c^\#}}$. However, it turns out that eventually, $\Psi_n\neq 0_{{}^*\mathbb{R}_c^\#}$.
That is,
Lemma 14.2.2. If $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence which does not #-tends to $0_{\mathbb{R}_c^\#}$, then there is an $N\in{}^*\mathbb{N}$ such that, for $n>N,\ \Psi_n\neq 0_{\mathbb{R}_c^\#}$.
We will now use it to complete the proof of Theorem 14.2.11.
Let $N\in\mathbb{N}^\#$ be such that $\Psi_n\neq 0_{\mathbb{R}_c^\#}$ for $n>N$. Define hyper infinite sequence Φ_n of hyperreal numbers from $\widetilde{\mathbb{R}_c^\#}$ as follows:
for $n\leq N,\Phi_n=0_{{}^*\mathbb{R}_c^\#}$, and for $n>N,\ \Phi_n=1_{{}^*\mathbb{R}_c^\#}/\Psi_n$:
$\{\Phi_n\}_{n=0}^{*\infty}=(0_{{}^*\mathbb{R}_c^\#},0_{{}^*\mathbb{R}_c^\#},\ldots,0_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#}/\Psi_{N+1},1_{{}^*\mathbb{R}_c^\#}/\Psi_{N+2},\ldots)$.
This makes sense since, for $n>N,$ an is a nonzero hyperreal number, so $1_{{}^*\mathbb{R}_c^\#}/\Psi_n$ exists.
Then $\Psi_n\times\Phi_n$ is equal to $\Psi_n\times 0_{{}^*\mathbb{R}_c^\#}=0_{{}^*\mathbb{R}_c^\#}$ for $n\leq N,$ and equals $\Psi_n\times\Phi_n=\Psi_n\times\ 1_{{}^*\mathbb{R}_c^\#}/\Psi_n=1_{{}^*\mathbb{R}_c^\#}$ for $n>N$
Well, then, if we look at the hyper infinite sequence

$$1_{\widetilde{{}^*\mathbb{R}_c^\#}}=(1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},\ldots),\tag{14.2.41}$$

we have $(1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},\ldots)-(\Psi_n\times\ \Phi_n)$ is the sequence which is $1_{\widetilde{{}^*\mathbb{R}_c^\#}}-0_{\widetilde{{}^*\mathbb{R}_c^\#}}=1_{\widetilde{{}^*\mathbb{R}_c^\#}}$ for $n\leq N$ and equals $1_{\widetilde{{}^*\mathbb{R}_c^\#}}-1_{\widetilde{{}^*\mathbb{R}_c^\#}}=0_{\widetilde{{}^*\mathbb{R}_c^\#}}$ for $n>N$. Since this hyper infinite sequence is eventually equal to $0_{\mathbb{R}_c^\#}$, it #-converges to $0_{\mathbb{R}_c^\#}$ as $n\to{}^*\infty$, and so $\left[\{\Psi_n\times\Phi_n\}_{n=0}^{*\infty}\right]=[(1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},1_{{}^*\mathbb{R}_c^\#},\ldots)]=1_{\widetilde{{}^*\mathbb{R}_c^\#}}\in\widetilde{{}^*\mathbb{R}_c^\#}$ or similarly $\left[\{\Psi_n\times\Phi_n\}_{n=0}^{*\infty}\right]=\breve{1}_{\widetilde{{}^*\mathbb{R}_c^\#}}\in\widetilde{{}^*\mathbb{R}_c^\#}$. This shows that $t=\left[\{\Phi_n\}_{n=0}^{*\infty}\right]$ is a multiplicative inverse (or similarly quasi inverse) to $s=\left[\{\Psi_n\}_{n=0}^{*\infty}\right]$.
Definition 14.2.41. Let $s\in\widetilde{{}^*\mathbb{R}_c^\#}$. Say that s is positive if $s\neq 0_{\widetilde{{}^*\mathbb{R}_c^\#}}$, and if $s=\left[\{\Psi_n\}_{n=0}^{*\infty}\right]$ for some Cauchy hyper infinite sequence such that for some N, $\Psi_n>0_{{}^*\mathbb{R}_c^\#}$ for all $n>N$. Given two hyperreal numbers $s,t\in\widetilde{{}^*\mathbb{R}_c^\#}$, say that $s>t$ if $s-t$ is positive.
Theorem 14.2.7. Let $s,t\in\widetilde{{}^*\mathbb{R}_c^\#}$ be hyperreal numbers such that $s>t,$ and let $r\in\widetilde{{}^*\mathbb{R}_c^\#}$. Then $s+r>t+r$.
Proof. Let $s=\left[\{\Psi_n\}_{n=0}^{*\infty}\right],t=\left[\{\Phi_n\}_{n=0}^{*\infty}\right]$, and $r=\left[\{\Theta_n\}_{n=0}^{*\infty}\right]$. Since $s>t,$ i.e. $s-t>0,$ we know that there is an N such that, for $n>N,\Psi_n-\Phi_n>0$. So $\Psi_n>\Phi_n$ for $n>N.$ Now, adding Θ_n to both sides of this inequality , we have $\Psi_n+\Theta_n>\Phi_n+\Theta_n$ for $n>N,$ or $(\Psi_n+\Theta_n)-(\Phi_n+\Theta_n)>0_{\mathbb{R}_c^\#}$ for $n>N$. Note also that $(\Psi_n+\Theta_n)-(\Phi_n+\Theta_n)=\Psi_n-\Phi_n$ does not #-converge to $0_{{}^*\mathbb{R}_c^\#}$ as $n\to{}^*\infty$, by the assumption that $s-t>0_{\widetilde{\mathbb{R}_c^\#}}$. Thus, by Definition 14.2.41, this means that:
$s+r=\left[\{\Psi_n+\Theta_n\}_{n=0}^{*\infty}\right]>\left[\{\Phi_n+\Theta_n\}_{n=0}^{*\infty}\right]=t+r.$

Definition 14.2.42. There is canonical imbeding

$$ {}^*\mathbb{R}_c^{\#} \hookrightarrow \widetilde{{}^*\mathbb{R}_c^{\#}} \tag{14.2.42}$$

defined by

$$ a \longmapsto \widetilde{a} \tag{14.2.43}$$

where $\widetilde{a}$ is hyper infinite sequence $\widetilde{a} = (a,a,\ldots.) \in \widetilde{{}^*\mathbb{R}_c^{\#}}, a \in {}^*\mathbb{R}_c^{\#}$.

Notation 14.2.5. $\widehat{a} = (a,a,\ldots.) \in \widetilde{{}^*\mathbb{R}_c^{\#}}, a \in \widetilde{{}^*\mathbb{R}_c^{\#}}$.

Definition 14.2.43. (i) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $\widetilde{{}^*\mathbb{R}_c^{\#}}, \{a_n\}_{n=0}^{k} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$.

We define external hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{k}} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$ by

$$\begin{gathered}\{A_n;k\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{k}} = \\ = \left(a_0,a_1,\ldots,a_m,\ldots,a_{k-1},k,\widehat{a_k}\right).\end{gathered} \tag{14.2.44}$$

(ii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $\widetilde{{}^*\mathbb{R}_c^{\#}} : \{a_n\}_{n=0}^{\infty} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$.

We define hyper infinite sequence $\{A_n\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{\infty}} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$ by

$$\begin{gathered}\{A'_n;\infty\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{\infty}} = \\ = \left(a_0,a_1,\ldots,a_k,\ldots\{a_n\}_{n=0}^{\infty},\widehat{\{a_n\}_{n=0}^{\infty}}\right).\end{gathered} \tag{14.2.45}$$

(iii) Let $\{a_n\}_{n=0}^{N}, N \in {}^*\mathbb{N}\backslash\mathbb{N}$ be external hyperfinite sequence in $\widetilde{{}^*\mathbb{R}_c^{\#}} : \{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$.

We define hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{N}} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$ by

$$\begin{gathered}\{A_n;N\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{N}} = \\ = \left(a_0,a_1,,\ldots,a_m,\ldots,a_{N-1},a_N,\widehat{a_N}\right) \in [[a_N]].\end{gathered} \tag{14.2.46}$$

Definition 14.2.44.(i) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $\widetilde{{}^*\mathbb{R}_c^{\#}}, \{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$.

We define external finite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n$ by

$$ Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n = \overbrace{\{c_n\}_{n=0}^{k}} = (c_0,c_1,\ldots,c_m,\ldots,c_k,\widehat{c}_k) \in [[c_k]] \tag{14.2.47}$$

where $c_0 = a_0, c_j = Ext\text{-}\sum_{n=0}^{n=j} a_n, 0 \leq j \leq k$.

(ii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $\widetilde{{}^*\mathbb{R}_c^{\#}} : \{a_n\}_{n=0}^{\infty} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$. We define external countable sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n$ by

$$\begin{gathered} Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n = \overbrace{\{c_n\}_{n=0}^{\infty}} = \\ = \left(c_0,c_1,\ldots,c_k,\ldots\{c_n\}_{n=0}^{\infty},\widehat{\{c_n\}_{n=0}^{\infty}}\right) \in \left[\left[\widehat{\{c_n\}_{n=0}^{\infty}}\right]\right]\end{gathered} \tag{14.2.48}$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n, k \in \mathbb{N}$.

(iii) Let $\{a_n\}_{n=0}^{n=N}, N \in {}^*\mathbb{N}\backslash\mathbb{N}$ be external hyperfinite sequence in $\widetilde{{}^*\mathbb{R}_c^{\#}} : \{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^{\#}}$.

We define external hyperfinite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = \overbrace{\{c_n\}_{n=0}^{n=N}} = (c_0, c_1, \ldots, c_k, \ldots, c_N, \widehat{c}_N) \in [[c_N]] \tag{14.2.49}$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n,\ 0 \leq k \leq N, c_N = Ext\text{-}\sum_{n=0}^{n=N} a_n.$

(iv) Let $\{a_n\}_{n=0}^{n=N}, N \in {}^*\mathbb{N}$ be external hyperfinite sequence in $\widetilde{{}^*\mathbb{R}_c^\#}$: $\{a_n\}_{n=0}^{N} \subset \widetilde{{}^*\mathbb{R}_c^\#}$ such that $a_n \equiv 0$ for all $n \in {}^*\mathbb{N}\backslash\mathbb{N}$. We assume that

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n. \tag{14.2.50}$$

Example 14.2.3.Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots. \alpha r^{N-1}, N \in {}^*\mathbb{N}, \alpha \in \widetilde{{}^*\mathbb{R}_c^\#}$, $r \in \widetilde{{}^*\mathbb{R}_c^\#}$ be the first term and the ratio of the G.P respectively. Then for any $N \in {}^*\mathbb{N}$ by Proposition 14.2.6 and Definition 14.2.44 one obtains that

$$Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1} \alpha r^{n-1} = \alpha \overbrace{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} =. \alpha \overbrace{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} - \alpha \overbrace{\frac{r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}}. \tag{14.2.51}$$

and

$$Ext\text{-}\widehat{\sum}_{n=1}^{\infty} \alpha r^{n-1} =. \alpha \overbrace{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} - \alpha \overbrace{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}}. \tag{14.2.52}$$

Example 14.2.4.Consider the G.P: $\alpha, \alpha r, \alpha r^2, \ldots. \alpha r^{N-1}, N \in {}^*\mathbb{N}, \alpha \in \widetilde{{}^*\mathbb{R}_c^\#}, r \in \widetilde{{}^*\mathbb{R}_c^\#}$, $r < 0_{\widetilde{{}^*\mathbb{R}_c^\#}},\ |r| < 1$. Note that

$$\begin{gathered}
\alpha \overbrace{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{{}^*\mathbb{R}_c^\#} - r}} = Ext\text{-}\widehat{\sum}_{n=1}^{n=N-1} \alpha r^{n-1} = \\
= Ext\text{-}\widehat{\sum}_{n=1}^{\infty} \alpha r^{n-1} + Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\backslash\mathbb{N}) \wedge (n \leq N-1)} \alpha r^{n-1} = \\
= \alpha \overbrace{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} - \alpha \overbrace{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} + Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\backslash\mathbb{N}) \wedge (n \leq N-1)} \alpha r^{n-1}.
\end{gathered} \tag{14.2.53}$$

From (14.2.53) we obtain

$$\begin{gathered}
Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\backslash\mathbb{N}) \wedge (n \leq N-1)} \alpha r^{n-1} = \alpha \overbrace{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r^N}{1_{{}^*\mathbb{R}_c^\#} - r}} - \alpha \overbrace{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}} + \alpha \overbrace{\left\{\frac{r^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} = \\
\alpha \overbrace{\left\{\frac{\left(-1_{\widetilde{{}^*\mathbb{R}_c^\#}}\right)^n |r|^n}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}\right\}_{n=1}^{\infty}} - \alpha \overbrace{\frac{r^N}{1_{\widetilde{{}^*\mathbb{R}_c^\#}} - r}}.
\end{gathered} \tag{14.2.54}$$

Assume that: (i) $r < 0_{\widetilde{{}^*\mathbb{R}_c^\#}},\ |r| < 1$ then from (14.2.54) we obtain

$$Ext\text{-}\widehat{\sum}_{(n \in {}^*\mathbb{N}\backslash\mathbb{N}) \wedge (n \leq N-1)} \alpha \left(-1_{\widetilde{{}^*\mathbb{R}_c^\#}}\right)^{n-1} |r|^{n-1} \neq 0_{\widetilde{{}^*\mathbb{R}_c^\#}}. \tag{14.2.55}$$

15.1.Basic analisys on external non-Archimedean field $\mathbb{R}_{\mathbf{c}}^\#$.

15.1.The #-limit of a function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_c^{\#}$

Definition 15.1.The (ε, δ) definition of the #-limit of a function $f : D \to \mathbb{R}_c^{\#}$ is as follows: Let f be a $\mathbb{R}_c^{\#}$-valued function defined on a subset $D \subset \mathbb{R}_c^{\#}$ of the Cauchy hyperreal numbers. Let c be a limit point of D and let L be a hyperreal number. We say that

$$\#\text{-}\lim_{x \to_{\#} c} f(x) = L \tag{15.1}$$

if for every $\varepsilon \approx 0, \varepsilon > 0$ there exists a $\delta \approx 0, \delta > 0$ such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$, symbolically:

$$\lim_{x \to_{\#} c} f(x) = L \Leftrightarrow (\forall \varepsilon(\varepsilon \approx 0 \wedge \varepsilon > 0) \exists \delta(\delta \approx 0 \wedge \delta > 0) \forall x \in D, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon. \tag{15.2}$$

Definition 15.2.The function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_c^{\#}$ is #-continuous (or micro continuous) at some
point c of its domain if the #-limit of $f(x)$, as x #-approaches c through the domain of f, exists and is equal to $f(c)$:

$$\#\text{-}\lim_{x \to_{\#} c} f(x) = f(c). \tag{15.3}$$

Theorem 15.1.If $\#\text{-}\lim_{x \to_{\#} x_0} f(x)$ exists; then it is unique that is; if $\#\text{-}\lim_{x \to_{\#} x_0} f(x) = L_1$ and $\#\text{-}\lim_{x \to_{\#} x_0} f(x) = L_2$, then $L_1 = L_2$.

Theorem 15.2. If $\#\text{-}\lim_{x \to_{\#} x_0} f_1(x) = L_1$ and $\#\text{-}\lim_{x \to_{\#} x_0} f_2(x) = L_2$ then

$$\begin{aligned} \#\text{-}\lim_{x \to_{\#} x_0} [f_1(x) \pm f_2(x)] &= L_1 \pm L_2, \\ \#\text{-}\lim_{x \to_{\#} x_0} [f_1(x) \times f_2(x)] &= L_1 \times L_2, \\ \#\text{-}\lim_{x \to_{\#} x_0} \frac{f_1(x)}{f_2(x)} &= \frac{L_1}{L_2}, L_2 \neq 0. \end{aligned} \tag{15.4}$$

Definition 15.3.(a) We say that $f(x)$ #-approaches the left-hand #-limit L as x #-approaches x_0 from the left,and write $\#\text{-}\lim_{x \to x_0 -} f(x) = L$, if $f(x)$ is defined on some #-open interval (a, x_0) and, for each $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $|f(x) - L| < \varepsilon$ if $x_0 - \delta < x < x_0$.

(b) We say that $f(x)$ #-approaches the right-hand #-limit L as x #-approaches x_0 from the
right, and write $\#\text{-}\lim_{x \to_{\#} x_0 +} f(x) = L$, if $f(x)$ is defined on some open interval (x_0, b) and, for
each $\varepsilon > 0$, there is a $\delta > 0, \delta \approx 0$ such that $|f(x) - L| < \varepsilon, \varepsilon > 0, \varepsilon \approx 0$ if $x_0 < x < x_0 + \delta$.

Left- and right-hand #-limits are also called one-sided #-limits. We will often simplify the
notation by writing $\#\text{-}\lim_{x \to_{\#} x_0 -} f(x) = f(x_0 -)$ and $\#\text{-}\lim_{x \to_{\#} x_0 +} f(x) = f(x_0 +)$.

Theorem 15.3. A function f has a #-limit at x_0 if and only if it has left- and right-hand #-limits at x_0; and they are equal. More specifically; $\#\text{-}\lim_{x \to_{\#} x_0} f(x) = L$ if and only if $f(x_0 +) = f(x_0 -) = L$.

Definition 15.4. We say that $f(x)$ approaches the #-limit L as x approaches $\infty^{\#}$, and write $\#\text{-}\lim_{x \to_{\#} \infty^{\#}} f(x) = L$, if f is defined on an interval $(a, \infty^{\#})$ and, for each $\varepsilon > 0, \varepsilon \approx 0$, there is a number β such that $|f(x) = L| < \varepsilon$ if $x > \beta$.

Definition 15.5. We say that $f(x)$ approaches $\infty^{\#}$ as x approaches x_0 from the left, and write

$$\#\text{-}\lim_{x\to_\# x_0-} f(x) = \infty^\# \text{ or } f(x_0-) = \infty^\# \tag{15.5}$$

if f is defined on an interval (a,x_0) and, for each hyperreal number $M,$ there is a $\delta \approx 0, \delta > 0$ such that $f(x) > M$ if $x_0 - \delta < x < x_0$.

Similarly we define: $\#\text{-}\lim_{x\to x_0-} f(x) = -\infty^\#, \#\text{-}\lim_{x\to_\# x_0+} f(x) = -\infty^\#, \#\text{-}\lim_{x\to_\# x_0+} f(x) = \infty^\#$.

Example 15.1. (i) $\#\text{-}\lim_{x\to_\# x_0-} x^{-1} = -\infty^\#$, (ii) $\#\text{-}\lim_{x\to_\# x_0+} x^{-1} = +\infty^\#$,

(iii) $\#\text{-}\lim_{x\to_\# -\infty^\#} x^2 = \#\text{-}\lim_{x\to_\# \infty^\#} x^2 = \infty^\#$.

Remark 15.1. Throughout this paper, $\#\text{-}\lim_{x\to_\# x_0} f(x)$ exists" will mean that $\#\text{-}\lim_{x\to_\# x_0} f(x) = L,$ where L is finite or hyperfinite.

To leave open the possibility that $L = \pm\infty^\#,$ we will say that

$\#\text{-}\lim_{x\to x_0} f(x)$ exists in the extended hyperreals.

This convention also applies to one-sided limits and limits as x approaches $\pm\infty^\#$.

15.2.Monotonic Functions $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$.

Definition 17.6.A function $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$ is nondecreasing on an interval $I \subset \mathbb{R}_c^\#$ if

$$f(x_1) \leq f(x_2) \tag{15.6}$$

whenever x_1 and x_2 are in I and $x_1 < x_2$, or nonincreasing on I if

$$f(x_1) \geq f(x_2) \tag{15.7}$$

whenever x_1 and x_2 are in I and $x_1 < x_2$.

In either case, f is on I. If $\leq$ can be replaced by $<$ in (15.6), f is increasing on I. If $\geq$ can be replaced by $>$ in (15.7), f is decreasing on I. In either of these two cases, f is strictly monotonic on I.

Theorem 15.4. Suppose that $f(x)$ is monotonic on (a,b) and define $\alpha = \inf_{a<x<b} f(x)$ and $\beta = \sup_{<x<b} f(x)$. Suppose that $\exists\alpha$ and $\exists\beta$, then:

(a) If f is nondecreasing, then $f(a+) = \alpha$ and $f(b-) = \beta$.

(b) If f is nonincreasing; then $f(a+) = \beta$ and $f(b-) = \alpha$.

Here $a+= -\infty^\#$ if $a = -\infty^\#$ and $b+= \infty^\#$ if $b = \infty^\#$.

(c) If $a < x_0 < b,$ then $f(x_0+)$ and $f(x_0-)$ exist and are finite or hyperfinite; moreover, $f(x_0+) \leq f(x_0) \leq f(x_0-)$ if f is nondecreasing, and $f(x_0+) \geq f(x_0) \geq f(x_0-)$ if f is nonincreasing:

Proof (a) We first show that $f(a+) = \alpha$. If $M > \alpha,$ there is an x_0 in (a,b) such that $f(x_0) < M$. Since f is nondecreasing, $f(x) < M$ if $a < x < x_0$. Therefore, if $\alpha = -\infty^\#$, then $f(a+) = -\infty^\#$. If $\alpha > -\infty^\#$, let $M = \alpha + \varepsilon,$ where $\varepsilon \approx 0, \varepsilon > 0$.

Then $\alpha \leq f(x) < \alpha + \varepsilon$, so (i) $|f(x) - \alpha| < \varepsilon$ if $a < x < x_0$.

If $a = -\infty^\#$, this implies that $f(-\infty^\#) = \alpha$. If $a > -\infty^\#$, let $\delta = x_0 - a$. Then (i) is equivalent to $|f(x) - \alpha| < \varepsilon$ if $a < x < a + \delta$,which implies that $f(a+) = \alpha$.

We now show that $f(b+) = \beta$. If $M < \beta,$ there is an x_0 in (a,b) such that $f(x_0) > M$. Since $f(x)$ is nondecreasing, $f(x) > M$ if $x_0 < x < b$. Therefore, if $\beta = \infty^\#$, then $f(b-) = \infty^\#$. If $\beta < \infty^\#$, let $M = \beta - \varepsilon,$ where $\varepsilon \approx \varepsilon > 0$. Then $\beta - \varepsilon < f(x) \leq \beta,$ so (ii) $|f(x) - \beta| < \varepsilon$ if $x_0 < x < b$.

If $b = \infty^\#$, this implies that $f(\infty^\#) = \beta$. If $b < \infty^\#$, let $\delta = b - x_0$. Then (ii) is equivalent to $f(x) <$ if $b - \delta < x < b$,which implies that $f(b-) = \beta$.

(b) The proof is similar to the proof of (a).

(c) Suppose that $f(x)$ is nondecreasing. Applying (a) to $f(x)$ on (a,x_0) and (x_0,b) separately shows that $f(x_0-) = \sup_{a<x<x_0} f(x)$ and $f(x_0+) = \inf_{x_0<x<b} f(x)$.

However, if $x_1 < x_0 < x_2$, then $f(x_1) \leq f(x_0) \leq f(x_2)$ and hence, $f(x_0 -) \leq f(x_0) \leq f(x_0 +)$.

15.3. #-Limits Inferior and Superior

Definition 15.7.We say that: (i) f is bounded on a set $S \subseteq \mathbb{R}_c^\#$ if there is a constant $M \in \mathbb{R}, M < \infty$ such that $f(x) \leq M$ for all $x \in S$, (ii) f is hyperbounded on a set $S \subseteq \mathbb{R}_c^\#$ if f is not bounded on a set S and there is a constant $M \in \mathbb{R}_c^\#/\mathbb{R}, M < \infty^\#$ such that $f(x) \leq M$ for all $x \in S$.

Definition 15.8. Suppose that f is bounded or hyperbounded on $[a, x_0)$, where x_0 may be finite or hyperfinite or $\infty^\#$. For $a \leq x < x_0$, define (i) $S_f(x; x_0) = \sup_{x \leq t < x_0} f(t)$ and

(ii) $I_f(x; x_0) = \inf_{x \leq t < x_0} f(t)$.

Then the left #-limit superior of $f(x)$ at x_0 is defined to be

$$\#\text{-}\overline{\lim}_{x \to_\# x_0 -} f(x) = \#\text{-}\lim_{x \to_\# x_0 -} S_f(x; x_0) \qquad (15.8)$$

and the left limit inferior of $f(x)$ at x_0 is defined to be

$$\#\text{-}\underline{\lim}_{x \to_\# x_0 -} f(x) = \#\text{-}\lim_{x \to_\# x_0 -} I_f(x; x_0). \qquad (15.9)$$

If $x_0 = \infty^\#$, we define $x_0 - = \infty^\#$.

Theorem 15.5. If $f(x)$ is bounded or hyperbounded on $[a, x_0)$, then $\beta = \#\text{-}\overline{\lim}_{x \to_\# x_0 -} f(x)$ exists and is the unique hyperreal number with the following properties:

(a) If $\varepsilon > 0, \varepsilon \approx 0$, there is an a_1 in $[a, x_0)$ such that

(i) $f(x) < \beta + \varepsilon$ if $a_1 \leq x < x_0$

(b) If $\varepsilon > 0, \varepsilon \approx 0$ and a_1 is in $[a, x_0)$, then

$f(\bar{x}) > \beta - \varepsilon$ for some $\bar{x} \in [a, x_0)$.

Proof. Since $f(x)$ is bounded or hyperbounded on $[a, x_0)$, $S_f(x; x_0)$ is nonincreasing and bounded or hyperbounded on $[a, x_0)$. By applying Theorem 17.4(b) to $S_f(x; x_0)$, we conclude that β exists finite or hyperfinite.

Therefore, if $\varepsilon > 0, \varepsilon \approx 0$, there is an $\bar{a}$ in $[a, x_0)$ such that

(ii) $\beta - \varepsilon/2 < S_f(x; x_0) < \beta + \varepsilon/2$ if $\bar{a} \leq x < x_0$.

Since $S_f(x; x_0)$ is an upper bound of $\{f(t) | x \leq t < x_0\}, f(x) < S_f(x; x_0)$. Therefore, the second inequality in (ii) implies the inequality (i) with $a_1 = \bar{a}$. This proves (a).

To prove (b),let a_1 be given and define $x_1 = \max\{a_1, \bar{a}\}$. Then the first inequality in (ii) implies that (iii) $S_f(x; x_0) > \beta - \varepsilon/2$. Since $S_f(x; x_0)$ is the supremum of $\{f(t) | x_1 \leq t < x_0\}$, there is an $\bar{x}$ in $[x_1, x_0)$ such that

$f(\bar{x}) > S_f(x; x_0) - \varepsilon/2$. This and (iii) imply that $f(\bar{x}) > \beta - \varepsilon/2$. Since $\bar{x}$ is in $[a_1, x_0)$, this proves (b).

Now we show that there cannot be more than one hyperreal number with properties (a) and (b). Suppose that $\beta_1 < \beta_2$ and β_2 has property (b); thus, if $\varepsilon \approx 0, \varepsilon > 0$ and a_1 is in $[a, x_0)$ there is an $\bar{x}$ in $[a_1, x_0)$ such that $f(\bar{x}) > \beta_2 - \varepsilon$. Letting $\varepsilon = \beta_2 - \beta_1$, we see that there is an $\bar{x}$ in $[a_1, b)$ such that $f(\bar{x}) > \beta_2 - (\beta_2 - \beta_1) = \beta_1$so β_1 cannot have property (a).Therefore, there cannot be more than one hyperreal number that satisfies both (a) and (b).

Theorem 15.6. If $f(x)$ is bounded or hyperbounded on $[a, x_0)$, then $\alpha = \underline{\lim}_{x \to x_0 -} f(x)$ exists and there is the unique hyperreal number with the following properties:

(a) If $\varepsilon \approx 0, \varepsilon > 0$ there is an a_1 in $[a, x_0)$ such that

$f(x) > \alpha - \varepsilon$ if $a_1 \leq x < x_0$.

(b) If $\varepsilon \approx 0, \varepsilon > 0$ and a_1 is in $[a, x_0)$, then

$f(\bar{x}) < \alpha + \varepsilon$ for some $\bar{x} \in [a, x_0)$.

Theorem 15.7. If $f(x)$ is bounded or hyperbounded on $[a, x_0)$, then
(i) #-$\underline{\lim}_{x \to_{\#} x_0-} f(x) \leq$ #-$\overline{\lim}_{x \to_{\#} x_0-} f(x)$;
(ii) #-$\underline{\lim}_{x \to_{\#} x_0-} f(-x) =$ #- $\overline{\lim}_{x \to_{\#} x_0-} f(x)$;
(iii) #-$\overline{\lim}_{x \to_{\#} x_0-} f(-x) =$ #- $\underline{\lim}_{x \to_{\#} x_0-} f(x)$;
(iv) #-$\underline{\lim}_{x \to_{\#} x_0-} f(x) =$ #-$\overline{\lim}_{x \to_{\#} x_0-} f(x)$ if and only if #-$\lim_{x \to_{\#} x_0-} f(x)$ exists, in which case #-$\lim_{x \to_{\#} x_0-} f(x) =$ #-$\underline{\lim}_{x \to_{\#} x_0-} f(x) =$ #-$\overline{\lim}_{x \to_{\#} x_0-} f(x)$

Theorem 15.8.Suppose that $f(x)$ and $g(x)$ are bounded or hyperbounded on $[a, x_0)$. Then: (i) #-$\overline{\lim}_{x \to_{\#} x_0-} (f+g)(x) \leq$ #-$\overline{\lim}_{x \to_{\#} x_0-} f(x) +$ #-$\overline{\lim}_{x \to x_0-} g(x)$;
(ii) #-$\underline{\lim}_{x \to_{\#} x_0-} (f+g)(x) \geq$ #-$\underline{\lim}_{x \to_{\#} x_0-} f(x) +$ #-$\underline{\lim}_{x \to_{\#} x_0-} g(x)$.

Theorem 15.9.The $\alpha = \lim_{x \to x_0-} f(x)$ exists i.e.,α is finite or hyperfinite if and only if for each $\varepsilon \approx 0, \varepsilon > 0$ there is a $\delta \approx 0, \delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ if $x_0 - \delta < x_1, x_2 < x_0$.

Theorem 15.10.(i) Suppose that $f(x)$ is bounded or hyperbounded on an interval $(x_0, b]$, then #-$\underline{\lim}_{x \to_{\#} x_0+} f(x) =$ #-$\overline{\lim}_{x \to_{\#} x_0+} f(x)$ if and only if #-$\lim_{x \to x_0+} f(x)$ exists, in which case #-$\lim_{x \to_{\#} x_0+} f(x) =$ #-$\underline{\lim}_{x \to_{\#} x_0+} f(x) =$ #-$\overline{\lim}_{x \to_{\#} x_0+} f(x)$.
(ii) Suppose that $f(x)$ is bounded or hyperbounded on an open interval containing x_0, then #-$\lim_{x \to_{\#} x_0} f(x)$ exists if and only if
#-$\overline{\lim}_{x \to_{\#} x_0-} f(x) =$ #-$\overline{\lim}_{x \to_{\#} x_0+} f(x) =$ #-$\underline{\lim}_{x \to_{\#} x_0-} f(x) =$ #-$\underline{\lim}_{x \to_{\#} x_0+} f(x)$.

15.4.The #-continuity of a function $f : \mathbb{R}_c^\# \to \mathbb{R}_{\mathbf{c}}^\#$.

Definition 15.9. (i) We say that a function $f : \mathbb{R}_c^\# \to \mathbb{R}_{\mathbf{c}}^\#$. is #-continuous at x_0 if f is defined on an open interval (a, b) containing x_0 and $\lim_{x \to_{\#} x_0-} f(x_0) = x_0$.
(ii) We say that f is #-continuous from the left at x_0 if f is defined on an open interval (a, x_0) and $f(x_0 -) = f(x_0)$.
(iii) We say that f is #-continuous from the right at x_0 if f is defined on an open interval (x_0, b) and $f(x_0 +) = f(x_0)$.

Theorem 15.11. (i) A function f is #-continuous at x_0 if and only if f is defined on an open
interval (a, b) containing x_0 and for each $\varepsilon \approx 0, \varepsilon > 0$ there is a $\delta \approx 0, \delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \tag{15.10}$$

whenever $|x - x_0| < \delta$.
(ii) A function f is #-continuous from the right at x_0 if and only if f is defined on an interval $[x_0, b)$ and for each $\varepsilon \approx 0, \varepsilon > 0$ there is a $\delta \approx 0, \delta > 0$ such that (17.10) holds whenever $x_0 \leq x < x_0 + \delta$.
(iii) A function f is #-continuous from the left at x_0 if and only if f is defined on an interval $(a, x_0]$ and for each $\varepsilon \approx 0, \varepsilon > 0$ there is a $\delta \approx 0, \delta > 0$ such that (15.10) holds whenever $x_0 - \delta < x \leq x_0$.
Note that from Definition 15.9 and Theorem 15.8, f is #-continuous at x_0 if and only if $f(x_0 +) = f(x_0 -) = f(x_0)$ or, equivalently, if and only if it is #-continuous from the right and left at x_0.

Definition 15.10. A function $f : \mathbb{R}_c^\# \to \mathbb{R}_{\mathbf{c}}^\#$ is #-continuous on an open interval (a, b) if it is
#-continuous at every point in (a, b). If, in addition,

$$f(b-) = f(b) \tag{15.11}$$

or

$$f(a+) = f(a) \tag{15.12}$$

then f is #-continuous on $(a,b]$ or $[a,b)$, respectively. If f is #-continuous on (a,b) and (15.11) and (15.12) both hold, then f is #-continuous on $[a,b]$. More generally, if S is a subset of $\mathbf{dom}(f)$ consisting of finitely or countably or hyper finitely or hyper infinitely many disjoint intervals, then f is #-continuous on S if f is #-continuous on every interval in S.

Definition 15.11. A function $f : \mathbb{R}_c^\# \to \mathbb{R}_{\mathbf{c}}^\#$ is piecewise #-continuous on $[a,b]$ if
(i) $f(x_0+)$ exists for all x_0 in $[a,b)$;
(ii) $f(x_0-)$ exists for all x_0 in $(a,b]$;
(iii) $f(x_0+) = f(x_0-) = f(x_0)$ for all but except finitely or hyper finitely many points x_0 in (a,b).

If (iii) fails to hold at some x_0 in (a,b), f has a jump #-discontinuity at x_0. Also, f has a jump #-discontinuity at a if $f(a+) \neq f(a)$ or at b if $f(b-) \neq f(b)$.

Theorem 15.12. If f and g are #-continuous on a set S, then so are $f \pm g$, and fg. In addition, f/g is #-continuous at each x_0 in S such that $g(x_0) \neq 0$.

By hyper infinite induction, it can be shown that if $\forall n \in \mathbb{N}^\#$ $f_n(x)$ are #-continuous on a set S, then so are $\sum_{i \leq n} f_n(x)$. Therefore, $\forall n,m \in \mathbb{N}^\#$ any rational function $r(x) = \sum_{i \leq n} a_i x^i / \sum_{i \leq m} b_i x^i, b_i \neq 0$ is #-continuous for all values of x except those for which its denominator vanishes.

15.5. Removable #-discontinuities.

Definition 15.12. Let $f(x)$ be defined on a deleted #-neighborhood of x_0 and #-discontinuous (perhaps even undefined) at x_0. Then we say that $f(x)$ has a removable #-discontinuity at x_0 if #-$\lim_{x\to x_0} f(x_0)$ exists. In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in \mathbf{dom}(f) \text{ and } x \neq x_0 \\ \\ \lim_{x\to x_0} f(x_0) & \text{if} \quad x = x_0 \end{cases} \tag{15.13}$$

is #-continuous at x_0.

15.6. Composite Functions $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$.

Definition 15.13. Suppose that $f : \mathbb{R}_c^\# \to \mathbb{R}_{\mathbf{c}}^\#$ and $g : \mathbb{R}_c^\# \to \mathbb{R}_{\mathbf{c}}^\#$ are functions with domains $\mathbf{dom}(f)$ and $\mathbf{dom}(g)$ correspondingly. If $\mathbf{dom}(g)$ has a nonempty subset T such that $g(x) \in \mathbf{dom}(g)$ whenever $x \in T$, then the composite function $f \circ g : \mathbb{R}_c^\# \to \mathbb{R}_{\mathbf{c}}^\#$ is defined on T by $(f \circ g)(x) = f(g(x))$

Theorem 15.10. Suppose that g is #-continuous at $x_0, g(x_0)$ is an #-interior point of $\mathbf{dom}(f)$ and f is #-continuous at $g(x_0)$. Then $f \circ g$ is #-continuous at x_0.

Proof. Suppose that $\varepsilon \approx 0, \varepsilon > 0$. Since $g(x_0)$ is an #-interior point of $\mathbf{dom}(f)$ and $f(x)$ is #-continuous at $g(x_0)$, there is a $\delta_1 \approx 0, \delta_1 > 0$ such that $f(t)$ is defined and

(i) $|f(t) - f(gx_0)| < \varepsilon$ if $|t - g(x_0)| < \delta_1$.
Since $g(x)$ is #-continuous at x_0, there is a $\delta \approx 0, \delta > 0$ such that $g(x)$ is defined and
(ii) $|g(x) - g(x_0)| < \delta_1$ if $|x - x_0| < \delta$.
Now (i) and (ii) imply that $|f(g(x)) - f(g(x_0))| < \delta$ if $|x - x_0| < \varepsilon$. Therefore, $f \circ g$ is #-continuous at x_0.

15.7. Bounded and Hyperbounded Functions $f : \mathbb{R}_c^{\#} \to \mathbb{R}_c^{\#}$.

Definition 15.14. (i) A function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_c^{\#}$ is bounded below on a set $S \subset \mathbb{R}_c^{\#}$ if there is a finite or hyperfinite hyperreal number $m \in \mathbb{R}_{c.\mathbf{fin}}^{\#}$ such that $f(x) \geq m$ for all $x \in S$. If in this case the set $V = \{f(x)|x \in S\}$ has infimum α, we write $\alpha = \inf_{x \in S} f(x)$. If there is a point $x_1 \in S$ such that $f(x_1) = \alpha_{\text{c}}$, we say that α is the minimum of $f(x)$ on S, and write $\alpha = \min_{x \in S} f(x)$
(ii) A function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_c^{\#}$ is bounded above on $S \subset \mathbb{R}_c^{\#}$ if there is a finite or hyperfinite hyperreal number $M \in \mathbb{R}_{c.\mathbf{fin}}^{\#}$ such that $f(x) \leq M$ for all $x \in S$. If in this case, V has a supremum β, we write $\beta = \sup_{x \in S} f(x)$. If there is a point $x_1 \in S$ such that $f(x_2) = \beta_{\text{c}}$, we say that β is the maximum of $f(x)$ on S, and write $\alpha = \max_{x \in S} f(x)$.
(iii) If f is bounded above and below on a set S, we say that f is bounded on S.

Theorem 15.11. If f is #-continuous on a finite or hyperfinite #-closed interval $[a,b]$, then f is bounded or hyperbounded on $[a,b]$.
Proof. Suppose that $t \in [a,b]$. Since f is #-continuous at t, there is an open interval I_t containing t such that

$$|f(x) - f(t)| < 1 if x \in I_t \cap [a,b] \tag{15.14}$$

To see this, set $\varepsilon = 1$ in (15.10), Theorem 15.11. The collection $H = \{I_t | a \leq t \leq b\}$ is an open covering of $[a,b]$. Since $[a,b]$ is #-compact, the generalized Heine–Borel theorem implies that there are hyper finitely many points $t_1, t_2, \ldots, t_n, n \in \mathbb{N}^{\#}$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b]$. According to (11.14) with $t = t_i$, $|f(x) - f(t_i)| < 1$ if $x \in I_{t_i} \cap [a,b]$. Therefore,

$$f(x) = |(f(x) - f(t_i)) + f(t_i)| \leq |f(x) - f(t_i)| + |f(t_i)| \leq 1 + |f(t_i)| \tag{15.15}$$

if $x \in I_{t_i} \cap [a,b]$. Let $M = 1 + \max_{1 \leq i \leq n} |f(t_i)|$. Since $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b])$,
(15.15) implies that $|f(x)| \leq M$ if $x \in [a,b]$.
Theorem 15.12. Suppose that f is #-continuous on a finite or hyperfinite closed interval $[a,b]$. Let $V_{a,b} = \{f(x)|x \in [a,b]\}$. Assume that the set $V_{a,b}$ is admissible above and below. Let

$$\alpha = \inf V_{a,b} = \inf_{a \leq x \leq b} f(x) \text{ and } \beta = \sup V_{a,b} = \sup_{a \leq x \leq b} f(x). \tag{15.16}$$

Then α and β are respectively the minimum and maximum of f on $[a,b]$; that is there are points x_1 and x_2 in $[a,b]$ such that $\alpha = f(x_1)$ and $\beta = f(x_2)$.
Proof. We show that x_1 exists. Note that $\alpha = \inf V_{a,b}$ and $\beta = \sup V_{a,b}$ exist since the set $V_{a,b}$ is admissible below and above. Suppose that there is no x_1 in $[a,b]$ such that $f(x_1) = \alpha_{\text{c}}$. Then $f(x) > \alpha_{\text{c}}$ for all $x \in [a,b]$. We will show that this leads to a contradiction. Suppose that $t \in [a,b]$. Then $f(t) > \alpha$, so $f(t) > [f(t) + \alpha]/2 > \alpha$. Since f is #-continuous at t, there is an open interval I_t about t such that

$$f(x) > \frac{f(t)+\alpha}{2} \qquad (15.17)$$

if $x \in I_t \cap [a,b]$. The collection $H = \{I_t | a \leq t \leq b\}$ is an open covering of $[a,b]$. Since $[a,b]$ is #-compact, the generalized Heine–Borel theorem implies that there are hyper finitely many points $t_1, t_2, \ldots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b]$. Define $\alpha_1 = \min_{1 \leq i \leq n}[f(t_i)+\alpha]/2$. Then, since $[a,b] \subset \bigcup_{i=1}^{n}(I_{t_i} \cap [a,b])$, (15.17) implies that $f(t) > \alpha_1, a \leq t \leq b$. But $\alpha_1 > \alpha$, so this contradicts the definition of α. Therefore, $f(x_1) = \alpha$ for some $x_1 \in [a,b]$.

15.8. Generalized Intermediate Value Theorem.

Theorem 15.13.(**Generalized Intermediate Value Theorem**) Suppose that:
(i) f is #-continuous on $[a,b]$, (ii) $f(a) \neq f(b)$ and $f(a) < \mu < f(b)$,
(iii) the set $S = \{x | (a \leq x \leq b) \wedge (f(x) \leq \mu)\}$ is admissible above. Then $f(c) = \mu$ for some $c \in (a,b)$.

Proof. Suppose that $f(a) < \mu < f(b)$. Note that $\sup S$ exists, since the set S is admissible above. Let $c = \sup S$. We will show that $f(c) = \mu$. If $f(c) > \mu$, then $c > a$ and, since f is #-continuous at c, there is an $\varepsilon > 0, \varepsilon \approx 0$ such that $f(x) > \mu$ if $c - \varepsilon < x \leq c$. Therefore, c is an upper bound for S, which contradicts the definition of c as the supremum of S. If $f(c) < \mu$, then $c < b$ and there is an $\varepsilon > 0, \varepsilon \approx 0$ such that $f(x) < \mu$ for $c \leq x < c - \varepsilon$, so c is not an upper bound for S. This is also a contradiction. Therefore, $f(c) = \mu$. The proof for the case where $f(b) < \mu < f(a)$ can be obtained by applying this result to $-f(x)$.

Lemma.15.1. If f is #-continuous at x_0 and $f(x_0) > \mu$, then $f(x) > \mu$ for all x in some #-neighborhood of x_0.

15.9. Uniform #-Continuity.

Definition 15.15. A function f is uniformly #-continuous on a subset S of its domain if, for every $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $|f(x) - f(x')| < \varepsilon$ whenever $|x - x'| < \delta$ and $x, x' \in S$.

We emphasize that in this definition δ depends only on and S and not on the particular choice of x and x', provided that they are both in S.

Theorem 15.14. If f is #-continuous on a #-closed and bounded or hyperbounded interval $[a,b]$, then f is uniformly #-continuous on $[a,b]$.

Proof. Suppose that $\varepsilon > 0, \varepsilon \approx 0$. Since f is #-continuous on $[a,b]$, for each $t \in [a,b]$ there is a positive number δ_t such that

$$|f(x) - f(t)| < \varepsilon/2 \qquad (15.18)$$

if $|x - t| < \delta_t$ and $x \in [a,b]$. If $I_t = (t - \delta_t, t + \delta_t)$, the collection $H = \{I_t | t \in [a,b]\}$ is an open covering of $[a,b]$. Since $[a,b]$ is #-compact, the generalized Heine–Borel theorem implies that there are hyper finitely many points $t_1, t_2, \ldots, t_n$ in $[a,b]$ such that $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b]$. Now define

$$\delta = \min\left\{\delta_{t_1}, \delta_{t_2}, \ldots, \delta_{t_n}\right\}. \qquad (15.19)$$

We will show that if

$$|x - x'| < \delta \text{ and } x, x' \in [a,b] \qquad (15.20)$$

then $|f(x)-f(x')|<\varepsilon$. From the triangle inequality one obtains:

$$|f(x)-f(x')| = |(f(x)-f(t_r))+(f(t_r)-f(x'))| \leq |f(x)-f(t_r)|+|f(t_r)-f(x')| \quad (15.21)$$

Since $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b]$, x must be in one of these intervals. Suppose that $x \in I_{t_r}$ that is,

$$|x-t_r| \leq \delta_{t_r}. \quad (15.22)$$

From (11.18) with $t = t_r$,

$$|f(x)-f(t_r)| \leq \frac{\varepsilon}{2}. \quad (15.23)$$

From (11.20), (11.22), and the triangle inequality,

$$|x'-t_r| = |(x'-x)+(x-t_r)| \leq |x'-x|+|x-t_r| < \delta+\delta_{t_r} \leq 2\delta_{t_r}. \quad (15.24)$$

Therefore, (11.18) with $t = t_r$ and x replaced by x' implies that

$$|f(x')-f(t_r)| \leq \frac{\varepsilon}{2}. \quad (15.25)$$

Thus (11.25),(11.21) and (11.23) imply that $|f(x')-f(t_r)| \leq \varepsilon/2$.

15.10. Monotonic External Functions $f : \mathbb{R}^{\#}_{c} \to \mathbb{R}^{\#}_{c}$.

Theorem 15.15. If f is monotonic and nonconstant on $[a,b]$, then f is #-continuous on $[a,b]$ if and only if its range $\mathbf{range}\,(f) = \{f(x)|x \in [a,b]\}$ is the #-closed interval with endpoints $f(a)$ and $f(b)$.

Theorem 15.16. Suppose that f is increasing and #-continuous on $[a,b]$ and let $f(a) = c$ and $f(b) = d$. Then there is a unique function g defined on $[c,d]$ such that

$$g(f(x)) = x, a \leq x \leq b, \quad (15.26)$$

and

$$f(g(y)) = y, c \leq y \leq d. \quad (15.27)$$

Moreover, g is #-continuous and increasing on $[c,d]$:

The function g of Theorem 15.16 is the inverse of f, denoted by f^{-1}. Since (15.26) and (15.27) are symmetric in f and g, we can also regard f as the inverse of g, and denote it by g^{-1}.

15.11. The #-derivative of a $\mathbb{R}^{\#}_{c}$-valued function $f : D \to \mathbb{R}^{\#}_{c}$.

A function $f : D \to \mathbb{R}^{\#}_{c}, D \subset \mathbb{R}^{\#}_{c}$ is differentiable at an #-interior point $x_0 \in D$ of its domain

$D \subset \mathbb{R}^{\#}_{c}$ if the difference quotient

$$\frac{f(x)-f(x_0)}{x-x_0}, x \neq x_0 \quad (15.28)$$

approaches a #-limit as x approaches x_0, in which case the #-limit is called the #-derivative

of f at x_0, and is denoted by $f^{\#\prime}(x_0)$ or by $f'^{\#}(x_0)$ or by $d^{\#}f(x_0)/d^{\#}x$ i.e.,

$$d^{\#}f(x_0)/d^{\#}x \triangleq f'^{\#}(x_0) = \#\text{-}\lim_{x \to_{\#} x_0} \frac{f(x)-f(x_0)}{x-x_0} \quad (15.29)$$

If f is defined on an #-open set $S \subset \mathbb{R}^{\#}_{c}$, we say that f is #-differentiable on S if f is #-differentiable at every point of S. If f is #-differentiable on S, then $f'^{\#}$ is a function on S.

We say that f is #-continuously #-differentiable on S if $f'^{\#}(x)$ is #-continuous on S. If f is

#-differentiable on a #-neighbourhood of x_0, it is reasonable to ask if $f'^{\#}(x)$ is #-differentiable at x_0. If so, we denote the #-derivative of $f'^{\#}$ at x_0 by $f''^{\#}(x_0)$. This is the second #-derivative of f at x_0, and it is also denoted by $f^{(2)\#}(x_0)$. Continuing inductively, if $f^{(n-1)\#}$ is defined on a #-neighborhood of x_0, then the n-th #-derivative of f at x_0, denoted by $f^{(n)\#}(x_0)$, where $n \in \mathbb{N}^{\#}$ or by $d^{n\#}f(x_0)/d^{\#}x^n$ is the #-derivative of $f^{(n-1)\#}(x)$ at x_0. For convenience we define the zeroth #-derivative of f to be f itself; thus $f^{(0)\#} = f$.

Example15.1 If $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ is a positive hyperinteger and $f(x), = x^n$ then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^n - x_0^n}{x - x_0} = \frac{x - x_0}{x - x_0}\left(Ext\text{-}\sum_{k=0}^{n-1} x^{n-k-1}\right). \tag{15.30}$$

Thus $f'^{\#}(x_0) = \#\text{-}\lim_{x\to_{\#} x_0} Ext\text{-}\sum_{k=0}^{n-1} x^{n-k-1} = nx^{n-1}$.

Lemma 15.2. If f is #-differentiable at x_0; then

$$f(x) = f(x_0) + \left[f'^{\#}(x_0) + E(x)\right](x - x_0), \tag{15.31}$$

where $E(x)$ is defined on a #-neighborhood of x_0 and $\#\text{-}\lim_{x\to_{\#} x_0} E(x) = E(x_0) = 0$.

Proof. Define

$$E(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} - f'^{\#}(x_0) & x \in \mathbf{Dom}(f) \text{ and } x \neq x_0 \\ 0 & x = x_0 \end{cases} \tag{15.32}$$

Solving (15.32) for $f(x)$ yields (15.31) if $x \neq x_0$, and (15.31) is obvious if $x = x_0$. Definition 15.29 implies that $\#\text{-}\lim_{x\to x_0} E(x) = 0$. We defined $E(x_0) = 0$ to make $E(x)$ #-continuous at x_0. Since the right side of (15.32) is #-continuous at x_0, so is the left. This yields the following theorem.

Theorem 15.17. If f is #-differentiable at x_0; then f is #-continuous at x_0.

Theorem 15.18. If f and g are #-differentiable at x_0, then so are $f \pm g$ and fg with

(a) $(f+g)'^{\#}(x_0) = f'^{\#}(x_0) + g'^{\#}(x_0)$;

(b) $(f-g)'^{\#}(x_0) = f'^{\#}(x_0) - g'^{\#}(x_0)$;

(c) $(fg)'^{\#}(x_0) = f'^{\#}(x_0)g(x_0) + f(x_0)g'^{\#}(x_0)$;

(d)The quotient f/g is #-differentiable at x_0 if $g(x_0) \neq 0$ with

$\left(\dfrac{f}{g}\right)'^{\#}(x_0) = \dfrac{f'^{\#}(x_0)g(x_0) - g'^{\#}(x_0)f(x_0)}{[g(x_0)]^2}$.

(e) If $n \in \mathbb{N}^{\#}$ and $f_i, 1 \leq i \leq n$ are #-differentiable at x_0, then so are $Ext\text{-}\sum_{i=1}^{n} f_i$ and

$\left(Ext\text{-}\sum_{i=1}^{n} f_i(x_0)\right)'^{\#} = Ext\text{-}\sum_{i=1}^{n} f_i'^{\#}(x_0)$.

(f) If $n \in \mathbb{N}^{\#}$ and $f^{(n)\#}(x_0), g^{(n)\#}(x_0)$ exist, then so does $(f \times g)^{(n)\#}(x_0)$ and

$(fg)^{(n)\#}(x_0) = Ext\text{-}\sum_{i=0}^{n}\binom{n}{i} f^{(i)\#}(x_0)g^{(n-i)\#}(x_0)$.

Proof. For the statements (a)-(d) the proof is straightforward. For the statements (e) and (f) immediately by hyper infinite induction.

Theorem 15.19. (The Chain Rule) Suppose that g is #-differentiable at x_0 and f is #-differentiable at $g(x_0)$. Then the composite function $h = f \circ g$ defined by $h(x) = f(g(x))$ is #-differentiable at x_0 with $h'^{\#}(x_0) = f'^{\#}(g(x_0))g'^{\#}(x_0)$.

Definition 15.16.If $f(x)$ is defined on $[x_0,b)$, the right-hand derivative of $f(x)$ at x_0 is defined to be

$$f_+^{\prime\#}(x_0) = \#\text{-}\lim_{x\to_\# x_0+} \frac{f(x)-f(x_0)}{x-x_0}, \tag{15.33}$$

if the #-limit exists, while if f is defined on $(a,x_0]$, the left-hand derivative of $f(x)$ at x_0 is defined to be

$$f_-^{\prime\#}(x_0) = \#\text{-}\lim_{x\to_\# x_0-} \frac{f(x)-f(x_0)}{x-x_0}, \tag{15.34}$$

if the #-limit exists.

Remark 15.2. Note that $f(x)$ is #-differentiable at x_0 if and only if $f_+^{\prime\#}(x_0)$ and $f_-^{\prime\#}(x_0)$ exist and are equal, in which case $f^{\prime\#}(x_0) = f_-^{\prime\#}(x_0) = f_+^{\prime\#}(x_0)$.

Definition 15.16$'$(1) We say that f is #-differentiable on the #-closed interval $[a,b]$ if f is #-differentiable on the #-open interval (a,b) and $f_+^{\prime\#}(a)$ and $f_-^{\prime\#}(b)$ both exist.
(2) We say that f is #-continuously #-differentiable on $[a,b]$ if f is #-differentiable on $[a,b]$, $f^{\prime\#}$ is #-continuous on (a,b),$f_+^{\prime\#}(a) = f^{\prime\#}(a+)$, and $f_-^{\prime\#}(b) = f^{\prime\#}(b+)$.

Definition 15.17.We say that $f(x_0)$ is a local extreme value of $f(x)$ if there is a $\delta > 0$, $\delta \approx 0$ such that $f(x)-f(x_0)$ does not change sign on

$$(x_0-\delta x_0+\delta) \cap \mathbf{dom}(f). \tag{15.35}$$

More specifically, $f(x_0)$ is a local maximum value of $f(x)$ if

$$f(x) \leq f(x_0) \tag{15.36}$$

or a local minimum value of $f(x)$ if

$$f(x) \geq f(x_0) \tag{15.37}$$

for all $x \in (x_0-\delta x_0+\delta) \cap \mathbf{dom}(f)$. The point x_0 is called a local extreme point of $f(x)$, or, more specifically, a local maximum or local minimum point of $f(x)$.

Theorem 15.20. If $f(x)$ is #-differentiable at a local extreme point $x_0 \in \mathbf{dom}(f)$ then

$$f^{\prime\#}(x_0) = 0.$$

Proof. We will show that x_0 is not a local extreme point of f if $f^{\prime\#}(x_0) \neq 0$. From Lemma 15.2 we get

$$\frac{f(x)-f(x_0)}{x-x_0} = f^{\prime\#}(x_0) + E(x), \tag{15.37$'$}$$

where $\#\text{-}\lim_{x\to_\# x_0} E(x) = 0$. Therefore, if $f^{\prime\#}(x_0) \neq 0$, there is a $\delta > 0, \delta \approx 0$, such that $|E(x)| < \left|f^{\prime\#}(x_0)\right|$,and the right side of (15.37$'$) must have the same sign as $f^{\prime\#}(x_0)$ for $|x-x_0| < \delta$. Since the same is true of the left side, $f(x)-f(x_0)$ must change sign in every neighborhood of x_0 (since $x-x_0$ does). Therefore, neither (15.36) nor (15.37) can hold for all x in any interval about x_0.

Theorem 15.21. (**Generalized Rolle's Theorem**) Suppose that:
(i) f is #-continuous on the #-closed interval $[a,b]$,
(ii) f is #-differentiable on the #-open interval (a,b),
(iii) the set $V_{a,b} = \{f(x)|x \in [a,b]\}$ is admissible above and below and
(iv) $f(a) = f(b)$.
Then $f^{\prime\#}(c) = 0$ for some $c \in (a,b)$.

Proof.Since f is #-continuous on $[a,b]$ and the set $V_{a,b} = \{f(x)|x \in [a,b]\}$ is admissible above and below, f attains a maximum and a minimum value on $[a,b]$ (Theorem 15.12). If these two extreme values are the same, then f is constant on (a,b), so $f'^{\#}(x) = 0$ for all $x \in [a,b]$. If the extreme values differ, then at least one must be attained at some point c in the #-open interval (a,b), and $f'^{\#}(c) = 0$, by Theorem 15.20.

Theorem 15.22. (**Intermediate Value Theorem for #-Derivatives**) Suppose that:
(i) $f(x)$ is #-differentiable on $[a,b]$,
(ii) the set $V_{a,b}[f] = \{f(x)|x \in [a,b]\}$ is admissible above and below,
(iii) $f'^{\#}(a) \neq f'^{\#}(b)$ and $f'^{\#}(a) < \mu < f'^{\#}(b)$. Then
$f'^{\#}(c) = \mu$ for some $c \in (a,b)$.

Proof.Suppose first that: (1) $f'^{\#}(a) < \mu < f'^{\#}(b)$ and define $g(x) = f(x) - \mu x$.
Then (2) $g'^{\#}(x) = f'^{\#}(x) - \mu, a \leq x \leq b$, and (1) implies that: (3) $g'^{\#}(a) < 0$ and $g'^{\#}(b) > 0$. Notice (ii) implies that $V_{a,b}[g] = \{x(x)|x \in [a,b]\}$ is admissible above and below. Since g is #-continuous on $[a,b]$, g attains a minimum at some point $c \in [a,b]$. Lemma 15.2 and (3) implies that there is a $\delta > 0, \delta \approx 0$, such that $g(x) < g(a), a < x < a+\delta$ and $g(x) < g(b), b-\delta < x < b$, and therefore $c \neq a$ and $c \neq b$. Hence, $a < c < b$, and therefore $g'^{\#}(c) = 0$ by Theorem 11.20.From (2) $f'^{\#}(c) = \mu$.
The proof for the case where $f'^{\#}(b) < \mu < f'^{\#}(a)$ can be obtained by applying this result to $-f(x)$.

Theorem 15.23. (**Generalized Mean Value Theorem**) Assume that:
(i) f and g are #-continuous on the #-closed interval $[a,b]$ and #-differentiable on the open interval (a,b), (ii) the set $V_{a,b}[f]$ and $V_{a,b}[g]$ are admissible above and below,
(iii) let $h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$, the set $V_{a,b}[h]$ admissible above and below, then

$$[g(b) - g(a)]f'^{\#}(c) = [f(b) - f(a)]g'^{\#}(c) \tag{15.38}$$

for some $c \in (a,b)$.

Proof.The function $h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$ is #-continuous on $[a,b]$ and #-differentiable on (a,b), and $h(a) = h(b) = g(b)f(a) - f(b)g(a)$. Note that the set $V_{a,b}[h]$ is admissible above and below. Therefore, Rolle's theorem (Theorem 11.21) implies that $h'^{\#}(c) = 0$ for some $c \in (a,b)$. Since
$h'^{\#}(c) = [g(b) - g(a)]f'^{\#}(c) - [f(b) - f(a)]g'^{\#}(c)$, this implies Eq.(15.38).
The following special case of Theorem 15.23 is important enough to be stated separately.

Theorem 15.24.(**Mean Value Theorem**) Assume that: (i) f is #-continuous on the #-closed interval $[a,b]$, (ii) #-differentiable on the #-open interval (a,b), (iii) the set $V_{a,b}[f]$ is admissible above and below, then

$$f'^{\#}(c) = \frac{f(b) - f(a)}{b - a} \tag{15.39}$$

for some $c \in (a,b)$.

Proof.Apply Theorem 15.24 with $g(x) = x$.
$h(x) = [b-a]f(x) - [f(b) - f(a)]x$
$h(a) = h(b) = bf(a) - f(b)g(a)$

Remark 15.3. Assume that the set $V_{a,b}[f]$ is admissible above and below. If f is #-differentiable on (a,b) and $x_1, x_2 \in (a,b)$ then f is #-continuous on the #-closed interval with endpoints x_1 and x_2 and #-differentiable on its interior. Hence, the mean value theorem (Theorem 15.24) implies that

$$f(x_2) - f(x_1) = f'^{\#}(c)(x_2 - x_1). \qquad (15.39')$$

for some c between x_1 and x_2. (This is true whether $x_1 < x_2$ or $x_2 < x_1$.) The next three theorems follow from (11.39′).

Theorem 15.25. Assume that the set $V_{a,b}[f]$ is admissible above and below. If $f'^{\#}(x) = 0$ for all $x \in (a,b)$, then f is constant on (a,b).

Remark 15.4.

Theorem 15.26. If $f'^{\#}(x)$ exists for all $x \in (a,b)$ and does not change sign on (a,b), then $f(x)$ is monotonic on (a,b) increasing, nondecreasing, decreasing, or nonincreasing as: (i) $f'^{\#}(x) > 0$, (ii) $f'^{\#}(x) \geq 0$, (iii) $f'^{\#}(x) < 0$, (iv) $f'^{\#}(x) \leq 0$, respectively, for all $x \in (a,b)$.

Theorem 15.27. If $\left|f'^{\#}(x)\right| < M, a < x < b$ then

$$|f(x) - f(x')| \leq M|x - x'|, \qquad (15.40)$$

where $x, x' \in (a,b)$.

Definition 15.18.A function that satisfies an inequality like (15.40) for all x and x' in an interval is said to satisfy a Lipschitz condition on the interval.

Theorem 15.28. (Generalized L'Hospital's Rule) Suppose that f and g are #-differentiable and $g'^{\#}$ has no zeros on (a,b). Let $\#\text{-}\lim_{x\to_{\#}b-} f(x) = \#\text{-}\lim_{x\to_{\#}b-} g(x)$ or $\#\text{-}\lim_{x\to_{\#}b-} f(x) = \pm\infty^{\#}$and $\#\text{-}\lim_{x\to_{\#}b-} g(x) = \pm\infty^{\#}$and suppose that

$$\# - \lim_{x\to_{\#}b-} \frac{f'^{\#}(x)}{g'^{\#}(x)} = L, \qquad (15.41)$$

where $L \in \mathbb{R}_c^{\#}$ or $L = \pm\infty^{\#}$. Then

$$\# - \lim_{x\to_{\#}b-} \frac{f(x)}{g(x)} = L, \qquad (15.42)$$

As we saw above in Lemma 15.2 if f is #-differentiable at x_0; then

$$f(x) = f(x_0) + f'^{\#}(x_0)(x - x_0) + E(x)(x - x_0), \qquad (15.43)$$

where $\#\text{-}\lim_{x\to_{\#}x_0} E(x) = 0$. To generalize this result, we first restate it: the polynomial $P_1(x) = f(x_0) + f'^{\#}(x_0)(x - x_0)$ which is of degree ≤ 1 and satisfies $P_1(x_0) = f(x_0)$, $P_1'^{\#}(x) = f'^{\#}(x_0)$, approximates $f(x)$ so well near x_0 such that

$$\#\text{-}\lim_{x\to_{\#}x_0} \frac{f(x) - P_1(x)}{x - x_0} = 0. \qquad (15.44)$$

Now suppose that f has n #-derivatives at x_0 and $P_n(x)$ is the polynomial of degree $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ such that

$$P_n^{(r)\#}(x_0) = f^{(r)\#}(x_0), 0 \leq r \leq n. \qquad (15.45)$$

Since $P_n(x)$ is a polynomial of hyperfinite degree n, it can be written as

$$P_n(x) = Ext\text{-}\sum_{i=0}^{n} a_i(x - x_0)^i \qquad (15.46)$$

where $a_0, \ldots, a_n \in \mathbb{R}_c^{\#}$ are constants. Differentiating (11.46) gives $P_n^{(r)\#}(x_0) = r!a_r$, $0 \leq r \leq n$, so (15.45) determines a_r uniquely as $a_r = f^{(r)\#}(x_0)/r!, 0 \leq r \leq n$. Therefore,

$$P_n(x) = Ext\text{-}\sum_{r=0}^{n} \frac{(x-x_0)^r f^{(r)\#}(x_0)}{r!}. \tag{15.47}$$

We call $P_n(x)$ the n-th Taylor hyper polynomial of $f(x)$ about x_0

Theorem 15.29.If $f^{(n)\#}(x_0)$ exists for some hyper integer $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ and $P_n(x)$ is the n-th Taylor hyper polynomial of f about x_0, then

$$\#\text{-}\lim_{x \to_{\#} x_0} \frac{f(x) - P_n(x)}{(x-x_0)^n} = 0. \tag{15.48}$$

Theorem 15.30. (Generalized Taylor's Theorem) Suppose that $f^{(n+1)\#}(x)$ exists on an #-open interval I about x_0, and let $x \in I$. Then the remainder $R_n(x) = f(x) - P_n(x)$ can be written as

$$R_n(x) = \frac{f^{(n+1)\#}(c)}{(n+1)!}(x-x_0)^n, \tag{15.49}$$

where c depends upon x and is between x and x_0.

15.12.The Riemann integral of a $\mathbb{R}_c^{\#}$-valued external function $f(x)$.

The Riemann integral is defined as #-limit of Riemann hyperfinite sums of functions with respect to tagged partitions of an interval $[a,b] \subset \mathbb{R}_c^{\#}$ A tagged hyperfinite partition P of a closed interval $[a,b]$ on the real line is a hyperfinite sequence

$$a = x_0 \leq t_1 \leq x_1 \leq t_2 \leq x_2 \leq \ldots \leq x_{n-1} \leq t_n \leq x_n = b, \tag{15.50}$$

where $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. This partitions the interval $[a,b]$ into n sub-intervals $[x_{i-1}, x_i]$ indexed by $i \in \mathbb{N}^{\#}$, each of which is "tagged" with a distinguished point $t_i \in [x_{i-1}, x_i]$. Thus, any set of $n+1 \in \mathbb{N}^{\#} \backslash \mathbb{N}$ points satisfying (15.50) defines a partition P of $[a,b]$, which we denote by $P = \{x_0, x_1, \ldots, x_n\}$. A Riemann hyperfinite sum of a function f with respect to such a tagged hyperfinite partition is defined as

$$I_n = \sum_{i=1}^{n} f(t_i)\Delta_i, \tag{15.51}$$

where $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. thus each term of the sum (15.51) is the area of a rectangle with height equal to the function value at the distinguished point of the given sub-interval, and width the same as the width of sub-interval, $\Delta_i = x_i - x_{i-1}$. The mesh(P) of such a tagged partition is the width of the largest sub-interval formed by the partition, $\max_{i=1\ldots n} \Delta_i$.

Definition 15.19. The Riemann integral of a function f over the interval $[a,b]$ is equal to I if for every $\varepsilon > 0, \varepsilon \approx 0$ there exists $\delta > 0, \delta \approx 0$ such that for any partition with distinguished points on $[a,b]$ whose mesh is less than δ.

Upper and Lower Integrals.

Definition 15.20. f is bounded (hyperbounded) on $[a,b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ is a hyperfinite partition of $[a,b]$, let

$$M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x) \tag{15.52}$$

and

$$m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x) \tag{15.53}$$

The upper external hyperfinite sum of f over P is

$$S(P) = Ext\text{-}\sum_{j=1}^{n} M_j(x_j - x_{j-1}) \tag{15.54}$$

and the upper external integral of f over $[a,b]$, denoted by

$$Ext\text{-}\overline{\int_a^b} f(x)d^{\#}x \tag{15.55}$$

is the infimum of all hyperfinite upper sums.
The lower external hyperfinite sum of f over P is

$$s(P) = Ext\text{-}\sum_{j=1}^{n} m_j(x_j - x_{j-1}) \tag{15.56}$$

and the lower external integral of f over $[a,b]$, denoted by

$$Ext\text{-}\underline{\int_a^b} f(x)d^{\#}x. \tag{15.57}$$

is the supremum of all lower hyperfinite sums. If $m \leq f(x) \leq M$ for all $x \in [a,b]$, then

$$m(b-a) \leq s(P) \leq S(P) \leq M(b-a) \tag{15.58}$$

for every hyperfinite partition P; thus, the set of upper hyperfinite sums of f over all partitions P of $[a,b]$ is bounded, as is the set of lower hyperfinite sums. Therefore, Theorems 15.3 and 15.8 imply that: if the quantity (15.55) and (15.57) exist then both are unique, and satisfy the inequalities

$$m(b-a) \leq Ext\text{-}\overline{\int_a^b} f(x)d^{\#}x \leq M(b-a) \tag{15.59}$$

and

$$m(b-a) \leq Ext\text{-}\underline{\int_a^b} f(x)d^{\#}x \leq M(b-a). \tag{15.60}$$

Theorem 15.31. Let f be bounded on $[a,b]$, and let P be a hyperfinite partition of $[a,b]$. Then (i) The upper hyperfinite sum $S(P)$ of f over P is the supremum of the set of all hyperfinite Riemann sums of f over P.
(ii) The lower hyperfinite sum $s(P)$ of f over P is the infimum of the set of all hyperfinite Riemann sums of f over P.

Proof (a) If $P = \{x_0, x_1, \ldots, x_n\}$, then $S(P) = Ext\text{-}\sum_{j=1}^{n} M_j(x_j - x_{j-1})$ where

$M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x)$.
An arbitrary hyperfinite Riemann sum of f over P is of the following form

$$\sigma = Ext\text{-}\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}), \tag{15.61}$$

where $x_{j-1} \leq c_j \leq x_j$. Since $f(c_j) \leq M_j$, it follows that $\sigma \leq S(P)$. Now let $\varepsilon > 0, \varepsilon \approx 0$ and choose $\bar{c}_j \in [x_{j-1}, x_j]$ so that

$$f(\bar{c}_j) > M_j - \frac{\varepsilon}{n(x_j - x_{j-1})}, \tag{15.62}$$

where $1 \leq j \leq n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. The hyperfinite Riemann sum $\bar{\sigma}$ produced in this way is

$$\bar{\sigma} = Ext\text{-}\sum_{j=1}^{n} f(\bar{c}_j)(x_j - x_{j-1}) > Ext\text{-}\sum_{j=1}^{n}\left[M_j - \frac{\varepsilon}{n(x_j - x_{j-1})}\right](x_j - x_{j-1}) = S(P) - \varepsilon. \tag{15.63}$$

Now Theorem 15.3 implies that $S(P)$ is the supremum of the set of hyperfinite

Riemann sums of f over P.

15.12.The Riemann–Stieltjes Integral of a $\mathbb{R}_c^{\#}$-valued external function $f(x)$.

Definition 15.21. Let f and g be defined on $[a,b]$. We say that f is Riemann–Stieltjes integrable with respect to g on $[a,b]$, if there is a number $L \in \mathbb{R}_c^{\#}$ with the following property: For every $\varepsilon > 0, \varepsilon \approx 0$, there is a $\delta > 0, \delta \approx 0$ such that

$$\left| Ext\text{-}\sum_{j=1}^{n} f(c_j)[g(x_j) - g(x_{j-1})] - L \right| < \varepsilon \tag{15.64}$$

provided only that $P = \{x_0, x_1, \ldots, x_n\}, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ is a hyperfinite partition of $[a,b]$ such that $\|P\| < \delta$ and $x_{j-1} \leq c_j \leq x_j, j \in n.$ In this case, we say that L is the external Riemann–Stieltjes integral of f with respect to g over $[a,b]$, and write

$$Ext\text{-}\int_a^b f(x) d^{\#} g(x) = L. \tag{15.65}$$

15.13 Existence of the integral of a $\mathbb{R}_c^{\#}$-valued external function $f(x)$.

Lemma 15.3 Suppose that

$$|f(x)| \leq M, a \leq x \leq b \tag{15.66}$$

and let P' be a hyperfinite partition of $[a,b]$ obtained by adding $r \in \mathbb{N}^{\#} \backslash \mathbb{N}$ points to a partition $P = \{x_0, x_1, \ldots, x_n\}, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ of $[a,b]$. Then

$$S(P) \geq S(P') \geq S(P) - 2Mr\|P\| \tag{15.67}$$

and

$$s(P) \leq s(P') \leq s(P) + 2Mr\|P\|. \tag{15.68}$$

Theorem 15.32. If $f(x)$ is bounded on $[a,b]$, then

$$Ext\text{-}\underline{\int_a^b} f(x) d^{\#} x \leq Ext\text{-}\overline{\int_a^b} f(x) d^{\#} x. \tag{15.69}$$

Theorem 15.33. If f is integrable on $[a,b]$, then

$$Ext\text{-}\underline{\int_a^b} f(x) d^{\#} x = Ext\text{-}\overline{\int_a^b} f(x) d^{\#} x = Ext\text{-}\int_a^b f(x) d^{\#} x. \tag{15.70}$$

Theorem 15.34.If f is bounded (or hyperbounded) on $[a,b]$ and

$$Ext\text{-}\underline{\int_a^b} f(x) d^{\#} x = Ext\text{-}\overline{\int_a^b} f(x) d^{\#} x = L, \tag{15.71}$$

then $f(x)$ is integrable on $[a,b]$ and

$$Ext\text{-}\int_a^b f(x) d^{\#} x = L. \tag{15.72}$$

Theorem 15.35.A bounded (hyperbounded) function f is integrable on $[a,b]$ if and only if

$$Ext\text{-}\underline{\int}_a^b f(x)d^\# x = Ext\text{-}\overline{\int}_a^b f(x)d^\# x. \qquad (15.73)$$

Theorem 15.36.If f is bounded (hyperbounded) on $[a,b]$, then f is integrable on $[a,b]$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is a partition P of $[a,b]$ for which

$$S(P) - s(P) < \varepsilon. \qquad (15.74)$$

Theorem 15.37.If f is #-continuous on $[a,b]$, then f is integrable on $[a,b]$.
Proof.
Theorem 15.38. If f is monotonic on $[a,b]$, then f is integrable on $[a,b]$.
Proof.
Theorem 15.39.(a) If f and g are integrable on $[a,b]$, then so is $f+g$, and

$$Ext\text{-}\int_a^b [f(x)+g(x)]d^\# x = Ext\text{-}\int_a^b f(x)d^\# x + Ext\text{-}\int_a^b g(x)d^\# x \qquad (15.75)$$

(b) If $f_i, 1 \leq i \leq n \in \mathbb{N}^\#$ are integrable on $[a,b]$, then so is $Ext\text{-}\sum_{j=1}^n f_i(x)$, and

$$Ext\text{-}\int_a^b \left[Ext\text{-}\sum_{j=1}^n f_i(x) \right] d^\# x. = Ext\text{-}\sum_{j=1}^n \left(Ext\text{-}\int_a^b f_i(x)d^\# x \right) \qquad (15.76)$$

Proof.
Theorem 15.40.(a) If f is integrable on $[a,b]$ and $c \in \mathbb{R}_c^\#$ is a constant, then cf is integrable on $[a,b]$ and

$$Ext\text{-}\int_a^b c\, f(x)d^\# x = c\left(Ext\text{-}\int_a^b f(x)d^\# x \right). \qquad (15.77)$$

(b) If $f_i, 1 \leq i \leq n \in \mathbb{N}^\#$ are integrable on $[a,b]$ and $c_i \in \mathbb{R}_c^\#$ are constants, then

$$Ext\text{-}\int_a^b \left[Ext\text{-}\sum_{j=1}^n c_i\, f_i(x) \right] d^\# x. = Ext\text{-}\sum_{j=1}^n c_i \left(Ext\text{-}\int_a^b f_i(x)d^\# x \right). \qquad (15.78)$$

Proof.
Theorem 15.41. If f is and g integrable on $[a,b]$ and $f(x) \leq g(x)$ for $x \in [a,b]$,then

$$Ext\text{-}\int_a^b f(x)d^\# x \leq Ext\text{-}\int_a^b g(x)d^\# x. \qquad (15.79)$$

Proof.
Theorem 15.42. If $f(x)$ is integrable on $[a,b]$, then so is $|f(x)|$, and

$$\left| Ext\text{-}\int_a^b f(x)d^\# x \right| \leq Ext\text{-}\int_a^b |f(x)| d^\# x. \qquad (15.80)$$

Theorem 15.43. If $f(x)$ and $g(x)$ are integrable on $[a,b]$, then so is the product $f(x)g(x)$.
Proof.

Theorem 15.44.(First Mean Value Theorem for Integrals) Suppose that $u(x)$ is #-continuous and $v(x)$ is integrable and nonnegative on $[a,b]$. Then

$$Ext\text{-}\int_a^b f(x)v(x)d^{\#}x = v(c)\left(Ext\text{-}\int_a^b f(x)d^{\#}x\right) \tag{15.81}$$

for some $c \in [a,b]$.

Proof.

Theorem 15.45.If $f(x)$ is integrable on $[a,b]$ and $a \leq a_1 < b_1 \leq b$, then $f(x)$ is integrable on $[a_1,b_1]$.

Proof.

Theorem 15.46.If $f(x)$ is integrable on $[a,b]$ and $[b,c]$ then $f(x)$ is integrable on $[a,b]$ and

$$Ext\text{-}\int_a^c f(x)d^{\#}x = Ext\text{-}\int_a^b f(x)d^{\#}x + Ext\text{-}\int_b^c f(x)d^{\#}x. \tag{15.82}$$

Proof.

Theorem 15.47.If $f(x)$ is integrable on $[a,b]$ and $a \leq c \leq b$, then the function $F(x)$ defined by

$$F(x) = Ext\text{-}\int_c^x f(t)d^{\#}t \tag{15.83}$$

satisfies a Lipschitz condition on $[a,b]$, and is therefore #-continuous on $[a,b]$.

Proof.

Theorem 15.48.If $f(x)$ is integrable on $[a,b]$ and $a \leq c \leq b$, then $F(x) = Ext\text{-}\int_c^x f(t)d^{\#}t$ is #-differentiable at any point $x_0 \in [a,b]$, where $f(x)$ is #-continuous, with

$$F'^{\#}(x_0) = f(x_0). \tag{15.84}$$

If $f(x)$ is #-continuous from the right at a, then $F'^{\#}_{+}(a) = f(a)$. If $f(x)$ is #-continuous from the left at b, then $F'^{\#}_{+}(b) = f(b)$.

Proof.We consider the case where $a < x_0 < b$. From the equality

$$\frac{1}{x - x_0}\int_{x_0}^{x} f(x_0)dt'^{\#} = f(x_0)$$

one obtains

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^{x}[f(t) - f(x_0)]dt'^{\#}.$$

From this one obtains

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \leq \frac{1}{|x - x_0|}\left|\int_{x_0}^{x}|f(t) - f(x_0)|dt'^{\#}\right|. \tag{15.84'}$$

Since f is #-continuous at x_0, there is for each $\varepsilon > 0, \varepsilon \approx 0$ a $\delta > 0, \delta \approx 0$ such that $|f(t) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$ and $x < t < x_0$. From (11.84′) we get

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| < \varepsilon \frac{|x - x_0|}{|x - x_0|} = \varepsilon,$$

where $0 < |x - x_0| < \delta$. Therefore $F'^{\#}(x_0) = f(x_0)$.

Theorem 15.49. Suppose that $F(x)$ is #-continuous on the #-closed interval $[a,b]$ and #-differentiable on the #-open interval (a,b), and $f^{Int}(x)$ is integrable on $[a,b]$. Suppose also that $F'^{\#}(x) = f(x), a < x < b$. Then

$$Ext\text{-}\int_a^b f(x)d^{\#}x = F(b) - F(a). \tag{15.85}$$

Proof. Let P by an partition $P = \{x_i\}_{i=0}^n$, $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ of $[a,b]$, then we get

$$F(b) - F(a) = Ext\text{-}\sum_{i=1}^n [F(x_i) - F(x_{i-1})]. \tag{15.85′}$$

From Theorem 2.3.11, there is in each #-open interval .(x_{j-1}, x_j) a point $c_j \in (x_{j-1}, x_j)$ such that $F(x_i) - F(x_{i-1}) = f(c_j)(x_j - x_{j-1})$. Hence, Eq.(11.85′) can be written as

$$F(b) - F(a) = Ext\text{-}\sum_{i=1}^n f(c_j)(x_j - x_{j-1}) = \sigma, \tag{15.85″}$$

where σ is a Riemann sum for f over P. Since f is integrable on $[a,b]$, there is for each $\varepsilon > 0, \varepsilon \approx 0$ a $\delta > 0, \delta \approx 0$ such that

$$\left|\sigma - \int_a^b f(x)d^{\#}x\right| < \varepsilon$$

if $\|P\| < \delta$. Therefore,

$$\left|F(b) - F(a) - \int_a^b f(x)d^{\#}x\right| < \varepsilon$$

for every $\varepsilon > 0, \varepsilon \approx 0$, which implies 15.85

Theorem 15.50. If $f'^{\#}(x)$ is integrable on $[a,b]$, then

$$Ext\text{-}\int_a^b f'^{\#}(x)d^{\#}x = f(b) - f(a). \tag{15.86}$$

Definition 15.22. A function $F(x)$ is an #-antiderivative of $f(x)$ on $[a,b]$ if $F(x)$ is #-continuous on $[a,b]$ and #-differentiable on $[a,b]$, with $F'^{\#}(x) = f(x), a < x < b$.

Theorem 15.50. If $F(x)$ is an #-antiderivative of $f(x)$ on $[a,b]$, then so is $F(x) + c$ for any constant c. Conversely, if $F_1(x)$ and $F_2(x)$ are #-antiderivatives of f on $[a,b]$, then $F_1(x) - F_2(x)$ is constant on $[a,b]$.

Theorem 15.51. (**Fundamental Theorem of Calculus**) If $f(x)$ is #-continuous on $[a,b]$, then $f(x)$ has an #-antiderivative on $[a,b]$. Moreover, if $F(x)$ is any #-antiderivative of f on $[a,b]$, then

$$Ext\text{-}\int_a^b f(x)d^\# x = F(b) - F(a). \quad (15.87)$$

Theorem 15.52. (Integration by Parts) If $u'^\#(x)$ and $v'^\#(x)$ are integrable on $[a,b]$, then

$$Ext\text{-}\int_a^b u(x)v'^\#(x)d^\# x = u(x)v(x)|_a^b - Ext\text{-}\int_a^b u(x)'^\# v(x)d^\# x. \quad (15.88)$$

Theorem 15.53.Suppose that the transformation $x = \varphi(t)$ maps the interval $c \leq t \leq d$ into the interval $a \leq x \leq b$, with $\varphi(c) = \alpha$ and $\varphi(c) = \beta,$ and let $f(x)$ be #-continuous on $[a,b]$. Let $\varphi'^\#(t)$ be integrable on $[c,d]$. Then

$$Ext\text{-}\int_\alpha^\beta f(x)d^\# x = Ext\text{-}\int_c^d f(\varphi(t))\varphi'^\#(t)d^\# t. \quad (15.89)$$

Theorem 15.54. Suppose that $\varphi'^\#(t)$ is integrable and $\varphi(t)$ is monotonic on $[c,d]$, and the transformation $x = \varphi(t)$ maps $[c,d]$ onto $[a,b]$. Let $f(x)$ be bounded (hyperbounded) on $[a,b]$. Then $g(t) = f(\varphi(t))\varphi'^\#(t)$ is integrable on $[c,d]$ if and only if $f(x)$ is integrable over $[a,b]$, and in this case

$$Ext\text{-}\int_a^b f(x)d^\# x = Ext\text{-}\int_c^d f(\varphi(t))\varphi'^\#(t)d^\# t. \quad (15.90)$$

15.14.Improper integrals.

Definition 15.22. We say $f(x)$ is locally integrable on an interval I if $f(x)$ is integrable on every finite or hyperfinite #-closed subinterval of I.

Definition 15.23.If f is locally integrable on $[a,b]$, we define

$$Ext\text{-}\int_a^b f(x)d^\# x = \#\text{-}\lim_{c \to_\# b-}\left(Ext\text{-}\int_a^c f(x)d^\# x\right). \quad (15.91)$$

Remark 11.3.The #-limit in (15.91) always exists if $[a,b]$ is finite or hyperfinite and f is locally integrable and bounded (hyperbounded) on $[a,b]$. In this case, Definitions 15.70 and 15.91 assign the same value to $Ext\text{-}\int_a^b f(x)d^\# x$ no matter how f is defined. However, the #-limit may also exist in cases where $b = \infty^\#$ or $b < \infty^\#$ and f is hyper unbounded as x approaches b from the left.

Definition 15.24.In the cases mentioned above, Definition 15.91 assigns a value to an integral that does not exist in the sense of Definition 15.70, and $Ext\text{-}\int_a^b f(x)d^\# x$ is said to be an improper integral that #-converges to the #-limit in (15.91). We also say in this case that f is integrable on $[a,b]$ and that $Ext\text{-}\int_a^b f(x)d^\# x$ exists.

If the #-limit in (15.91) does not exist (finite or hyperfinite), we say that the improper integral $Ext\text{-}\int_a^b f(x)d^\# x$ #-diverges, and f is nonintegrable on $[a,b)$. In particular, if $\#\text{-}\lim_{c \to_\# b-}\left(Ext\text{-}\int_a^c f(x)d^\# x\right) = \pm\infty^\#$ we say that #-diverges to $\infty^\#,$ and we write

$$Ext\text{-}\int_a^b f(x)d^{\#}x = \infty^{\#} \tag{15.92}$$

or

$$Ext\text{-}\int_a^b f(x)d^{\#}x = -\infty^{\#}, \tag{15.93}$$

whichever the case may be.Similar comments apply to the next two definitions.
Definition 15.25.If $f(x)$ is locally integrable on $(a,b]$, we define

$$Ext\text{-}\int_a^b f(x)d^{\#}x = \#\text{-}\lim_{c\to_{\#}a+}\left(Ext\text{-}\int_c^b f(x)d^{\#}x\right). \tag{15.94}$$

provided that the #-limit exists (finite or hyperfinite). To include the case where $a = -\infty^{\#}$, we adopt the convention that $-\infty^{\#} += -\infty^{\#}$.
Definition 15.26.If $f(x)$ is locally integrable on (a,b), we define

$$Ext\text{-}\int_a^b f(x)d^{\#}x = Ext\text{-}\int_a^{\alpha} f(x)d^{\#}x + Ext\text{-}\int_{\alpha}^b f(x)d^{\#}x, \tag{15.95}$$

where $a < \alpha < b$, provided that both improper integrals on the right exist i.e.,finite or hyperfinite.
Remark 15.4.Note that the existence and value of $Ext\text{-}\int_a^b f(x)d^{\#}x$ according to Definition 15.26 do not depend on the particular choice of $\alpha \in (a,b)$.
Remark 15.5.When we wish to distinguish between improper integrals and integrals in the sense of Definition 11.70, we will call the latter proper integrals.
Theorem 15.55.Suppose that $f_1,f_2,\ldots,f_n$ are locally integrable on $[a,b)$ and that $Ext\text{-}\int_a^b f_1(x)d^{\#}x,\ldots,Ext\text{-}\int_a^b f_n(x)d^{\#}x$ #-converge. Let $c_1,c_2,\ldots,c_n$ be constants.Then $Ext\text{-}\int_a^b\left(Ext\text{-}\sum_{i=1}^n c_i f_i(x)\right)d^{\#}x$ #-converges and

$$Ext\text{-}\int_a^b\left(Ext\text{-}\sum_{i=1}^n c_i f_i(x)\right)d^{\#}x = Ext\text{-}\sum_{i=1}^n c_i\left(Ext\text{-}\int_a^b f_i(x)d^{\#}x\right). \tag{15.96}$$

15.15.Improper integrals of nonnegative functions $f : D \to \mathbb{R}_c^{\#}$.Absolute Integrability.

Theorem 15.56. If $f(x)$ is nonnegative and locally integrable on $[a,b)$, then $Ext\text{-}\int_a^b f(x)d^{\#}x$ converges if the function

$$F(x) = Ext\text{-}\int_a^x f(x)d^{\#}x \tag{15.97}$$

is bounded (hyperbounded) on $[a,b)$, and $Ext\text{-}\int_a^b f(x)d^{\#}x = \infty^{\#}$ if it is not.
Theorem 15.57.(Comparison Test) If f and g are locally integrable on $[a,b)$ and $0 \leq f(x) \leq g(x), 0 \leq x < b$, then (a) $Ext\text{-}\int_a^b f(x)d^{\#}x < \infty^{\#}$ if $Ext\text{-}\int_a^b g(x)d^{\#}x < \infty^{\#}$ and (b) $Ext\text{-}\int_a^b f(x)d^{\#}x = \infty^{\#}$ if $Ext\text{-}\int_a^b g(x)d^{\#}x = \infty^{\#}$.
Theorem 15.58. Suppose that f and g are locally integrable on $[a,b)$, $g(x) > 0$ and

$f(x) \geq 0$ on some subinterval $[a_1, b) \subset [a, b)$, and

$$\#\text{-}\lim_{c \to_\# b-} \frac{f(x)}{g(x)} = M. \tag{15.98}$$

(a) If $0 < M < \infty^\#$, then $Ext\text{-}\int_a^b f(x)d^\# x$ and $Ext\text{-}\int_a^b g(x)d^\# x$ converge or diverge together.

(b) If $M = \infty^\#$ and $Ext\text{-}\int_a^b g(x)d^\# x = \infty^\#$, then $Ext\text{-}\int_a^b f(x)d^\# x = \infty^\#$.

(c) If $M = 0$ and $Ext\text{-}\int_a^b g(x)d^\# x < \infty^\#$, then $Ext\text{-}\int_a^b f(x)d^\# x < \infty^\#$.

Definition 15.27. We say that *f* is absolutely integrable on $[a, b)$ if *f* is locally inte grable on $[a, b)$ and $Ext\text{-}\int_a^b |f(x)|d^\# x < \infty^\#$. In this case we also say that $Ext\text{-}\int_a^b f(x)d^\# x$ #-converges absolutely or is absolutely #-convergent.

Theorem 11.59.If f is locally integrable on $[a, b)$ and $Ext\text{-}\int_a^b |f(x)|d^\# x < \infty^\#$, then $Ext\text{-}\int_a^b f(x)d^\# x$ #-converges: that is, an absolutely #-convergent integral is #-convergent.

Theorem 15.60. (Dirichlet's Test) Suppose that *f* is #-continuous and its #-antiderivative $F(x) = Ext\text{-}\int_a^x f(x)d^\# x$ is bounded (hyperbounded) on $[a, b)$.
Let $g'^\#$ be absolutely integrable on $[a, b)$, and suppose that

$$\#\text{-}\lim_{c \to_\# b-} g(x) = 0. \tag{15.99}$$

Then $Ext\text{-}\int_a^x f(x)g(x)d^\# x$ #-converges.

Theorem 15.61. Suppose that $u(x)$ is #-continuous on $[a, b)$ and $Ext\text{-}\int_a^x u(x)d^\# x$ #-diverges. Let $v(x)$ be positive and #-differentiable on $[a, b)$, and suppose that $\#\text{-}\lim_{c \to_\# b-} v(x) = \infty^\#$ and $v'^\#/v^2$ is absolutely integrable on $[a, b)$. Then $Ext\text{-}\int_a^x u(x)v(x)d^\# x$ #-diverges.

Theorem 15.62.Suppose that $g(x)$ is monotonic on $[a, b)$ and $Ext\text{-}\int_a^b f(x)d^\# x = \infty^\#$.
Let $f(x)$ be locally integrable on $[a, b)$ and

$$Ext\text{-}\int_{x_j}^{x_{j+1}} |f(x)|d^\# x \geq \rho, j \geq 0 \tag{15.100}$$

for some positive ρ where $\{x_j\}_{j \in \mathbb{N}^\#}$ is an increasing hyper infinite sequence of points in $[a, b)$ such that $\#\text{-}\lim_{j \to_\# \infty^\#} x_j = b$ and $x_{j+1}xj \leq M, j \geq 0$, for some M. Then

$$Ext\text{-}\int_a^b |f(x)g(x)|d^\# x = \infty^\#. \tag{15.101}$$

15.16.Change of Variable in an Improper Integral

Theorem 11.63.Suppose that $\varphi(t)$ is monotonic and $\varphi'^\#(t)$ is locally integrable on either of the half-open intervals $I = [c, d)$ or $(c, d]$, and let $x = \varphi(t)$ map I onto either of the half-open intervals $J = [a, b)$ or $J = (a, b]$. Let f be locally integrable on J. Then the improper integrals

$$Ext\text{-}\int_a^b f(x)d^\# x \text{ and } Ext\text{-}\int_a^b f(\varphi)|\varphi'^\#(t)|d^\# t \tag{15.102}$$

#-diverge or #-converge together, in the latter case to the same value. The same conclusion holds if $\varphi(t)$ and $\varphi'^\#(t)$ have the stated properties only on the #-open interval (a, b), the transformation $x = \varphi(t)$ maps (c, d) onto (a, b), and *f* is locally integrable on (a,b).

15.17.Generalized integrability criterion due to Lebesgue.

The main result of this section is an integrability criterion due to Lebesgue that does not require computation, but has to do with how badly #-discontinuous a function may be and still be integrable.

Definition 15.28. If $f(x)$ is bounded (hyperbounded) on $[a,b]$, the oscillation $W_f[a,b]$ of $f(x)$ on $[a,b]$ is defined by

$$W_f[a,b] = \sup_{a \leq x,x' \leq b} |f(x) - f(x')| \tag{15.103}$$

which can also be written as

$$W_f[a,b] = \sup_{a \leq x \leq b} f(x) - \inf_{a \leq x \leq b} f(x). \tag{15.104}$$

Definition 15.29. If $a < x < b$, the oscillation $w_f(x)$ of $f(x)$ at x is defined by

$$w_f(x) = \#\text{-}\lim_{h \to_\# 0+} W_f(x-h,x+h) \tag{15.105}$$

The corresponding definitions for $x = a$ and $x = b$ are

$$w_f(a) = \#\text{-}\lim_{h \to_\# 0+} W_f(a,a+h) \text{ and } w_f(b) = \#\text{-}\lim_{h \to_\# 0+} W_f(b-h,b). \tag{15.106}$$

Note that for a fixed $x \in (a,b)$, $W_f(x-h,x+h)$ is a nonnegative and nondecreasing function of h for $0 < h < \min\{x-a,b-x\}$,therefore, $w_f(x)$ exists and is nonnegative.

Theorem 15.64.Let f be defined on $[a,b]$. Then f is #-continuous at $x_0 \in [a,b]$ if and only if $w_f(x) = 0$; #-continuity at a or b means #-continuity from the right or left, respectively.

Definition 15.30. A subset S of the $\mathbb{R}_c^\#$ is of Lebesgue measure zero if for every $\varepsilon > 0, \varepsilon \approx 0$, there is a hyperfinite or hyper infinite sequence of open intervals $I_1, I_2, \ldots$ such that

$$S \subset \bigcup_j I_j \tag{15.107}$$

and

$$Ext\text{-}\sum_{j=1}^{n} L(I_j) < \varepsilon, n \geq 1. \tag{15.108}$$

Note that any subset of a set of Lebesgue measure zero is also of Lebesgue measure zero.

Example 15.1. Any hyperfinite set $S = \{x_i\}_{i \in \mathbf{n}}, \mathbf{n} \in \mathbb{N}^\# \backslash \mathbb{N}$ is of Lebesgue measure zero, since we can choose #-open intervals $I_1, I_2, \ldots, I_\mathbf{n}$ such that $x_j \in I_j$ and $L(I_j) < \varepsilon/\mathbf{n}$, $1 \leq j \leq \mathbf{n}$.

Definition 15.31.An infinite set $S \subset \mathbb{R}_c^\#$ is hyper denumerable if its members can be listed in a hyper infinite sequence (that is, in a one-to-one correspondence with the positive hyper integers); thus, $S = \{x_i\}_{i \in \mathbb{N}^\#}$.An infinite set that does not have this property is hyper non hyper denumerable.

Example 15.2. Any denumerable set $S = \{x_i\}_{i \in \mathbb{N}^\#}$is of Lebesgue measure zero, since if $\varepsilon > 0, \varepsilon \approx 0$, it is possible to choose open intervals $I_1, I_2, \ldots,$ so that $x_j \in I_j$ and $L(I_j) < 2^{-j}\varepsilon, j \geq 1$. Then (15.108) holds since

$Ext\text{-}\sum_{j=1}^{\mathbf{n}} 2^{-j} = 1 - 2^{-\mathbf{n}} < 1.$

Theorem 15.64. If $w_f(x) < \varepsilon, \varepsilon \approx 0,$ for $a \leq x \leq b,$ then there is a $\delta > 0, \delta \approx 0$ such that $W_f[a,b] < \varepsilon,$ provided that $a_1, b_1 \subset [a,b]$ and $b_1 - a_1 < \delta.$

Theorem 15.65.Let f be bounded (hyperbounded) on $[a,b]$ and define $E_\rho = \{x \in [a,b] | w_f(x) > \rho\}$. Then E_ρ is #-closed; and f is integrable on $[a,b]$ if and only if for every pair of positive numbers ρ and δ, E_ρ can be covered by hyper finitely many open intervals $I_1, I_2, \ldots, I_{\mathbf{p}}, \mathbf{p} \in \mathbb{N}^\# \backslash \mathbb{N}$ such that

$$Ext\text{-}\sum_{j=1}^{\mathbf{p}} L(I_j) < \delta. \tag{15.109}$$

Theorem 15.66. A bounded (hyperbounded) function f is integrable on a finite or hyperfinite interval $[a,b]$ if and only if the set S of #-discontinuities of f in $[a,b]$ is of Lebesgue measure zero.

16.Hyper infinite external sequences and series

16.1.Hyper infinite external sequences

An hyper infinite sequence (or hypersequence) of $\mathbb{R}_c^\#$-real numbers is a $\mathbb{R}_c^\#$-valued function defined on a set of hyperintegers $\{n | n \in \mathbb{N}^\# \wedge n \geq k \in \mathbb{N}\}$. We call the values of the function the terms of the hypersequence. We denote a hypersequence by listing its terms in order; thus, $\{s_n\}_k^{\infty^\#} = \{s_k, s_{k+1}, \ldots\}$. We often write $\{s_n\}_{n \in \mathbb{N}^\#}$or simple $\{s_n\}$ for a shot.

Definition 16.1. A hyper infinite sequence $\{s_n\}_k^{\infty^\#}$ converges to a limit $s \in \mathbb{R}_c^\#$ if for every $\varepsilon \approx 0, \varepsilon > 0$ there is an hyperinteger $N \in \mathbb{N}^\# \backslash \mathbb{N}$ such that

$$|s_n - s| < \varepsilon \text{ if } n \geq N \tag{16.1}$$

In this case we say that $\{s_n\}$ is #-convergent and write

$$\#\text{-}\lim_{n \to_\# \infty^\#} s_n = s. \tag{16.2}$$

A hyper infinite sequence that does not #-converge diverges, or is #-divergent.

Theorem 16.1. The #-limit of a #-convergent hypersequence is unique:

Proof. Suppose that $\#\text{-}\lim_{n \to_\# \infty^\#} s_n = s_1$ and $\#\text{-}\lim_{n \to_\# \infty^\#} s_n = s_2$. We must show that $s = s'$. Let $\varepsilon \approx 0, \varepsilon > 0$. From Definition 10.1, there are hyperintegers N_1 and N_2 such that $|s_n - s_1| < \varepsilon$ if $n \geq N_1$, and $|s_n - s_2| < \varepsilon$ if $n \geq N_2$. These inequalities both hold if $n \geq N = \max(N_1, N_2)$, which implies that: $|s_1 - s_2| < 2\varepsilon$. Since this inequality holds for every $\varepsilon \approx 0, \varepsilon > 0$ and $|s_1 - s_2|$ is independent of ε, we conclude that $|s_1 - s_2| = 0$; that is, $s_1 = s_2$.

Definition 16.2.A hypersequence $\{s_n\}$ is bounded above if there is a hyperreal number $b \in \mathbb{R}_c^\#$ such that $s_n \leq b$ for all $n \in \mathbb{N}^\#$; bounded below if there is a real number $a \in \mathbb{R}_c^\#$ such that $s_n \geq a$ for all $n \in \mathbb{N}^\#$;or bounded if there is a real number $r \in \mathbb{R}_c^\#$ such that $|s_n| \leq r$ for all $n \in \mathbb{N}^\#$.

Theorem 16.2. Any #-convergent hypersequence $\{s_n\}$ is bounded or hyperbounded.

Proof. By taking $\varepsilon = 1$ in Eq.(16.1), we see that if $\#\text{-}\lim_{n \to_\# \infty^\#} s_n = s$, then there is an hyperinteger $N \in \mathbb{N}^\# \backslash \mathbb{N}$ such that $|s_n - s| < 1$ if $n \geq N$. Therefore,

$s_n = |(s_n - s) + s| \leq |s_n - s| + |s| < 1 + |s|$ if $n \geq N$; and
$s_n \leq \max\{(\max_{1 \leq i \leq N-1}\{|s_0|, |s_1|, \ldots, |s_{N-1}|\}), 1 + |s|\}$ for all $n \in \mathbb{N}^\#$, so $\{s_n\}$ is bounded.

Definition 16.3.(Sequences Diverging to $\pm\infty^\#$). We say that

$$\#\text{-}\lim_{n \to_\# \infty^\#} s_n = +\infty^\#$$

if for any hyperreal number $a, s_n > a$ for any $n \geq N \in \mathbb{N}^\# \backslash \mathbb{N}$. Similarly,

$$\#\text{-}\lim_{n \to_\# \infty^\#} s_n = -\infty^\#$$

if for any hyperreal number $a, s_n < a$ for any $n \geq N \in \mathbb{N}^\# \backslash \mathbb{N}$. However, we do not regard $\{s_n\}$ as #-convergent unless $\#\text{-}\lim_{n \to_\# \infty^\#} s_n$
is finite or hyperfinite, as required by Definition 16.1. To emphasize this distinction, we say that $\{s_n\}$ diverges to $\infty^\#(-\infty^\#)$ if $\#\text{-}\lim_{n \to_\# \infty^\#} s_n = \infty^\#(-\infty^\#)$.

Theorem 16.3. Assume that a nonempty set $S \subset \mathbb{R}_c^\#$ of real $\mathbb{R}_c^\#$-numbers has a supremum $\sup(S)$, then $\sup S$ is the unique hyperreal number $\beta \in \mathbb{R}_c^\#$ such that
(a) $x \leq \beta$ for all $x \in S$
(b) if $\varepsilon > 0, \varepsilon \approx 0$ (no matter how infinite small) there is an $x_0 \in S$ such that $x_0 > \beta - \varepsilon$.

Proof. We first show that $\beta = \sup S$ has properties (a) and (b). Since β is an upper bound of S, it must satisfy (a). Since any hyperreal number α less than β can be written as $\alpha = \beta - \varepsilon$ with $\varepsilon = \beta - \alpha > 0$, (b) is just another way of saying that no number less than β is an upper bound of S. Hence, $\beta = \sup S$ satisfies (a) and (b). Now we show that there cannot be more than one hyperreal number with properties (a) and (b).
Suppose that $\beta_1 < \beta_2$ and β_2 has property (b); thus, if $\varepsilon > 0$, there is an $x_0 \in S$ such that $x_0 > \beta_2 - \varepsilon$. Then, by taking $\varepsilon = \beta_2 - \beta_1$, we see that there is an $x_0 \in S$ such that $x_0 > \beta_2 - (\beta_2 - \beta_1) = \beta_1$, so β_1 cannot have property (a). Therefore, there cannot be more than one hyperreal number that satisfies both (a) and (b).

Definition 16.4. A hypersequence $\{s_n\}_{n \in \mathbb{N}^\#}$ is nondecreasing if $s_n \geq s_{n-1}$ for all $n \in \mathbb{N}^\#$, or nonincreasing if $s_n \leq s_{n-1}$ for all $n \in \mathbb{N}^\#$. A monotonic hyper infinite sequence is a hyper infinite sequence that is either nonincreasing or nondecreasing. If $s_n > s_{n-1}$ for all $n \in \mathbb{N}^\#$, then $\{s_n\}_{n \in \mathbb{N}^\#}$ is increasing, while if $s_n < s_{n-1}$ for all $n \in \mathbb{N}^\#$, $\{s_n\}_{n \in \mathbb{N}^\#}$ is decreasing.

Theorem 16.4.(a) If $\{s_n\}_{n \in \mathbb{N}^\#}$ is nondecreasing and there exists $\sup\{s_n | n \in \mathbb{N}^\#\}$ then

$$\#\text{-}\lim_{n \to_\# \infty^\#} s_n = \sup\{s_n | n \in \mathbb{N}^\#\}.$$

(b) If $\{s_n\}_{n \in \mathbb{N}^\#}$ is nonincreasing and there exists $\inf\{s_n | n \in \mathbb{N}^\#\}$ then

$$\#\text{-}\lim_{n \to_\# \infty^\#} s_n = \inf\{s_n | n \in \mathbb{N}^\#\}.$$

Proof. (a) Let $\beta = \sup\{s_n | n \in \mathbb{N}^\#\}$. . If $\beta < +\infty^\#$, Theorem 16.3 implies that if $\varepsilon > 0$ then $\beta - \varepsilon < s_N \leq \beta$ for some hyperinteger $N \in \mathbb{N}^\# \backslash \mathbb{N}$. Since $s_N \leq s_n \leq \beta$ if $n \geq N$, it follows that $\beta - \varepsilon < s_n \leq \beta$ if $n \geq N$. This implies that $|s_n - \beta| < \varepsilon$ if $n \geq N$, so $\#\text{-}\lim_{n \to_\# \infty^\#} s_n = \beta$, by definition of the #-limit. If $\beta = +\infty^\#$
and b is any hyperreal number, then $s_N > b$ for some hyperinteger N. Then $s_n > b$ for $n \geq N$, so $\#\text{-}\lim_{n \to_\# \infty^\#} s_n = +\infty^\#$.

Theorem 16.5.(Generalized Nested Intervals Theorem)
Let $\{I_n\}_{n \in \mathbb{N}^\#} = \{[a_n, b_n]\}_{n \in \mathbb{N}^\#}, [a_n, b_n] \subset \mathbb{R}_c^\#$ be a hyper infinite sequence of #-closed intervals satisfying each of the following conditions:
(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots$,
(ii) $b_n - a_n \to_\# 0$ as $n \to \infty^\#$.

Then $\bigcap_{n=1}^{\infty^{\#}} I_n$ consists of exactly one hyperreal number $x \in \mathbb{R}_c^{\#}$. Moreover both hyper infinite sequences $\{a_n\}$ and $\{b_n\}$ #-converge to x.
Proof. See proof to Theorem 8.11.
Theorem 16.6.(Generalized Bolzano-Weierstrass Theorem) Every bounded (hyperbounded) hyper infinite sequence $\{s_n\}_{n\in\mathbb{N}^{\#}}$ has a #-convergent sub hyper infinite sequence.
Proof.Let $\{s_n\}_{n\in\mathbb{N}^{\#}}$ be a bounded hyper infinite sequence. Then, there exists an interval $[a_1, b_1]$ such that: (i) $a_1, b_1 \in \mathbb{Q}^{\#}$ and (ii) $a_1 \leq s_n \leq b_1$ for all $n \in \mathbb{N}^{\#}$.
Either $\left[a_1, \frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2}, b_1\right]$ contains hyperinfinitely many terms of $\{s_n\}_{n\in\mathbb{N}^{\#}}$.
That is, there exists hyper infinitely many $n \in \mathbb{N}^{\#}$ such that a_n is in $\left[a_1, \frac{a_1+b_1}{2}\right]$, or there exists hyper infinitely many $n \in \mathbb{N}^{\#}$ such that a_n is in $\left[\frac{a_1+b_1}{2}, b_1\right]$.
If $\left[a_1, \frac{a_1+b_1}{2}\right]$ contains hyper infinitely many terms of $\{s_n\}_{n\in\mathbb{N}^{\#}}$, let $[a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$.Otherwise, let $[a_2, b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$.
Either $\left[a_2, \frac{a_2+b_2}{2}\right]$ or $\left[\frac{a_2+b_2}{2}, b_2\right]$ contains hyper infinitely many terms of $\{s_n\}_{n\in\mathbb{N}^{\#}}$.
If $\left[a_2, \frac{a_2+b_2}{2}\right]$ contains hyper infinitely many terms of $\{s_n\}_{n\in\mathbb{N}^{\#}}$, let $[a_3, b_3] = \left[a_2, \frac{a_2+b_2}{2}\right]$.Otherwise, let $[a_3, b_3] = \left[\frac{a_2+b_2}{2}, b_2\right]$.
By hyper infinite induction, we can continue this construction and obtain a hyper infinite sequence of intervals $\{[a_n, b_n]\}_{n\in\mathbb{N}^{\#}}$ such that:
(i) for each $n \in \mathbb{N}^{\#}$, interval $[a_n, b_n]$ contains hyper infinitely many terms of $\{s_n\}_{n\in\mathbb{N}^{\#}}$,
(ii) for each $n \in \mathbb{N}^{\#}, [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and
(iii) for each $n \in \mathbb{N}^{\#}, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.
The nested intervals theorem implies that the intersection $\bigcap\limits_{n\in\mathbb{N}^{\#}} [a_n, b_n]$ of all of the intervals $[a_n, b_n]$ is a single point s. We will now construct a sub hyper infinite sequence of
$\{s_n\}_{n\in\mathbb{N}^{\#}}$ which will #-converge to s.
Since $[a_1, b_1]$ contains hyper infinitely many terms of $\{s_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_1 \in \mathbb{N}^{\#}$ such that s_{k_1} is in $[a_1, b_1]$. Since $[a_2, b_2]$ contains hyper infinitely many terms of $\{s_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_2 \in \mathbb{N}^{\#}, k_2 > k_1$ such that s_{k_2} is in $[a_2, b_2]$. Since $[a_3, b_3]$ contains hyper infinitely many terms of $\{s_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_3 \in \mathbb{N}^{\#}, k_3 > k_2$ such that s_{k_3} is in $[a_3, b_3]$. Continuing this process by hyper infinite induction, we obtain a hyper infinite sequencev$\{s_{k_n}\}_{n\in\mathbb{N}^{\#}}$ such that $s_{k_n} \in [a_n, b_n]$ for each $n \in \mathbb{N}^{\#}$. The hypersequence $\{s_{k_n}\}_{n\in\mathbb{N}^{\#}}$ is a sub hyper infinite sequence of $\{s_n\}_{n\in\mathbb{N}^{\#}}$ since $k_{n+1} > k_n$ for each $n \in \mathbb{N}^{\#}$. Since #-$\lim_{n\to\infty^{\#}} a_n = s$ and #-$\lim_{n\to\infty^{\#}} b_n = s$ and $a_n \leq s_n \leq b_n$ for each $n \in \mathbb{N}^{\#}$, the squeeze theorem implies that that #-$\lim_{n\to\infty^{\#}} s_n = s$.

16.2.Hyper infinite external series of constant.

Definition 16.5. If $\{a_n\}_k^{\infty^{\#}}$ is an hyper infinite external sequence of Cauchy hyperreal numbers, the symbol

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n \tag{16.3}$$

is an hyper infinite series, and a_n is the n-th term of the hyper infinite series.

We say that $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ #-converges to the sum $A \in \mathbb{R}_c^{\#}$, and write

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = A \tag{16.4}$$

if the hyper infinite sequence $\{A_n\}_k^{\infty^{\#}}$ defined by

$$A_n = Ext\text{-}\sum_{i=k}^{i=n} a_n \tag{16.5}$$

$n \in \mathbb{N}^{\#}$, #-converges to A. The hyperf inite sum A_n is the n-th partial sum of $Ext\text{-} \sum_{n=k}^{\infty^{\#}} a_n$

If $\{A_n\}_k^{\infty^{\#}}$ diverges, we say that $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ diverges; in particular, if $\mathbf{lim}_{n\to_{\#}\infty^{\#}} A_n = \infty^{\#}$ or $-\infty^{\#}$, we say that $Ext\text{-} \sum_{n=k}^{\infty^{\#}} a_n$ #-diverges to $\infty^{\#}$ or $-\infty^{\#}$, and write

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#} \text{ or } Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = -\infty^{\#}. \tag{16.6}$$

A divergent hyperinfinite series that does not diverge to $\pm\infty^{\#}$ is said to oscillate, or be oscillatory.

Example 16.1 Consider the hyper infinite series

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}} r^n, -1 < r < 1. \tag{16.7}$$

Here $a_n = r^n, n \geq 0, n \in \mathbb{N}^{\#}$ and

$$A_n = 1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r} \tag{16.8}$$

which #-converges to $1 = 1/(1-r)$ as $n \to \infty^{\#}$; thus, we write

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}} r^n = 1/(1-r), -1 < r < 1.$$

An hyperinfinite series can be viewed as a generalization of a gyperfinite sum $A_N = Ext\text{-}\sum_{n=k}^{N} a_n$ Therefore, #-$\mathbf{lim}_{N\to\infty^{\#}} A_N = A$.

Theorem 16.7. The sum of a #-convergent hyper infinite series is unique.

Theorem 16.8. Let $\sum_{n=k}^{\infty^{\#}} a_n = A$ and $\sum_{n=k}^{\infty^{\#}} b_n = B$ where A and B are finite or hyperfinite.

Then

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} (a_n \pm b_n) = A \pm B \tag{16.9}$$

and

$$Ext\text{-}\sum_{n=k}^{\infty^\#}(c\times a_n) = c\times A \tag{16.10}$$

if $c\in\mathbb{R}_c^\#$ is a constant.

Theorem 16.9. (Cauchy's #-convergence criterion for hyper infinite series) A hyper infinite series $Ext\text{-}\sum_{n=k}^{\infty^\#}a_n$ #-converges if and only if for every $\varepsilon>0,\varepsilon\approx 0$ there is an gyperinteger $N\in\mathbb{N}^\#\backslash\mathbb{N}$ such that

$$\left|Ext\text{-}\sum_n^m a_n\right| < \varepsilon \tag{16.11}$$

if $m\geq n\geq N$.

Corollary 16.1. If $Ext\text{-}\sum_{n=k}^{\infty^\#}a_n$ #-converges; then #-$\mathbf{lim}_{N\to\infty^\#}a_n = 0$.

Corollary 16.2. If $Ext\text{-}\sum_n^{\infty^\#}a_n$ #-converges; then for each $\varepsilon>0,\varepsilon\approx 0$ there is an gyperinteger $K\in\mathbb{N}^\#\backslash\mathbb{N}$ such that $\left|Ext\text{-}\sum_{n=k}^{\infty^\#}a_n\right|<\varepsilon$ if $k\geq K,$ that is

$$\#\text{-}\lim_{k\to\infty^\#}\left(Ext\text{-}\sum_{n=k}^{\infty^\#}a_n\right) = 0. \tag{16.12}$$

16.3.Hyper Infinite Series of Nonnegative Terms.

The theory of series $Ext\text{-}\sum\limits_{n=k}^{\infty^\#}a_n$ with terms that are nonnegative for sufficiently large $n\in\mathbb{N}^\#\backslash\mathbb{N}$ is simpler than the general theory, since such a series either #-converges to a

finite or hyperfinite #-limit or diverges to $\infty^\#$, as the next theorem shows.

Theorem 16.10.If $a_n\geq 0$ for $n\geq k$, then $Ext\text{-}\sum\limits_{n=1}^{\infty^\#}a_n$ #-converges if its partial sums are bounded or hyper bounded, or #-diverges to $\infty^\#$ if they are not. These are the only possibilities and, in either case, $Ext\text{-}\sum\limits_{n=k}^{\infty^\#}a_n = \{A_n|n\geq k\}$,where $A_n = Ext\text{-}\sum\limits_{i=k}^{n}a_i$.

Theorem 16.11.(The Comparison Test) Suppose that

$$0\leq a_n\leq b_n, n\geq k. \tag{16.13}$$

Then

(a) $Ext\text{-}\sum\limits_{n=k}^{\infty^\#}a_n<\infty^\#$ if $Ext\text{-}\sum\limits_{n=k}^{\infty^\#}b_n<\infty^\#$.(b) $Ext\text{-}\sum\limits_{n=k}^{\infty^\#}a_n = \infty^\#$ if $Ext\text{-}\sum\limits_{n=k}^{\infty^\#}b_n = \infty^\#$.

Theorem 16.12.(The Integral Test) Let

$$c_n = f(n), n\geq k, \tag{16.14}$$

where f is positive; nonincreasing; and locally #-integrable on $[k,\infty^\#)$. Then

$$Ext\text{-}\sum_{n=k}^{\infty^\#}a_n < \infty^\# \tag{16.15}$$

if and only if

$$Ext\text{-}\int_k^{\infty^\#}f(x)d^\#x < \infty^\#. \tag{16.16}$$

Example 16.2. The integral test implies that the hyper infinnite series $Ext\text{-}\sum_{n=k}^{\infty^{\#}} n^{-p}$ converge if $p > 1$ and diverge if $0 < p \leq 1$, because the same is true of the integral $Ext\text{-}\int_{a}^{\infty^{\#}} x^{-p}d^{\#}x, a > 1.$

The next theorem is often applicable where the integral test is not.

Theorem 16.13.Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k.$ Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n < \infty^{\#}$ and $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_n}{b_n} < \infty^{\#}$.

(b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n = \infty^{\#}$ and $\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_n}{b_n} > 0.$

Corollary 16.3. Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k$, and $\#\text{-}\lim_{n\to\infty^{\#}} \frac{a_n}{b_n} = L.$ where $0 < L < \infty^{\#}$.Then $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ and $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n$ #-converge or #-diverge together.

Theorem 16.14.Suppose that $a_n > 0, b_n > 0,$ and

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}. \tag{16.17}$$

Then (a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n < \infty^{\#}$. (b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n = \infty^{\#}$.

Theorem 16.15.(The Ratio Test) Suppose that $a_n > 0$ for $n \geq k.$ Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} < 1.$ (b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} > 1.$ If

$$\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} \leq 1 \leq \overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} \tag{16.18}$$

then the test is inconclusive; that is, $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ may #-converge or #-diverge.

Proof.(a) If $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} < 1,$ there is a number *r* such that $0 < r < 1$ and $\frac{a_{n+1}}{a_n} < r$ for $n \in \mathbb{N}^{\#}$ sufficiently large. This can be rewritten as $\frac{a_{n+1}}{a_n} < \frac{r^{n+1}}{r^n}$

Since $Ext\text{-}\sum_{n=k}^{\infty^{\#}} r^n < \infty^{\#}$Theorem 16.14 (a) with $b_n = r^n$ implies that $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$.

(b) If $\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} > 1,$there is a number *r* such that $r > 1$ and $\frac{a_{n+1}}{a_n} > r$ for $n \in \mathbb{N}^{\#}$ sufficiently large. This can be rewritten as $\frac{a_{n+1}}{a_n} > \frac{r^{n+1}}{r^n}$.Since $Ext\text{-}\sum_{n=k}^{\infty^{\#}} r^n = \infty^{\#}$Theorem 16.14 (b) with $b_n = r^n$ implies that $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$.

To see that no conclusion can be drawn if (12.18) holds,consider hyper infinite series

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = Ext\text{-}\sum_{n=k}^{\infty^{\#}} n^{-p}. \tag{16.19}$$

This series #-converges if $p > 1$ or #-diverges if $p \leq 1$, however,

$$\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} = \overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} = 1. \tag{16.20}$$

Corollary 16.4.Suppose that $a_n > 0$ for $n \geq k$ and $\#\text{-}\lim_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} = L$. Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $L < 1$. (b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $L > 1$.

The test is inconclusive if $L = 1$.

Theorem 16.16.(Generalized Raabe's Test) Suppose that $a_n > 0$ for large $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$.
Let $M = \overline{\#\text{-}\lim}_{n\to\infty^{\#}}\left(\frac{a_{n+1}}{a_n} - 1\right)$ and $m = \underline{\#\text{-}\lim}_{n\to\infty^{\#}}\left(\frac{a_{n+1}}{a_n} - 1\right)$. Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $M < -1$. (b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $m > -1$.

The test is inconclusive if $m \leq -1 \leq M$.

Theorem 16.17.(Generalized Cauchy's Root Test)bSuppose that $a_n \geq 0$ for $n \geq k \in \mathbb{N}^{\#}\backslash\mathbb{N}$, then

(a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \sqrt[n]{a_n} < 1$. (b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \sqrt[n]{a_n} > 1$.

The test is inconclusive if $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \sqrt[n]{a_n} = 1$.

16.4.Absolute and Conditional #-Convergence.

Definition 16.6.A series $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ #-converges absolutely, or is absolutely #-convergent if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} |a_n| < \infty^{\#}$.

Theorem 12.18. If $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ #-converges absolutely; then $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ #-converges.

Theorem 12.19. (Dirichlet's Test for Hyper Infinite Series) The hyper infinite series $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n b_n$ is #-converges if the following conditions are satisfied

(i) $\#\text{-}\lim_{n\to\infty^{\#}} a_n = 0$,

(ii)

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} |a_{n+1} - a_n| < \infty^{\#} \tag{16.21}$$

and

(iii) for all $n \geq k$

$$Ext\text{-}\sum_{i=k}^{n} b_n \leq M \tag{16.22}$$

for some constant M.

Proof. Let $B_n, n \geq k$ be the partial sum

$$B_n = Ext\text{-}\sum_{i=k}^{n} b_n \tag{16.23}$$

Let us consider the partial sums $S_n, n \geq k$ of $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n b_n$, where

$$S_n = Ext\text{-}\sum_{i=k}^{n} a_n b_n \tag{16.24}$$

By substituting $b_k = B_k$ and $b_n = B_n - B_{n-1}, n \geq k+1$, into (16.24), we obtain

$$S_n = a_k b_k + Ext\text{-}\sum_{i=k+1}^{n} a_i (B_i - B_{i-1}), \tag{16.25}$$

which we rewrite as

$$S_n = a_n B_n + Ext\text{-}\sum_{i=k}^{n-1} (a_i - a_{i+1}) B_i. \tag{16.26}$$

Now (16.26) can be viewed as

$$S_n = T_{n-1} + a_n B_n, \tag{16.27}$$

where $T_{n-1} = Ext\text{-}\sum_{i=k}^{n-1} (a_i - a_{i+1}) B_i$; that is, $\{T_n\}$ is the hyper infinite sequence of partial sums of the hyper infinite series

$$Ext\text{-}\sum_{i=k}^{\infty^\#} (a_i - a_{i+1}) B_i. \tag{16.28}$$

Since $|(a_i - a_{i+1}) B_i| \leq M |a_i - a_{i+1}|$ from (16.22), the comparison test and (16.21) imply that the series (16.28) #-converges absolutely. Theorem 12.18 now implies that $\{T_n\}_{n \in \mathbb{N}^\#}$ #-converges. Let $T = \#\text{-}\lim_{n \to \infty^\#} T_n$. Since B_n is bounded (hyperbounded) and $\#\text{-}\lim_{n \to \infty^\#} a_n = 0$, we infer from (16.27) that

$$\#\text{-}\lim_{n \to \infty^\#} S_n = \#\text{-}\lim_{n \to \infty^\#} T_{n-1} + \#\text{-}\lim_{n \to \infty^\#} a_n B_n = T. \tag{16.29}$$

Therefore, $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n b_n$ is #-converges.

Corollary 16.4.(Abel's Test for Hyper Infinite Series) The series $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n b_n$ #-converges if $a_{n+1} \leq a_n$ for $n \geq k$, $\#\text{-}\lim_{n \to \infty^\#} a_n = 0$ and $Ext\text{-}\sum_{i=k}^{n} b_n \leq M$, for some constant M.

Corollary 16.5.(Alternating Hyper Infinite Series Test) The series $Ext\text{-}\sum_{n=0}^{\infty^\#} (-1)^n a_n$ #-converges if $0 \leq a_{n+1} \leq a_n$ and $\#\text{-}\lim_{n \to \infty^\#} a_n = 0$.

Proof. Let $b_n = (-1)^n$, then $\{|B_n|\}_{n \in \mathbb{N}^\#}$ is a hyper infinite sequence of zeros and ones and therefore bounded. The conclusion now follows from Abel's test.

16.5.Grouping Terms in a Hyper Infinite Series.

The terms of a hyper finite sum can be grouped arbitrarily by it hyper finite (but not by countable set of it finite subsets) subsets by inserting corresponding parentheses, see Appendix C. According to the next theorem, the same is true of an hyper infinite series that #-converges or #-diverges to $\pm\infty^{\#}$.

Theorem 16.20.Suppose that $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = A,$ where $-\infty^{\#} \leq A \leq \infty^{\#}.$ Let $\{n_j\}_{n\in\mathbb{N}^{\#}}$ be an increasing hyper infite sequence of integers, with $n_1 \geq k.$ Define

$$\begin{gathered} b_1 = Ext\text{-}\sum_{n=k}^{n_1} a_n, \\ b_2 = Ext\text{-}\sum_{n=n_1+1}^{n_2} a_n, \\ \cdots\cdots\cdots\cdots\cdots \\ b_r = Ext\text{-}\sum_{n=n_{r-1}+1}^{n_r} a_n \end{gathered} \tag{16.30}$$

Then

$$Ext\text{-}\sum_{j=1}^{\infty^{\#}} b_{n_j} = A. \tag{12.31}$$

16.6.Rearrangement of hyper infite series.

A hyperfinite sum is not changed by rearranging its terms ,see Appendix C. According to the next theorem, we see that every rearrangement of an absolutely #-convergent hyper infite series has the same sum, but that conditionally #-convergent series fail, spectacularly, to have this property.

Theorem 16.21.If $Ext\text{-}\sum_{n=1}^{\infty^{\#}} b_n$ is a rearrangement of an absolutely #-convergent series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} a_n$ then $Ext\text{-}\sum_{n=1}^{\infty^{\#}} b_n$ also #-converges absolutely, and to the same sum.

Theorem 16.22.If $\{a_{n_i}\}_{i\in\mathbb{N}^{\#}}$ and $\{a_{m_j}\}_{j\in\mathbb{N}^{\#}}$ are respectively the subsequences of all positive and negative terms in a conditionally #-convergent series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} a_n$ then

$$Ext\text{-}\sum_{i=1}^{\infty^{\#}} a_{n_i} = \infty^{\#} \text{ and } Ext\text{-}\sum_{j=1}^{\infty^{\#}} a_{m_j} = -\infty^{\#}. \tag{16.32}$$

Theorem 16.23.Suppose that $Ext\text{-}\sum_{n=1}^{\infty^{\#}} a_n$ is conditionally #-convergent and μ and ν are arbitrarily given in the extended hyperreals; with $\mu \leq \nu.$ Then the terms of

$Ext\text{-}\sum_{n=1}^{\infty^\#} a_n$ can be rearranged to form a series $Ext\text{-}\sum_{n=1}^{\infty^\#} b_n$ with partial sums $B_n = Ext\text{-}\sum_{i=1}^{n} b_i$ such that

$$\overline{\lim_{n\to_\# \infty^\#}}\ B_n = \nu \text{ and } \underline{\lim_{n\to_\# \infty^\#}}\ B_n = \mu. \tag{16.33}$$

16.7.Multiplication of hyper infite Series.

Given two hyper infite series $Ext\text{-}\sum_{n=0}^{\infty^\#} a_n$ and $Ext\text{-}\sum_{n=0}^{\infty^\#} b_n$ we can arrange all possible products $a_i b_j,\ i,j \geq 0$ in a two-dimensional array:

$$\begin{array}{ccccc} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \end{array} \tag{16.34}$$

where the subscript on a is constant in each row and the subscript on b is constant in each column. Any sensible definition of the product

$$\left(Ext\text{-}\sum_{n=0}^{\infty^\#} a_n\right)\left(Ext\text{-}\sum_{n=0}^{\infty^\#} b_n\right) \tag{12.35}$$

clearly must involve every product in this array exactly once; thus, we might define the product of the two series to be the series $Ext\text{-}\sum_{n=0}^{\infty^\#} p_n$, where $\{p_n\}_{i\in\mathbb{N}^\#}$ is a hyper infite sequence obtained by ordering the products in (12.34) according to some method that chooses every product exactly once.

Theorem 16.24.Let $Ext\text{-}\sum_{n=0}^{\infty^\#} a_n = A$ and $Ext\text{-}\sum_{n=0}^{\infty^\#} b_n = B,$ where A and B are finite or hyperfinite, and at least one term of each series is nonzero. Then $Ext\text{-}\sum_{n=0}^{\infty^\#} p_n = A \times B$ for every hyper infinite sequence $\{p_n\}_{i\in\mathbb{N}^\#}$ obtained by ordering the products in (16.34) if and only if $Ext\text{-}\sum_{n=0}^{\infty^\#} a_n$ and $Ext\text{-}\sum_{n=0}^{\infty^\#} b_n$ #-converge absolutely:

Moreover, in this case, $Ext\text{-}\sum_{n=0}^{\infty^\#} p_n$ #-converges absolutely.

Definition 16.7.The Cauchy product of $Ext\text{-}\sum_{n=0}^{\infty^\#} a_n$ and $Ext\text{-}\sum_{n=0}^{\infty^\#} b_n$ is $Ext\text{-}\sum_{n=0}^{\infty^\#} c_n,$

where

$$c_n = Ext\text{-}\sum_{j=0}^{n} a_j b_{n-j}. \tag{16.36}$$

Thus, c_n is the external sum of all products $a_l b_k$, where $i \geq 0, j \geq 0$, and $i + j = n$; thus,

$$c_n = Ext\text{-}\sum_{j=0}^{n} a_j b_{n-j} = Ext\text{-}\sum_{j=0}^{n} b_j a_{n-j}. \tag{16.37}$$

Theorem 16.25. If $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n$ and $Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n$ #-converge absolutely to sums A and B, then the Cauchy product $Ext\text{-}\sum_{j=0}^{n} a_j b_{n-j}$ #-converges absolutely to AB.

Theorem 16.26. Let $f(\alpha) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{\alpha^n}{n!}$ and $f(\beta) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{\beta^n}{n!}$, then

$$f(\alpha)f(\beta) = f(\alpha + \beta). \tag{16.38}$$

Proof. From Eq.(16.37) we obtain

$$c_n = Ext\text{-}\sum_{n=0}^{m} \frac{\alpha^{n-m}\beta^m}{(n-m)!m!} = \frac{1}{n!}\left(Ext\text{-}\sum_{n=0}^{m} \binom{n}{m} \alpha^{n-m}\beta^m \right) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{(\alpha+\beta)^n}{n!} \tag{16.39}$$

Thus

$$f(\alpha)f(\beta) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{(\alpha+\beta)^n}{n!} = f(\alpha + \beta). \tag{16.40}$$

16.8.Double Hyper Infinite Sequences.

Definition 16.8. A double hyper infinite sequence of hyperreal numbers $\mathbb{R}_c^{\#}$ (complex numbers $\mathbb{C}_c^{\#} = \mathbb{R}_c^{\#} + i\mathbb{R}_c^{\#}$) is a $\mathbb{R}_c^{\#}$-valued ($\mathbb{C}_c^{\#}$-valued) function $s : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{R}_c^{\#}$ or $s : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{C}_c^{\#}$. We shall use the notation $\{s_{n,m}\}_{n,m \in \mathbb{N}^{\#}}$ or simply s_{nm}.

Definition 16.9. We say that a double hyper infinite sequence $s_{n,m}$ #-converges to $a \in \mathbb{C}_c^{\#}$ and we write $\#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} s_{n,m} = a$, if the following condition is satisfied: for every $\varepsilon > 0, \varepsilon \approx 0$, there exists $N \in \mathbb{N}^{\#}$ such that $|s_{n,m} - a| < \varepsilon$ if $n, m \geq N$.

Theorem 16.27. (Uniqueness of Double #-Limits). A double hyper infinite $\mathbb{C}_c^{\#}$-valued sequence has at most one #-limit.

Definition 16.10. A double hyper infinite sequence $s_{n,m}$ is called bounded (hyper bounded) if there exists finite (hyperfinite) number $M \in \mathbb{R}_c^{\#}$, $M > 0$ such that $|s_{n,m}| \leq M, \forall n, m \in \mathbb{N}^{\#}$.

Theorem16.28. A #-convergent double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence is bounded or hyper bounded.

Definition 16.11.A double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence $s_{n,m}$ is called a Cauchy sequence if and only if for every $\varepsilon > 0, \varepsilon \approx 0$, there exists a hypernatural number $N \in \mathbb{N}^{\#} \backslash \mathbb{N}$ such that $|s_{p,q} - s_{n,m}| < \varepsilon$, $\forall p(p \geq n \geq N)$ and $\forall q(q \geq m \geq N)$.

Theorem 16.29.(Cauchy Convergence Criterion for Double hyper infinite Sequences). A double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence $s_{n,m}, n, m \in \mathbb{N}^{\#}$ #-converges if and only if it is a Cauchy sequence.

Definition 16.12.Let $s_{n,m}$ be a double $\mathbb{R}_c^{\#}$-valued hyper infinite sequence.
(i) If $s_{n,m} \leq s_{j,k},\ \forall n\forall j\forall m\forall k(n \leq j \wedge m \leq k),\ n,m,j,k \in \mathbb{N}^{\#},$ we say the sequence $s_{n,m}$ is increasing.
(ii) $s_{n,m} \geq s_{j,k},\ \forall n\forall j\forall m\forall k(n \leq j \wedge m \leq k),\ n,m,j,k \in \mathbb{N}^{\#},$ we say the sequence $s_{n,m}$ is decreasing.
(ii) If $s_{n,m}$ is either increasing or decreasing, then we say it is monotone.
Definition 16.13.For a double sequence $s_{n,m}$, the #-limits
$\#\text{-}\lim_{n\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{n,m}\right)$
and $\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}\right)$ are called repeated #-limits.
Theorem 16.30.Let $\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} s_{n,m} = a.$Then $\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}\right) = a$ if and only if $\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}$ exists for each $m \in \mathbb{N}^{\#}$.
Theorem 16.31. Let $\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} s_{n,m} = a.$Then the repeated #-limits $\#\text{-}\lim_{n\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{n,m}\right)$ and $\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}\right)$ exist and both are equal to a if and only if (i) $\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}$ exists for each $m \in \mathbb{N}^{\#}$,and (ii) $\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{n,m}$ exists for each $n \in \mathbb{N}^{\#}$.
Theorem 16.32.If $s_{n,m}$ is a double sequence such that the repeated #-limit $\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}\right) = a$ and the #-limit $\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}$ exists uniformly in $m \in \mathbb{N}^{\#}$,then the double #-limit $\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} s_{n,m} = a.$
Theorem 16.33.(Monotone Convergence Theorem). A monotone double $\mathbb{R}_c^{\#}$-valued hyper infinite sequence is #-convergent if and only if it is bounded (hyper bounded).
Further: (i) If $s_{n,m}$ is increasing and bounded (hyper bounded) above, then
$\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}\right) = \#\text{-}\lim_{n\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{n,m}\right) = \#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} s_{n,m}.$
(ii) If $s_{n,m}$ is decreasing and bounded (hyper bounded) below, then
$\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}\right) = \#\text{-}\lim_{n\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{n,m}\right) = \#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} s_{n,m}.$
Theorem16.34.(The Sandwich Theorem). Suppose that $x_{n,m}, s_{n,m},$ and $y_{n,m}$ are double $\mathbb{R}_c^{\#}$-valued hyper infinite sequences such that
$x_{n,m} \leq s_{n,m} \leq y_{n,m}, \forall n,m \in \mathbb{N}^{\#}$,and $\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} x_{n,m} = \#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} y_{n,m}.$
Then $s_{n,m}$ is #-convergent and $\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} x_{n,m} = \#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} y_{n,m} = \#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} s_{n,m}.$
Definition 16.14. Let $s_{n,m}$ be a double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence and let $(k_1,r_1) < (k_2,r_2) <\ldots< (k_n,r_n) <\ldots$ be a strictly increasing sequences of pairs of hypernatural numbers. Then the sequence s_{k_n,r_m} is called a subsequence of $s_{n,m}$.
Theorem16.35.If a double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence $s_{n,m}$ #-converges to number $a \in \mathbb{C}_c^{\#}$, then any hyper infinite subsequence of $s_{n,m}$ also #-converges to a.
Theorem16.36.If the repeated #-limits of a double sequence $s_{n,m}$ exist and satisfy $\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}\right) = \#\text{-}\lim_{n\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{n,m}\right) = a,$then the

repeted #-limits for any subsequence s_{p_n,q_m} exist and satisfy
$\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{p_n,q_m}\right) = \#\text{-}\lim_{n\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{p_n,q_m}\right) = a.$
Theorem16.37. Every double $\mathbb{R}_c^{\#}$-valued hyper infinite sequence has a monotone hyper infinite subsequence.
Theorem16.38.(Bolzano-Weierstrass Theorem). A bounded (hyper bounded) double $\mathbb{R}_c^{\#}$-valued hyper infinite sequence sequence has a #-convergent monotone subsequence.

16.9.External Double Hyper Infinite Series.

Definition 16.15. Let $z : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{C}_c^{\#}$ be external hyper infinite double sequence of complex numbers $\mathbb{C}_c^{\#}$ and let $s_{n,m}$ be the double hyper infinite sequence defined by the equation

$$s_{n,m} = Ext\text{-}\sum_{i=1}^{n}\left(Ext\text{-}\sum_{j=1}^{m} z_{i,j}\right). \quad (16.41)$$

The pair (z,s) is called a double hyper infinite series and is denoted by the symbol

$$Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m} \quad (16.42)$$

or, more briefly by $Ext\text{-}\sum_{n,m=1}^{\infty^{\#}} z_{n,m}$. Each number $z_{n,m}$ is called a term of the double series and each $s_{n,m}$ is called a partial sum.

Definition 16.16.We say that the double series $Ext\text{-}\sum_{n,m=1}^{\infty^{\#}} z_{n,m}$ is #-convergent to the sum s if $\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} s_{n,m} = s$. If no such #-limit exists, we say that the double series $Ext\text{-}\sum_{n,m=1}^{\infty^{\#}} z_{n,m}$ is #-divergent.

Definition 16.17. The hyper infinite series

$$Ext\text{-}\sum_{n=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{m=1}^{\infty^{\#}} z_{n,m}\right) \quad (16.43)$$

and

$$Ext\text{-}\sum_{m=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{n=1}^{\infty^{\#}} z_{n,m}\right) \quad (16.44)$$

are called repeated hyper infinite series.

Theorem 16.38.If the double hyper infinite series $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}$ is #-convergent, then

$$\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} z_{n,m} = 0. \quad (12.45)$$

Theorem 16.39.(Cauchy #-Convergence Criterion for Double hyper infinite Series.)

A double hyper infinite series $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}$ #-converges if and only if its sequence of partial sums $s_{n,m}$ is Cauchy.

Theorem16.40.If the double series $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}$ #-converges to s_1 and $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} u_{n,m}$ #-converges to s_2, then: (i) $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m} + Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} u_{n,m} = s_1 + s_2$.

(ii) $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} c \times z_{n,m} = c \times \left(Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}\right)$.

Theorem16.41.Suppose that the double series $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}$ is #-convergent, with sum s. Then the repeated series $Ext\text{-}\sum_{n=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{m=1}^{\infty^{\#}} z_{n,m}\right)$ and $Ext\text{-}\sum_{m=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{n=1}^{\infty^{\#}} z_{n,m}\right)$ are both #-convergent with sum s if and only if for every $m \in \mathbb{N}^{\#}$, the series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} z_{n,m}$ is #-convergent, and for every $n \in \mathbb{N}^{\#}$, the series $Ext\text{-}\sum_{m=1}^{\infty^{\#}} z_{n,m}$ is #-convergent.

16.10.Interchanging the order of summation of hyper infinite sum.

Theorem 12..Assum that

$$Ext\text{-}\sum_{i=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{k=1}^{\infty^{\#}}|a_{jk}|\right) < \infty^{\#}. \tag{16.46}$$

Then

$$Ext\text{-}\sum_{i=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{k=1}^{\infty^{\#}}|a_{jk}|\right) = Ext\text{-}\sum_{k=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{j=1}^{\infty^{\#}}|a_{jk}|\right) \tag{16.47}$$

17.Hyper infinite sequences and series of $\mathbb{R}_c^{\#}$-valued functions.

17.1.Uniform #-Convergence

If $f_1,\ldots,f_k,f_{k+1},\ldots,f_n,\ldots,n \in \mathbb{N}^{\#}$ are $\mathbb{R}_c^{\#}$-valued functions defined on a subset $D \subset \mathbb{R}_c^{\#}$ of the hyperreals, we say that $\{f_n(x)\}_{n\in\mathbb{N}^{\#}}$ is an hyper infinite sequence of functions on D. If the sequence of values $\{f_n(x)\}_{n\in\mathbb{N}^{\#}}$ #-converges for each x in some subset S of D, then $\{f_n\}_{n\in\mathbb{N}^{\#}}$ defines a #-limit function on S. The formal definition is as follows.

Definition 17.1. Suppose that $\{f_n(x)\}_{n\in\mathbb{N}^{\#}}$ is a hyper infinite sequence of functions on $D \subset \mathbb{R}_c^{\#}$ and the hyper infinite sequence of values $\{f_n(x)\}_{n\in\mathbb{N}^{\#}}$ #-converges for each x in some subset S of D. Then we say that $\{f_n\}_{n\in\mathbb{N}^{\#}}$ #-converges pointwise on S to the #-limit function f, defined by

$$f(x) = \#\text{-}\lim_{n\to\infty^{\#}} f_n(x), x \in S. \tag{17.1}$$

Definition 17.2.Let f be a function defined on $S \subset \mathbb{R}_c^{\#}$ and there exist $\sup_{x\in S}|f(x)|$, then we set

$$\|f\|_S = \sup_{x\in S}|f(x)|. \tag{17.2}$$

Lemma 17.1. If g and h are defined on S, then $\|g+h\|_S \leq \|g\|_S + \|h\|_S$ and $\|g \times h\|_S \leq \|g\|_S \times \|h\|_S$. Moroever if either g or h is bounded on S, then $\|g-h\|_S \geq \|g\|_S - \|h\|_S$.

Definition 17.2. A hyper infinite sequence $\{f_n\}_{n\in\mathbb{N}^\#}$ of functions defined on a set S #-converges uniformly to the #-limit function f on S if $\#\text{-}\lim_{n\to\infty^\#}\|f_n - f\|_S = 0$.
Thus, f_n #-converges uniformly to f on S if for each $\varepsilon > 0, \varepsilon \approx 0,$ there is an integer $N \in \mathbb{N}^\#\backslash\mathbb{N}$ such that

$$\|f_n - f\| < \varepsilon \text{ if } n \geq N. \tag{17.3}$$

Theorem 17.1. Let $f_n, n \in \mathbb{N}^\#$ be hyper infinite sequence defined on S. Then
(a) f_n #-converges pointwise to f on S if and only if there is, for each $\varepsilon > 0, \varepsilon \approx 0,$ and $x \in S,$an integer $N \in \mathbb{N}^\#\backslash\mathbb{N}$ which may depend on x as well as ε such that $|f_n(x) - f(x)| < \varepsilon$ if $n \geq N$;
(b) f_n #-converges uniformly to f on S if and only if there is for each $\varepsilon > 0, \varepsilon \approx 0,$ an integer $N \in \mathbb{N}^\#\backslash\mathbb{N}$ which depends only on and not on any particular x in S such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$ if $n \geq N$.

Theorem 17.2. If f_n #-converges uniformly to f on S, then f_n #-converges pointwise to f on S. The converse is false; that is pointwise #-convergence does not imply uniform #-convergence.

Theorem 17.3. (Cauchy's Uniform #-Convergence Criterion) A sequence of functions f_n #-converges uniformly on a set S if and only if for each $\varepsilon > 0, \varepsilon \approx 0,$ there is an integer $N \in \mathbb{N}^\#\backslash\mathbb{N}$ such that

$$\|f_n - f_m\|_S < \varepsilon \text{ if } n,m \geq N. \tag{17.4}$$

Theorem 17.4. If f_n #-converges uniformly to f on S and each f_n is #-continuous at a point $x_0 \in S$; then so is f. Similar statements hold for #-continuity from the right and left.

Theorem 17.5. Suppose that f_n #-converges uniformly to f on $S = [a,b]$. Assume that f and all f_n are #-integrable on $[a,b]$. Then

$$Ext\text{-}\int_a^b f(x)d^\#x = \#\text{-}\lim_{n\to\infty^\#}\left(Ext\text{-}\int_a^b f_n(x)d^\#x\right). \tag{17.5}$$

Proof. Since

$$\left|Ext\text{-}\int_a^b f(x)d^\#x - Ext\text{-}\int_a^b f_n(x)d^\#x\right| \leq Ext\text{-}\int_a^b |f(x) - f_n(x)|d^\#x \leq (b-a)\|f - f_n\|_S \tag{17.6}$$

and $\#\text{-}\lim_{n\to\infty^\#}\|f - f_n\|_S = 0,$the Eq.(17.5) follows.

Theorem 17.6. Suppose that $f_n(x)$ #-converges pointwise to f and each $f_n(x)$ is #-integrable on $[a,b]$.Then
(a) If the #-convergence is uniform, then $f(x)$ is #-integrable on $[a,b]$ and (13.5) holds.
(b) If the sequence $\|f_n\|_{[a,b]}$ is bounded and $f(x)$ is #-integrable on $[a,b]$, then (13.5) holds.

Theorem 17.7. Suppose that $f_n^{\prime\#}(x)$ is #-continuous on $[a,b]$ for all $n \in \mathbb{N}^\#$ and $\left\{f_n^{\prime\#}\right\}_{n\in\mathbb{N}^\#}$ #-converges uniformly on $[a,b]$ Suppose also that $\left\{f_n(x_0)\right\}_{n\in\mathbb{N}^\#}$ #-converges for some $x_0 \in [a,b]$.Then $\left\{f_n(x)\right\}_{n\in\mathbb{N}^\#}$ #-converges uniformly on $[a,b]$ to a #-differentiable #-limit function $f(x)$ and

$$f'^{\#}(x) = \#\text{-}\lim_{n\to\infty^\#} f_n'^{\#}(x), x \in (a,b), \tag{17.7}$$

while

$$f_+'^{\#}(a) = \#\text{-}\lim_{n\to\infty^\#} f_n'^{\#}(a+), f_-'^{\#}(b) = \#\text{-}\lim_{n\to\infty^\#} f_n'^{\#}(b-). \tag{17.8}$$

17.2.Hyper Infinite Series of Functions.

Definition 17.3. If $\{f_j(x)\}_{j=k}^{\infty^\#}$ is a hyper infinite sequence of $\mathbb{R}_c^\#$-valued functions defined on a set $D \subset \mathbb{R}_c^\#$ of hyperreals, then

$$Ext\text{-}\sum_{j=k}^{\infty^\#} f_j(x) \tag{17.9}$$

is an hyper infinite series of functions on D. The partial sums of , $Ext\text{-}\sum_{j=k}^{\infty^\#} f_j(x)$ are defined by

$$F_n(x) = Ext\text{-}\sum_{j=k}^{n} f_j(x), n \in \mathbb{N}^\#. \tag{17.10}$$

If $F_n(x)$ #-converges pointwise to a function F on a subset $S \subset D$, we say that $Ext\text{-}\sum_{j=k}^{n} f_j(x)$ #-converges pointwise to the sum $F(x)$ on S, and write

$$F(x) = Ext\text{-}\sum_{j=k}^{\infty^\#} f_j(x). \tag{17.11}$$

If $F_n(x)$ #-converges uniformly to $F(x)$ on S, we say that $Ext\text{-}\sum_{j=k}^{n} f_j(x)$ #-converges uniformly to $F(x)$ on S.

Example 17.1. The functions $f_j(x) = x^j, j \in \mathbb{N}^\#$ define the hyper infinite series $Ext\text{-}\sum_{j=0}^{\infty^\#} x^j$ on $D = (-\infty^\#, \infty^\#)$. The n-th partial sum of the series is $F_n(x) = Ext\text{-}\sum_{j=0}^{n} x^j$, or, in closed form,

$$F_n(x) = \begin{cases} \dfrac{1-x^n}{1-x} & x \neq 1 \\ n+1 & x = 1 \end{cases} \tag{17.12}$$

Therefore $\{F_n\}$ #-converges pointwise toif $|x| < 1$ and #-diverges if $|x| > 1$, hence, we get $F(x) = Ext\text{-}\sum_{j=0}^{\infty^\#} x^j = (1-x)^{-1}, -1 < x < 1.$ Since the difference $F(x) - F_n(x) =$ $= \dfrac{x^n}{1-x}$ can be made arbitrarily infinite large by taking x infinite close to 1, $\|F - F_n\|_{(-1,1)} = \infty^\#$ so the #-convergence is not uniform on $(-1,1)$. Neither is it uniform on any interval $(-1,r]$ with $1 < r < 1$, since $\|F - F_n\|_{[-r,r]} = r^n/(1-r)$ and $\#\text{-}\lim_{n\to\infty^\#} r^n = \infty^\#$. Put another way, the series #-converges uniformly on #-closed subsets of $(-1,1)$.

Theorem 17.8.(Cauchy's Uniform #-Convergence Criterion) A hyper infinite series

$Ext\text{-}\sum_{i=0}^{\infty^\#} f_i(x)$ #-converges uniformly on a set $S \subset \mathbb{R}_c^\#$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is an hyperinteger $N \in \mathbb{N}^\#$ such that

$$\left\| Ext\text{-}\sum_{n}^{m} f_i(x) \right\|_S < \varepsilon \tag{17.13}$$

if $m \geq n \geq N$.

Corollary 17.1. If $Ext\text{-}\sum_{i=0}^{\infty^\#} f_i(x)$ #-converges uniformly on S, then $\#\text{-}\lim_{n\to\infty^\#} \|f_n\|_S = 0$.

Theorem 17.9.(Weierstrass's Test) The hyper infinite series $Ext\text{-}\sum_{i=0}^{\infty^\#} f_i(x)$ #-converges uniformly on S if

$$\|f_n\|_S \leq M_n, n \geq k, \tag{17.14}$$

where $Ext\text{-}\sum_{n=k}^{\infty^\#} M_n < \infty^\#$.

Theorem 17.10.(Dirichlet's Test for Uniform #-Convergence) The hyper infinite series $Ext\text{-}\sum_{n=k}^{\infty^\#} f_n(x)g_n(x)$ #-converges uniformly on S if f_n #-converges uniformly to zero on S, $Ext\text{-}\sum_{n=k}^{\infty^\#}(f_{n+1}(x) - f_n(x))$ #-converges absolutely uniformly on S, and

$$\left\| Ext\text{-}\sum_{i=k}^{n} g_i(x) \right\|_S \leq M, \tag{17.14}$$

where $n \geq k$, for some constant M.

Corollary 17.2.The hyper infinite series $Ext\text{-}\sum_{n=k}^{\infty^\#} f_n(x)g_n(x)$ #-converges uniformly on S if $f_{n+1}(x) \leq f_n(x), x \in S, n \geq k, \{f_n\}$ #-converges uniformly to zero on S, and

$$\left\| Ext\text{-}\sum_{i=k}^{n} g_i(x) \right\|_S \leq M, \tag{17.15}$$

for some constant M.

17.3.#-Continuity, #-Differentiability, and Integrability of hyper infinite Series.

Theorem 17.11.If $Ext\text{-}\sum_{n=k}^{\infty^\#} f_n(x)$ #-converges uniformly to $F(x)$ on S and each f_n is #-continuous at a point x_0 in S, then so is $F(x)$. Similar statements hold for #-continuity from the right and left.

Theorem 17.12.Suppose that $Ext\text{-}\sum_{n=k}^{\infty^\#} f_n(x)$ #-converges uniformly to $F(x)$ on $S = [a,b]$ Assume that $F(x)$ and $f_n(x), n \geq k$, are integrable on $[a,b]$. Then

$$Ext\text{-}\int_a^b F(x)d^\# x = Ext\text{-}\sum_{n=k}^{\infty^\#}\left(Ext\text{-}\int_a^b f_n(x)d^\# x \right). \tag{17.16}$$

Theorem 17.13.Suppose that f_n is #-continuously #-differentiable on $[a,b]$ for each $n \geq k,\ Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n(x_0)$ #-converges for some $x_0 \in [a,b]$ and $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n'^{\#}(x)$ #-converges uniformly on $[a,b]$. $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n(x)$ #-converges uniformly on $[a,b]$ to a #-differentiable function $F(x)$, and $F'^{\#}(x) = Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n'^{\#}(x), a < x < b,$ while $F'^{\#}(a+) = Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n'^{\#}(a+)$ and $F'^{\#}(b-) = Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n'^{\#}(b-)$.

18.Hyper Infinite Power Series.

18.1.The convergence properties of hyper infinite power series.

Definition 18.1. A hyper infinite series of the form

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n (x - x_0)^n \tag{18.1}$$

where $x_0 \in \mathbb{R}_c^{\#}$ and $a_n \in \mathbb{R}_c^{\#}, n \in \mathbb{N}^{\#}$ is called a hyper infinite power series in $(x - x_0)$.
The following theorem summarizes the #-convergence properties of hyper infinite power series.

Theorem 18.1.In connection with the hyper infinite power series (14.1) define R in the extended hyperreals by

$$\frac{1}{R} = \overline{\#\text{-}\lim_{n\to\infty^{\#}}} \sqrt[n]{|a_n|} \tag{18.2}$$

In particular, $R = 0$ if $\overline{\#\text{-}\lim_{n\to\infty^{\#}}} \sqrt[n]{|a_n|} = \infty^{\#}$, and $R = \infty^{\#}$ if $\overline{\#\text{-}\lim_{n\to\infty^{\#}}} \sqrt[n]{|a_n|} = 0.$ Then the hyper infinite power series #-converges:
(a) only for $x = x_0$ if $R = 0$
(b) for all $x \in \mathbb{R}_c^{\#}$ if $R = \infty^{\#}$, and absolutely uniformly in every bounded set;
(c) for $x \in (x_0 - R, x_0 + R)$ if $0 < R < 1$, and absolutely uniformly in every closed subset of this interval.
The series #-diverges if $|x - x_0| > R.$ No general statement can be made concerning #-convergence at the endpoints $x = x_0 + R$ and $x = x_0 + R$: the series may #-converge absolutely or conditionally at both; #-converge conditionally at one and #-diverge at the other; or #-diverge at both.

Theorem 18.2. The radius of #-convergence of $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n (x - x_0)^n$ is given by

$$\frac{1}{R} = \#\text{-}\lim_{n\to\infty^{\#}} \left| \frac{a_{n+1}}{a_n} \right| \tag{18.3}$$

if the #-limit exists in the extended hyperreals.

Example 18.1. For the hyper infinite power series

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{x^n}{n!} \tag{18.4}$$

one obtains that

$$\#\text{-}\lim_{n\to\infty^\#}\left|\frac{a_{n+1}}{a_n}\right| = \#\text{-}\lim_{n\to\infty^\#}\frac{n!}{(n+1)!} = \#\text{-}\lim_{n\to\infty^\#}\frac{1}{n+1} = 0. \qquad (18.4')$$

Therefore, $R = \infty^\#$; that is, the series #-converges for all $x \in \mathbb{R}_c^\#$, and absolutely uniformly

in every bounded set.

Theorem 18.3. A hyper infinite power series

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^\#} a_n(x-x_0)^n \qquad (18.5)$$

with positive radius of #-convergence R is #-continuous and #-differentiable in its interval of #-convergence; and its #-derivative can be obtained by #-differentiating term by term; that is;

$$f'^\#(x) = Ext\text{-}\sum_{n=0}^{\infty^\#} na_n(x-x_0)^{n-1} \qquad (18.6)$$

which can also be written as

$$f'^\#(x) = Ext\text{-}\sum_{n=0}^{\infty^\#}(n+1)a_{n+1}(x-x_0)^n \qquad (18.7)$$

This hyper infinite series also has radius of #-convergence R.

Theorem 18.4. A hyper infinite power series

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^\#} a_n(x-x_0)^n \qquad (18.8)$$

with positive radius of #-convergence R has #-derivatives of all orders in its interval of #-convergence, which can be obtained by repeated term by term #-differentiation thus,

$$\begin{aligned} f^{(n)\#}(x) &= Ext\text{-}\sum_{n=k}^{\infty^\#} n(n-1)\cdots(n-k+1)a_n(x-x_0)^n = \\ &= Ext\text{-}\sum_{n=k}^{\infty^\#}\left[\left(Ext\text{-}\prod_{j=n-k+1}^{n} j\right)a_n(x-x_0)^n\right]. \end{aligned} \qquad (18.9)$$

The radius of #-convergence of each of these hyper infinite series is R.

Corollary 18.1. (Uniqueness of hyper infinite Power Series) If

$$Ext\text{-}\sum_{n=0}^{\infty^\#} a_n(x-x_0)^n = Ext\text{-}\sum_{n=0}^{\infty^\#} b_n(x-x_0)^n \qquad (18.10)$$

for all x in some interval $(x_0 - r, x_0 + r)$ then

$$a_n = b_n, n \geq 0. \qquad (18.11)$$

Corollary 18.2. If

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^\#} a_n(x-x_0)^n, |x-x_0| < R \qquad (18.12)$$

then

$$a_n = \frac{f^{(n)\#}(x)}{n!}. \qquad (18.13)$$

Theorem 18.5. If x_1 and x_2 are in the interval of #-convergence of

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n(x-x_0)^n \tag{18.14}$$

Then

$$Ext\text{-}\int_{x_1}^{x_2} f(x)d^{\#}x = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{a_n}{n+1}\left[(x_2-x_0)^{n+1}-(x_1-x_0)^{n+1}\right] \tag{18.15}$$

that is, a hyper infinite power series may be integrated term by term between any two points in its interval of #-convergence.

Theorem 18.6.Suppose that $f(x)$ is hyper infinitely #-differentiable on an interval I and

$$\#\text{-}\lim_{n\to\infty^{\#}} \frac{r^n}{n!}\left\| f^{(n)\#}(x)\right\|_I = 0. \tag{18.16}$$

Then, if $x_0 \in I^0$, the hyper infinite Taylor series

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{f^{(n)\#}(x)}{n!}(x-x_0)^n \tag{18.17}$$

#-converges uniformly to $f(x)$ on $I_r = I \cap [x_0-r, x_0+r]$.

Theorem 18.7.If

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n(x-x_0)^n, |x-x_0| < R_1 \tag{18.18}$$

and

$$g(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n(x-x_0)^n, |x-x_0| < R_2 \tag{18.19}$$

and α and β are constants, then

$$\alpha f(x)+\beta g(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (\alpha a_n+\beta b_n)(x-x_0)^n, |x-x_0| < R, \tag{18.20}$$

where $R \geq \min\{R_1, R_2\}$.

Theorem 18.8.If $f(x)$ and $g(x)$ are given by Eq.(18.19) and Eq.(18.20) correspondingly, then

$$f(x)g(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} c_n(x-x_0)^n, |x-x_0| < R, \tag{18.21}$$

where

$$c_n = Ext\text{-}\sum_{j=0}^{n} a_j b_{n-j} = \sum_{j=0}^{n} a_{n-j} b_j, \tag{18.22}$$

$n \in \mathbb{N}^{\#}$ and $R \geq \min\{R_1, R_2\}$.

Theorem 18.9.(Generalized Abel’s Theorem) Let $f(x)$ be defined by a hyper infinite power series

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n(x-x_0)^n, |x-x_0| < R \tag{18.23}$$

with finite or hyperfinite radius of #-convergence $R \in \mathbb{R}_c^{\#}$.

(a) If $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n R^n$ #-converges, then

$$\#\text{-}\lim_{x \to_{\#} (x_0+R)-} f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n R^n. \tag{18.24}$$

(b) If $Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n a_n R^n$ #-converges, then

$$\#\text{-}\lim_{x \to_{\#} (x_0-R)+} f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n a_n R^n. \tag{18.25}$$

18.2.The $\mathbb{R}_c^{\#}$-valued #-exponential $Ext\text{-}\exp(x)$

We define the #-exponential $Ext\text{-}\exp(x)$ function as the solution of the differential equation

$$f'^{\#}(x) = f(x), f(0) = 1. \tag{18.26}$$

We solve it by setting

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n x^n, f'^{\#}(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} n a_n x^n. \tag{18.27}$$

Therefore

$$Ext\text{-}\exp(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{x^n}{n!} \tag{18.28}$$

From Eq.(18.40) and Eq.(18.28) we get

$$(Ext\text{-}\exp(x))(Ext\text{-}\exp(y)) = Ext\text{-}\exp(x+y), \tag{18.29}$$

for any $x, y \in \mathbb{R}_c^{\#}$. We often denote #-exponential $Ext\text{-}\exp(x)$ by $Ext\text{-}e^x$

$$Ext\text{-}e^x . \tag{18.30}$$

18.3.The $\mathbb{R}_c^{\#}$-valued Trigonometric Functions $Ext\text{-}\sin(x)$ and $Ext\text{-}\cos(x)$.

We define the $\mathbb{R}_c^{\#}$-valued Trigonometric Functions $Ext\text{-}\sin(x)$ and $Ext\text{-}\cos(x)$ by

$$Ext\text{-}\sin(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{18.31}$$

and

$$Ext\text{-}\cos(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{18.32}$$

It can be shown that the series (18.30) and (18.31) #-converge for all $x \in \mathbb{R}_c^{\#}$ and #-differentiating yields

$$[Ext\text{-}\sin(x)]'^{\#} = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n \frac{x^{2n}}{(2n)!} = Ext\text{-}\cos(x) \tag{18.33}$$

and

$$[Ext\text{-}\cos(x)]'^{\#} = Ext\text{-}\sum_{n=1}^{\infty^{\#}} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = -Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \tag{18.34}$$

$$= -[Ext\text{-}\sin(x)].$$

18.4. $\mathbb{R}_c^{\#}$-valued functions of several variables.

In this subsection we study $\mathbb{R}_c^{\#}$-valued functions defined on subsets of the n-dimensional external linear space $\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$ which consists of all external and internal hyperfinite (or finite) sequences (called a vector) $\mathbf{X} = \{x_i\}_{i=1}^{i=n} = \{x_i\}_{i \in n}$ of hyperreal numbers, called the coordinates or components of vector $\mathbf{X}$.

Definition 18.2. The vector sum of $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$ and $\mathbf{Y} = \{y_i\}_{i=1}^{i=n}$ is

$$\mathbf{X} + \mathbf{Y} = \{x_i + y_i\}_{i=1}^{i=n}. \tag{18.35}$$

If $a \in \mathbb{R}_c^{\#}$ is a hyperreal number, the scalar multiple of $\mathbf{X}$ by a is

$$a \cdot \mathbf{X} = \{ax_i\}_{i=1}^{i=n}. \tag{18.36}$$

Theorem 18.10.If $\mathbf{X}, \mathbf{Y},$ and $\mathbf{Z}$ are in $\mathbb{R}_c^{\#n}$ and $a, b \in \mathbb{R}_c^{\#}$ are hyperreal numbers, then
(i) $\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}$ - vector addition is commutative
(ii) $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$ - vector addition is associative
(iii) There is a unique vector $\mathbf{0}$, called the zero vector, such that $\mathbf{X} + 0 = \mathbf{X}$ for all $\mathbf{X} \in \mathbb{R}_c^{\#n}$
(iv) For each $\mathbf{X} \in \mathbb{R}_c^{\#n}$ there is a unique vector $-\mathbf{X}$ such that $\mathbf{X} + (-\mathbf{X}) = \mathbf{0}$
(v) $a \cdot (b \cdot \mathbf{X}) = (ab) \cdot \mathbf{X}$
(vi) $(a + b) \cdot \mathbf{X} = a \cdot \mathbf{X} + b \cdot \mathbf{X}$
(vii) $a \cdot (\mathbf{X} + \mathbf{Y}) = a \cdot \mathbf{X} + a \cdot \mathbf{Y}$
(viii) $1 \cdot \mathbf{X} = \mathbf{X}$.
Clearly, $\mathbf{0} = \{0\}_{i=1}^{i=n}$ and, if $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$, then $-\mathbf{X} = \{-x_i\}_{i=1}^{i=n}$.
We write $\mathbf{X} + (-\mathbf{Y})$ as $\mathbf{X} - \mathbf{Y}$. The point $\mathbf{0}$ is called the origin.

Definition 18.3. The length of the vector $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$ is

$$\|\mathbf{X}\| = \left(Ext\text{-}\sum_{i=1}^{n} x_i^2\right)^{1/2}. \tag{18.37}$$

The distance between points $\mathbf{X}$ and $\mathbf{Y}$ is $\|\mathbf{X} - \mathbf{Y}\|$; in particular, $\|\mathbf{X}\|$ is the distance between $\mathbf{X}$ and the origin. If $\|\mathbf{X}\| = 1$, then $\mathbf{X}$ is a unit vector.

Definition 18.4.The inner product $\mathbf{X} \cdot \mathbf{Y}$ of $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$ and $\mathbf{Y} = \{y_i\}_{i=1}^{i=n}$ is

$$\mathbf{X} \cdot \mathbf{Y} = Ext\text{-}\sum_{i=1}^{n} x_i y_i. \tag{18.38}$$

Theorem 18.11. (Schwarz's Inequality) If $\mathbf{X}, \mathbf{Y} \in \mathbb{R}_c^{\#n}$ then

$$\|\mathbf{X} \cdot \mathbf{Y}\| \leq \|\mathbf{X}\| \|\mathbf{Y}\|, \tag{18.39}$$

with equality if and only if one of the vectors is a scalar multiple of the other:

Theorem 18.12. (Triangle Inequality) If $\mathbf{X}, \mathbf{Y} \in \mathbb{R}_c^{\#n}$ then

$$\|\mathbf{X} + \mathbf{Y}\| \leq \|\mathbf{X}\| + \|\mathbf{Y}\|, \tag{18.40}$$

with equality if and only if one of the vectors is a nonnegative multiple of the other.

Corollary 18.3. If $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}_c^{\#n}$, then

$$\|\mathbf{X} - \mathbf{Z}\| \leq \|\mathbf{X} - \mathbf{Y}\| + \|\mathbf{Y} - \mathbf{Z}\|. \tag{18.41}$$

Corollary 18.4. If $\mathbf{X}, \mathbf{Y} \in \mathbb{R}_c^{\#n}$, then

$$\|\mathbf{X} - \mathbf{Y}\| \geq |\|\mathbf{X}\| - \|\mathbf{Y}\||. \tag{18.42}$$

Theorem 18.13.If $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}_c^{\#n}$ and $a \in \mathbb{R}_c^{\#}$ is a scalar, then
(i) $\|a\mathbf{X}\| = |a|\|\mathbf{X}\|$
(ii) $\|\mathbf{X}\| \geq 0$, with equality if and only if $\mathbf{X} = \mathbf{0}$

(iii) $\|\mathbf{X}-\mathbf{Y}\| \geq 0$, with equality if and only if $\mathbf{X} = \mathbf{Y}$
(iv) $\mathbf{X} \cdot \mathbf{Y} = \mathbf{Y} \cdot \mathbf{X}$
(v) $\mathbf{X} \cdot (Y+Z) = \mathbf{X} \cdot \mathbf{Y} + \mathbf{X} \cdot \mathbf{Z}$
(vi) $(c\mathbf{X}) \cdot \mathbf{Y} = \mathbf{X} \cdot (c\mathbf{Y}) = c(\mathbf{X} \cdot \mathbf{Y})$

Definition 18.5.Non-Archimedian metric space (X,d) is a set X together with a $\mathbb{R}_c^{\#}$-valued function $d : X \times X \to \mathbb{R}_c^{\#}$ (called a metric or non-Archimedian distance function) which assigns a hyperreal number $d(x,y)$ to every pair x,y belongs X satisfying the properties:
1.$d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$,
2.$d(x,y) = d(y,x)$,
3.$d(x,y) + d(y,z) \geq d(x,z)$.

Remark 18.1. Note that external linear space $\mathbb{R}_c^{\#n}$ endroved with distance function $d(\mathbf{X},\mathbf{Y}) = \|\mathbf{X}-\mathbf{Y}\|$ satisfying the properties 1-3 mentioned above in Definition 14.5.

18.5.Line Segments in $\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$.

Definition 18.6. Suppose that $\mathbf{X}_0, \mathbf{U} \in \mathbb{R}_c^{\#n}$ and $\mathbf{U} \neq \mathbf{0}$. Then the line through $\mathbf{X}_0$ in the direction of $\mathbf{U}$ is the set of all points in $\mathbb{R}_c^{\#n}$of the form

$$\mathbf{X}(\mathbf{X}_0,\mathbf{U}) = \mathbf{X}_0 + t\mathbf{U}, -\infty^{\#} < t < \infty^{\#}. \tag{18.43}$$

A set of points of the form

$$\mathbf{X} = \mathbf{X}_0 + t\mathbf{U}, t_1 \leq t \leq t_2 \tag{18.44}$$

is called a line segment. In particular, the line segment from $\mathbf{X}_0$ to $\mathbf{X}_1$ is the set of points of the form

$$\mathbf{X} = \mathbf{X}_0 + t(\mathbf{X}_1 - \mathbf{X}_0) = t\mathbf{X}_1 + (1-t)\mathbf{X}_0, 0 \leq t \leq 1. \tag{18.45}$$

Definition 18.7. A hyper infinite sequence of points $\mathbf{X}_n, n \in \mathbb{N}^{\#}$ in $\mathbb{R}_c^{\#n}$ #-converges to the #-limit $\overline{\mathbf{X}}$ if

$$\#\text{-}\lim_{n\to_{\#}\infty^{\#}} \|\mathbf{X}_n - \overline{\mathbf{X}}\| = 0. \tag{18.46}$$

In this case we write $\#\text{-}\lim_{n\to_{\#}\infty^{\#}} \mathbf{X}_n = \overline{\mathbf{X}}$.

Theorem 18.14. Let $\overline{\mathbf{X}} = \{x_i\}_{i=1}^{i=n}$ and $\mathbf{X}_m = \{x_{i_m}\}_{i=1}^{i=n}, m \geq 1$. Then $\#\text{-}\lim_{n\to_{\#}\infty^{\#}} \mathbf{X}_m = \overline{\mathbf{X}}$ if and only if $\#\text{-}\lim_{m\to_{\#}\infty^{\#}} x_{i_m} = \bar{x}_i, 1 \leq i \leq n;$ that is a hyper infinite sequence $\{\mathbf{X}_m\}$ of points in $\mathbb{R}_c^{\#n}$ #-converges to a #-limit $\overline{\mathbf{X}}$ if and only if the hyper infinite sequences of components of $\{\mathbf{X}_m\}$ #-converge to the respective components of $\overline{\mathbf{X}}$.

Theorem 18.15.(Cauchy's #-Convergence Criterion) A hyper infinite sequence $\{\mathbf{X}_m\}$ in $\mathbb{R}_c^{\#n}$ #-converges if and only if for each $\varepsilon > 0, \varepsilon \approx 0$, there is an hyperinteger $N \in \mathbb{N}^{\#}\backslash\mathbb{N}$ such that

$$\|\mathbf{X}_n - \mathbf{X}_m\| < \varepsilon \tag{18.47}$$

if $n,m \geq N$.

Definition 18.8.If A is a subset of a metric space $\mathbb{R}_c^{\#n}$ then x is a #-limit point of A if it is the #-limit of an eventually non-constant hyper infinite sequence $\{a_i\}_{i\in\mathbb{N}^{\#}}$ of points of A.

Definition 18.9.A subset A is said to be a #-closed subset of $\mathbb{R}_c^{\#n}$ if it contains all its #-limit points.

Example 18.1.(i) $\mathbb{R}_c^{\#}$ with the canonical metric $d(x,y) = |x-y|$, since in $\mathbb{R}_c^{\#}$ every hyperreal number is a #-limit point of the hyper infinite sequence $\{q_i\}_{i\in\mathbb{N}^{\#}}$ of

hyperrationals $q_i \in \mathbb{Q}^{\#}, i \in \mathbb{N}^{\#}$.
(ii) The empty set is #-closed.
(iii) Any finite set is #-closed.
(iv) Any hyperfinite set is #-closed.
(v) The closed interval $[a,b]$, where $a,b \in \mathbb{R}_c^{\#}$, is #-closed subset of $\mathbb{R}_c^{\#}$ with its canonical metric.
(vi) Let Δ be a set $\Delta = \{\varepsilon | |\varepsilon| \approx 0\}$. A set Δ is #-closed subset of $\mathbb{R}_c^{\#}$, since in Δ every hyperreal number $\delta \in \Delta$ is a #-limit point of the hyper infinite sequence $\{q_i\}_{i \in \mathbb{N}^{\#}}$ of hyperrationals $q_i \in \Delta \cap \mathbb{Q}^{\#}, i \in \mathbb{N}^{\#}$.

Definition 18.10.An #-neighbourhood of a point p in a metric space (X,d) is the set $N_\varepsilon(p) = \{x \in X | d(x,p) < \varepsilon, \varepsilon \approx 0\}$

Definition 18.11.A subset A of a metric space (X,d) is called #-open in X if every point of A has an #-neighbourhood which lies completely in A.

Example 18.2. (i) Any open interval (a,b) is an #-open set in $\mathbb{R}_c^{\#}$ with its canonical metric $d(x,y) = |x-y|$.
(ii) A set $\Delta = \{\varepsilon | |\varepsilon| \approx 0\}$ is #-open subset of $\mathbb{R}_c^{\#}$, since every point of Δ obviously has a #-neighbourhood which lies completely in Δ.

Remark 18.2.Note that a set $\Delta = \{\varepsilon | |\varepsilon| \approx 0\}$ are #-open and #-closed simultaneously.

Definition 18.12.A subset A of a non-Archimedian metric space X is admissible if A is exactly #-closed or exactly #-open but not #-open and #-closed simultaneously.

Theorem 18.16.(i) The union (of an arbitrary number) of #-open admissible sets is #-open.(ii) The intersection of finitely or hyper finitely many #-open admissible sets is #-open.

Proof. (i) Let $x \in \cup A_i = A$. Then $x \in A_i$ for some i. Since this is #-open, x has an #-neighbourhood lying completely inside A_i and this is also inside A.
(ii) It is enough to show this for just two #-open sets A and B. So suppose $x \in A \cap B$. Then $x \in A$ and so has an #-neighbourhood $N_{\varepsilon_1}(p), \varepsilon_1 \approx 0$ lying in A. Similarly x has an #-neighbourhood $N_{\varepsilon_2}(p), \varepsilon_2 \approx 0$ lying in B. So if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ the #-neighbourhood $N_\varepsilon(p)$ lies in both A and B and hence in $A \cap B$. By hyper infinite induction statement (ii) holds in general.

Theorem 18.17. Any admissible subset A of a metric space X is #-closed if and only if its complement $X \backslash A$ is admissible and is #-open subset of a metric space X.

Proof. 1.Suppose A is admissible and A is #-closed. We need to show that $X \backslash A$ is #-open.
So suppose that x belongs $X \backslash A$. Then some #-neighbourhood of x does not meet A (otherwise x would be a #-limit point of A and hence in A). Thus this #-neighbourhood of x lies completely in $X \backslash A$ which is what we needed to prove.
2.Conversely, suppose that $X \backslash A$ is #-open. We need to show that A contains all its #-limit points. So suppose x is a #-limit point of A and that $x \notin A$. Then $x \in X \backslash A$ and hence has an #-neighbourhood subset $X \backslash A$. But this is an #-neighbourhood that does not meet A and we have a contradiction.

Definition 18.13.If $\mathbf{S}$ is a nonempty subset of $\mathbb{R}_c^{\#n}$, then

$$d(\mathbf{S}) = \sup\{\|\mathbf{X} - \mathbf{Y}\| | \mathbf{X}, \mathbf{Y} \in \mathbf{S}\} \qquad (18.48)$$

is the diameter of $\mathbf{S}$. If $d(S) < \infty^{\#}, \mathbf{S}$ is bounded or hyperbounded.If $d(S) = \infty^{\#}, \mathbf{S}$ is hyperunbounded.

Theorem 18.18. (Principle of Nested Sets) If $\mathbf{S}_1, \mathbf{S}_2, \ldots,$ are #-closed nonempty subsets of $\mathbb{R}_c^{\#n}$ such that

$$\forall r(r \in \mathbb{N}^{\#})[\mathbf{S}_{r+1} \subset \mathbf{S}_r] \tag{18.49}$$

and

$$\#\text{-}\lim_{r \to_{\#} \infty^{\#}} d(\mathbf{S}_r) = 0, \tag{18.50}$$

then the intersection

$$\Lambda = \bigcap_{r=1}^{\infty^{\#}} \mathbf{S}_r \tag{18.51}$$

contains exactly one point:

Proof.Let $\{\mathbf{X}_r\}$ be a hyper infinite sequence such that $\mathbf{X}_r \in \mathbf{S}_r, r \geq 1$. Because of (18.49), $\mathbf{X}_r \in \mathbf{S}_k$ if $r \geq k$, so $\|\mathbf{X}_r - \mathbf{X}_s\| < d(\mathbf{S}_k)$ if $r,s \geq k$.

From (18.50) and Theorem 14.15., $\mathbf{X}_r$ #-converges to a #-limit $\overline{\mathbf{X}}$. Since $\overline{\mathbf{X}}$ is a #-limit point of every $\mathbf{S}_k$ and every $\mathbf{S}_k$ is #-closed, $\overline{\mathbf{X}}$ is in every $\mathbf{S}_k$. Therefore, $\overline{\mathbf{X}} \in \Lambda$, so $\Lambda \neq \varnothing$.

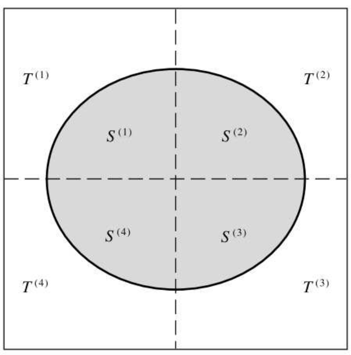
Moreover, $\overline{\mathbf{X}}$ is the only point in Λ, since if $\mathbf{Y} \in \Lambda$, then $\|\overline{\mathbf{X}} - \mathbf{Y}\| < d(\mathbf{S}_k), k \geq 1,$and (18.50) implies that $\mathbf{Y} = \overline{\mathbf{X}}$.

Definition 18.14. If $\mathbf{S}$ is a nonempty admissible subset of $\mathbb{R}_c^{\#n}$ we say that $\mathbf{S}$ is a #-compact set in $\mathbb{R}_c^{\#n}$ if is a #-closed and bounded or hyperbounded set.

Definition 18.15. Collection $\mathbf{H}$ of admissible #-open sets is an #-open covering of a set S if $S \subset \cup\{H|H \in \mathbf{H}\}$.

Theorem 18.19.(Heine-Borel Theorem) If H is an #-open covering of a #-compact subset S, then S can be covered by hyper finitely many sets from H.

Proof.The proof is by contradiction. We first consider the case where $n = 2$.Suppose that there is a #-open covering $\mathbf{H}$ for $\mathbf{S}$ from which it is impossible to select a hyperfinite subcovering. Since $\mathbf{S}$ is bounded or hyperbounded, $\mathbf{S}$ is contained in a #-closed square $T = \{\{x,y\}|a_1 \leq x \leq a_1 + L, a_2 \leq xa_2 + L\}$ with sides of length L (Pic. 14.5.1).



Pic.18.5.1.

Bisecting the sides of T as shown by the dashed lines in Figure 14.5.1 leads to four #-closed squares, $T^{(1)}, T^{(2)}, T^{(3)},$, and $T^{(4)}$, with sides of length $L/2$. Let $S^{(i)} = \mathbf{S} \cap T, 1 \leq i \leq 4.$Each $S^{(i)}$, being the intersection of admissible #-closed sets,is

#-closed, and $S = \cup_{i=1}^{4} S^{(i)}$. Moreover, $\mathbf{H}$ covers each $S^{(i)}$, but at least one $S^{(i)}$cannot be covered by any finite or hyperfinite subcollection of $\mathbf{H}$, since if all the $S^{(i)}$could be, then so could S. Let S_1 be a set with this property, chosen from $S^{(1)}, S^{(2)}, S^{(3)}$, and $S^{(4)}$.
We are now back to the situation we started from: a #-compact set S_1 covered by $\mathbf{H}$, but not by any hyperfinite subcollection of $\mathbf{H}$. However, S_1 is contained in a square T_1 with sides of length $L/2$ instead of L. Bisecting the sides of T_1 and repeating the argument, we obtain a subset S_2 of S_1 that has the same properties as S, except that it is contained in a square with sides of length $L/4$. Continuing in this way produces a hyper infinite sequence of nonempty #-closed sets $S_0 = S, S_1, S_2, \ldots,$ such that $S_k \supset S_{k+1}$ and $d(S_k) \leq L/2^{k-1/2}, k \geq 0$. From Theorem 14.18, there is a point $\overline{\mathbf{X}}$ in $\cap_{k=1}^{\infty^{\#}} S_k$. Since $\overline{\mathbf{X}} \in S$, there is an open set H in $\mathbf{H}$ that contains $\overline{\mathbf{X}}$, and this H must also contain some #-neighborhood of $\overline{\mathbf{X}}$. Since every $\mathbf{X}$ in S_k satisfies the inequality $\|\mathbf{X} - \overline{\mathbf{X}}\| < 2^{-k+1/2}L$, it follows that $S_k \subset H$ for $k \in \mathbb{N}^{\#}/\mathbb{N}$ sufficiently large. This contradicts our assumption on $\mathbf{H}$, which led us to believe that no S_k could be covered by a hyperfinite
number of sets from $\mathbf{H}$. Consequently, this assumption must be false: $\mathbf{H}$ must have a finite or hyperfinite subcollection that covers S. This completes the proof for $n = 2$.
The idea of the proof is the same for $n > 2$. The counterpart of the square T is the hypercube with sides of length L :

$$T = \{(x_1, x_2, \ldots, x_n) | a_i \leq x_i \leq a_i + L, 1 \leq i \leq n\}.$$

Halving the intervals of variation of the n coordinates $x_1, x_2, \ldots, x_n$ divides T into 2^n closed hypercubes with sides of length $L/2$:

$$T^{(i)} = \{(x_1, x_2, \ldots, x_n) | b_i \leq x_i \leq b_i + L/2, 1 \leq i \leq n\},$$

where $b_i = a_i$ or $b_i = a_i + L/2$. If no hyperfinite subcollection of $\mathbf{H}$ covers S, then at least one of these smaller hypercubes must contain a subset of S that is not covered by any hyperfinite subcollection of S. Now the proof proceeds as for $n = 2$.

18.6.#-Neighborhoods and #-open sets in $\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$. Connected Sets and Regions in $\mathbb{R}_c^{\#n}$.

Definition 18.16. Assum that A is admissible subset of $\mathbb{R}_c^{\#n}$.
(i) The #-interior #-$int(A)$ of a set A is the largest open subset A,
(ii) The #-closure #-$cl(A)$ of a set A is the smallest #-closed set containing A.
Theorem 18.20. 1. #-$cl(\varnothing) = \varnothing$
2. $A \subset$ #-$cl(A)$ for any set A.
3. #-$cl(A \cup B) =$ #-$cl(A) \cup$ #-$cl(B)$ for any sets A and B.
4. $cl($#-$cl(A)) =$ #-$cl(A)$ for any set A.
Proof.1. and 2. follow from the definition.
To prove 3 note that #-$cl(A) \cup$ #-$cl(B)$ is a #-closed set which contains $A \cup B$ and so #-$cl(A) \subset$ #-$cl(A \cup B)$. Similarly, #-$cl(B) \subset$ #-$cl(A \cup B)$ and so #-$cl(A) \cup$ #-$cl(B) \subset$ #-$cl(A \cup B)$ and the result follows.
To prove 4 we have #-$cl(A) \subset$ #-$cl($#-$cl(A))$ from 2. Also #-$cl(A)$ is a #-closed set which contains #-$cl(A)$ and hence it contains #-$cl($#-$cl(A))$.
Example 18.3.For $\mathbb{R}_c^{\#}$ with its usual topology induced by its canonical metric $d(x,y) = |x - y|$, #-cl$((a,b)) = [a,b]$ and #-$int([a,b]) = (a,b)$.
Definition 18.17. If $\varepsilon > 0, \varepsilon \approx 0$, the ε-neighborhood of a point $\mathbf{X}_0$ in $\mathbb{R}_c^{\#n}$ is the set

$$N_\varepsilon(\mathbf{X}_0) = \{\mathbf{X}|\|\mathbf{X}-\mathbf{X}_0\| < \varepsilon\}. \tag{18.52}$$

Definition 18.18.If $\mathbf{X}_0$ is a point in in $\mathbb{R}_c^{\#n}$ and $r > 0$, the sphere of radius r about $\mathbf{X}_0$ is the set $S_r(\mathbf{X}_0) = \{\mathbf{X}|\|\mathbf{X}-\mathbf{X}_0\| = r\}$

Definition 18.19.If $\mathbf{X}_0$ is a point in in $\mathbb{R}_c^{\#n}$ and $r > 0$, the #-open n-ball of radius r about $\mathbf{X}_0$ is the set $B_r(\mathbf{X}_0) = \{\mathbf{X}|\|\mathbf{X}-\mathbf{X}_0\| < r\}$. Thus, ε-neighborhoods are #-open n-balls.

If $\mathbf{X}_1$ is in $B_r(\mathbf{X}_0)$ and $\|\mathbf{X}-\mathbf{X}_1\| < \varepsilon = r - \|\mathbf{X}-\mathbf{X}_0\|$, then $\mathbf{X}$ is in $B_r(\mathbf{X}_0)$.

Thus, $B_r(\mathbf{X}_0)$ contains an ε-neighborhood of each of its points, and is therefore #-open.

The #-closure of $B_r(\mathbf{X}_0)$ is the #-closed n-ball of radius r about $\mathbf{X}_0$, defined by $\#\text{-}cl(B_r(\mathbf{X}_0)) = \{\mathbf{X}|\|\mathbf{X}-\mathbf{X}_0\| < r\}, r = \|\mathbf{X}_1-\mathbf{X}_0\|$.

Proposition18.1. If $\mathbf{X}_1$ and $\mathbf{X}_2$ are in $\mathrm{S}_r(X_0)$ for some $r > 0$, then so is every point on the line segment from $\mathbf{X}_1$ to $\mathbf{X}_2$.

Definition 18.20.A subset $S \subset \mathbb{R}_c^{\#n}$ is #-connected if it is impossible to represent S as the union of two disjoint nonempty sets such that neither contains a #-limit point of the other; that is, if S cannot be expressed as $S = A \cup B$, where

$$A \neq \varnothing, B \neq \varnothing, \#\text{-}cl(A) \cap B = \varnothing, \#\text{-}cl(B) \cap A = \varnothing. \tag{18.53}$$

If S can be expressed in this way, then S is #-disconnected.

Definition 18.21. A region S in $\mathbb{R}_c^{\#n}$ is the union of an #-open #-connected set with some, all, or none of its #-boundary; thus, $\#\text{-}int(S)$ is #-connected, and every point of S is a #-limit point of $\#\text{-}int(S)$.

18.7.The #-limits and #-continuity $\mathbb{R}_c^{\#}$-valued functions of $n \in \mathbb{N}^{\#}$ variables.

We denote the domain of a function f by $\mathbf{D}_f$ and the value of f at a point $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$ by $f(\mathbf{X})$ or $f\left(\{x_i\}_{i=1}^{i=n}\right)$.

Definition 18.22. We say that $f(\mathbf{X})$ #-approaches the #-limit L as $\mathbf{X}$ #-approaches $\mathbf{X}_0$ and write

$$\#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0} f(\mathbf{X}) = L \tag{18.54}$$

if $\mathbf{X}_0$ is a #-limit point of $\mathbf{D}_f$ and, for every $\varepsilon > 0, \varepsilon \approx 0$, there is a $\delta > 0, \delta \approx 0$, such that $|f(\mathbf{X}) - L < \varepsilon|$ for all $\mathbf{X} \in \mathbf{D}_f$ such that $0 < \|\mathbf{X}-\mathbf{X}_0\| < \delta$.

Theorem 18.21.If $\#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0} f(\mathbf{X})$ exists, then it is unique.

Theorem 18.22. Suppose that f and g are defined on a set $D \subset \mathbb{R}_c^{\#n}$, $\mathbf{X}_0$ is a #-limit point

of D, and $\#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0} f(\mathbf{X}) = L_1, \#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0} g(\mathbf{X}) = L_2$. Then

$$\begin{gathered}\#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0}(f \pm g)(\mathbf{X}) = L_1 \pm L_2,\\ \#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0}(f \times g)(\mathbf{X}) = L_1 L_2,\\ \text{and}\\ \#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0}(f/g)(\mathbf{X}) = L_1 L_2,\end{gathered} \tag{18.55}$$

if $L_2 \neq 0$.

Definition 18.23. We say that $f(\mathbf{X})$ #-approaches $\infty^{\#}$ as $\mathbf{X}$ #-approaches $\mathbf{X}_0$ and write

$$\#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0} f(\mathbf{X}) = \infty^{\#} \tag{14.56}$$

if $\mathbf{X}_0$ is a #-limit point of $\mathbf{D}_f$ and, for every hyperreal number M, there is a $\delta > 0$,

$\delta \approx 0,$ such that $f(\mathbf{X}) > M$ whenever $0 < \|\mathbf{X} - \mathbf{X}_0\| < \delta$ and $\mathbf{X} \in \mathbf{D}_f$. We say that

$$\#\text{-}\lim_{\mathbf{X}\to_\#\mathbf{X}_0} f(\mathbf{X}) = -\infty^\# \tag{18.57}$$

if $\#\text{-}\lim_{\mathbf{X}\to_\#\mathbf{X}_0}[-f(\mathbf{X})] = \infty^\#$.

Definition 18.24. If $\mathbf{D}_f$ is hyperunbounded, we say that

$$\#\text{-}\lim_{\|\mathbf{X}\|\to_\#\infty^\#} f(\mathbf{X}) = L, \tag{18.58}$$

where L finite or hyperfinite if for every $\delta > 0, \delta \approx 0,$ there is a number $R \in \mathbb{R}_c^\#$ such that $|f(\mathbf{X}) - L < \varepsilon|$ whenever $\|\mathbf{X}\| > R$ and $\mathbf{X} \in \mathbf{D}_f$.

Definition 18.25. If $\mathbf{X}_0 \in \mathbf{D}_f$ and is a #-limit point of $\mathbf{D}_f$, then we say that f is #-continuous at $\mathbf{X}_0$ if

$$\#\text{-}\lim_{\mathbf{X}\to_\#\mathbf{X}_0} f(\mathbf{X}) = f(\mathbf{X}_0). \tag{18.59}$$

Theorem 18.23.Suppose that $\mathbf{X}_0 \in \mathbf{D}_f$ and is a #-limit point of $\mathbf{D}_f$. Then f is #-continuous at $\mathbf{X}_0$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $|f(\mathbf{X}) - f(\mathbf{X}_0)| < \varepsilon$ whenever $\|\mathbf{X} - \mathbf{X}_0\| < \delta$ and $\mathbf{X} \in \mathbf{D}_f$.

Definition 18.26.We will say that f is #-continuous on S if f is #-continuous at every point of S.

Theorem 18.24. If f and g are #-continuous on a set $S \subset \mathbb{R}_c^{\#n}$, then so are $f \pm g,$ and fg. Also, f/g is #-continuous at each $\mathbf{X}_0 \in S$ such that $g(\mathbf{X}_0) \neq 0$.

Definition 18.27. Suppose that $g_1, g_2, \ldots, g_n, n \in \mathbb{N}^\#$ are $\mathbb{R}_c^\#$-valued functions defined on a subset $T \subset \mathbb{R}_c^{\#n}$, and define the vector-valued function G on T by

$$G(\mathbf{U}) = (g_1(\mathbf{U}), g_2(\mathbf{U}), \ldots, g_n(\mathbf{U})), \mathbf{U} \in T. \tag{18.60}$$

Then $g_1, g_2, \ldots, g_n$ are the component functions of $G = (g_1, g_2, \ldots, g_n)$.
We say that

$$\#\text{-}\lim_{\mathbf{U}\to_\#\mathbf{U}_0} G(\mathbf{U}) = \mathbf{L} = (L_1, \ldots, L_n) \tag{18.61}$$

if $\#\text{-}\lim_{\mathbf{U}\to_\#\mathbf{U}_0} g_i(\mathbf{U}) = L_i, 1 \leq i \leq n$ and that G is #-continuous at $\mathbf{U}_0$ if $g_1, g_2, \ldots, g_n$ are each #-continuous at $\mathbf{U}_0$.

Theorem 18.25.For a vector-valued function G, $\#\text{-}\lim_{\mathbf{U}\to_\#\mathbf{U}_0} G(\mathbf{U}) = \mathbf{L}$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $\|G(\mathbf{U}) - \mathbf{L}\| < \varepsilon$ whenever $0 < \|\mathbf{U} - \mathbf{U}_0\| < \delta$ and $\mathbf{U} \in \mathbf{D}_G$.Similarly, G is #-continuous at $\mathbf{U}_0$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $\|G(\mathbf{U}) - G(\mathbf{U}_0)\| < \varepsilon$ whenever $\|\mathbf{U} - \mathbf{U}_0\| < \delta$ and $\mathbf{U} \in \mathbf{D}_G$.

Theorem 18.26.Let f be a $\mathbb{R}_c^\#$-valued function defined on a subset of $\mathbb{R}_c^{\#n}$, and let the vector-valued function $G = (g_1, g_2, \ldots, g_n)$ be defined on a domain $\mathbf{D}_G$ in $\mathbb{R}_c^{\#n}$. Let the set $T = \{\mathbf{U} | \mathbf{U} \in \mathbf{D}_G$ and $G(U) \in D_f\}$, be nonempty; and define the $\mathbb{R}_c^\#$-valued composite function $h = f \circ G$ on T by $h(\mathbf{U}) = f(G(\mathbf{U})), \mathbf{U} \in T$. Now suppose that $\mathbf{U}_0 \in T$ and is a #-limit point of T, G is #-continuous at $\mathbf{U}_0$, and f is #-continuous at $\mathbf{X}_0 = G(\mathbf{U}_0)$. Then h is #-continuous at $\mathbf{U}_0$.

Theorem 18.27.If f is #-continuous on a #-compact set $S \subset \mathbb{R}_c^{\#n}$, then f is bounded or hyperbounded on S.

Theorem 18.28.Let f be #-continuous on a compact set $S \subset \mathbb{R}_c^{\#n}$ and $\alpha = \inf_{\mathbf{X}\in S} f(\mathbf{X})$, $\beta = \sup_{\mathbf{X}\in S} f(\mathbf{X})$. Then $f(\mathbf{X}_1) = \alpha$ and $f(\mathbf{X}_2) = \beta$ for some $\mathbf{X}_1$ and $\mathbf{X}_2$ in S.

Theorem 18.29.(Intermediate Value Theorem) Let f be #-continuous on a region $S \subset \mathbb{R}_c^{\#n}$. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are in S and $f(\mathbf{A}) < u < f(\mathbf{B})$. Then $f(\mathbf{C}) = u$

for some $\mathbf{C} \in S$.

Definition 18.28. f is uniformly #-continuous on a subset S of its domain in $\mathbb{R}_c^{\#n}$ if for every $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $|f(X) - f(X_0)| < \varepsilon$ whenever $\left\| \mathbf{X} - \mathbf{X}_0 \right\| < \delta$ and $\mathbf{X}, \mathbf{X}_0 \in S$.

Theorem 18.30.If f is #-continuous on a #-compact set $S \subset \mathbb{R}_c^{\#n}$ then f is uniformly #-continuous on S.

18.8.Partial #-Derivatives and the #-Differential

Definition 18.29. Let Φ be a unit vector and $\mathbf{X}$ a point in $\mathbb{R}_c^{\#\mathbf{n}}$. The directional #-derivative

of $f(\mathbf{X})$ at $\mathbf{X}$ in the direction of Φ is defined by

$$\frac{\partial'^{\#} f(\mathbf{X})}{\partial'^{\#}\Phi} = \#\text{-}\lim_{t \to_{\#} 0} \frac{f(\mathbf{X} + t\Phi) - f(\mathbf{X})}{t} \tag{18.62}$$

if the #-limit exists. That is, $\partial'^{\#} f(\mathbf{X})/\partial'^{\#}\Phi$ is the ordinary derivative of the function $H(t) = f(\mathbf{X} + t\Phi)$ at $t = 0$, if $H'^{\#}(t)$ exists.The directional #-derivatives that we are most interested in are those in the directions of the unit vectors $\mathbf{E}_i, 1 \leq i \leq n$,where all components of $\mathbf{E}_i$ are zero except for the i-th, which is 1.

Definition 18.30. Since $\mathbf{X}$ and $\mathbf{X} + t\mathbf{E}_i$ differ only in the i-th coordinate, $\partial'^{\#} f(\mathbf{X})/\partial'^{\#}\mathbf{E}_i$ is called the partial #-derivative of f with respect to x_i at $\mathbf{X}$. It is also denoted by $\partial'^{\#} f(\mathbf{X})/\partial'^{\#} x_i$ or $f'^{\#}_{x_i}(\mathbf{X})$, thus,

$$\frac{\partial'^{\#} f(\mathbf{X})}{\partial'^{\#} x_i} = f'^{\#}_{x_i}(\mathbf{X}) = \#\text{-}\lim_{t \to_{\#} 0} \frac{f(\{x_i + t\}_{i \in n}) - f(\{x_i\}_{i \in n})}{t} \tag{18.63}$$

If $\mathbf{X} = (x, y)$, then we denote the partial #-derivatives accordingly; thus,

$$\frac{\partial'^{\#} f(x,y)}{\partial'^{\#} x} = f'^{\#}_{x}(x,y) = \#\text{-}\lim_{h \to_{\#} 0} \frac{f(x+h,y) - f(x,y)}{h} \tag{18.64}$$

and

$$\frac{\partial'^{\#} f(x,y)}{\partial'^{\#} y} = f'^{\#}_{y}(x,y) = \#\text{-}\lim_{h \to_{\#} 0} \frac{f(x,y+h) - f(x,y)}{h}. \tag{18.65}$$

Theorem 18.31.If $f'^{\#}_{x_i}(\mathbf{X})$ and $g'^{\#}_{x_i}(\mathbf{X})$ exist, then

$$\frac{\partial'^{\#}(f+g)(\mathbf{X})}{\partial'^{\#} x_i} = f'^{\#}_{x_i}(\mathbf{X}) + g'^{\#}_{x_i}(\mathbf{X}), \frac{\partial'^{\#}(f \times g)(\mathbf{X})}{\partial'^{\#} x_i} = f'^{\#}_{x_i}(\mathbf{X})g(\mathbf{X}) + g'^{\#}_{x_i}(\mathbf{X})f(\mathbf{X}), \tag{18.66}$$

and, if $g(\mathbf{X}) \neq 0$,

$$\frac{\partial'^{\#}(f/g)(\mathbf{X})}{\partial'^{\#} x_i} = \frac{g(\mathbf{X})f'^{\#}_{x_i}(\mathbf{X}) - f(\mathbf{X})g'^{\#}_{x_i}(\mathbf{X})}{[g(\mathbf{X})]^2}. \tag{18.67}$$

If $f'^{\#}_{x_i}(\mathbf{X})$ exists at every point of a set $D \subset \mathbb{R}_c^{\#\mathbf{n}}$, then it defines a function $f'^{\#}_{x_i}(\mathbf{X})$ on D. If this function has a partial #-derivative with respect to x_j on a subset of D, we denote the partial #-derivative by

$$\frac{\partial'^{\#}}{\partial'^{\#} x_j}\left(\frac{\partial'^{\#} f(\mathbf{X})}{\partial'^{\#} x_i}\right) = \frac{\partial^{2'\#} f(\mathbf{X})}{\partial'^{\#} x_j \partial'^{\#} x_i} = f'^{\#}_{x_i x_j}(\mathbf{X}). \tag{14.68}$$

The function obtained by differentiating $f(\mathbf{X})$ successively with respect to $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ is denoted by

$$\frac{\partial^{r'\#} f(\mathbf{X})}{\partial'^{\#} x_{i_r} \partial'^{\#} x_{i_{r-1}} \ldots \partial'^{\#} x_{i_1}} = f'^{\#}_{x_{i_1} \ldots x_{i_{r-1}} x_{i_r}}(\mathbf{X}) \tag{18.69}$$

it is an r th-order partial derivative of $f(\mathbf{X})$.

Theorem 18.32.Suppose that $f, f'^{\#}_{x}, f'^{\#}_{y}$, and $f'^{\#}_{xy}$ exist on a #-neighborhood Ω of (x_0, y_0), and $f'^{\#}_{xy}$ is #-continuous at (x_0, y_0). Then $f'^{\#}_{yx}$ (x_0, y_0) exists, and

$$f'^{\#}_{yx}(x_0, y_0) = f'^{\#}_{xy}(x_0, y_0). \tag{18.70}$$

Theorem 18.33.Suppose that f and all its partial #-derivatives of order r are #-continuous on an #-open subset S of $\mathbb{R}^{\#\mathbf{n}}_c$. Then

$$f'^{\#}_{x_{i_1}, x_{i_2}, \ldots, x_{i_r}}(\mathbf{X}) = f'^{\#}_{x_{j_1}, x_{j_2}, \ldots, x_{j_r}}(\mathbf{X}), \mathbf{X} \in \mathbf{S}, \tag{18.71}$$

if each of the variables $x_1, x_2, \ldots, x_n$ appears the same number of times in $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ and $\{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}$. If this number is r_k, we denote the common value of the two sides of (18.71) by

$$\frac{\partial^{r'\#} f(\mathbf{X})}{\partial'^{\#} x_{i_r} \partial'^{\#} x_{i_{r-1}} \ldots \partial'^{\#} x_{i_1}}. \tag{18.72}$$

Definition 18.31. A function $f(\mathbf{X})$ is #-differentiable at $\mathbf{X}_0 = (x_{10}, x_{20}, \ldots, x_{n0})$ if there are constants $m_1, m_2, \ldots, m_n$ such that

$$\#\text{-}\lim_{\|\mathbf{X}-\mathbf{X}_0\| \to_{\#} 0} \frac{f(\mathbf{X}) - f(\mathbf{X}_0) - \left(Ext\text{-}\sum_{i=1}^{n} m_i (x_i - x_{i0})\right)}{\|\mathbf{X} - \mathbf{X}_0\|} = 0. \tag{18.73}$$

Theorem 18.34. If f is differentiable at at $\mathbf{X}_0 = (x_{10}, x_{20}, \ldots, x_{n0})$, then $f'^{\#}_{x_{i0}}(\mathbf{X}_0)$, $1 \leq i \leq n$, exist and the constants $m_i, 1 \leq i \leq n$, in Eq.(18.73) are given by

$$m_i = f'^{\#}_{x_{i0}}(\mathbf{X}_0). \tag{18.74}$$

Theorem 18.35.If f is #-differentiable at $\mathbf{X}_0$, then f is #-continuous at $\mathbf{X}_0$.

Definition 18.32. A linear function $L : \mathbb{R}^{\#n}_c \to \mathbb{R}^{\#}_c$ is a $\mathbb{R}^{\#}_c$-valued function of the form

$$L(\mathbf{X}) = Ext\text{-}\sum_{i=1}^{n} m_i x_i, \tag{18.75}$$

where $m_i, 1 \leq i \leq n$ are constants. From Definition 14.31, f is #-differentiable at $\mathbf{X}_0$ if and only if there is a linear function L such that $f(\mathbf{X}) - f(\mathbf{X}_0)$ can be approximated so well near $\mathbf{X}_0$ by

$$f(\mathbf{X}) - f(\mathbf{X}_0) = L(\mathbf{X} - \mathbf{X}_0) + E(\mathbf{X})\|\mathbf{X} - \mathbf{X}_0\|, \tag{18.76}$$

where

$$\#\text{-}\lim_{\|\mathbf{X}-\mathbf{X}_0\| \to_{\#} 0} E(\mathbf{X}) = 0. \tag{18.77}$$

Remark 18.3.Theorem 18.34 implies that if f is #-differentiable at $\mathbf{X}_0$, then there is exactly one linear function L that satisfies (18.76) and (18.77).

This function is called the #-differential of f at $\mathbf{X}_0$. We will denote it by $d'^{\#}_{\mathbf{X}_0} f$ and its value by $(d'^{\#}_{\mathbf{X}_0} f)(\mathbf{X})$; thus,

$$(d'^{\#}_{\mathbf{X}_0} f)(\mathbf{X}) = Ext\text{-}\sum_{i=1}^{n} f'^{\#}_{x_{i0}}(\mathbf{X}_0) x_i. \tag{18.78}$$

For convenience in writing $d'^{\#}_{\mathbf{X}_0} f$, and to conform with standard notation, we introduce the function $d'^{\#} x_i : \mathbb{R}^{\#n}_c \to \mathbb{R}^{\#}_c$ defined by $dx_i(\mathbf{X}) = x_i$. That is, $d'^{\#} x_i$ is the function whose value at a point in $\mathbb{R}^{\#n}_c$ is the i-th coordinate of the point. It is the #-differential of the function $g_i(X) = x_i$. From Eq.(18.78)

$$d'^{\#}_{\mathbf{X}_0} f = Ext\text{-}\sum_{i=1}^{n} f'^{\#}_{x_{i0}}(\mathbf{X}_0) d'^{\#} x_i. \tag{18.78}$$

19.#-Analytic functions $f : \mathbb{C}^{\#}_c \to \mathbb{C}^{\#}_c$.

19.1. $\mathbb{C}_c^{\#}$-valued #-analytic functions $f : \mathbb{C}_c^{\#} \to \mathbb{C}_c^{\#}$.

The class of #-analytic functions is formed by the complex functions of a complex variable $z \in \mathbb{C}_c^{\#} = \mathbb{R}_c^{\#} + i\mathbb{R}_c^{\#}$ which possess a #-derivative wherever the function is defined. The term #-holomorphic function is used with identical meaning. For the purpose of this preliminary investigation the reader may think primarily of functions which are defined in the whole plane $\mathbb{C}_c^{\#}$.

The definition of the #-derivative can be written in the form

$$f'^{\#}(z) = \#\text{-}\lim_{h\to_{\#}0} \frac{f(z+h)-f(z)}{h} \tag{19.1}$$

As a first obvious consequence $f(z)$ is necessarily #-continuous. Indeed, from $f(z+h)-f(z) = h \times (f(z+h)-f(z))/h$ one obtains $\#\text{-}\lim_{h\to_{\#}0}(f(z+h)-f(z)) = 0 \times f'^{\#}(z) = 0.$ If we write $f(z) = u(z)+iv(z)$ it follows, moreover, that $u(z)$ and $v(z)$ are both #-continuous.

Remark 19.1. When we consider the #-derivative of a $\mathbb{C}_c^{\#}$-valued function, defined on a set A$\subset \mathbb{C}_c^{\#}$ in the complex plane $\mathbb{C}_c^{\#}$, it is of course understood that $z \in A$ and that the limit is with respect to values h such that $z+h \in A$. The existence of the #-derivative will therefore have a different meaning depending on whether z is an interior point or a #-boundary point of A. The way to avoid this is to insist that all #-analytic functions be defined on open sets.

Definition 19.1. A $\mathbb{C}_c^{\#}$-valued function $f(z)$, defined on an open set Ω, is said to be $\mathbb{C}_c^{\#}$-analytic in Ω if it has a #-derivative at each point of Ω. And more explicitly that $f(z)$ is #-analytic function. A commonly used synonym is #-holomorphic function.

Definition 19.2.A function $f(z)$ is #-analytic on an arbitrary point set A if it is the restriction to A of a function which is #-analytic in some open set containing A.

Remark 19.2. Note that the real and imaginary parts of an #-analytic function in Ω satisfy the generalized Cauchy-Riemann equations

$$\frac{\partial^{\#}u}{\partial^{\#}x} = \frac{\partial^{\#}v}{\partial^{\#}y}; \frac{\partial^{\#}u}{\partial^{\#}y} = -\frac{\partial^{\#}v}{\partial^{\#}x}. \tag{19.2}$$

Conversely, if u and v satisfy these equations in Ω, and if the partial #-derivatives are #-continuous, then $u+iv$ is an #-analytic function in Ω.

Theorem 19.1. An #-analytic function f in a region Ω whose #-derivative vanishes identically must reduce to a constant. The same is true if either the real part, the imaginary part, the modulus, or the argument is constant.

19.2. The $\mathbb{C}_c^{\#}$-valued #-Exponential $Ext\text{-}\exp(z)$.

We define the #-exponential $Ext\text{-}\exp(z)$ function as the solution of the differential equation

$$f'^{\#}(z) = f(z), f(0) = 1. \tag{19.3}$$

We solve it by setting

$$f(z) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n z^n, f'^{\#}(z) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} n a_n z^n. \tag{19.4}$$

If Eq.(15.4) is to be satisfied, we must have $a_{n-1} = na_n, n \in \mathbb{N}^{\#}$ and the initial condition gives $a_0 = 1$. It follows by hypee infinite induction that $a_n = 1/n!$.

Abbreviation 19.1. The solution of the Eq.(15.4) is denoted by $Ext\text{-}e^z$ or $Ext\text{-}\exp(z)$ or $Ext\text{-}\exp z$. Thus finally we obtain

$$Ext\text{-}\exp(z) = Ext\text{-}\sum_{n=0}^{\infty^\#} \frac{z^n}{n!}. \tag{19.5}$$

19.3.The $\mathbb{C}_c^\#$-valued Trigonometric Functions $Ext\text{-}\sin(z)$, $Ext\text{-}\cos(z)$.

The $\mathbb{C}_c^\#$-valued trigonometric functions $Ext\text{-}\sin(z), Ext\text{-}\cos(z)$ are defined by

$$Ext\text{-}\sin(z) = \frac{1}{2}(Ext\text{-}\exp(iz) - Ext\text{-}\exp(-iz)) \tag{19.6}$$

and

$$Ext\text{-}\cos(z) = \frac{1}{2}(Ext\text{-}\exp(iz) + Ext\text{-}\exp(-iz)). \tag{19.7}$$

Substitution (19.)-(19.) in (19.) gives that

$$Ext\text{-}\sin(z) = \tag{19.8}$$

and

$$Ext\text{-}\cos(z) = \tag{19.9}$$

From (14) we obtain generalized Euler's formula

$$Ext\text{-}\exp(iz) = Ext\text{-}\cos(z) + i(Ext\text{-}\sin(z)) \tag{19.10}$$

and as well as the identity

$$(Ext\text{-}\sin(z))^2 + (Ext\text{-}\cos(z))^2 = 1. \tag{19.11}$$

19.4. The periodicity of the #-exponential $Ext\text{-}\exp(iz)$.

Definition 19.4.We say that $f(z)$ has the period c if $f(z+c) = f(z)$ for all $z \in \mathbb{C}_c^\#$.
Thus a period of $Ext\text{-}e^z$ satisfies $Ext\text{-}e^{z+c} = Ext\text{-}e^z$, or $Ext\text{-}e^c = 1$. It follows that $c = i\omega$ with real $\omega \in \mathbb{R}_c^\#$ we prefer to say that ω is a period of $Ext\text{-}e^{iz}$. We shall show that there are periods, and that they are all integral multiples of a positive period ω_0.From $(Ext\text{-}\sin(y))'^\# = Ext\text{-}\cos(y) \leq 1$ and $Ext\text{-}\sin(0) = 0$ one obtains $Ext\text{-}\sin(y) < y$ for $y > 0$, either by integration or by use of the generalized mean-value theorem. In the same way $(Ext\text{-}\cos(y))'^\# = -Ext\text{-}\sin(y) > -y$ and $Ext\text{-}\cos(0) = 1$ gives $Ext\text{-}\cos(y) > 1 - y^2/2$, which in turn leads to $Ext\text{-}\sin(y) > y - y^3/6$ and finally to $Ext\text{-}\cos(y) < 1 - y^2/2 + y^4/24$. This inequality shows that $Ext\text{-}\cos\left(\sqrt{3}\right) < 0$, and therefore there is a y_0 such that $0 < y_0 < \sqrt{3}$ and $Ext\text{-}\cos(y_0) = 0$. Because $(Ext\text{-}\sin(y_0))^2 + (Ext\text{-}\cos(y_0))^2 = 1$,we have $Ext\text{-}\sin(y_0) = \pm 1$, that is, $Ext\text{-}e^{iy_0} = \pm i$, and hence $Ext\text{-}e^{4iy_0} = 1$. We have shown that $4y_0$ is a period. Actually, it is the smallest positive period. To see this, take $0 < y < y_0$.
Then $Ext\text{-}\sin(y) > y(1 - y^2/6) > y/2 > 0$, which shows that $Ext\text{-}\cos(y)$ is strictly decreasing. Because $Ext\text{-}\sin(y)$ is positive and $(Ext\text{-}\sin(y))^2 + (Ext\text{-}\cos(y))^2 = 1$ it follows that $Ext\text{-}\sin(y)$ is strictly increasing, and hence $Ext\text{-}\sin(y) < Ext\text{-}\sin(y_0) = 1$.
The double inequality $0 < Ext\text{-}\sin(y) < 1$ guarantees that $Ext\text{-}e^{iy}$ is neither ± 1 nor $\pm i$. Therefore $Ext\text{-}e^{4iy} \neq 1$, and $4y_0$ is indeed the smallest positive period.We denote

it by ω_0. Consider now an arbitrary period ω_0. There exists an integer n such that $n\omega_0 \leq \omega < (n+1)\omega_0$. If w were not equal to $n\omega_0$, then ω - $n\omega_0$ would be a positive period $< \omega_0$. Since this is not possible, every period must be an integral multiple of ω_0.

Abbreviation 19.2.The smallest positive period of $Ext\text{-}e^{iz}$ is denoted by $2\pi_\#$.

Remark 19.3.Note that $\mathbf{st}(\pi_\#) = \pi \in \mathbb{R}$.

19.5.The $\mathbb{C}_c^\#$-valued Logarithm.

Together with the exponential function $Ext\text{-}e^{iz}$ we must also introduce its inverse function, the $\mathbb{C}_c^\#$-valued logarithm. By definition, $z = Ext\text{-}\log w$ is a root of the equation $Ext\text{-}e^{iz} = w$. First of all, since $Ext\text{-}e^{iz}$ is always $\neq 0$, the number 0 has no logarithm. For $w \neq 0$ the equation $Ext\text{-}e^{x+iv} = w$ is equivalent to

$$Ext\text{-}e^{iz} = |w|, Ext\text{-}e^{iy} = \frac{w}{|w|}. \tag{19.12}$$

The first equation has a unique solution $x = Ext\text{-}\log|w|$, the $\mathbb{R}_c^\#$-valued logarithm of the positive number $|w| \in \mathbb{R}_c^\#$. The right-hand member of the second equation (15.12) is a complex number in $\mathbb{C}_c^\#$ of absolute value 1. Therefore, as we have just seen, it has one and only one solution in the interval $0 \leq y < 2\pi_\#$. In addition, it is also satisfied by all y that differ from this solution by an integral multiple of $2\pi_\#$. We see that every complex number other than 0 has hyper infinitely many logarithms which differ from each other by multiples of $2\pi_\# i$.

The imaginary part of $Ext\text{-}\log w$ is also called the argument of w, $Ext\text{-}\arg w$, and it is interpreted geometrically as the angle, measured in radians, between the positive real axis and the half line from 0 through the point w. According to this definition the argument has hyper infinitely many values which differ by multiples of $2\pi_\#$, and

$$Ext\text{-}\log w = Ext\text{-}\log|w| + i\arg w. \tag{19.13}$$

Remark 19.4.The addition property of the exponential function $Ext\text{-}e^{iz}$ implies

$$\begin{aligned} Ext\text{-}\log(z_1 \times z_2) &= Ext\text{-}\log z_1 + Ext\text{-}\log z_2, \\ Ext\text{-}\arg(z_1 \times z_2) &= Ext\text{-}\arg z_1 + Ext\text{-}\arg z_2, \end{aligned} \tag{19.14}$$

but only in the sense that both sides represent the same hyper infinite set of complex numbers.The inverse of $Ext\text{-}\cos(z)$ is obtained by solving the equation

$$Ext\text{-}\cos(z) = \frac{1}{2}(Ext\text{-}e^{iz} + Ext\text{-}e^{-iz}) = w. \tag{19.15}$$

This is a quadratic equation in $Ext\text{-}e^{iz}$ with the roots

$$Ext\text{-}e^{iz} = w \pm \sqrt{w^2 - 1} \tag{19.16}$$

and therefore

$$z = Ext\text{-}\arccos(w) = -i\left(Ext\text{-}\log\left(w \pm \sqrt{w^2+1}\right)\right), \tag{19.17}$$

or in the form

$$Ext\text{-}\arccos(w) = \pm i\left(Ext\text{-}\log\left(w + \sqrt{w^2+1}\right)\right) \tag{19.18}$$

The hyper infinitely many values of $Ext\text{-}\arccos(w)$ reflect the evenness and periodicity of $Ext\text{-}\cos(w)$. The inverse sine is most easily defined by formula

$$Ext\text{-}\arcsin(w) = \frac{\pi_\#}{2} - (Ext\text{-}\arccos(w)). \tag{19.19}$$

20. Complex Integration of the $\mathbb{C}_c^{\#}$-valued function $f(t)$.

20.1. Definition and basic properties of the complex integral.

If $f(t) = u(t) + iv(t)$ is a #-continuous function, defined in an interval (a,b), we set by definition

$$Ext\text{-}\int_a^b f(t)d^{\#}t = Ext\text{-}\int_a^b u(t)d^{\#}t + i\left(Ext\text{-}\int_a^b v(t)d^{\#}t \right). \qquad (20.1)$$

This integral has most of the properties of the real integral. In particular, if $c = \alpha + i\beta$ is a complex constant we obtain

$$Ext\text{-}\int_a^b cf(t)d^{\#}t = c\left(Ext\text{-}\int_a^b f(t)d^{\#}t \right). \qquad (20.2)$$

The fundamental inequality

$$\left| Ext\text{-}\int_a^b f(t)d^{\#}t \right| \leq Ext\text{-}\int_a^b |f(t)|d^{\#}t. \qquad (20.3)$$

holds for arbitrary $\mathbb{C}_c^{\#}$-valued function $f(t)$.

We consider now a piecewise #-differentiable arc γ with the equation $z = z(t), a \leq t \leq b$.

If the function $f(z)$ is defined and #-continuous on γ, then $f(z(t))$ is also #-continuous and we can set

$$v \quad Ext\text{-}\int_\gamma f(z)d^{\#}z = Ext\text{-}\int_a^b f(z(t))z'^{\#}(t)d^{\#}t. \qquad (20.4)$$

The most important property of the integral (20.4) is its invariance under a change of parameter. A change of parameter is determined by an increasing function $t = t(\tau)$ which maps an interval $\alpha \leq \tau \leq \beta$ onto $a \leq t \leq b$; we assume that $t(\tau)$ is piecewise #-differentiable. By the rule for changing the variable of integration we get

$$Ext\text{-}\int_a^b f(z(t))z'^{\#}(t)d^{\#}t = Ext\text{-}\int_\alpha^\beta f(z(t(\tau)))z'^{\#}(t(\tau))t'^{\#}(\tau)d^{\#}\tau. \qquad (20.5)$$

We defined the opposite arc $-\gamma$ by the equation $z = z(-t), -b \leq t \leq -a.$ We have thus

$$Ext\text{-}\int_{-\gamma} f(z)d^{\#}z = -\left(Ext\text{-}\int_\gamma f(z)d^{\#}z \right). \qquad (20.6)$$

The integral (20.4) has also a very obvious additive property. It is clear what is meant by subdividing an arc γ into a finite or hyperfinite number of subarcs. A subdivision can be indicated by a symbolic equation: $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n, n \in \mathbb{N}^{\#}$, and the corresponding integrals satisfy the relation

$$Ext\text{-}\int_{\gamma_1+\gamma_2+\ldots+\gamma_n} f(z)d^{\#}z = Ext\text{-}\sum_{i=1}^{n}\left(Ext\text{-}\int_{\gamma_i} f(z)d^{\#}z\right). \quad (20.7)$$

Finally, the integral over a closed curve is also invariant under a shift of parameter. The old and the new initial point determine two subarcs $\gamma_1, \gamma_2,$ and the invariance follows from the fact that the integral over $\gamma_1 + \gamma_2$ is equal to the integral over $\gamma_2 + \gamma_1$
In addition to integrals of the form (20.4) we can also consider line integrals with respect to $\bar{z}$. The most convenient definition is by double conjugation

$$Ext\text{-}\int_{\gamma} f(z)\overline{d^{\#}z} = \overline{Ext\text{-}\int_{\gamma} f(z)d^{\#}z}. \quad (20.8)$$

Using notation (20.7), line integrals with respect to x or y can be introduced by

$$\begin{aligned} Ext\text{-}\int_{\gamma} f(z)d^{\#}x &= \frac{1}{2}\left(Ext\text{-}\int_{\gamma} f(z)d^{\#}z + Ext\text{-}\int_{\gamma} f(z)\overline{d^{\#}z}\right), \\ Ext\text{-}\int_{\gamma} f(z)d^{\#}y &= \frac{1}{2i}\left(Ext\text{-}\int_{\gamma} f(z)d^{\#}z - Ext\text{-}\int_{\gamma} f(z)\overline{d^{\#}z}\right). \end{aligned} \quad (20.9)$$

With $f = u + iv$ we find that the integral (16.4) can be written in the form

$$Ext\text{-}\int_{\gamma}(ud^{\#}x - vd^{\#}y) + i\left(Ext\text{-}\int_{\gamma}(ud^{\#}y + vd^{\#}x)\right). \quad (16.10)$$

Of course we could just as well have started by defining integrals of the form

$$Ext\text{-}\int_{\gamma}(pd^{\#}x + qd^{\#}y), \quad (20.11)$$

in which case formula (20.10) would serve as definition of the integral (20.4).
An essentially different line integral is obtained by integration with respect to arc length. Two notations are in common use, and the definition is

$$Ext\text{-}\int_{\gamma} fd^{\#}s = Ext\text{-}\int_{\gamma} f(z)|d^{\#}z| = Ext\text{-}\int_{\gamma} f(z(t))|z'^{\#}(t)|d^{\#}t. \quad (20.12)$$

This integral is again independent of the choice of parameter. In contrast to (20.6) we get

$$Ext\text{-}\int_{-\gamma} f(z)|d^{\#}z| = Ext\text{-}\int_{\gamma} f(z)|d^{\#}z|, \quad (20.13)$$

while (20.7) remains valid in the same form. The inequality

$$\left|Ext\text{-}\int_{\gamma} f(z)d^{\#}z\right| \leq Ext\text{-}\int_{\gamma}|f(z)||d^{\#}z| \quad (20.14)$$

is a consequence of (20.3).
Remark 20.1.For $f \equiv 1$ the integral (20.3) Generalized Cauchy's Theorem for a

Rectangle reduces to $\int_\gamma |dz|$ which is by definition the length of γ. As an example we compute the length of a circle. From the parametric equation $z = z(t) = a + \rho(Ext\text{-}e^{it}),\ 0 \leq t \leq 2\pi_\# r,$ of a full circle we obtain $z'^\#(t) = i\rho(Ext\text{-}e^{it})$ and hence

$$\int_0^{2\pi_\#} |z'^\#(t)|d^\#t = \int_0^{2\pi_\#} \rho d^\#t = 2\pi_\#\rho \tag{20.15}$$

as expected.

20.2.Line Integrals as Functions of Arcs.

Remind that the length of an arc can also be defined as the least upper bound of all hyperfinite sums

$$Ext\text{-}\sum_{i=1}^{n} |z(t_i) - z(t_{i-1})|, \tag{20.16}$$

$n \in \mathbb{N}^\#/\mathbb{N}$, where $a = t_0 < t_1 < \ldots < t_n = b$. If this least upper bound is finite or hyperfinite we say that the arc is rectifiable. It is quite easy to show that piecewise #-differentiable arcs are rectifiable, and that the two definitions of length coincide. It is clear that the sums (20.16) and the corresponding sums

$$Ext\text{-}\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|; Ext\text{-}\sum_{i=1}^{n} |y(t_i) - y(t_{i-1})|, \tag{20.17}$$

where $z(t) = x(t) + iy(t)$, are bounded or hyperbounded at the same time. When the latter sums are bounded (or hyperbounded), one says that the functions $x(t)$ and $y(t)$ are of bounded (or hyperbounded) variation. An arc $z = z(t)$ is rectifiable if and only if the real and imaginary parts of $z(t)$ are of bounded (or hyperbounded) variation.
If γ is rectifiable and $f(z)$ #-continuous on γ it is possible to define integrals of type (20.12) as a #-limit

$$Ext\text{-}\int_\gamma fd^\#s = \#\text{-}\lim_{n\to\infty^\#}\left(Ext\text{-}\sum_{k=1}^{n} f(z(t_k))|z(t_i) - z(t_{i-1})| \right). \tag{20.18}$$

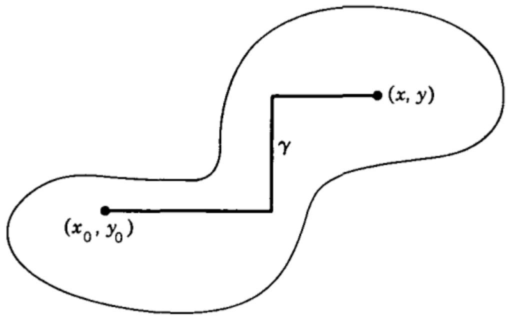
General line integral of the form $Ext\text{-}\int_\gamma (pd^\#x + qd^\#y)$ can be considered as functional of the arc γ. It is then assumed that p and q are defined and #-continuous in a region Ω and that γ is free to vary in Ω. An important class of integrals is characterized by the property that the integral over an arc depends only on its end points. In other words, if γ_1 and γ_2 have the same initial point and the same end point, we require that

$$Ext\text{-}\int_{\gamma_1} (pd^\#x + qd^\#y) = Ext\text{-}\int_{\gamma_2} (pd^\#x + qd^\#y). \tag{20.19}$$

To say that an integral depends only on the end points is equivalent to saying that the integral over any closed curve is zero. Indeed, if γ is a closed curve, then γ and $-\gamma$ have the same end points, and if the integral depends only on the end points, we obtain

$$Ext\text{-}\int_\gamma (pd^\#x + qd^\#y) = Ext\text{-}\int_{-\gamma} (pd^\#x + qd^\#y) = -\left(Ext\text{-}\int_\gamma (pd^\#x + qd^\#y) \right) \tag{20.20}$$

and consequently $\int_\gamma (pd^\# x + qd^\# y) = 0$. Conversely, if γ_1 and γ_2 have the same end points, then $\gamma_1 - \gamma_2$ is a closed curve, and if the integral over any closed curve vanishes, it follows that $Ext\text{-}\int_{\gamma_1} (pd^\# x + qd^\# y) = Ext\text{-}\int_{\gamma_2} (pd^\# x + qd^\# y)$.



Pic. 20.1.

The following theorem gives a necessary and sufficient condition under which a line integral depends only on the end points.

Theorem 20.1.The line integral $Ext\text{-}\int_\gamma (pd^\# x + qd^\# y)$, defined in Ω, depends only on the end points of γ if und only if there exists a function $U(x,y)$ in Ω with the partial #-derivatives $\partial^\# u/\partial^\# x = p, \partial^\# u/\partial^\# y = q$.

The sufficiency follows at once, for if the condition is fulfilled we can write, with the usual notations,

$$Ext\text{-}\int_\gamma (pd^\# x + qd^\# y) = Ext\text{-}\int_a^b \left[\frac{\partial^\# U}{\partial^\# x} x'^\#(t) + \frac{\partial^\# U}{\partial^\# y} y'^\#(t) \right] d^\# t = \\ Ext\text{-}\int_a^b \frac{d^\#}{d^\# t} U(x(t),y(t)) d^\# t = U(x(b),y(b)) - U(x(a),y(a)). \tag{20.21}$$

and the value of this difference depends only on the end points. To prove the necessity we choose a fixed point $(x_0,y_0) \in \Omega$, join it to (x,y) by a polygon γ, contained in Ω, whose sides are parallel to the coordinate axes (Pic.1) and define a function $U(x,y)$ by

$$U(x,y) = Ext\text{-}\int_\gamma (pd^\# x + qd^\# y). \tag{20.22}$$

Since the integral depends only on the end points, the function is well defined. Moreover, if we choose the last segment of γ horizontal, we can keep y constant and let x vary without changing the other segments. On the last segment we can choose x for parameter and obtain

$$U(x,y) = Ext\text{-}\int^x p(x,y) d^\# x + const., \tag{20.23}$$

the lower limit of the integral being irrelevant. From Eq.(20.23) it follows at once that

$\frac{\partial^\# U}{\partial^\# x} = p$. In the same way, by choosing the last segment vertical, we can show that $\frac{\partial^\# U}{\partial^\# y} = q$. It is customary to write $d^\# U = (\partial^\# U/\partial^\# x)d^\# x + (\partial^\# U/\partial^\# y)d^\# y$ and to say that an expression $pd^\# x + qd^\# y$ which can be written in this form is an exact #-differential. Thus an integral depends only on the end points if and only if the integrand is an exact differential. Observe that p, q and U can be either real or complex. The function U, if it exists, is uniquely determined up to an additive constant, for if two functions have the same partial #-derivatives their #-difference must be constant.
When is $f(z)d^\# z = f(z)d^\# x + if(z)d^\# y$ an exact #-differential? According to the definition there must exist a function $F(z)$ in Ω with the partial #-derivatives

$$\frac{\partial^\# F(z)}{\partial^\# x} = f(z), \frac{\partial^\# F(z)}{\partial^\# y} = if(z). \tag{20.24}$$

If this is so, $F(z)$ fulfills the generalized Cauchy-Riemann equation

$$\frac{\partial^\# F(z)}{\partial^\# x} = i\frac{\partial^\# F(z)}{\partial^\# y}, \tag{20.25}$$

since $f(z)$ is by assumption #-continuous $F(z)$ is #-analytic with the #-derivative $f(z)$. The integral $Ext\text{-}\int_\gamma fd^\# z$, with #-continuous f, depends only on the end points of γ if and only iff is the derivative of an analytic function in Ω. Under these circumstances we shall prove later that $f(z)$ is itself #-analytic.
As an immediate application of the above result we find that

$$\int_\gamma (z-a)^n d^\# z = 0 \tag{20.26}$$

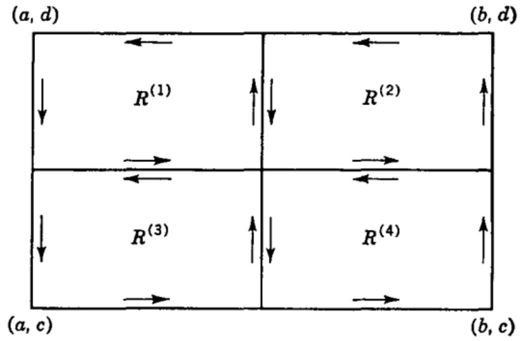
for all closed curves γ, provided that the integer $n \in \mathbb{N}^\#$ is ≥ 0. In fact, $(z-a)^n$ is the #-derivative of $(z-a)^{n+1}/(n+1)$, a function which is #-analytic in the whole plane $\mathbb{C}_c^\#$. If n is negative, but $\neq -1$, the same result holds for all closed curves which do not pass through a, for in the complementary region of the point a the indefinite integral is still #-analytic and single-valued. For $n = -1$, Eq.(20.26) does not always hold. Consider a circle C with the center a, represented by the equation $z = a + \rho(Ext\text{-}e^{it})$, $0 \leq t \leq 2\pi_\#$. We obtain

$$\int_\gamma \frac{d^\# z}{(z-a)} = \int_0^{2\pi_\#} id^\# t = 2\pi_\# i. \tag{20.27}$$

This result shows that it is impossible to define a single-valued branch of $Ext\text{-}\log(z-a)$ in an annulus $\rho_1 < |z-a| < \rho_2$. On the other hand, if the closed curve γ is contained in a half plane which does not contain a, the integral vanishes, for in such a half plane a single-valued and #-analytic branch of $Ext\text{-}\log\,(z-a)$ can be defined.

20.3.Generalized Cauchy's Theorem for a Rectangle.

We consider, specifically, a rectangle $R \subset \mathbb{C}_c^\#$ defined by inequalities $a \leq x \leq b$, $c \leq y \leq d$. Its perimeter can be considered as a simple closed curve consisting of four line segments whose direction we choose so that R lies to the left of the directed segments. The order of the vertices is thus $(a,c),(b,c),(b,d),(a,d)$. We refer to this closed curve as the boundary curve or contour of R, and we denote it by $\partial^\# R$



Pic.20.2. Bisection of rectangle.

Theorem 16.2. If the function $f(z)$ is #-analytic on R, then

$$Ext\text{-}\int\limits_{\partial^{\#}R} f(z)d^{\#}z = 0. \tag{20.28}$$

Proof. The proof is based on the method of bisection. Let us introduce the notation

$$\eta(R) = Ext\text{-}\int\limits_{\partial^{\#}R} f(z)d^{\#}z\,. \tag{20.29}$$

If R is divided into four congruent rectangles $R^{(1)},R^{(2)},R^{(3)},R^{(4)}$, we get

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)}). \tag{20.30}$$

for the integrals over the common sides cancel each other,see Pic.201.It follows from Eq.(16.30) that at least one of the rectangles $R^{(k)}, k = 1,2,3,4,$ must satisfy the condition $|\eta(R^{(k)})| \geq |\eta(R)|/4$. This process can be repeated inductively by hyper infinite induction,

and we obtain a hyper infinite sequence of nested rectangles $R \supset R_1 \supset R_2 \ldots \supset$ $\supset R_n \ldots \supset \ldots$ with the property $|\eta(R_n)| \geq 4^{-n}|\eta(R_{n-1})|, n \in \mathbb{N}^{\#}$. Thus

$$|\eta(R_n)| \geq 4^{-n}|\eta(R)|. \tag{20.31}$$

The rectangles R_n converge to a point $z^* \in R$ in the sense that R_n will be contained in a prescribed neighborhood $|z - z^*| < \delta$ as soon as $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ is sufficiently large. First of all, we choose δ so small that $f(z)$ is defined and #-analytic in $|z - z^*| < \delta$, $\delta \approx 0$. Secondly, if $\varepsilon > 0, \varepsilon \approx 0$ is given, we can choose δ such that

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'^{\#}(z^*) \right| < \varepsilon, \tag{20.32}$$

and therefore

$$\left| f(z) - f(z^*) - (z - z^*)f'^{\#}(z^*) \right| < \varepsilon|z - z^*|. \tag{20.33}$$

for $|z - z^*| < \delta$. We assume that δ satisfies both conditions and that R_n is contained in $|z - z^*| < \delta$. We make now the observation that

$$Ext\text{-}\int\limits_{\partial^{\#}R_n} d^{\#}z = 0, Ext\text{-}\int\limits_{\partial^{\#}R_n} zd^{\#}z = 0 \tag{20.34}$$

By virtue of the equations (20.34) we are able to write

$$|\eta(R_n)| = Ext\text{-}\int_{\partial^\# R_n} \left|f(z) - f(z^*) - (z - z^*)f'^\#(z^*)\right| d^\# z \tag{20.35}$$

and it follows by (20.33) that

$$|\eta(R_n)| \leq \varepsilon\left(Ext\text{-}\int_{\partial^\# R_n} |z - z^*| \times |d^\# z|\right). \tag{20.36}$$

In the last integral $|z - z^*|$ is at most equal to the length d_n of the diagonal of R_n.
If L_n denotes the length of the perimeter of R_n, the integral is hence $\leq d_n L_n$. But if d and L are the corresponding quantities for the original rectangle R, it is clear that $d_n = 2^{-n}d$ and $L_n = 2^{-n}L$. By (20.36) we have hence

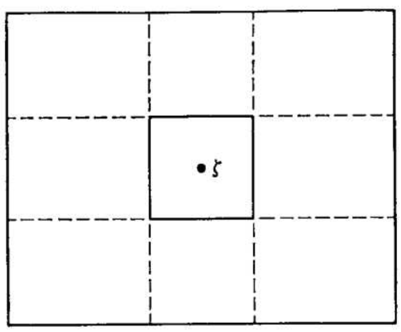
$$|\eta(R_n)| \leq 4^{-n} dL\varepsilon \tag{20.37}$$

and comparison with (20.31) yields

$$|\eta(R)| \leq dL\varepsilon. \tag{20.38}$$

Since $\varepsilon \approx 0$ is arbitrary, we can only have $\eta(R) \equiv 0$, and the theorem is proved.
Theorem 20.3.Let $f(z)$ be #-analytic on the set R' obtained from a rectangle R by omitting a finite or hyperfinite number of interior points ζ_j If it is true that #-$\lim_{z \to_\# \zeta_j}(z - \zeta_j)f(z) = 0$ for all $j \in \mathbb{N}^\#$, then $Ext\text{-}\int_{\partial^\# R} f(z)d^\# z = 0$.
Proof. It is sufficient to consider the case of a single exceptional point ζ, for evidently R can be divided into smaller rectangles which contain at most one ζ_j. We divide now R into nine rectangles, as shown in Pic.20.2, and apply Theorem 20.2 to all but the rectangle R_0 in the center. If the corresponding equations (20.28) are added, we obtain,
after cancellations,

$$Ext\text{-}\int_{\partial^\# R} f(z)d^\# z = Ext\text{-}\int_{\partial^\# R_0} f(z)d^\# z \tag{20.39}$$



Pic. 20. 3.

If $\varepsilon > 0, \varepsilon \approx 0$ we can choose the rectangle R_0 so infinite small that $|f(z)| \leq \varepsilon|z - \zeta|$ on $\partial^\# R_0$. By (16.39) we have thus

$$\left|Ext\text{-}\int_{\partial^\# R} f(z)d^\# z\right| = \varepsilon\left(Ext\text{-}\int_{\partial^\# R_0} \frac{|d^\# z|}{|z - \zeta|}\right) \tag{16.40}$$

If we assume, as we may, that R_0 is a square of center ζ, elementary estimates show that

$$Ext\text{-}\int_{\partial^{\#}R_0} \frac{|d^{\#}z|}{|z-\zeta|} < 8. \tag{20.41}$$

Thus finally we obtain

$$\left| Ext\text{-}\int_{\partial^{\#}R} f(z)d^{\#}z \right| < 8\varepsilon. \tag{20.42}$$

and since ε is arbitrary the theorem follows.We conclude that the hypothesis of the theorem is certainly fulfilled if $f(z)$ is #-analytic and bounded or hyperbounded on R'.

20.4.Generalized Cauchy's Theorem in a Disk.

It is not true that the integral of an #-analytic function over a closed curve is always zero. For example

$$\int_C \frac{d^{\#}z}{|z-a|} = 2i\pi_{\#}. \tag{20.43}$$

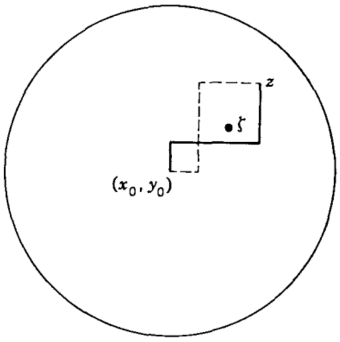
Theorem 20.4.If $f(z)$ is #-analytic in an open disk Δ, then 20

$$Ext\text{-}\int_{\gamma} f(z)d^{\#}z = 0 \tag{20.44}$$

for every closed curve $\gamma \subset \Delta$.
Proof. We define a function $F(z)$ by

$$F(z) = Ext\text{-}\int_{\sigma} f(z)d^{\#}z, \tag{20.45}$$

where σ consists of the horizontal line segment from the center (x_0,y_0) to (x,y_0) and the vertical segment from (x,y_0) to (x,y); it is immediately seen that $\partial^{\#}F/\partial^{\#}y = if(z)$. On the other hand, by Theorem 20.2 σ can be replaced by a path consisting of a vertical segment followed by a horizontal segment. This choice defines the same function $F(z)$, and we obtain $\partial^{\#}F/\partial^{\#}x = f(z)$. Hence $F(z)$ is #-analytic in Δ. with the #-derivative $f(z)$, and $f(z)d^{\#}z$ is an exact #-differential.
Theorem 20.5. Let $f(z)$ be #-analytic in the region Δ' obtained by omitting a finite or hyperfinite number of points ζ_j from an open disk Δ. If $f(z)$ satisfies the condition $\#\text{-}\lim_{z\to_{\#}\zeta_j}(z-\zeta_j)f(z) = 0$ for all j, then (20.44) holds for any closed curve $\gamma \subset \Delta'$.



Pic.20.4.

The proof must be modified, for we cannot let rr pass through the exceptional points. Assume first that no ζ_j lies on the lines $x = x_0$ and $y = y_0$. It is then possible to avoid the exceptional points by letting σ consist of three segments (Pic.20.4). By an obvious application of Theorem 20.3 we find that the value of $F(z)$ in (20.44) is independent of the choice of the middle segment; moreover, the last segment can be either vertical or horizontal. We conclude as before that $F(z)$ is an indefinite integral of $f(z)$, and the theorem follows..

20.5.Generalized Cauchy's integral formula.

Through a very simple application of the generalized Cauchy's theorem it becomes possible to represent an #-analytic function $f(z)$ as a line integral in which the variable $z \in \mathbb{R}_c^{\#}$ enters as a parameter. This representation, known in classical case as Cauchy's integral formula,has numerous important applications. Above all, it enables us to study the local properties of an #-analytic function in full detail.

Lemma 20.1. If the piecewise #-differentiable closed curve γ does not pass through the point a, then the value of the integral

$$\int_{\gamma} \frac{d^{\#}z}{|z-a|}. \tag{20.46}$$

is a multiple of $2i\pi_{\#}$.

Definition 20.1.We define the index of the point a with respect to the curve γ by the equation

$$n(\gamma, a) = \frac{1}{2\pi_{\#}i} \int_{\gamma} \frac{d^{\#}z}{z-a}. \tag{20.47}$$

The index (20.47) is also called the winding number of γ with respect to a. It is clear that $n(-\gamma, a) = -n(\gamma, a)$.The following property is an immediate consequence of Theorem 20.4.

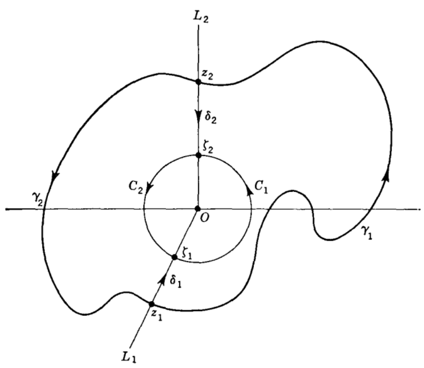
(i) If γ lies inside of a circle, then $n(\gamma, a) = 0$ for all points a outside of the same circle. As a point set γ is #-closed and bounded (or hyperbounded). Its complement is #-open and can be represented as a union of disjoint regions, the components of the complement. We shall say, for short, that γ determines these regions.

If the complementary regions are considered in the extended plane, there is exactly one which contains the point at infinity. Consequently, γ determines one and

only one unbounded region.
(ii) As a function of a the index $n(\gamma,a)$ is constant in each of the regions determined by γ, and zero in the unbounded region.
Any two points in the same region determined by γ can be joined by a polygon which does not meet γ. For this reason it is sufficient to prove that $n(\gamma,a) = n(\gamma,b)$ if γ does not meet the line segment from a to b. Outside of this segment the function $(z-a)/(z-b)$ is never real and ≤ 0. For this reason the principal branch of $Ext\text{-}\log[(z-a)/(z-b)]$ is #-analytic in the complement of the segment. Its derivative is equal to $(z-a)^{-1}-(z-b)^{-1}$, and if γ does not meet the segment we get

$$Ext\text{-}\int_{\gamma}\left(\frac{1}{z-a}-\frac{1}{z-b}\right)d^{\#}z = 0; \qquad (20.48)$$

hence $n(\gamma,a) = n(\gamma,b)$. If lal is sufficiently large, γ is contained in a disk $|z| < \rho < |a|$ and we conclude by (i) that $n(\gamma,a) = 0$. This proves that $n(\gamma,a) = 0$ in the unbounded region.
We shall find the case $n(\gamma,a) = 1$ particularly important, and it is desirable to formulate a geometric condition which leads to this consequence.
For simplicity we take $a = 0$.
Lemma 20.2. Let z_1,z_2 be two points on a closed curve γ which does not pass through the origin. Denote the subarc from z_1 to z_2 in the direction of the curve by γ_1, and the subarc from z_2 to z_1 by γ_2. Suppose that z_1 lies in the lower half plane and z_2 in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis, then $n(\gamma,0) = 1$.
For the proof we draw the half lines L_1 and L_2 from the origin through z_1 and z_2 (Pic. 4-5). Let s_1,s_2 be the points in which L_1,L_2 intersect a circle C about the origin. If C is described in the positive sense, the arc C_1 from s_1 to s_2 does not intersect the negative axis, and the arc C_2 from s_2 to s_1 does not intersect the positive axis. Denote the directed line segments from z_1 to s_1 and from z_2 to s_2 by δ_1,δ_2. Introducing the closed curves $\sigma_1 = \gamma_1+\delta_2-C_1-\delta_1$, $\sigma_2 = \gamma_2+\delta_1 - C_2 - \delta_2$ we get that $n(\gamma,0) = n(C,0)+n(\sigma_1,0)+n(\sigma_2,0)$ because of cancellations. But σ_1 does not meet the negative axis. Hence the origin belongs to the unbounded region determined by σ_1, and we obtain $n(\sigma_1,0) = 0$. For a similar reason $n(\sigma_2,0) = 0$, and we conclude that $n(\gamma,0) = n(C,0) = 1$.



Pic.20.5

Let $f(z)$ be #-analytic in an open disk Δ. Consider a closed curve $\gamma \subset \Delta$. and a point $a \in \Delta$
which does not lie on γ. We apply Cauchy's theorem to the function

$$F(z) = \frac{f(z) - f(a)}{z - a}. \tag{20.49}$$

This function is analytic for $z \neq a$. For $z = a$ it is not defined, but it satisfies the condition #-$\lim_{z \to_\# a}[(z-a)F(z)] =$ #-$\lim_{z \to_\# a}[f(z) - f(a)] = 0,$ which is the condition of Theorem 20.5. We conclude that

$$Ext\text{-}\int_{\gamma} \frac{f(z) - f(a)}{z - a} d^{\#}z = 0. \tag{20.50}$$

This equation can be rewritten in the form

$$Ext\text{-}\int_{\gamma} \frac{f(z)d^{\#}z}{z - a} = f(a)\left(Ext\text{-}\int_{\gamma} \frac{d^{\#}z}{z - a} \right), \tag{20.51}$$

and we observe that the integral in the right-hand member is by definition $2\pi_{\#}in(\gamma, a)$.
Theorem 20.6. Suppose that $f(z)$ is #-analytic in an open disk Δ, and let γ be a closed curve in Δ. For any point a such that $a \notin \gamma$

$$n(\gamma, a) \times f(a) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{\gamma} \frac{f(z)d^{\#}z}{z - a} \right), \tag{20.52}$$

where $n(\gamma, a)$ is the index of a with respect to γ.
In this statement we have suppressed the requirement that a be a point in Δ. We have done so in view of the obvious interpretation of the formula (16.52) for the case that a is not in Δ. Indeed, in this caseb$n(\gamma, a)$ and the integral in the right-hand member are both zero.
It is clear that Theorem 16.6 remains valid for any region Ω to which Theorem 16.5 can be applied. The presence of exceptional points ζ_j is permitted,provided none of them coincides with a.
The most common application is to the case where $n(\gamma, a) = 1$. We have then

$$f(a) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{\gamma} \frac{f(z)d^{\#}z}{z - a} \right) \tag{20.53}$$

and this we interpret as a representation formula. Indeed, it permits us to compute $f(a)$ as soon as the values of $f(z)$ on γ are given, together with the fact that $f(z)$ is #-analytic in Δ. In (20.53) we may let a take different values, provided that the order of a with respect to γ remains equal to 1. We may thus treat a as a variable, and it is #-convenient to change the notation and rewrite (20.53) in the form

$$f(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{\gamma} \frac{f(\zeta)d^{\#}\zeta}{\zeta - z} \right). \tag{20.54}$$

It is this formula which is usually referred to as Cauchy's integral formula. We must remember that it is valid only when $n(\gamma, z) = 1$, and that we have proved it only when $f(z)$ is #-analytic in a disk.

The representation formula (20.54) gives us a tool for the study of the local properties of #-analytic functions. In particular we can now show that an #-analytic function has #-derivatives of all orders $n \in \mathbb{N}^{\#}$, which are then also #-analytic.
We consider a function $f(z)$ which is #-analytic in an arbitrary region Ω. To a point $a \in \Omega$ we determine a δ-neighborhood $\Delta \subset \Omega$, and in Δ a circle C about a.
Theorem 20.6 can be applied to $f(z)$ in Δ. Since $n(C,a) = 1$ we have $n(C,z) = 1$ for all points z inside of C. For such z we obtain by (20.54)

$$f(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_C \frac{f(\zeta)d^{\#}\zeta}{\zeta - z}\right) \tag{20.55}$$

Provided that the integral in (20.) can be #-differentiated under the sign of integration we find

$$f'^{\#}(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_C \frac{f(\zeta)d^{\#}\zeta}{(\zeta - z)^2}\right) \tag{20.56}$$

and

$$f^{(n)\#}(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_C \frac{f(\zeta)d^{\#}\zeta}{(\zeta - z)^n}\right) \tag{20.57}$$

If the #-differentiations can be justified, we shall have proved the existence of all #-derivatives at the points inside of C. Since every point in Ω lies inside of some such circle, the existence will be proved in the whole region Ω.
Lemma 20.3. Suppose that $\varphi(\zeta)$ is #-continuous on the arc γ. Then the function

$$F_n(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_\gamma \frac{\varphi(\zeta)d^{\#}\zeta}{(\zeta - z)^n}\right) \tag{20.58}$$

is #-analytic in each of the regions determined by γ, and its #-derivative is $F_n'^{\#}(z) = nF_{n+1}(z)$.
It is clear that Lemma 20.3 is just what is needed in order to deduce (20.55) and (20.56) in a rigorous way. We have thus proved that an #-analytic function has #-derivatives of all orders which are #-analytic and can be represented by the formula (20.57).
Theorem 20.7. (Generalized Morera's theorem) If $f(z)$ is defined and #-continuous in a region Ω, and if $Ext\text{-}\int_\gamma f(z)d^{\#}z = 0$ for all closed curves γ in Ω, then $f(z)$ is #-analytic in Ω.

20.6.Generalized Liouville's theorem.

Theorem 20.8. (Generalized Liouville's theorem) A function $f(z)$ which is #-analytic and bounded in the whole plane $\mathbb{C}_c^{\#}$ must reduce to a constant.
Proof. We make use of a simple estimate derived from (20.57). Let the radius of C be r, and assume that $|f(z)| \leq M$ on C. If we apply (20.57) with $z = a$, we obtain

$$\left|f^{(n)\#}(a)\right| \leq Mn!r^{-n} \tag{20.59}$$

We need only the case $n = 1$. The hypothesis means that $|f(z)| \leq M$ on all circles. Hence we can let r tend to $\infty^{\#}$, and (16.59) leads to $f'^{\#}(a) = 0$ for all a. We conclude that the function is constant.

20.7.Generalized fundamental theorem of algebra.

Liouville's theorem leads to an almost trivial proof of the generalized fundamental theorem of algebra.

Theorem 20.9. (Generalized fundamental theorem of algebra) Suppose that $P(z)$ is external polynomial of degree $n \in \mathbb{N}^{\#}$.The equation $P(z) = 0$ must have a root $\xi \in \mathbb{C}_c^{\#}$.

Proof.Suppose that $P(z)$ is a polynomial of degree $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. If $P(z)$ were never zero, the function $1/P(z)$ would be #-analytic in the whole plane $\mathbb{C}_c^{\#}$. We know that $P(z) \to \infty^{\#}$, and therefore $1/P(z)$ tends to zero. This implies boundedness (the absolute value is #-continuous on the Riemann sphere and has thus a finite or hyperfinite maximum), and by Liouville's theorem $1/P(z)$ would be constant. Since this is not so, the equation $P(z) = 0$ has a root.

21.The local properties of #-analytic function.

21.1.Removable Singnlarities. Taylor's Theorem.

Theorem 21.1. Suppose that $f(z)$ is #-analytic in the region Δ' obtained by omitting a point a from a region Δ. A necessary and sufficient condition that there exist an #-analytic function in Δ which coincides with $f(z)$ in Δ' is that #-$\lim_{z \to_{\#} a}(z-a)f(z) = 0$. The extended function is uniquely determined.

Proof. The necessity and the uniqueness are trivial since the extended function must be #-continuous at a. To prove the sufficiency we draw a circle C about a so that C and its inside are contained in Δ. Cauchy's formula is valid, and therefore we have

$$f(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{C} \frac{f(\zeta)d^{\#}\zeta}{\zeta - z}\right) \tag{21.1}$$

for all $z \neq a$ inside of C. But the integral in the right-hand member represents an #-analytic function of z throughout the inside of C. Consequently, the function which is equal to $f(z)$ for $z \neq a$ and which has the value

$$\frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{C} \frac{f(\zeta)d^{\#}\zeta}{\zeta - z}\right). \tag{21.2}$$

for $z = a$ is #-analytic in Δ. It is natural to denote the extended function by $f(z)$ and the value (21.2) by $f(a)$.We apply this result to the function $F(z) = [f(z) - f(a)]/(z-a)$. It is not defined for $z = a$, but it satisfies the condition #-$\lim_{z \to_{\#} a}(z-a)F(z) = 0$. The #-limit of $F(z)$ as $z \to_{\#} a$ is $f'^{\#}(a)$. Hence there exists an #-analytic function which is equal to $F(z)$ for $z = a$ and equal to $f'^{\#}(a)$ f

Part II. $\mathbb{R}_c^{\#}$-Valued Lebesgue Integral.

1.External $\mathbb{R}_c^{\#}$-Valued Lebesgue Measure.

Let us consider a bounded interval $I \subset \mathbb{R}_c^{\#}$ with endpoints a and b $(a < b)$.The length of this bounded interval I is defined by $l(I) = b - a$. In contrast,the length of an unbounded interval, such as $(a, \infty^{\#}), (-\infty^{\#}, b)$ or $(-\infty^{\#}, \infty)$, is defined to be gyperinfinite. Obviously, the length of a line segment is easy to quantify.However, what should we do if we want to measure an arbitrary subset of $\mathbb{R}_c^\#$? Given a set $E \subset \mathbb{R}_c^\#$ of gyperreal numbers, we denote the Lebesgue measure of set E by $\mu(E)$. To correspond with the length of a line segment, the measure of a set $A \subset \mathbb{R}_c^\#$ should keep the following properties:

(1) If A is an interval, then $\mu(A) = l(A)$.

(2) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(3) Given $A \subseteq \mathbb{R}_c^\#$ and $x_0 \in \mathbb{R}_c^\#$, define $A + x_0 = \{x + x_0 : x \in A\}$. Then $\mu(A) = \mu(A + x_0)$.

(4) If A and B are disjoint sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$. If $\{A_i\}_{i \in \mathbb{N}^\#}$ is a hyperinfinite sequence of disjoint sets, then $\mu(\bigcup_{i \in \mathbb{N}^\#} A_i) = \sum_{i=1}^{\infty^\#} \mu(A_i)$.

2.External $\mathbb{R}_c^\#$-Valued Lebesgue outer measure

Definition 2.1. Let E be a subset of $\mathbb{R}_c^\#$. Let $\{I_k\} \triangleq \{I_k\}_{k \in \mathbb{N}^\#}$ be a hyperinfinite sequence

of open intervals such that $E \subseteq \bigcup_{k \in \mathbb{N}^\#} A_k$ and let Σ be a set of the all such hyperinfinite sequences. The external Lebesgue outer measure of E is defined by

$$\mu^*(E) = \inf_{\{I_k\} \in \Sigma} \left\{ \sum_{k=1}^{\infty^\#} l(I_k) \right\}. \tag{2.1}$$

Note that $0 \leq \mu^*(E) \leq \infty^\#$.

Definition 2.2. A set E is #-countable if there exists an injective function f from E to the

gypernatural numbers $\mathbb{N}^\#$. If such an f can be found that is also surjective (and therefore

bijective), then E is called #-countably infinite or gyperinfinite, i.e. a set is #-countably infinite if it has one-to-one correspondence with the set $\mathbb{N}^\#$.

Theorem 2.1.The external Lebesgue outer measure has the following properties:

(a) If $E_1 \subseteq E_2$, then $\mu^*(E_1) \leq \mu^*(E_2)$.

(b) The external Lebesgue outer measure of any #-countable set is zero.

(c) The external Lebesgue outer measure of the empty set is zero.

(d) The external Lebesgue outer measure is invariant under translation, that is, $\mu^*(E + x_0) = \mu^*(E)$.

(e) Lebesgue outer measure is #-countably sub-additive, that is,

$$\mu^* \left(\bigcup_{i=1}^{\infty^\#} E_i \right) \leq \sum_{i=1}^{\infty^\#} \mu^*(E_i). \tag{2.2}$$

(f) For any interval $I, \mu^*(I) = l(I)$.

Proof. Part (a) is trivial.

For part (b) and (c), let $E = \{x_k | k \in \mathbb{Z}_+^\#\}$ be a #-countably hyper infinite set.

Let $\varepsilon > 0, \varepsilon \approx 0$ and let ε_k be a hyper infinite sequence of positive numbers such that $\sum_{k=1}^{\infty^\#} \varepsilon_k = \varepsilon/2$. Since $E \subseteq \cup_{k=1}^{\infty^\#} (x_k - \varepsilon_k, x_k + \varepsilon_k)$, it follows that $\mu^*(E) \leq \varepsilon$. Hence, $\mu^*(E) = 0$. Since $\varnothing \subseteq E$, then $\mu^*(\varnothing) = 0$.

For part (d), since each cover of E by open intervals can generate a cover of $E + x_0$ by open intervals with the same length, then $\mu^*(E + x_0) \leq \mu^*(E)$. Similarly, $\mu^*(E + x_0) \geq \mu^*(E)$, since $E + x_0$ is a translation of E Therefore, $\mu^*(E + x_0) = \mu^*(E)$.

For part (e), if $\sum_{i=1}^{\infty^\#} \mu^*(E_i) = \infty^\#$, then the statement is trivial. Suppose that the sum is hyperfinite and let $\varepsilon > 0, \varepsilon \approx 0$. For each $i \in \mathbb{N}^\#$, there exists a hyperinfinite sequence $\{I_i^k\}$ of open intervals such that $E_i \subseteq \cup_{k=1}^{\infty^\#}$ and $\sum_{k=1}^{\infty^\#} l(I_i^k) < \mu^*(E_i) + \varepsilon/2^i$.

Now $\{I_i^k\}$ is a double-indexed sequence of open intervals such that
$\cup_{i=1}^{\infty^\#} E_i \subseteq \cup_{i=1}^{\infty^\#} \ \cup_{k=1}^{\infty^\#} I_i^k$ and
$\sum_{i=1}^{\infty^\#} \sum_{k=1}^{\infty^\#} l(I_i^k) \leq \sum_{i=1}^{\infty^\#} (\mu^*(E_i) + \varepsilon/2^i) = \sum_{i=1}^{\infty^\#} \mu^*(E_i) + \varepsilon.$
Therefore, $\mu^*\left(\cup_{i=1}^{\infty^\#} E_i\right) \leq \sum_{i=1}^{\infty^\#} \mu^*(E_i) + \varepsilon.$ The result follows since $\varepsilon > 0, \varepsilon \approx 0$ was arbitrary.
For part (f), we need to prove $\mu^*(I) \leq l(I)$ and $\mu^*(I) \geq l(I)$ respectively.
We can assume that $I = [a,b]$ where $a,b \in \mathbb{R}_c^\#$.
First, we want to prove $\mu^*(I) \leq l(I)$. Let $\varepsilon > 0, \varepsilon \approx 0,$ we have
$I \subseteq (a,b) \cup (a-\varepsilon, a+\varepsilon) \cup (b-\varepsilon, b+\varepsilon).$
Thus,$\mu^*(I) \leq l(a,b) + l(a-\varepsilon, a+\varepsilon) + l(b-\varepsilon, b+\varepsilon) =$
$= (b-a) + 2\varepsilon + 2\varepsilon = b - a + 4\varepsilon.$
As $\varepsilon > 0, \varepsilon \approx 0$ is arbitrary, we conclude that $\mu^*(I) \leq b-a = l(I).$
Then, we want to prove that $\mu^*(I) \geq l(I).$ Let $\{I_k\}$ be any sequence of open intervals that covers $I.$ Since I is compact, by the generalized Heine-Borel theorem, there is a gyperfinite subcollection $\{J_i | 1 \leq i \leq n\}, n \in \mathbb{N}^\#$ of I_k that still covers $I.$ By reordering and deleting if necessary, we can assume that
$a \in J_1 = (a_1, b_1), b_1 \in J_2 = (a_2, b_2), \ldots, b_{n-1} \in J_n = (a_n, b_n),$where $b_{n-1} \leq b < b_n.$
We then can compute that

$$b - a < b_n - a_1 = Ext\text{-}\sum_{i=2}^{n} (b_i - b_{i-1}) + (b_1 - a_1) < Ext\text{-}\sum_{i=1}^{n} l(J_i) \leq Ext\text{-}\sum_{i=1}^{\infty^\#} l(I_k).$$

Therefore, $l(I) \leq \mu^*(I).$ We can now conclude that $\mu^*(I) = l(I).$ This proves the result for closed and bounded intervals.
Suppose that $I = (a,b)$ is an open and bounded interval. Then, $\mu^*(I) \leq l(I)$ as above and $b - a = \mu^*([a,b]) \ \leq \mu^*((a,b)) \ + \mu^*(a) + \mu^*(b) = \mu^*((a,b)).$
Hence $l(I) \leq \mu^*(I).$ The proof for half-open intervals is similar.
Finally, suppose that I is an hyper infinite interval and let $M > 0.$ There exists a bounded interval $J \subseteq I$ such that $\mu^*(J) = l(J) = M$ and it follows that
$\mu^*(I) \geq \mu^*(J) = M.$ Since $M > 0$ was arbitrary, $\mu^*(I) = \infty^\# = l(I).$
This completes the proof.

2.2.External Lebesgue inner measure

In previous subsection, we have discussed external Lebesgue outer measure. There is
another external measure named external Lebesgue inner measure. Let's define the external inner measure and see some basic properties.

Definition 2.2. Let E be a subset of $\mathbb{R}^\#$. The external inner measure of E is defined by

$$\mu^*(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is \#-closed}\} \tag{2.3}$$

iff supremum in RHS of the (2.3) exists.
Recall that external Lebesgue outer measure of a set E uses an infimum of the union of a sequence open sets that cover the set $E,$ while external Lebesgue inner measure of a set E uses a supremum of a set inside the set $E.$ Then, it is obvious that

$$\mu_*(E) \leq \mu^*(E) \tag{2.4}$$

for any set $E.$ Also, for $A \subseteq B, \mu_*(A) \leq \mu^*(B).$

Theorem 2.2. Let A and E be subsets of $\mathbb{R}_c^\#.$

(i) Suppose that $\mu^*(E) < \infty^\#$. Then E is measurable if and only if $\mu_*(E) = \mu^*(E)$.
(ii) If E is measurable and $A \subseteq E$, then $\mu(E) = \mu_*(A) + \mu^*(E \backslash A)$.
Proof. For part (i), suppose that E is a measurable set and let $\varepsilon > 0, \varepsilon \approx 0$. According to Theorem 2.9, there exists a #-closed set K such that $K \subseteq E$ and $\mu(E \backslash K) < \varepsilon$.
Thus,$\mu^*(E) \geq \mu_*(E) \geq \mu(K) > \mu(E) - \varepsilon = \mu^*(E) - \varepsilon,$which implies that the external inner
measure and external outer measure of E are equal. Now let's prove the reverse direction. Suppose that $\mu_*(E) = \mu^*(E)$. Let $\varepsilon > 0, \varepsilon \approx 0$. Then
there exists a #-closed set K and an #-open set G such that $K \subseteq E \subseteq G$ and
$\mu(K) > \mu_*(E) - \varepsilon/2$ and $\mu(G) < \mu^*(E) + \varepsilon/2.$Then we find that
$\mu^*(G \backslash E) \leq \mu^*(G \backslash K) = \mu(G \backslash K) = \mu(G) - \mu(K) < \varepsilon.$
According to Theorem 2.9, the set E is measurable.
For part (2), let $\varepsilon > 0, \varepsilon \approx 0$. There exists a #-closed set $K \subseteq A$ such that
$\mu(K) > \mu_*(A) - \varepsilon$. Then, $\mu(E) = \mu(K) + \mu(E \backslash K) > \mu_*(A) - \varepsilon + \mu^*(E \backslash A)$
and it follows that $\mu(E) \geq \mu_*(A) + \mu^*(E \backslash A)$. According to Theorem 2.9, there exists a measurable set B such that $E \backslash A \subseteq B \subseteq E$ and $\mu(B) = \mu^*(E \backslash A)$.
Since $E \backslash B \subseteq A$, it follows that $\mu_*(E \backslash B) \geq \mu_*(A)$. Thus,
$\mu(E) = \mu(B) + \mu(E \backslash B) = \mu^*(E \backslash A) + \mu(E \backslash B) \leq \mu^*(E \backslash A) + \mu_*(A)$.
By combining these two inequalities, we can obtain $\mu(E) = \mu_*(A) + \mu^*(E \backslash A)$.

2.3.External Lebesgue measure

Definition 2.3. A set $E \subseteq \mathbb{R}_c^\#$ is Lebesgue measurable if for each set $A \subseteq \mathbb{R}_c^\#$, the equality $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement})$ is satisfied. If E is a Lebesgue measurable set, then the external Lebesgue measure of E is its external Lebesgue outer measure and will be written as $\mu(E)$.
Since the external Lebesgue outer measure satisfies the property of subadditivity, then we always have $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*\left(A \cap E^{\complement}\right)$,$E^{\complement} = \mathbb{R}_c^\# \backslash E$ and we only need to check the reverse inequality.
Note that there is always a set E that can divide A into two mutually exclusive sets, $A \cap E$ and $A \cap E^{\complement}$. But only when $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement})$ holds, the set E is Lebesgue measurable. The latter theorem will show some properties of measurable sets.
Theorem 2.3. The collection of measurable sets defined on $\mathbb{R}_c^\#$ has the following properties:
(a) Both $\varnothing$ and $\mathbb{R}_c^\#$ are measurable.
(b) If E is measurable, then $E^{\complement}$ is measurable,where $E^{\complement} = \mathbb{R}_c^\# \backslash E$.
(c) If $\mu^*(E) = 0$, then E is measurable.
(d) If E_1 and E_2 are measurable, then $E_1 \cup E_2$ and $E_2 \cap E_2$ are measurable.
(e) If E is measurable, then $E + x_0$ is measurable.
Proof. For part (a), let $A \subseteq \mathbb{R}_c^\#$.Then
$\mu^*(A \cap \varnothing) + \mu^*(A \cap \varnothing^{\complement}) = \mu^*(\varnothing) + \mu^*(A) = 0 + \mu^*(A) = \mu * (A)$,
$\mu * (A \cap \mathbb{R}_c^\#) + \mu * (A \cap \mathbb{R}_c^{\#\complement}) = \mu^*(A) + \mu^*(\varnothing) = \mu^*(A) + 0 = \mu^*(A)$.
For part (b), if E is measurable, then for every set $A \subseteq \mathbb{R}_c^\#$, such that
$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement})$. Then,
$\mu^*(A \cap E^{\complement}) + \mu^*(A \cap (E^{\complement}) = \mu^*(A \cap E^{\complement}) + \mu^*(A \cap E) = \mu^*(A)$.
For part (c), let $A \subseteq \mathbb{R}_c^\#$. Since $\mu^*(E) = 0$ and $A \cap E \subseteq E$, then $\mu^*(A \cap E) = 0$.

We can obtain that μ∗(A) ≥ μ∗(A ∩ E)= μ∗(A ∩ E) + μ∗(A ∩ E),which implies that μ∗(A) = μ∗(A ∩ E) + μ∗(A ∩ E) by Theorem 2.1 part (e).
For part (d), let $A \subseteq \mathbb{R}_c^{\#}$. Note that
$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2) = (A \cap E_1) \cup (A \cap E_1 \cap E_2)$
Then, by De Morgan Law and Theorem 14.1 part (e), we know that
$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1) =$
$= \mu^*(A \cap E_1) + \mu^*(A \cap E_1 \cap E_2) + \mu^*(A \cap E_1 \cap E_2) \geq \mu^*(A \cap (E_1 \cup E_2)) +$
$+\mu^*(A \cap (E_1 \cup E_2)),$
showing that $E_1 \cup E_2$ is measurable. Since $E_1 \cap E_2 = (E_1 \cup E_2)$, then the set $E_1 \cap E_2$ is measurable by Theorem 12.1 part (b).
For part (e), let $A \subseteq \mathbb{R}_c^{\#}$. Then,
$\mu^*(A) = \mu^*(A - x_0) = \mu^*((A - x_0) \cap E) + \mu^*((A - x_0) \cap E^{\complement}) =$
$\mu^*(((A - x_0) \cap E) + x_0) + \mu^*(((A - x_0) \cap E^{\complement}) + x_0) =$
$\mu^*(A \cap (E + x_0)) + \mu^*(A \cap (E^{\complement} + x_0)).$
Therefore, $E + x_0$ is measurable.
Lemma 2.1. Let $E_i : 1 \leq i \leq n \in \mathbb{N}^{\#}$ be a gyperfinite collection of disjoint measurable sets. If $A \subseteq \mathbb{R}_c^{\#}$, then
$\mu^*\left(\bigcup_{i=1}^{n}(A \cap E_i)\right) = \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = Ext\text{-}\sum_{i=1}^{n} \mu^*(A \cap E_i).$
Proof. We will prove this by the principle of mathematical induction. When $n = 1$, the equality holds. Suppose that the statement is valid for $n - 1$ disjoint measurable sets when $n > 1$. Then, when there are n disjoint measurable sets,

$\mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) =$
$= \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n\right) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n^{\complement}\right) =$
$= \mu^*(A \cap E_n) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right) =$
$= \mu^*(A \cap E_n) + Ext\text{-}\sum_{i=1}^{n-1} \mu^*(A \cap E_i) = Ext\text{-}\sum_{i=1}^{n} \mu^*(A \cap E_i).$
Note that when $A = \mathbb{R}_c^{\#}$, $\mu\left(\bigcup_{i=1}^{n} E_i\right) = Ext\text{-}\sum_{i=1}^{n} \mu(E_i).$
Theorem 2.4. If $\{E_i\}_{i=1}^{\infty^{\#}}$ is a hyper infinite sequence of disjoint measurable sets, then
$$\mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right) = Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(E_i). \tag{2.5}$$
Proof. According to Lemma 2.1, $Ext\text{-}\sum_{i=1}^{n} \mu(E_i) = \mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right)$
for each positive integer $n \in \mathbb{N}^{\#}$, which implies that $Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right)$.
By #-countably subadditive property,$Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(E_i) \geq \mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right)$.
Therefore, $Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(E_i) = \mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right)$.
The previous theorem shows that if A and B are disjoint measurable sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$. If $\{A_i\}_{i \in \mathbb{N}^{\#}}$ is a hyper infinite sequence of disjoint measurable sets,then $\mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right) = Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(A_i)$. As so far, we have already seen that when the sets are measurable, Lebesgue measure satisfies property

(1),(2),(3) and (4). But what kinds of sets are measurable? Certainly every interval is measurable.

Theorem 2.5. Every interval $[a,b] \subset \mathbb{R}_c^{\#}$ is measurable.

Theorem 2.6. If $\{E_i\}_{i\in\mathbb{N}^{\#}}$ is a hyper infinite sequence of measurable sets, then

are measurable sets.

Definition 2.4. Let *f* be a function from $E \subset \mathbb{R}_c^{\#}$ into $\mathbb{R}_c^{\#} \cup (-\infty^{\#}, \infty^{\#})$. The function *f* is (Lebesgue) measurable if

.

3.External Lebesgue Integral

Let $(\mathbb{R}, B, \mu)$ be the standard Lebesgue space on $\mathbb{R}$. Our internal starting point could be the internal measure space $({}^*\mathbb{R}, {}^*B, {}^*\mu)$. By transfer we can write down internal Lebesgue integrals

$$\mathcal{L}[{}^*f(t)] = \int_A {}^*f(t)\,d{}^*\mu(t),$$

where $A \in {}^*B$ and $f : \mathbb{R} \to \mathbb{R}$.

3.1.Lebesgue Integral of a $\mathbb{R}_c^{\#}$-valued external function $f(x)$.

First, in particular, we need external function that can help us distinguish whether a given value x is in the measurable set A_i. We call this function the characteristic function. The following statement is the formal definition of characteristic function and introduces the simple function.

Definition 3.1. For any set A, the function

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

is called the characteristic function of set A. A linear combination of characteristic functions,

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x) \tag{3.2}$$

is called a simple function if the sets A_i are measurable.

For a function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_c^{\#}$ defined on a measurable set A that takes no more than gyper finitely many distinct values $a_1, \ldots, a_n, n \in \mathbb{N}^{\#}$ the function *f* can always be written as a simple function

$$f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x), \tag{3.3}$$

where $A_i = \{x \in A |\, f(x) = a_i\}$. That is a simple function of the first kind. Therefore, simple functions can be thought of as dividing the range of *f*, where resulting sets A_i may or may not be intervals.

Let us pause for a second. We want to ask ourselves: is the simple function $\phi(x)$ unique? The answer is no. Because we might define different disjoint sets that have a same function value. The simplest expression is

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x) \tag{3.4}$$

where $A_i = \{x \in A | \phi(x) = a_i\}$. At this case, the constants a_i are distinct, the sets A_i are disjoint and we call that representation the canonical representation of φ.

Then, for simple functions, we define the Lebesgue integral as follows:

Definition 3.2. If $\phi(x) = \sum_{i=1}^{n} a_i\chi_{A_i}(x)$ is a simple function and $\mu(A_i)$ is gyperfinite for all i, then the Lebesgue integral of $\phi(x)$ is defined as

$$\int_E \phi(x) = \sum_{i=1}^{n} a_i\chi_{A_i}(x). \tag{3.5}$$

Definition 3.3. Suppose $f : \mathbb{R}_c^{\#} \to \mathbb{R}_c^{\#}$ is a bounded function defined on a measurable set E with giperfinite measure. We define the upper and lower Lebesgue integrals if exist, respectively, as

$$I_L^{\#}(f) = \int_E \inf\left\{\phi(x)|\phi \text{ is simple and } \phi \geq f\right\} \tag{3.6}$$

and

$$I_{\#L}(f) = \int_E \sup\left\{\phi(x)|\phi \text{ is simple and } \phi \leq f\right\}. \tag{3.7}$$

If (i) the quantity $I_L^{\#}(f)$ and $I_{\#L}(f)$ exist and (ii) $I_L^{\#}(f) = I_{\#L}(f)$, then the function f is called Lebesgue integrable over set E and the external Lebesgue integral of f over set E is denoted by $I_L(f) = \int_E f dx$.

The Lebesgue Integral for Simple Functions of the second kind

Let $\varphi(x)$ be some simple external function of the second kind which takes on the gyperinfinitely many distinct values $y_1,\ldots,y_n,\ldots,n \in \mathbb{N}^{\#}, y_i \neq y_j$ for $i \neq j$.

It is natural to define the integral of the function $\varphi(x)$ over the set E by the equation

$$\int_E \varphi(x)d^{\#}\mu = \sum_{n\in\mathbb{N}^{\#}} y_n\mu\{x|x \in E, \varphi(x) = y_n\}. \tag{3.8}$$

Definition 3.4.The simple function $\varphi(x)$ of the second kind is called integrable (with respect to the measure μ) over the set E if the gyperinfinite series (15.8) #-converges absolutely.

If $\varphi(x)$ is #-integrable, then the sum of the series (15.8) is called the integral of $\varphi(x)$ over the set E.

Remark 3.1. Note that in definition 15.4 we assume that all the y_n are different. One can, however, represent the value of the integral of a simple function as a sum of products of the form $c_k\mu(B_k)$ and not assume that all the c_k are different.

Lemma 3.1. Let $A = \cup_k B_k, B_i \cap B_j = \varnothing$ for$i \neq j$, and assume that on each set B_k the function $f(x)$ takes on only one value c_k. Then

$$\int_A \varphi(x)d^{\#}\mu = \sum_{k\in\mathbb{N}^{\#}} c_k\mu(B_k). \tag{3.9}$$

moreover, the function $f(x)$ is integrable over A if and only if the gyper infinite series (3.9) #-converges absolutely.

Proof. It is easy to see that every set $A_n = \{x|x \in A, f(x) = y_n\}$ is the union of those B_k for which $c_k = y_n$. Therefore $\sum_{n\in\mathbb{N}^{\#}} y_n\mu(A_n) = \sum_{n\in\mathbb{N}^{\#}} y_n \sum_{c_k=y_k} \mu(B_k) = \sum_{k\in\mathbb{N}^{\#}} c_k\mu(B_k)$. Since the measure is non-negative, $\sum_{n\in\mathbb{N}^{\#}}|y_n|\mu(A_n) = \sum_{n\in\mathbb{N}^{\#}}|y_n|\sum_{c_k=y_k} \mu(B_k) = \sum_{k\in\mathbb{N}^{\#}}|c_k|\mu(B_k)$.

i.e., the series $\sum_{n\in\mathbb{N}^{\#}} y_n\mu(A_n)$ and $\sum_{k\in\mathbb{N}^{\#}}|c_k|\mu(B_k)$ both either #-converge absolutely or #-diverge.

Let us consider some properties of the external Lebesgue integral for simple external functions:

$$\int_A f(x)d^{\#}\mu + \int_A g(x)d^{\#}\mu = \int_A [f(x) + g(x)]d^{\#}\mu \tag{3.10}$$

moreover, from the existence of the integrals on the left-hand side it follows that the integrals on the right-hand side exist.

To prove this assume that $f(x)$ takes on the values f_i, on the sets $F_i \subseteq A$, and $g(x)$ the values g_i, on the sets $G_i \subseteq A$, since

$$J_1 = \int_A f(x)d^\# \mu = \sum_{i\in\mathbb{N}^\#} f_i \mu(F_i) \tag{3.11}$$

and

$$J_2 = \int_A g(x)d^\# \mu = \sum_{i\in\mathbb{N}^\#} g_i \mu(G_j). \tag{3.12}$$

Then, by the Lemma 2.1 we get

$$J = \int_A [f(x) + g(x)]d^\# \mu = \sum_{i\in\mathbb{N}^\#} \sum_{j\in\mathbb{N}^\#} [f_i + g_j]\mu(F_i \cap G_j), \tag{3.13}$$

where

$$\mu(F_i) = \sum_{j\in\mathbb{N}^\#} \mu(F_i \cap G_j), \mu(G_j) = \sum_{i\in\mathbb{N}^\#} \mu(F_i \cap G_j). \tag{3.14}$$

From the absolute #-convergence of the series (3.11)-(3.12) it follows the absolute #-convergence of the series (3.13); here $J = J_1 + J_2$.

For any constant $k \in \mathbb{R}_c^\#$

$$k \int_A f(x)d^\# \mu = \int_A [kf(x)]d^\# \mu \tag{3.15}$$

moreover, the existence of the integral on the left-hand side implies the existence of the integral on the right.A simple function $f(x)$ which is bounded on the set $A \subset \mathbb{R}_c^\#$ is #-integrable over A; moreover, if $|f(x)| \leq M \in \mathbb{R}_c^\#$ on A, then

$$\left| \int_A f(x)d^\# \mu \right| \leq M\mu(A). \tag{3.16}$$

4.General Definition and Basic Properties of the external Lebesgue Integral.

Definition.4.1. We shall say that the function $f(z)$ is #-integrable over the set $A \subset \mathbb{R}_c^\#$, if there exists a hyper infinite sequence of simple functions $f_n(z), n \in \mathbb{N}^\#$ which are #-integrable over A and #-converge uniformly to $f(x)$. We shall denote the #-limit

$$J = \#\text{-}\mathbf{lim}_{n\to\infty^\#} \int_A f_n(x)d^\# \mu \tag{4.1}$$

by

$$\int_A f(x)d^\# \mu. \tag{4.2}$$

and call it the integral of the external function $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$ over the set A.

This definition 4.1 is correct if the following conditions are satisfied:

1.The #-limit (4.1) for any uniformly #-convergent hyperinfinite sequence of simple functions which are #-integrable over A exists.

2.This #-limit for a given function f(x) does not depend on the choice of the hyperinfinite sequence $\{f_n(x)\}_{n\in\mathbb{N}^\#}$.

3.For simple functions the definitions of #-integrability and #-integral are equivalent to those given in section 3.

Notice that all these conditions are indeed satisfied.

To prove the first it suffices to note that by properties for #-integrals of simple functions,

$$\left|\int_A f_n(x)d^\# \mu - \int_A f_m(x)d^\# \mu\right| \leq \mu(A)\sup_{x\in A}|f_n(x) - f_m(x)|. \tag{4.3}$$

To prove the second condition, we must consider the two hyperinfinite sequences $\{f_n(x)\}_{n\in\mathbb{N}^\#}$ and $\{f_n'(x)\}_{n\in\mathbb{N}^\#}$, and use the inequality

$$\left|\int_A f_n(x)d^\# \mu - \int_A f_n'(x)d^\# \mu\right| \leq \mu(A)\left[\sup_{x\in A}|f_n(x) - f(x)| + \sup_{x\in A}\left|f_n'(x) - f(x)\right|\right]. \tag{4.4}$$

Finally, to prove the third condition it suffices to consider the hyperinfinite sequence $f_n(x) = f(x)$.

The basic properties of the external Lebesgue #-**integral**.

Theorem 4.1.

$$\int_A 1 \cdot d^\# \mu = \mu(A). \tag{4.5}$$

Proof. Immediately from the definition of the #-integral.

Theorem 4.2.For any constant $k \in \mathbb{R}_c^\#$

$$k\int_A f(x)d^\# \mu = \int_A [kf(x)]d^\# \mu \tag{4.6}$$

where the existence of the #-integral on the left-hand side implies the existence of the #-integral on the right.

Proof. The proof is obtained from property (8.15) by proceeding to the #-limit for an #-integral of simple functions.

Theorem 4.3. Assume that $f(x)$ and $g(x)$ are #-integrable over A then $f(x) + g(x)$ #-integrable over A and

$$\int_A f(x)d^\# \mu + \int_A g(x)d^\# \mu = \int_A [f(x) + g(x)]d^\# \mu \tag{4.7}$$

Let $\{f_i(x)\}_{i=1}^n, n \in \mathbb{N}^\#$ be a hyperfinite sequence such that any $f_i(x)$ is #-integrable over A

then $\sum_{i=1}^n f_i(x)$ is #-integrable over A and

$$\sum_{i=1}^n \int_A f_i(x)d^\# \mu = \int_A \left[\sum_{i=1}^n f_i(x)\right]d^\# \mu \tag{4.8}$$

where the existence of the #-integrals on the left implies the existence of the #-integral on the right.

Proof. The proof of (4.7) is obtained from property A) by proceeding to the #-limit for an

#-integral of simple functions.

Theorem 4.4. A function $f : A \to \mathbb{R}_c^\#$ which is hyperbounded on the set A is #-integrable over A.

Proof. The proof is obtained from property C) by proceeding to the limit for an integral of

simple functions.

Theorem 4.5. If $f(x) \geq 0,$ then

$$\int_A f(x)d^\# \mu \geq 0 \tag{4.9}$$

assuming that the #-integral exists.

Proof. For simple functions this follows immediately from the definition; for the general case the proof is based on the possibility of approximating non-negative functions by non-negative simple functions

Corollary 4.1. If $f(x) \geq g(x)$, then

$$\int_A f(x)d^{\#}\mu \geq \int_A g(x)d^{\#}\mu. \tag{4.10}$$

Chapter III. ${}^*\mathbb{R}_c^{\#}$-Valued abstract measures

1. $\sigma^{\#}$-algebras

Definition 1.1 ($\sigma^{\#}$-algebra). Let X be any set. We denote by $2^X = P(X) = \{A : A \subset X\}$ the set of all subsets of X. A family $\mathcal{F} \subset 2^X$ is called a $\sigma^{\#}$-algebra (on X) if:

(i) $\varnothing \in \mathcal{F}$;

(ii) $\mathcal{F}$ is closed under complements, i.e. $A \in \mathcal{F}$ implies $X\backslash A \in \mathcal{F}$;

(iii) $\mathcal{F}$ is closed under hypercountable unions, i.e. if $(A_n)_{n\in\mathbb{N}^{\#}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcup_{n\in\mathbb{N}^{\#}} A_n \in \mathcal{F}$.

Proposition 1.1. If $\mathcal{F}$ is a $\sigma^{\#}$-algebra on X then:

1. $\mathcal{F}$ is closed under hypercountable intersections, i.e. if $(A_n)_{n\in\mathbb{N}^{\#}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcap\limits_{n\in\mathbb{N}^{\#}} A_n \in \mathcal{F}$.
2. $X \in \mathcal{F}$.
3. $\mathcal{F}$ is closed under hyperfinite unions and hyperfinite intersections.
4. $\mathcal{F}$ is closed under set differences.
5. $\mathcal{F}$ is closed under symmetric differences.

Proposition 1.2. Suppose $\mathcal{F} \subset 2^X$ is a family of subsets satisfying the following:

1. $\varnothing \in \mathcal{F}$;
2. $\mathcal{F}$ is closed under complements;
3. $\mathcal{F}$ is closed under hyperinfinite intersections.

Then $\mathcal{F}$ is a $\sigma^{\#}$-algebra.

Proposition 1.3. If $(\mathcal{F}_\alpha)_{\alpha\in I}$ is a collection of $\sigma^{\#}$-algebras on X, then $\bigcap\limits_{\alpha}\mathcal{F}_\alpha$ is also a $\sigma^{\#}$-algebra on X.

Proposition 1.4. ($\sigma^{\#}$-algebra generated by subsets). Let K be a collection of subsets of X. There exists a $\sigma^{\#}$-algebra, denoted $\sigma^{\#}(K)$ such that $K \subset \sigma^{\#}(K)$ and for every other $\sigma^{\#}$ algebra $\mathcal{F}$ such that $K \subset \mathcal{F}$ we have that $\sigma^{\#}(K) \subset \mathcal{F}$

We call $\sigma^{\#}(K)$ the $\sigma^{\#}$-algebra generated by K.

Proof. Define $\sigma^{\#}(K) \triangleq \bigcap\{\mathcal{F}|\mathcal{F}$ is a $\sigma^{\#}$-algebra on $X, K \subset \mathcal{F}\}$.

This is a $\sigma^{\#}$-algebra with the required properties.

Proposition 1.5. If $K \subset \mathcal{L}$ then $\sigma^{\#}(K) \subset \sigma^{\#}(\mathcal{L})$. Also, if $K \subset \mathcal{F}$ and $\mathcal{F}$ is a $\sigma^{\#}$-algebra, then $\sigma^{\#}(K) \subset \mathcal{F}$.

Definition 1.2. (Borel $\sigma^{\#}$-algebra). Given a topological space X, the Borel $\sigma^{\#}$-algebra is the $\sigma^{\#}$-algebra generated by the open sets. It is denoted $B^{\#}(X)$.

Specifically in the case $X = {}^*\mathbb{R}_c^{\#d}, d \in \mathbb{N}^{\#}$ we have that

$B_d^{\#} \triangleq B^{\#}({}^*\mathbb{R}_c^{\#d}) = \sigma^{\#}(U|U$ is an #-open set).

A Borel-#-measurable set, i.e. a set in $B^{\#}(X)$, is called a #-Borel set.

Measurable functions. Let f be a ${}^*\mathbb{R}_c^{\#}$-valued function defined on a set X. We suppose that some $\sigma^{\#}$-algebra $\Omega \subseteq P(X)$ is fixed.

Definition 1.3. We say that f is #-measurable, if $f^{-1}([a,b]) \in \Omega$ for any hyperreals

$a,b \in {}^*\mathbb{R}_c^\#$ such that $a < b$.
The following three propositions are obvious.
Proposition 1.7. Let $f : X \to {}^*\mathbb{R}_c^\#$ be a function. Then the following conditions are equivalent:
(a) f is #-measurable;
(b) $f^{-1}([0,b)) \in \Omega$ for any hyperreal $b \in {}^*\mathbb{R}_c^\#$;
(c) $f^{-1}((b,\infty)) \in \Omega$ for any hyperreal $b \in {}^*\mathbb{R}_c^\#$;
(d) $f^{-1}(B) \in \Omega$ for any $B \in B(R)$.
Proposition 1.8 Let f and g be #-measurable functions, then
(a) $\alpha \times f + \beta \times g$ is #-measurable for any $\alpha, \beta \in {}^*\mathbb{R}_c^\#$;
(b) functions $\max\{f,g\}$ and $f \times g$ are #-measurable.
In particular, functions $f^+ := \max\{f,0\}, f^- := (-f)^+$, and $|f| := f^+ + f^-$ are #-measurable.   .

§2.#-Measures and measure #-space

Definition 2.1. A pair $(X,\mathcal{F})$ where $\mathcal{F}$ is a $\sigma^\#$-algebra on X is call a #-measurable space. Elements of $\mathcal{F}$ are called #-measurable sets.
Given a #-measurable space $(X,\mathcal{F})$, a function $\mu^\# : \mathcal{F} \to [0,\infty^\#]$ is called a #-measure on $(X,\mathcal{F})$ if
1. $\mu^\#(\varnothing) = 0$;
2. (Hyper infinite additivity) For all hyper infinite sequences $(A_n)_{n\in\mathbb{N}^\#} \subset \mathcal{F}$ of pairwise disjoint sets in $\mathcal{F}$, we have that $\mu^\#\left(\bigcup_{n\in\mathbb{N}^\#} A_n\right) = Ext\text{-}\sum_{n\in\mathbb{N}^\#} \mu^\#(A_n)$.
$(X,\mathcal{F},\mu^\#)$ is called a #-measure space.
Definition 2.2. A measure space $(X,\mathcal{F},\mu^\#)$ is called: (a) hyperfinite if $\mu^\#(X) < \infty^\#$.
(b)It is called $\sigma^\#$-hyperfinite if $X = \bigcup_{n\in\mathbb{N}^\#} A_n$ where $A_n \in \mathcal{F}$ and $\mu^\#(A_n) < \infty^\#$ for all $n \in \mathbb{N}^\#$.
Definition 2.3. Let Σ be a $\sigma^\#$-algebra of subsets of a set X, and let $E = (E, \|\cdot\|_\#)$ be a non-Archimedean Banach space.A function $\mu^\# : \Sigma \to E \cup \{{}^*\infty\}$ is called a vector-valued #-measure (or E-valued measure) if
(a) $\mu^\#(\varnothing) = 0$;
(b) $\mu^\#\left(\bigcup_{n\in\mathbb{N}^\#} A_n\right) = Ext\text{-}\sum_{n\in\mathbb{N}^\#} \mu^\#(A_n)$ for any pairwise disjoint sequence $A_n, n \in \mathbb{N}^\#$, $A_n \subseteq \Sigma$;
(c) for any $S \in \Sigma,\ \mu^\#(S) = \infty$, there exists $B \in \Sigma$ such that $B \subseteq S$ and $0 < \|\mu^\#(B)\|_\# < {}^*\infty$.
Definition 2.4.(a) A function $\mu^\# : \mathcal{F} \to {}^*\mathbb{C}_c^\# \cup \{{}^*\infty\}$ is called a complex #-measure if
1.$\mu^\#(\emptyset) = 0$,
2.$\mu^\#\left(\bigcup_{n\in\mathbb{N}^\#} A_n\right) = Ext\text{-}\sum_{n\in\mathbb{N}^\#} \mu^\#(A_n)$ for any sequence $A_n, n \in \mathbb{N}^\#$ of pairwise disjoint sets from $\mathcal{F}$, and, for any $A \in \mathcal{F}, \mu^\#(A) = {}^*\infty$, there exists $B \in \mathcal{F}$ such that $B \subseteq A$ and $0 < |\mu^\#(B)|_\# < {}^*\infty$.
(b) A function $\mu^\# : \mathcal{F} \to {}^*\mathbb{R}_c^\# \cup \{{}^*\infty\}$ is called a signed #-measure if

$\mu^{\#}(\emptyset) = 0$

$\mu^{\#}\left(\bigcup_{n\in\mathbb{N}^{\#}} A_n\right) = Ext\text{-}\sum_{n\in\mathbb{N}^{\#}} \mu^{\#}(A_n)$ for any sequence $A_n, n \in \mathbb{N}^{\#}$ of pairwise disjoint sets from $\mathcal{F}$, , and, for any $A \in \mathcal{F}, \mu^{\#}(A) = {}^{*}\infty,$ there exists $B \in \mathcal{F}$ such that $B \subseteq A$ and $0 < |\mu^{\#}(B)| < {}^{*}\infty$.

Definition 2.5. If a certain property involving the points of #-measure space is true, except a subset having #-measure zero, then we say that this property is true #-almost everywhere (abbreviated as #-a.e.).

Proposition 2.5. Let $\mu^{\#}$ be a #-measure on a $\sigma^{\#}$-algebra $\mathcal{F}, A_n \in \mathcal{F}$, and $A_n \to A$. Then $A \in \mathcal{F}$ and $\mu^{\#}(A) = \#\text{-}\lim_{n\to{}^{*}\infty} \mu^{\#}(A_n)$. In particular, if $(B_n)_{n=1}^{{}^{*}\infty}$ is a decreasing hyper infinite sequence of elements of $\mathcal{F}$ such that $\bigcap_{n=1}^{{}^{*}\infty} B_n = \emptyset,$ then $\mu^{\#}(B_n) \to_{\#} 0$.

Definition 2.6. If $\mathcal{F}$ is a $\sigma^{\#}$-algebra of subsets of X and $\mu^{\#}$ is a #-measure on $\mathcal{F}$, then the triple $(X, \mathcal{F}, \mu)$ is called a #-measure space. The sets belonging to $\mathcal{F}$ are called #-measurable sets because the #-measure is defined for them. .

§2.1.#-Convergence of functions and the generalized Egoroff theorem.

Definition 2.1.1. Let $f_n, n \in \mathbb{N}^{\#}$ be a hyper infinite sequence of ${}^{*}\mathbb{R}_c^{\#}$-valued functions defined on X. We say that:

1. $f_n \to_{\#} f$ pointwise, if $f_n(x) \to_{\#} f(x)$ for all $x \in X$;
2. $f_n \to_{\#} f$ almost #-everywhere (#-a.e.), if $f_n(x) \to_{\#} f(x)$ for all $x \in X$ except a set of #-measure 0;
3. $f_n \to_{\#} f$ uniformly, if for any $\varepsilon > 0, \varepsilon \approx 0$ there is $n(\varepsilon)$ such that $\sup\{|f_n(x) - f(x)| : x \in X\} \leq \varepsilon$ for all $n \geq n(\varepsilon)$.

Theorem 2.1.1. (generalized Egoroff 's theorem) Suppose that $\mu^{\#}(X) < {}^{*}\infty$, $\{f_n\}_{n=1}^{{}^{*}\infty}$ and f are #-measurable functions on X such that $f_n \to_{\#} f$ #-a.e. Then, for every $\varepsilon \approx 0, \varepsilon > 0,$ there exists $E \subseteq X$ such that $\mu^{\#}(E) < \varepsilon$ and $f_n \to_{\#} f$ uniformly on $E^c = X \backslash E$.

Proof: Without loss of generality, we may assume that $f_n \to_{\#} f$ everywhere on X and (by replacing f_n with $f_n - f$) that $f \equiv 0$. For $k,\ n \in {}^{*}\mathbb{N},$ let

$E_n(k) := \bigcup_{m=n}^{{}^{*}\infty} \{x : |f_m(x)| \geq k-1\}.$ Then, for a fixed $k, E_n(k)$ decreases as n increases, and $\bigcap_{n=1}^{{}^{*}\infty} E_n(k) = \emptyset$. Since $\mu^{\#}(X) < {}^{*}\infty$, we conclude that $\mu^{\#}(E_n(k)) \ \to_{\#} 0$ as $n \to {}^{*}\infty$.

Given $\varepsilon \approx 0, \varepsilon > 0$ and ${}^{*}\mathbb{N}$, choose n_k such that $\mu^{\#}(En_k(k)) \ < \varepsilon \times 2^{-k}$, and set

$E = \bigcup_{n=1}^{{}^{*}\infty} En_k(k).$ Then $\mu^{\#}(E) < \varepsilon,$ and we have $|f_n(x)| < k^{-1}(\forall n > n_k, x \notin E)$.

Thus $f_n \to_{\#} 0$ uniformly on $X \backslash E$.

Generalized exhaustion argument.

Let $(X, \Sigma, \mu^{\#})$ be a $\sigma^{\#}$-finite #-measure space. Given a hyper infinite sequence $(U_n)_{n=1}^{{}^{*}\infty} \subseteq \Sigma,$ a set $A \in \Sigma$ is called $(U_n)_n$-bounded if there exists $n \in {}^{*}\mathbb{N}$ such that $A \subseteq U_n$ $\mu^{\#}$-almost everywhere.

Theorem 2.1.2. (Generalized Exhaustion theorem) Let $(Y_n)_{n=1}^{{}^{*}\infty} \subseteq \Sigma$ be a

hyper infinite sequence satisfying $Y_n \uparrow X$ and $\mu^{\#}(Y_n) < {}^*\infty$ for all $n \in {}^*\mathbb{N}$.
Let P be some property of $(Y_n)_n$-bounded
#-measurable sets, such that $A \in P$ iff $B \in P$ for all $B, \mu^{\#}(A\Delta B) = 0$. Suppose that any $(Y_n)_n$-bounded set A, $\mu^{\#}(A) > 0$, has a subset $B \in \Sigma, \mu^{\#}(B) > 0$ with the property P. Moreover, assume that either
(a) $A_1 \cup A_2 \in P$ for every $A_1, A_2 \in P$, or
(b) $\bigcup_{n \in {}^*\mathbb{N}} B_n \in P$ for every at most hyper infinite family $(B_n)_n$ of pairwise disjoint sets possessing the property P.
Then there exists hyper infinite sequence $(X_n)_{n=1}^{*\infty} \subseteq \Sigma$ such that $X_n \uparrow X$, and
$P \ni X_n \subseteq Y_n$
for all $n \in {}^*\mathbb{N}$. Moreover, there exists a pairwise disjoint sequence $(A_n)_{n=1}^{*\infty} \subseteq \Sigma$ such that $\bigcup_{n \in {}^*\mathbb{N}} A_n = X$ and $A_n \in P$ for all $n \in {}^*\mathbb{N}$.
Proof: Let A be a $(Y_n)_n$-bounded set with $\mu^{\#}(A) > 0$. Denote
$P_A := \{B \in P : B \subseteq A\} \bigwedge m(A) := \sup\{\mu^{\#}(B) : B \in P_A\}$.
I(a) Suppose P satisfies (a). Then there exists a sequence $(F_n)_{n=1}^{*\infty} \subseteq P_A$ such that $m(A) = \#\text{-}\lim_{n \to {}^*\infty} \mu^{\#}(F_n)$, We may assume, that $F_n \uparrow$. By Proposition 2.5 the set $F = \bigcup_{n=1}^{*\infty} F_n$ satisfies $\mu^{\#}(F) = m(A)$. We show that $\mu^{\#}(A) = m(A)$. If not then $\mu^{\#}(A \backslash F) > 0$. The set $A \backslash F$ has a subset of positive #-measure $F_0 \in P$.
Then $F_n \cup F_0 \in P_A$ and $\mu^{\#}(F_n \cup F_0) > m(A)$ for a sufficiently large $n \in {}^*\mathbb{N}$, which contradicts to the definition of $m(A)$. Therefore, $\mu^{\#}(A) = m(A)$.
Now we apply this for $A = Y_n$. Thus, there exists hyper infinite sequence $(X'_n)_n \subseteq \Sigma$ such that $X'_n \subseteq Y_n$, $X'_n, n \in P$, and $\mu^{\#}(Y_n \backslash X'_n) < n^{-1}$for all $n \in {}^*\mathbb{N}$. By (a), we may assume that $X'_n \uparrow$. The set $X'_0 = \bigcup_{n=1}^{*\infty} X'_n$ satisfies $Y_n \backslash X'_0 \subseteq Y_n \backslash X'_n$, so $\mu^{\#}(Y_n \backslash X'_0) < n^{-1}$ for all $n \in {}^*\mathbb{N}$. Then $\mu^{\#}(Y_n \backslash X'_0) = 0$, and $\mu^{\#}((\bigcup_{n=1}^{*\infty} Y_n) \backslash X'_0) = 0$, or $\mu^{\#}(X \backslash X'_0) = 0$.
Let $X_n = (X'_n \cup (X \backslash X'_0) \cap Y_n$, then the hyper infinite sequence $(X_n)_n$ has the required properties. The desired pairwise disjoint sequence $(A_n)_{n=1}^{*\infty}$ is given recurrently by $A_1 = X_1$and $A_{k+1} = X_{k+1} \backslash \bigcup_{i=1}^{k} A_i$.
I(b) Suppose P satisfies (b). Let F_A be the family of all pairwise disjoint families of elements of P_A of nonzero measure. Then F_A is ordered by inclusion and, obviously, satisfies the conditions of the Zorn lemma. Therefore, we have a maximal element in F_A, say Δ. Then Δ is at most hyper infinite family, say $\Delta = \{D_n\}_n$. By (b), its union $D = \bigcup_n D_n$ is an element of P_A as well. If D is a proper subset of A, then $\mu^{\#}(A \backslash D) > 0$. The set $A \backslash D$ has a subset $F \in P$ of the positive measure. Then $\Delta_1 := \Delta \cup \{F\}$ is an element of F_A which is strictly greater then Δ. The obtained contradiction, shows that $A \in P$ for every $(Y_n)_n$-bounded set A. So, we may take $X_n = Y_n$ for each $n \in {}^*\mathbb{N}$.
Now we apply this for $A = Z_m = Y_m \backslash \bigcup_{k=1}^{m-1} Y_k$ be a pairwise disjoint union, where $D_n^m \in P$ for all $n, m \in {}^*\mathbb{N}$. The family $\{D_n^m\}_{n,m}$ is an at most hyper infinite disjoint decomposition of X, say $\{D_n^m\}_{n,m} = (A_n)_{n=1}^{*\infty}$. The sequence $(A_n)_{n=1}^{*\infty}$ satisfies the required properties.
Theorem 2.1.3.(The generalized Borel-Cantelli lemma) Let $(X, \Sigma, \mu^{\#})$ be a #-measure space. Assume that $\{A_n\}_n \subseteq \Sigma$ and $Ext\text{-}\sum_{n=1}^{*\infty} \mu(A_n) < {}^*\infty$ then $\limsup_{n \to {}^*\infty} \mu^{\#}(A_n) = 0$.   .

§2.2.Vector-valued #-measures

In this section, we extend the notion of a measure. Then we study the basic operations with signed measures and present the Jordan decomposition theorem.

2.2.1. Vector-valued, signed and complex #-measures.

Let $\Sigma^{\#}$ be a $\sigma^{\#}$-algebra of subsets of a set X, and let $E^{\#} = (E^{\#}, \|\cdot\|_{\#})$ be a non-Archimedean Banach space.

Definition 2.2.1 A function $\mu^{\#} : \Sigma^{\#} \to E^{\#} \cup \{{}^*\infty\}$ is called a vector-valued #-measure (or $E^{\#}$-valued measure) if

(a) $\mu^{\#}(\emptyset) = 0$;

(b) $\mu^{\#}(\cup_{k=1}^{*\infty} A_k) = Ext\text{-}\sum_{k=1}^{*\infty} \mu^{\#}(A_k)$ for any pairwise disjoint sequence $(A_k)_k \subseteq \Sigma^{\#}$;

(c) for any $A \in \Sigma^{\#}, \mu^{\#}(A) = {}^*\infty,$ there exists $B \in \Sigma^{\#}$ such that $B \subseteq A$ and $0 < \|\mu^{\#}(B)\|_{\#} < {}^*\infty$.

Example 2.2.1 Take $\Sigma^{\#} = P({}^*\mathbb{N})$, and $c_0^{\#}$ is the non-Archimedean Banach space of all #-convergent $\mathbb{C}_c^{\#}$-valuedhyper infinite sequences with a fixed element $(\alpha_n)_n \in c_0^{\#}$. Define now for any $A \subseteq \mathbb{N}\psi(A) := (\beta_n)_n$, where $\beta_n = \alpha_n$ if $n \in A$ and $\beta_n = 0$ if $n \notin A$. Then ψ is a $c_0^{\#}$-valued #-measure on $P({}^*\mathbb{N})$.

Example 2.2.2 Let X be a set and let Ω be a $\sigma^{\#}$-algebra in $P(X)$. Then for any family $\{\mu_k\}_{k=1}^m$of finite #-measures on Ω and for any family $\{w_k\}_{k=1}^m$ of vectors of $\mathbb{R}_c^{\#n}$, the $\mathbb{R}_c^{\#n}$-valued #-measure Ψ on Ω is defined by the formula $\Psi(E) = Ext\text{-}\sum_{k=1}^m \mu_k(E) \times w_k, (E \in \Omega)$.

Example2.2.3 Let X be a set and let Ω be a $\sigma^{\#}$-algebra in $P(X)$. Then for any family $\{\mu_k\}_{k=1}^m$of finite #-measures on Ω, for any family $\{A_k\}_{k=1}^m$ of pairwise disjoint sets in Ω, and for any family $\{w_k\}_{k=1}^m$ of $\mathbb{R}_c^{\#n}, n \in {}^*\mathbb{N}$, the $\mathbb{R}_c^{\#n}$-valued #-measure Φ on Ω is defined by the formula $\Phi(E) = Ext\text{-}\sum_{k=1}^m \mu_k(E \cap A_k) \times w_k$, $(E \in \Omega)$.

§3. The Lebesgue #-Integral

In the following consideration, we fix a $\sigma^{\#}$-finite #-measure space $(X, \mathcal{F}, \mu^{\#})$.

Definition 3.1. Let $A_i \in \mathcal{F}, i = 1,\ldots,n \in {}^*\mathbb{N}$, be such that $\mu^{\#}(A_i) < {}^*\infty$ for all i, and $A_i \cap A_j = \emptyset$ for all $i \neq j$. The external function

$$f(x) = Ext\text{-}\sum_{i=1}^n \lambda_i \chi_{A_i}(x), \tag{3.1}$$

$\lambda_i \in {}^*\mathbb{R}_c^{\#}$, is called a simple external function. The Lebesgue external integral (Lebesgue #-integral) of a simple external function $f(x)$ is defined as

$$Ext\text{-}\int_X f(x) d^{\#}\mu^{\#} = Ext\text{-}\sum_{i=1}^n \lambda_i \mu^{\#}(A_i). \tag{3.2}$$

The Lebesgue external integral of a simple function is well defined.

Notation 3.1. Let $A_i \in \mathcal{F}, i = 1,\ldots,n \in {}^*\mathbb{N}$, be such that $\mu^{\#}(A_i) < {}^*\infty$ for all i, and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Let $f_1(x), f_2(x)$ be a simple external function such that (i) $0 \leq f_1(x) \leq f_2(x)$ and (ii) $f_1(x) = Ext\text{-}\sum_{i=1}^n \lambda_{1,i}\chi_{A_i}(x), f_2(x) = Ext\text{-}\sum_{i=1}^n \lambda_{2,i}\chi_{A_i}(x)$.

$$Ext\text{-}\sum_{i=1}^n \lambda_{1,i} \leq Ext\text{-}\sum_{i=1}^n \lambda_{2,i}, \tag{3.3}$$

then we will write $f_1(x) \leq_s f_2(x)$.

Definition 3.2. Suppose that $\mu^{\#}$ is hyperfinite. Let $f : X \to {}^*\mathbb{R}_c^{\#}$ be an arbitrary nonnegative bounded in ${}^*\mathbb{R}_c^{\#}$ #-measurable external function and let $(f_n)_{n \in {}^*\mathbb{N}}$, be a hyper infinite sequence of simple external functions which #-converges uniformly to f. Then the Lebesgue #-integral of f is

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = \#\text{-}\lim_{n\to^*\infty}\left(Ext\text{-}\int_X f_n(x)d^\#\mu^\#\right). \tag{3.4}$$

Remark 3.1.It can be easily shown that the #-limit in Definition 3.2 exists and does not depend on the choice of a hyper infinite sequence $(f_n)_{n\in^*\mathbb{N}}$, and moreover, the hyper infinite sequence $(f_n)_{n\in^*\mathbb{N}}$can be chosen such that $0 \leq f_n \leq f$ for all $n \in {}^*\mathbb{N}$.

Notation 3.2.Let $f_1 : X \to {}^*\mathbb{R}_c^\#$ and $f_2 : X \to {}^*\mathbb{R}_c^\#$ be an arbitrary nonnegative bounded in ${}^*\mathbb{R}_c^\#$ #-measurable external functions and let $(f_{1,n})_{n\in^*\mathbb{N}}$ and $(f_{2,n})_{n\in^*\mathbb{N}}$be a hyper infinite sequences of simple external functions which #-converges uniformly to f_1 and to f_2 correspondingly. We assume that for all $n \in {}^*\mathbb{N}$ the inequality (3.3) is satisfied, then we will write $f_1(x) \leq_s f_2(x)$.

Definition 3.3. Let $f : X \to {}^*\mathbb{R}_c^\#$ be a #-measurable function. Then the Lebesgue #-integral of f is defined by

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = Ext\text{-}\int_X f^+(x)d^\#\mu^\# - Ext\text{-}\int_X f^-(x)d^\#\mu^\#. \tag{3.5}$$

If both of these terms are finite or hyperfinite then the function f is called #-integrable. In this case we write $f \in L_1^\# = L_1^\#(X,\mathcal{F},\mu^\#)$.

Notation 3.3.We will use the following notation. For any $A \in \mathcal{F}$:

$$Ext\text{-}\int_A f(x)d^\#\mu^\# = Ext\text{-}\int_X f(x)\chi_A(x)d^\#\mu^\#. \tag{3.6}$$

Lemma 3.1.(1) Let $f : X \to {}^*\mathbb{R}_c^\#$ be an arbitrary nonnegative #-measurable function then

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = \sup\left\{Ext\text{-}\int_X \varphi(x)d^\#\mu^\# \middle| \varphi \text{ is a simple function such that } 0 \leq \varphi(x) \leq_s f(x)\right\}. \tag{3.7}$$

(2) If $f,g : X \to {}^*\mathbb{R}_c^\#$ are #-measurable, g is #-integrable, and $|f(x)| \leq_s g(x)$, then f is #-integrable and

$$\left|Ext\text{-}\int_X f(x)d^\#\mu^\#\right| \leq Ext\text{-}\int_X g(x)d^\#\mu^\#. \tag{3.8}$$

(3) $Ext\text{-}\int_X |f(x)|d^\#\mu^\# = 0$ if and only if $f(x) = 0$ #-a.e.

(4) If $f_1,f_2,\ldots f_n : X \to {}^*\mathbb{R}_c^\#, n \in {}^*\mathbb{N}$ are integrable then, for $\lambda_1,\lambda_2,\ldots,\lambda_n \in {}^*\mathbb{R}_c^\#$, the linear combination $Ext\text{-}\sum_{i=1}^n \lambda_i f_i$ is #-integrable and

$$Ext\text{-}\int_X \left(Ext\text{-}\sum_{i=1}^n \lambda_i f_i\right)d^\#\mu^\# = Ext\text{-}\sum_{i=1}^n\left(Ext\text{-}\int_X \lambda_i f_i d^\#\mu^\#\right). \tag{3.9}$$

(5) Let $f \in L_1^\#(X,\mathcal{F},\mu^\#)$, then the formula

$$\nu^\#(A) = Ext\text{-}\int_A f(x)d^\#\mu^\# = Ext\text{-}\int_X f(x)\chi_A(x)d^\#\mu^\# \tag{3.10}$$

defines a signed #-measure on the $\sigma^\#$-algebra $\mathcal{F}$.

Remark 3.2. Assume that $f,g : X \to {}^*\mathbb{R}_c^\#$ are #-integrable functions and such that $0 \leq f \leq_s g$ #-a.e., then

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} \le Ext\text{-}\int_X g(x)d^{\#}\mu^{\#}.$$

#-Convergence theorem

Definition 3.4. We say that a hyper infinite sequence $\{f_n\}_{n=1}^{*\infty}$ of #-integrable functions $L_1^{\#}$-#-converges to f (or #-converges in $L_1^{\#}(X,\mathcal{F},\mu^{\#})$) if

$$Ext\text{-}\int_X |f_n - f| d^{\#}\mu^{\#} \to_{\#} 0 \text{ as } n \to {}^*\infty. \tag{3.11}$$

Theorem 3.1 (The monotone #-convergence theorem) If $\{f_n\}_{n=1}^{*\infty}$ is a hyper infinite sequence in $L_1^{\#+}(X,\mathcal{F},\mu^{\#})$ such that $f_j \le_s f_{j+1}$ for all j and $f(x) = \sup_{n\in{}^*\mathbb{N}} f_n(x)$ then

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} = \#\text{-}\lim_{n\to{}^*\infty} Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}. \tag{3.12}$$

Proof: The #-limit of the increasing sequence

$$\left(Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}\right)_{n=1}^{*\infty}$$

($*$-finite or $*$-infinite) exists. Moreover by (3.2),

$$Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#} \le Ext\text{-}\int_X f(x)d^{\#}\mu^{\#}$$

for all $n \in {}^*\mathbb{N}$, so

$$\#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}\right) \le Ext\text{-}\int_X f(x)d^{\#}\mu^{\#}.$$

To establish the reverse inequality, fix $\alpha \in (0,1)$, let φ be a simple function with $0 \le \varphi \le f$, and let $E_n = \{x : f_n(x) \ge \alpha\varphi(x)\}$. Then $(E_n)_{n=1}^{*\infty}$ is an increasing hyper infinite sequence of #-measurable sets whose union is X, and we have

$$Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#} \ge Ext\text{-}\int_{E_n} f_n(x)d^{\#}\mu^{\#} \ge \alpha\left(Ext\text{-}\int_{E_n} \varphi(x)d^{\#}\mu^{\#}\right) \tag{3.13}$$

By (3.10) and by Proposition 2.5,

$$\#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_{E_n} \varphi(x)d^{\#}\mu^{\#}\right) = Ext\text{-}\int_X \varphi(x)d^{\#}\mu^{\#}, \tag{3.14}$$

and hence

$$\#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_{E_n} f_n(x)d^{\#}\mu^{\#}\right) \ge \alpha\left(Ext\text{-}\int_X \varphi(x)d^{\#}\mu^{\#}\right). \tag{3.15}$$

Since this is true for all $\alpha, 0 < \alpha < 1$, it remains true for $\alpha = 1$:

$$\#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_{E_n} f_n(x)d^{\#}\mu^{\#}\right) \ge Ext\text{-}\int_X \varphi(x)d^{\#}\mu^{\#}. \tag{3.16}$$

Using Lemma 3.1.(1), we may take the supremum over all simple functions φ, $0 \le \varphi \le_s f$. Thus

$$\#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_{E_n} f_n(x)d^{\#}\mu^{\#}\right) \ge Ext\text{-}\int_X f(x)d^{\#}\mu^{\#}. \tag{3.17}$$

Proofs of the following two corollaries of Theorem 3.1 are straightforward.

Corollary 3.1 If $(f_n)_{n=1}^{*\infty}$ is a hyper infinite sequence in $L_+^1(X)$ and $f = Ext\text{-}\sum_{n=1}^{*\infty} f_n$ pointwise then

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} = Ext\text{-}\sum_{n=1}^{*\infty}\left(Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}\right). \tag{3.18}$$

Corollary 3.2 If $(f_n)_{n=1}^{*\infty}$ is a hyper infinite sequence in $L_+^1(X)$, f $\in L_+^1(X)$, and $f_n \to_{\#} f$ $\mu^{\#}$-a.e., then

$$Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#} \to_{\#} Ext\text{-}\int_X f(x)d^{\#}\mu^{\#}. \tag{3.19}$$

Theorem 3.2 (Generalized Fatou's lemma) If $(f_n)_{n=1}^{*\infty}$is any hyper infinite sequence in $L_+^1(X)$ then

$$Ext\text{-}\int_X \#\text{-}\liminf_{n\to^*\infty}(f_n(x))d^{\#}\mu^{\#} \leq \#\text{-}\liminf_{n\to^*\infty}\left(Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}\right). \tag{3.20}$$

Theorem 3.3 (The dominated #-convergence theorem) Let f and g be #-measurable, let f_n be #-measurable for any $n \in {}^*\mathbb{N}$ such that $|f_n(x)| \leq_s g(x)$ #-a.e., and $f_n \to_{\#} f$ #-a.e. If g is #-integrable then f and f_n are also #-integrable and

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} = \#\text{-}\lim_{n\to^*\infty} Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}. \tag{3.21}$$

Proof: f is #-measurable and, since $|f| \leq_s g$ $\mu^{\#}$-a.e., we have $f \in L_+^1(X)$. We have that $g + f_n \geq 0$ $\mu^{\#}$-a.e. and $g - f_n \geq 0$ so, by Fatou's lemma,

$$\begin{gathered}
Ext\text{-}\int_X gd^{\#}\mu^{\#} + Ext\text{-}\int_X fd^{\#}\mu^{\#} \leq \#\text{-}\liminf_{n\to^*\infty}\left(Ext\text{-}\int_X [g+f_n]d^{\#}\mu^{\#}\right) = \\
Ext\text{-}\int_X gd^{\#}\mu^{\#} + \#\text{-}\liminf_{n\to^*\infty}\left(Ext\text{-}\int_X f_n d^{\#}\mu^{\#}\right), \\
Ext\text{-}\int_X gd^{\#}\mu^{\#} - Ext\text{-}\int_X fd^{\#}\mu^{\#} \leq \#\text{-}\liminf_{n\to^*\infty}\left(Ext\text{-}\int_X [g-f_n]d^{\#}\mu^{\#}\right) = \\
= Ext\text{-}\int_X gd^{\#}\mu^{\#} - \#\text{-}\limsup_{n\to^*\infty}\left(Ext\text{-}\int_X f_n d^{\#}\mu^{\#}\right)
\end{gathered} \tag{3.22}$$

Therefore

$$\#\text{-}\liminf_{n\to^*\infty}\left(Ext\text{-}\int_X f_n d^{\#}\mu^{\#}\right) \geq Ext\text{-}\int_X fd^{\#}\mu^{\#} \geq \#\text{-}\limsup_{n\to^*\infty}\left(Ext\text{-}\int_X f_n d^{\#}\mu^{\#}\right) \tag{3.23}$$

and the required result follows from (3.23).

§ 4. #-Convergence in #-measure.

Definition 4.1. We say that a hyper infinite sequence $(f_n)_{n=1}^{*\infty}$ of #-measurable functions on $(X, M, \mu^{\#})$ is Cauchy in #-measure if, for every $\varepsilon \approx 0, \varepsilon > 0$,

$$\mu^{\#}(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) \to_{\#} 0 \text{ as } m,n \to {}^*\infty, \tag{4.1}$$

and that $(f_n)_{n=1}^{*\infty}$ #-converges in #-measure to f if, for every $\varepsilon \approx 0, \varepsilon > 0$,

$$\mu^{\#}(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to_{\#} 0 \text{ as } n \to {}^*\infty. \tag{4.2}$$

Proposition 4.1. If $f_n \to_{\#} f$ in L^1 then $f_n \to_{\#} f$ in #-measure.

Proof. Let $E_{n,\varepsilon} = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Then

$$Ext\text{-}\int_X |f_n - f| d\mu^{\#} \geq Ext\text{-}\int_{E_{n,\varepsilon}} |f_n - f| d\mu^{\#} \geq \varepsilon\mu^{\#}(E_{n,\varepsilon}),$$

so $\mu(E_{n,\varepsilon}) \leq \varepsilon^{-1} Ext\text{-}\int_X |f_n - f| d\mu^{\#} \to_{\#} 0$.

Theorem 3.1. Suppose that $(f_n)_{n=1}^{*\infty}$ is Cauchy in #-measure. Then there is a #-measurable function f such that $f_n \to_{\#} f$ in #-measure, and there is a hyper infinite subsequence $(f_{n_j})_{j\in^*\mathbb{N}}$ that #-converges to f #-a.e. Moreover, if $f_n \to_{\#} g$ in #-measure then $g = f$ #-a.e.

Proof. We can choose a hyper infinite subsequence $(g_j)_j = (f_{n_j})_j$ of $(f_n)_{n=1}^{*\infty}$ such that if $E_j = \{x : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}$ then $\mu^{\#}(E_j) \leq 2^{-j}$. If $F_k = \bigcup_{j=k}^{*\infty} E_j$ then

$\mu^{\#}(F_k) \leq Ext\text{-}\sum_{j=k}^{*\infty} 2^{-j} = 2^{1-k}$, and if $x \notin F_k$ we have for $i \geq j \geq k$

$$|g_j(x) - g_i(x)| \leq Ext\text{-}\sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq Ext\text{-}\sum_{l=j}^{i-1} \leq 2^{1-j}. \qquad (4.3)$$

Thus $(g_j)_j$ is pointwise Cauchy on F_k^c. Let $F = \bigcap_{k=1}^{*\infty} F_k = \limsup_j E_j$. Then $\mu^{\#}(F) = 0$, and if we set $f(x) = \lim_{j \to {}^*\infty} g_j(x)$ for $x \notin F$, and $f(x) = 0$ for $x \in F$, then f is #-measurable and $g_j \to_{\#} f$ a.e. By (4.3), we have that $|g_j(x) - f(x)| \leq 2^{1-j}$ for $x \notin F_k$ and $j \geq k$. Since $\mu^{\#}(F_k) \to_{\#} 0$ as $k \to {}^*\infty$, it follows that $g_j \to_{\#} f$ in #-measure, because

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq \{x : |f_n(x) - g_j(x)| \geq (1/2)\varepsilon\} \cup \{x : |g_j(x) - f(x)| \geq (1/2)\varepsilon\}, \quad (4.4)$$

and the sets on the right both have infinte small #-measure when n and j are infinte large. Likewise, if $f_n \to_{\#} g$ in #-measure

$$\{x : |f(x) - g(x)| \geq \varepsilon\} \subseteq \{x : |f(x) - f_n(x)| \geq (1/2)\varepsilon\} \cup \{x : |f_n(x) - g(x)| \geq (1/2)\varepsilon\} \qquad (4.5)$$

for all $n \in {}^*\mathbb{N}$, hence $\mu^{\#}(\{x : |f(x) - g(x)| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$, and $f = g$ #-a.e.

Theorem 3.2 Let $f_n \to_{\#} f$ in $L_1^{\#}$ then there is a hyper infinite subsequence $(f_{n_k})_k$ such that $f^{n_k} \to_{\#} f$ #-a.e.

Proof. Let $E_{n,\varepsilon} = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Then

$$Ext\text{-}\int_X |f_n - f| d^{\#}\mu \geq Ext\text{-}\int_{E_{n,\varepsilon}} |f_n - f| d^{\#}\mu \geq \varepsilon\mu^{\#}(E_{n,\varepsilon}),$$

so $\mu^{\#}(E_{n,\varepsilon}) \to_{\#} 0$. Then, by Theorem 3.1, there is a hyper infinite subsequence $(f_{n_k})_k$ such that $f_{n_k} \to f$ #-a.e.

§ 5.The Extension of #-Measure

§ 5.1.Outer #-measures.

Definition 5.1.1. Let X be a nonempty set. An outer #-measure (or #-submeasure) on X is a function $\xi^{\#} : \widetilde{P}(X) \to [0, {}^*\infty], \widetilde{P}(X) \subset P(X)$ that satisfies:

(a) $\xi^{\#}(\emptyset) = 0$;

(b) $\xi^{\#}(A) \leq \xi(B)$ if $A \subseteq B$;

(c) $\xi^{\#}\left(\bigcup_{j=1}^{*\infty} A_j\right) \leq Ext\text{-}\sum_{j=1}^{*\infty} \xi^{\#}(A_j)$ for all hyper infinite sequences $(A_j)_{j=1}^{*\infty}$ in $\widetilde{P}(X)$.

The common way to obtain an outer #-measure is to start with a family G of “elementary sets” on which a notion of measure is defined (such as rectangles or cubes in ${}^*\mathbb{R}_c^{\#n}$and then approximate arbitrary sets from the outside by hyper infinite unions of members of G.

Proposition 5.1.1 Let $G \subseteq \widetilde{P}(X)$ be a set such that $\emptyset \in G, X \in G$ and let $\rho : G \to [0, {}^*\infty]$ be a function such that $\rho(\emptyset) = 0$. For any $A \subseteq X$, define

$$\xi^{\#}(A) = \rho^*(A) = \inf\left\{Ext\text{-}\sum_{j=1}^{*\infty} \rho(G_j) : G_j \in G \text{ and } A \subseteq \bigcup_{j=1}^{*\infty} G_j\right\}. \qquad (5.1.1)$$

if $\rho^*(A)$ exists. Then $\xi^{\#}$ is an outer #-measure.

Definition 5.1.2.We will say that $A \subseteq X$ is admissible if $\rho^*(A)$ exists.

Proof. For any admissible $A \subseteq X$, $\xi^{\#}(A)$ is well defined. Obviously $\xi^{\#}(\emptyset) = 0$.
To prove $*$-countable subadditivity, suppose $\{A_j\}_{j=1}^{*\infty} \subseteq \widetilde{P}(X)$ and $\varepsilon \approx, \varepsilon > 0$.
For each $j \in {}^*\mathbb{N}$, there exists $\{G_k^j\}_{k=1}^{*\infty} \subseteq G$ such that $A_j \subseteq \bigcup_{k=1}^{*\infty} G_k^j$ and
$Ext\text{-}\sum_{k=1}^{*\infty} \rho(G_k^j) \leq \xi^{\#}(A_j) + \varepsilon 2^{-j}$. Then if $A = \bigcup_{j=1}^{*\infty} A_j$, we have $A \subseteq \bigcup_{j,k=1}^{*\infty} G_k^j$ and
$Ext\text{-}\sum_{j,k=1}^{*\infty} \rho(G_k^j) \leq \sum_{j=1}^{*\infty} \xi(A_j) + \varepsilon$, whence $\xi^{\#}(A) \leq Ext\text{-}\sum_{j=1}^{*\infty} \xi^{\#}(A_j) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have done.

Definition 5.1.3. A set $A \subseteq X$ is called $\xi^{\#}$-measurable if $\rho^*(A)$ exists and $\forall B \subseteq X$ such that $\rho^*(B)$ exists the equality (5.1.2) holds

$$\xi^{\#}(B) = \xi^{\#}(B \cap A) + \xi^{\#}(B \cap (X \setminus A)). \tag{5.1.2}$$

Of course, the inequality $\xi^{\#}(B) \leq \xi^{\#}(B \cap A) + \xi^{\#}(B \cap (X \setminus A))$ holds for any (admissible) set A and B.
So, to prove that A is $\xi^{\#}$-measurable, it suffices to prove the reverse inequality, which is trivial if $\xi^{\#}(B) = {}^*\infty$. Thus, we see that A is $\xi^{\#}$-measurable iff for any admissible $B \subseteq X, \xi^{\#}(B) < {}^*\infty$

$$\xi^{\#}(B) \geq \xi^{\#}(B \cap A) + \xi^{\#}(B \cap (X \setminus A)). \tag{5.1.3}$$

Theorem 5.1.1 (Generalized Caratheodory's theorem) Let $\xi^{\#}$ be an outer #-measure on X. Then the family Σ of all $\xi^{\#}$-measurable sets is a $\sigma^{\#}$-algebra, and the restriction of $\xi^{\#}$ to Σ is a complete #-measure.
Proof: First, we observe that Σ is closed under complements, since the definition of $\xi^{\#}$-measurability of A is symmetric in A and $A^c \triangleq X \setminus A$. Next, if $A, B \in \Sigma$ and $E \subseteq X$,
$\xi^{\#}(E) = \xi^{\#}(E \cap A) + \xi^{\#}(E \cap A^c) = \xi^{\#}(E \cap A \cap B) + \xi^{\#}(E \cap A \cap B^c) + \xi^{\#}(E \cap Ac \cap B) +$
$+\xi^{\#}(E \cap A^c \cap B^c)$.
But $(A \cup B) = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$ so, by subadditivity,
$\xi^{\#}(E \cap A \cap B) + \xi^{\#}(E \cap A \cap B^c) + \xi^{\#}(E \cap A^c \cap B) \geq \xi^{\#}(E \cap (A \cup B))$,
and hence $\xi^{\#}(E) \geq \xi^{\#}(E \cap (A \cup B)) + \xi^{\#}(E \cap (A \cup B)^c)$.
It follows that $A \cup B \in \Sigma$, so Σ is an algebra. Moreover, if $A, B \in \Sigma$ and $A \cap B = \emptyset, \xi^{\#}(A \cup B) = \xi^{\#}((A \cup B) \cap A) + \xi^{\#}((A \cup B) \cap A^c) = \xi^{\#}(A) + \xi^{\#}(B)$, so $\xi^{\#}$ is hyperfinitely additive on Σ.
To show that Σ is a $\sigma^{\#}$-algebra, it suffices to show that Σ is closed under $*$-countable disjoint unions. If $(A_j)_{j=1}^{*\infty}$ is a sequence of disjoint sets in Σ, set
$B_n = \bigcup_{j=1}^{n} A_j \wedge B = \bigcup_{j=1}^{*\infty} A_j$. Then, for any admissible $E \subseteq X$,
$\xi^{\#}(E \cap B_n) = \xi^{\#}(E \cap B_n \cap A_n) + \xi^{\#}(E \cap B_n \cap A_n^c) = \xi^{\#}(E \cap A_n) + \xi^{\#}(E \cap B_{n-1})$,
so a hyperfinite induction shows that $\xi^{\#}(E \cap B_n) = Ext\text{-}\sum_{j=1}^{n} \xi^{\#}(E \cap A_j)$. Therefore

$$\xi^{\#}(E) = \xi^{\#}(E \cap B_n) + \xi^{\#}(E \cap B_n^c) \geq Ext\text{-}\sum_{j=1}^{n} \xi^{\#}(E \cap A_j) + \xi(E \cap B^c)$$

and, letting $n \to {}^*\infty$, we obtain
$\xi^{\#}(E) \geq Ext\text{-}\sum_{j=1}^{*\infty} \xi^{\#}(E \cap A_j) + \xi^{\#}(E \cap B^c) \geq \xi^{\#}\left(\bigcup_{j=1}^{*\infty} E \cap A_j\right) + \xi^{\#}(E \cap B^c) = \xi^{\#}(E \cap B) +$

$+\xi^{\#}(E \cap B^c) \geq \xi^{\#}(E)$.
Thus the inequalities in this last calculation become equalities. It follows $B \in \Sigma$.
Taking $E = B$ we have $\xi^{\#}(B) = Ext\text{-}\sum_{j=1}^{{}^*\infty} \xi^{\#}(A_j)$, so $\xi^{\#}$ is $\sigma^{\#}$-additive on Σ. Finally, if $\xi^{\#}(A) = 0$ then we have for any admissible set $E \subseteq X$
$\xi^{\#}(E) \leq \xi^{\#}(E \cap A) + \xi^{\#}(E \cap A^c) = \xi^{\#}(E \cap A^c) \leq \xi^{\#}(E)$, so $A \in \Sigma$.
Therefore $\xi^{\#}(E \cap A) = 0$ and $\xi^{\#}|_{\Sigma}$ is a complete #-measure.
Combination of Proposition 5.1.1 and Theorem 5.1.1 gives the following corollary which is also called generalized Caratheodory's theorem.

Corollary 5.1.1 Let $G \subseteq P(X)$ be a set such that $\emptyset \in G, X \in G$, and let $\rho : G \to [0, {}^*\infty]$ satisfy $\rho(\emptyset) = 0$. Then the family Σ of all ρ^* #-measurable sets (where $\rho^*$is given by (5.1.1)) is a $\sigma^{\#}$-algebra, and the restriction $\rho^*|_{\Sigma}$ of ρ^* to Σ is a complete #-measure.

Definition 5.1.4 Let $\bar{A}$ be an algebra of subsets of X, i.e. $\bar{A}$ contains $\emptyset$ and X, and $\bar{A}$ is closed under hyperfinite intersections and complements. A function $\zeta : \bar{A} \to [0, {}^*\infty]$ is called a #-premeasure if $\zeta(\emptyset) = 0$ and $\zeta\left(\bigcup_{j=1}^{{}^*\infty} A_j\right) = Ext\text{-}\sum_{j=1}^{{}^*\infty} \zeta(A_j)$ for any disjoint sequence $(A_j)_{j \in {}^*\mathbb{N}}$ of elements of $\bar{A}$ such that $\bigcup_{j=1}^{{}^*\infty} A_j \in \bar{A}$.

Theorem 5.1.2 If ζ is a #-premeasure on an algebra $\bar{A} \subseteq P(X)$ and $\zeta^* : P(X) \to [0, {}^*\infty]$ is given by (5.1.1) then $\zeta^*|_A = \zeta$ and every $A \in \bar{A}$ is ζ^* #-measurable.

§ 5.2.The Lebesgue and Lebesgue – Stieltjes #-measure on ${}^*\mathbb{R}_c^{\#}$.

The most important application of generalized Caratheodory's theorem is the construction of the Lebesgue #-measure on ${}^*\mathbb{R}_c^{\#}$. Take G as the set of all intervals $[a,b]$, where $a,b \in {}^*\mathbb{R}_c^{\#} \cup \{-{}^*\infty, +{}^*\infty\}$ and $[a,b] = \emptyset$ if $a > b$. Define the function $\rho : G \to {}^*\mathbb{R}_c^{\#} \cup \{{}^*\infty\}$ by

$$\forall a \forall b (a \leq b)\ [\rho([a,b]) \ = b - a] \text{ and } \forall a \forall b (a > b)[\rho([a,b]) \ = 0]. \qquad (5.2.1)$$

The function ρ has the obvious extension (which we denote also by ρ) to the algebra A generated by all intervals, and this extension is a #-premeasure on A. The $\sigma^{\#}$-algebra Σ given by Corollary 5.1.1 is called the the Lebesgue $\sigma^{\#}$-algebra in R, and the restriction of ρ^* to $\Sigma = \Sigma({}^*\mathbb{R}_c^{\#})$ is called the Lebesgue #-measure on ${}^*\mathbb{R}_c^{\#}$ and is denoted by $\mu^{\#}$. By Theorem 5.1.2, $\mu^{\#}$ is the unique extension of ρ. By the construction, $B^{\#}({}^*\mathbb{R}_c^{\#}) \subseteq \Sigma({}^*\mathbb{R}_c^{\#})$. Hence the Lebesgue #-measure is a Borel #-measure. It can be shown that $B^{\#}({}^*\mathbb{R}_c^{\#}) \neq \Sigma({}^*\mathbb{R}_c^{\#})$ and that the Lebesgue #-measure can be obtained also as the completion of any Borel #-measure $\omega^{\#}$ such that $\omega^{\#}([a,b]) \ = b - a(\forall a \leq b)$.
The notion of the Lebesgue measure on ${}^*\mathbb{R}_c^{\#}$ has the following generalization. Suppose that $\mu^{\#}$ is a $\sigma^{\#}$-finite Borel measure on ${}^*\mathbb{R}_c^{\#}$, and let $\forall x \in^* \mathbb{R}_c^{\#}$

$$F(x) = \mu^{\#}((-{}^*\infty, x]) \qquad (5.2.2)$$

Then F is increasing and right #-continuous . Moreover, if $b > a$, $(-{}^*\infty, b] = (-{}^*\infty, a] \cup (a,b]$, so $\mu^{\#}((a,b]) \ = F(b) - F(a)$.

Our procedure used above can be to turn this process around and construct a measure μ starting from an increasing, right-continuous function F. The special case $F(x) = x$ will yield the usual Lebesgue #-measure. As building blocks we can use the left-#-open, right-#-closed intervals in ${}^*\mathbb{R}_c^\#$ i.e. sets of the form $(a,b]$ or $(a,{}^*\infty)$ or $\emptyset$, where $-{}^*\infty \leq a < b < {}^*\infty$. We call such sets h-intervals. The family A of all finite disjoint unions of h-intervals is an algebra, moreover, the $\sigma^\#$-algebra generated by A is the #-Borel algebra $B^\#({}^*\mathbb{R}_c^\#)$.

Lemma 5.2.1. Given an increasing and right #-continuous function $F :{}^* \mathbb{R}_c^\# \to {}^*\mathbb{R}_c^\#$, if $(a_j,b_j](j = 1,\ldots,n), n \in {}^*\mathbb{N}$ are disjoint h-intervals, let

$$\mu_0^\#\left(\bigcup_{j=1}^n (a_j,b_j]\right) = Ext\text{-}\sum_{j=1}^n [F(b_j) - F(a_j)], \quad (5.2.3)$$

and let $\mu_0^\#(\emptyset) = 0.$ Then $\mu_0^\#$ is a #-premeasure.

Lemma 5.2.2.f Assume that $\{(a_\alpha,b_\alpha)|\alpha \in G\}$ is a hyperfinite or $*$-countable family of intervals in ${}^*\mathbb{R}_c^\#$ such that $[0,1] \subseteq \cup_{\alpha\in G} (a_\alpha,b_\alpha)$ then $Ext\text{-}\sum_{\alpha\in G} |a_\alpha - b_\alpha| > 1.$

Theorem 5.2.1 If $F : {}^*\mathbb{R}_c^\# \to {}^*\mathbb{R}_c^\#$ is any increasing, right #-continuous function, there is a unique Borel #-measure $\mu_F^\#$ on ${}^*\mathbb{R}_c^\#$ such that $\forall a \forall b(a,b \in {}^*\mathbb{R}_c^\#)$

$$\mu_F^\#((a,b]) = F(b) - F(a).$$

If G is another such function then $\mu_F^\# = \mu_G^\#$ iff $F - G$ is constant.
Conversely, if $\mu^\#$ is a Borel #-measure on ${}^*\mathbb{R}_c^\#$ that is gyperfinite on all #-bounded #-Borel sets, and we define $F(x) = \mu^\#((0,x])$ if $x > 0, F(x) = 0$ if $x = 0$, $F(x) = -\mu^\#((x,0])$ if $x < 0$,
then F is increasing and right #-continuous function, and $\mu^\# = \mu_F^\#$.

Proof: Each F induces a #-premeasure on $B^\#({}^*\mathbb{R}_c^\#)$ by Lemma 5.1.1. It is clear that F and G induce the same #-premeasure iff $F - G$ is constant, and that these #-premeasures are $\sigma^\#$-finite (since ${}^*\mathbb{R}_c^\# = \bigcup_{-{}^*\infty}^{{}^*\infty} (j,j+1]$). The first two assertions follow now from Lemma 5.2.2. As for the last one, the monotonicity of $\mu^\#$ implies the monotonicity of F, and the #-continuity of $\mu^\#$ from above and from below implies the right #-continuity of F for $x \geq 0$ and $x < 0.$ It is evident that $\mu^\# = \mu_F^\#$ on algebra A, and hence $\mu^\# = \mu_F^\#$ on $B^\#({}^*\mathbb{R}_c^\#)$ (accordingly to Lemma 5.2.4).
Lebesgue – Stieltjes #-measures possess some important and useful regularity properties.
Let us fix a complete Lebesgue – Stieltjes #-measure $\mu^\#$ on ${}^*\mathbb{R}_c^\#$ associated to an increasing, right #-continuous function $F.$ We denote by $\Sigma_{\mu^\#}$ the Lebesgue algebra correspondent to $\mu^\#$. Thus, for any $E \in \Sigma_{\mu^\#}$,

$$\begin{aligned}\mu^\#(E) &= \inf\left\{Ext\text{-}\sum_{j=1}^{{}^*\infty}[F(b_j) - F(a_j)] \,\middle|\, E \subseteq \bigcup_{j=1}^{{}^*\infty}(a_j,b_j]\right\} = \\ &= \inf\left\{Ext\text{-}\sum_{j=1}^{{}^*\infty} \mu_F^\#((a_j,b_j]) \,\middle|\, E \subseteq \bigcup_{j=1}^{{}^*\infty}(a_j,b_j]\right\}\end{aligned} \quad (5.2.4)$$

if infinum in RHS of (5.2.4) exists. Since $B^\#({}^*\mathbb{R}_c^\#) \subseteq \Sigma_{\mu^\#}$, we may replace in the second formula for $\mu^\#(E)$ h-intervals by #-open intervals, namely

Lemma 5.2.3 For any $E \in \Sigma_{\mu^\#}$,

$$\mu^{\#}(E) = \inf\left\{ Ext\text{-}\sum_{j=1}^{*\infty} \mu_F^{\#}((a_j, b_j)) \middle| E \subseteq \bigcup_{j=1}^{*\infty}(a_j, b_j) \right\}. \qquad (5.2.5)$$

Theorem 5.2.2 If $E \in \Sigma_{\mu^{\#}}$ then

$$\begin{gathered} E \in \Sigma_{\mu^{\#}} = \inf\{\mu^{\#}(U) : U \supseteq E \text{ and } U \text{ is } \#-\text{open}\} = \\ = \sup\{\mu^{\#}(K) : K \subseteq E \text{and } K \text{ is } \#-\text{compact}\}. \end{gathered} \qquad (5.2.6)$$

Proof. By Lemma 5.2.2, for any $\varepsilon \approx, \varepsilon > 0,$ there exist intervals (a_j, b_j) such that $E \subseteq \bigcup_{j=1}^{*\infty}(a_j, b_j)$ and $\mu^{\#}(E) \leq Ext\text{-}\sum_{j=1}^{*\infty} \mu^{\#}((a_j, b_j)) + \varepsilon.$ If $U = \bigcup_{j=1}^{*\infty}(a_j, b_j)$ then U is #-open, $E \subseteq U$, and $\mu^{\#}(U) \leq \mu^{\#}(E) + \varepsilon.$ On the other hand, $\mu^{\#}(U) \geq \mu^{\#}(E)$ whenever $E \subseteq U$ so the first equality is valid.

For the second one, suppose first that E is bounded in ${}^*\mathbb{R}_c^{\#}$. If E is #-closed then E is #-compact and the equality is obvious. Otherwise, given $\varepsilon \approx, \varepsilon > 0,$ we can choose an #-open $U,$ $(\#\text{-}\bar{E})\backslash E \subseteq U,$ such that $\mu^{\#}(U) \leq \mu^{\#}((\#\text{-}\bar{E}) \backslash E) + \varepsilon.$

Let $K = (\#\text{-}\bar{E}) \backslash U.$ Then K is #-compact, $K \subseteq E,$ and

$\mu^{\#}(K) = \mu^{\#}(E) - \mu^{\#}(E \cap U) = \mu^{\#}(E) - [\mu^{\#}(U) - \mu^{\#}(U\backslash E)] \geq$

$\geq \mu^{\#}(E) - \mu^{\#}(U) + \mu^{\#}((\#\text{-}\bar{E})\backslash E) \geq \mu^{\#}(E) - \varepsilon.$

If E is unbounded in ${}^*\mathbb{R}_c^{\#}$, let $E_j = E \cap (j, j+1]$. By the preceding argument, for any $\varepsilon \approx, \varepsilon > 0$, there exist a #-compact $K_j \subseteq E_j$ with $\mu^{\#}(K_j) \geq \mu^{\#}(E_j) - \varepsilon 2^{-j}$. Let $H_n = \bigcup_{j=-n}^{j=n} K_j$. Then H_n is #-compact, $H_n \subseteq E$, and $\mu^{\#}(H_n) \geq \mu^{\#} \bigcup_{j=-n}^{j=n}(E_j) - \varepsilon.$

Since $\mu^{\#}(E) = \#\text{-}\lim_{n \to {}^*\infty} \mu^{\#}\left(\bigcup_{j=-n}^{j=n} E_j\right)$, the result follows.

Theorem 5.2.3. If $E \subseteq {}^*\mathbb{R}_c^{\#}$, the following are equivalent:

(a) $E \in \Sigma_{\mu^{\#}}$;

(b) $E = V\backslash N_1,$ where V is a $G_{\delta^{\#}}$–set and $\mu^{\#}(N_1) = 0$;

(c) $E = H \cup N_2,$ where H is an $F_{\sigma^{\#}}$–set and $\mu^{\#}(N_2) = 0.$

Theorem 5.2.4. If E $\in \Sigma_{\mu^{\#}}$ and $\mu^{\#}(E) < {}^*\infty$ then, for every $\varepsilon \approx, \varepsilon > 0,$ there is a set A that is a hyperfinite union of #-open intervals such that $\mu^{\#}(E \Delta A) < \varepsilon.$

Lemma 5.2.4 Let $A \subseteq P(X)$ be an algebra, let $\mu_0^{\#}$ be a $\sigma^{\#}$-finite #-premeasure on A, and let Ω be the $\sigma^{\#}$-algebra generated by A. Then there exists a unique extension of $\mu_0^{\#}$ to a #-measure $\mu^{\#}$ on Ω.

§ 5.3. Product #-measures.

Definition 5.3.1. Let $\{(X_\alpha, \mathcal{F}_\alpha, \mu_\alpha^{\#})\}_{\alpha \in \Delta}$ be a nonempty family of #-measure spaces. We define the family Ω of blocks:

$$\begin{gathered} A(A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}) := \\ = A_{\alpha_1} \times A_{\alpha_2} \times \cdots \times A_{\alpha_n} \times Ext\text{-}\prod_{\alpha \in \Delta, \alpha \neq \alpha_k, 1 \leq k \leq n} X_\alpha, \end{gathered} \qquad (5.3.1)$$

where $A_{\alpha_k} \in \mathcal{F}_{\alpha_k}$ and define a function

$$\mu^{\#}_{\Omega} : \Omega \to {}^{*}\mathbb{R}^{\#}_{c} \cup \{{}^{*}\infty\} :=$$

$$\mu^{\#}\left(A_{\alpha_1}\right) \times \mu^{\#}(A_{\alpha_2}) \times \cdots \times \mu^{\#}(A_{\alpha_n}) \times \left[Ext\text{-}\prod_{\alpha\in\Delta,\alpha\neq\alpha_k,1\leq k\leq n} \mu^{\#}(X_{\alpha}) \right]. \quad (5.3.2)$$

This function possesses an extension (by #-additivity) on the #-algebra A generated by Ω. It is easily to show that $\mu^{\#}_{\Omega}$ is a #-premeasure on A.

Definition 5.3.2 The #-measure $\widehat{\mu}^{\#}$ on the $\sigma^{\#}$-algebra Σ generated by A accordingly to **Theorem 2.1.3** is called the product #-measure of $\{\mu^{\#}_{\alpha}\}_{\alpha\in\Delta}$, and the triple $\left(\prod_{\alpha\in\Delta} X_{\alpha}, \Sigma, \widehat{\mu}^{\#}\right)$ is called the product of #-measure spaces $(X_{\alpha}, \Sigma_{\alpha}, \mu^{\#}_{\alpha})$.

We denote the $\sigma^{\#}$-algebra Σ by $\bigotimes_{\alpha\in\Delta}\Sigma_{\alpha}$, and the #-measure $\widehat{\mu}^{\#}$ by $\bigotimes_{\alpha\in\Delta}\mu^{\#}_{\alpha}$.

Definition 5.3.3. If $E \subseteq X_1 \times X_2$ and $x_1 \in X_1, x_2 \in X_2$, we define

$$E_{x_1} = \{x \in X_2 : (x_1, x) \in E\} \text{ and } E^{x_2} = \{x \in X_1 : (x, x_2) \in E\}.$$

If $f : X_1 \times X_2 \to {}^{*}\mathbb{R}^{\#}_{c}$ is a function, we define $f_{x_1} : X_2 \to {}^{*}\mathbb{R}^{\#}_{c}$ and $f^{x_2} : X_1 \to {}^{*}\mathbb{R}^{\#}_{c}$ by $f_{x_1}(x) = f(x_1, x)$ and $f^{x_2}(x) = f(x, x_2)$.

Theorem 5.3.1. (The generalized Fubini's theorem) Let $\mu^{\#}_1, \mu^{\#}_2$ be $\sigma^{\#}$-hyperfinite #-measures on $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$,

$$(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu^{\#}_1 \otimes \mu^{\#}_2) = (X_1, \mathcal{F}_1, \mu^{\#}_1) \times (X_2, \mathcal{F}_2, \mu^{\#}_2), \quad (5.3.3)$$

and let $f \in L^{\#}_1(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu^{\#}_1 \otimes \mu^{\#}_2)$. Then $f_{x_1} \in L^{\#}_1(X_2, \mathcal{F}_2, \mu^{\#}_2)$ $\mu^{\#}_1$-#-a.e., and $f^{x_2} \in L^{\#}_1(X_1, \mathcal{F}_1, \mu^{\#}_1)$ $\mu^{\#}_2$-#-a.e., and

$$\begin{aligned} Ext\text{-}\int_{X_1\times X_2} fd^{\#}(\mu^{\#}_1 \otimes \mu^{\#}_2) &= Ext\text{-}\int_{X_2}\left[Ext\text{-}\int_{X_1} f^{x_2}d^{\#}\mu^{\#}_1\right]d^{\#}\mu^{\#}_2 = \\ &= Ext\text{-}\int_{X_1}\left[Ext\text{-}\int_{X_2} f_{x_1}d^{\#}\mu^{\#}_2\right] d^{\#}\mu^{\#}_1 \end{aligned} \quad (5.3.4)$$

Lemma 5.3.1. Let $(X_1, \Sigma_1, \mu^{\#}_1)$ and $(X_2, \Sigma_2, \mu^{\#}_2)$ be #-measure spaces, $E \in \Sigma_1 \otimes \Sigma_2$, and let f be a $\Sigma_1 \otimes \Sigma_2$-measurable function on $X_1 \times X_2$, then:

(a) $E_{x_1} \in \Sigma_2$ for all $x_1 \in X_1$ and $E_{x_2} \in \Sigma_1$ for all $x_2 \in X_2$;

(b) f_{x_1} is Σ_2-measurable and f_{x_2} is Σ_1-measurable for all $x_1 \in X_1$ and $x_2 \in X_2$.

Proof. Denote by A the collection of all $A \subseteq X_1 \times X_2$ such that $A_{x_1} \in \Sigma_2$ and $A^{x_2} \in \Sigma_1 (\forall x_1 \in X_1, x_2 \in X_2)$.

The family A contains all rectangles. Thus, since

$$\left[\bigcup_{n=1}^{*\infty} A_n\right]_{x_1} = \bigcup_{n=1}^{*\infty} [A_n]_{x_1}, [B_n]^{x_2} = [B_n]^{x_2} \quad (5.3.5)$$

and

$$[X_1 \times X_2 \backslash A]x_1 = X_2 \backslash A_{x_1}, [X_1 \times X_2 \backslash A]^{x_2} = X_1 \backslash A^{x_2}, \quad (5.3.6)$$

A is a $\sigma^{\#}$-algebra. So $\Sigma_1 \otimes \Sigma_2 \subseteq A$, and (a) is proved. Now the part (b) follows from (a) due to $f^{-1}_{x_1}(A) = [f^{-1}(A)]_{x_1}$ and $[f^{x_2}]^{-1}(A) = [f^{-1}(A)]^{x_2} (\forall A \subseteq {}^{*}\mathbb{R}^{\#}_{c})$.

Definition 5.3.4 A family $M \subseteq P(X)$ is called a monotone class if M is closed under $*$-countable increasing unions and $*$-countable decreasing intersections.

Lemma 5.3.2. If $A \subseteq P(X)$ is an algebra then the monotone class generated by A coincides with the $\sigma^{\#}$-algebra generated by A.

Lemma 5.3.3. Let $(X_1,\Sigma_1,\mu_1^{\#})$ and $(X_2,\Sigma_2,\mu_2^{\#})$ be #-measure spaces, $E \in \Sigma_1 \otimes \Sigma_2$. Then the functions $x_1 \to \mu_2^{\#}(E_{x_1})$ and $x_2 \to \mu_1^{\#}(E^{x_2})$ are #-measurable on (X_1,Σ_1) and (X_2,Σ_2), and

$$\mu_1^{\#} \otimes \mu_2^{\#}(E) = Ext\text{-}\int_{X_2} \mu_1^{\#}(E^{x_2})d^{\#}\mu_2^{\#} = Ext\text{-}\int_{X_1} \mu_2^{\#}(E_{x_1})d^{\#}\mu_1^{\#}. \tag{5.3.7}$$

Proof. First we consider the case when $\mu_1^{\#}$ and $\mu_2^{\#}$ are finite. Let A be the family of all $E \in \Sigma_1 \otimes \Sigma_2$ for which (5.3.7) is true. If $E = A \times B$, then $\mu_1^{\#}(E^{x_2}) = \mu_1^{\#}(A)\chi_B(x_2)$ and $\mu_2^{\#}(E_{x_1}) = \mu_2^{\#}(B)\chi_A(x_1)$, so $E \in A$. By additivity, it follows that gyperfinite disjoint unions of rectangles are in A so, by Lemma 5.3.2,bit will suffice to show that A is a monotone class. If $(E_n)_{n=1}^{*\infty}$ is an increasing hyper infinite sequence in A and $E = \bigcup_{n=1}^{*\infty} E_n$, then the function $f_n(x_2) = \mu_1^{\#}((E_n)^{x_2})$ are #-measurable and increase pointwise to $f(y) = \mu_1^{\#}(E^{x_2})$. Hence f is #-measurable and, by the monotone convergence theorem,

$$\begin{gathered} Ext\text{-}\int_{X_2} \mu_1^{\#}(E^{x_2})d\mu_2^{\#} = \#\text{-}\lim_{n\to{}^*\infty} ExtX_1 \int_{X_2} \mu_1^{\#}((E_n)^{x_2})d\mu_2^{\#} = \\ \#\text{-}\lim_{n\to{}^*\infty} \mu_1^{\#} \times \mu_2^{\#}(E_n) = \mu_1^{\#} \times \mu_2^{\#}(E). \end{gathered} \tag{5.3.8}$$

Likewise $\mu_1^{\#} \times \mu_2^{\#}(E) = Ext\text{-}\int_{X_1} \mu_2^{\#}(E_x)d\mu_1^{\#}$, so $E \in A$. Similarly, if $(E_n)_{n=1}^{*\infty}$ is a decreasing hyper infinite sequence in A and $E = \bigcap_{n=1}^{*\infty} E_n$, the function $x_2 \to \mu_1^{\#}((E_1)^{x_2})$ is in $L_1^{\#}(\mu_2^{\#})$ because $\mu_1^{\#}((E_1)x_2) \leq \mu_1^{\#}(X_1) < {}^*\infty$ and $\mu_2^{\#}(X_2) < {}^*\infty$, so the dominated convergence theorem can be applied to show that $E \in A$. Thus, A is a monotone class, and the proof is complete for the case of finite #-measure spaces.
Finally, if $\mu_1^{\#}$ and $\mu_2^{\#}$ are $\sigma^{\#}$-finite, we can write $X_1 \times X_2$ as the union of an increasing hyper infinite sequence $(X_1^j \times X_2^j)_{j=1}^{*\infty}$ of rectangles of finite or hyperfinite #-measure. If $E \in \Sigma_1 \otimes \Sigma_2$, the preceding argument applies to $E \cap (X_1^j \times X_2^j)$ for each j gives us

$$\mu_1^{\#} \times \mu_2^{\#}(E \cap (X_1^j \times X_2^j)) = Ext\text{-}\int_{X_2} \mu_1^{\#}(E^{x_2} \cap X_1^j)\mu_2^{\#} = Ext\text{-}\int_{X_1} \mu_2^{\#}(E_{x_1} \cap X_2^j)\mu_1^{\#}. \tag{5.3.9}$$

The application of the monotone convergence theorem then yields the desired result.

Lemma 5.3.3. (Generalized Tonelli's theorem) Let $(X_1,\Sigma_1,\mu_1^{\#})$ and $(X_2,\Sigma_2,\mu_2^{\#})$ be #-measure spaces, and $f : X_1 \times X_2 \to {}^*\mathbb{R}_{c+}^{\#}$ be a $\Sigma_1 \otimes \Sigma_2$-#-measurable function.Then the functions

$$f_{\mu_2^{\#}}(x_1) = Ext\text{-}\int_{X_2} f_{x_1}d^{\#}\mu_2^{\#} \text{ and } f_{\mu_1^{\#}}(x_2) = Ext\text{-}\int_{X_1} f^{x_2}d^{\#}\mu_1^{\#} \tag{5.3.10}$$

are Σ_1-#-measurable and Σ_2-#-measurable, respectively, and

$$Ext\text{-}\int_{X_1\times X_2} f d^{\#}\mu_1^{\#}\otimes\mu_2^{\#} = Ext\text{-}\int_{X_2}\left[Ext\text{-}\int_{X_1} f^{x_2} d^{\#}\mu_1^{\#}\right] d^{\#}\mu_2^{\#} = \tag{5.3.11}$$

$$= Ext\text{-}\int_{X_1}\left[Ext\text{-}\int_{X_2} f_{x_1} d^{\#}\mu_2^{\#}\right] d^{\#}\mu_1^{\#}.$$

Proof: In the case when f is a characteristic function, the statement of this lemma follows from Lemma 5.3.3. Therefore, by linearity, it holds also for nonnegative simple functions. If a nonnegative #-measurable function f is arbitrary, there exists a sequence of nonnegative simple functions which increase pointwise to f, say $(f_n)_{n=1}^{*\infty}$. By the monotone convergence theorem,

$$Ext\text{-}\int_{X_1} f_{\mu_2^{\#}} d^{\#}\mu_1^{\#} = \#\text{-}\lim_{n\to{}^*\infty}\left[Ext\text{-}\int_{X_1} f^{n}_{\mu_2^{\#}} d^{\#}\mu_1^{\#}\right] = \tag{5.3.12}$$

$$= \#\text{-}\lim_{n\to{}^*\infty}\left[Ext\text{-}\int_{X_1\times X_2} f_n d^{\#}\mu_1^{\#}\otimes\mu_2^{\#}\right]$$

and

$$Ext\text{-}\int_{X_2} f_{\mu_1^{\#}} d^{\#}\mu_2^{\#} = \#\text{-}\lim_{n\to{}^*\infty}\left[Ext\text{-}\int_{X_2} f^{n}_{\mu_1^{\#}} d^{\#}\mu_2^{\#}\right] = \tag{5.3.13}$$

$$= \#\text{-}\lim_{n\to{}^*\infty}\left[Ext\text{-}\int_{X_1\times X_2} f_n d^{\#}\mu_1^{\#}\otimes\mu_2^{\#}\right],$$

where

$$f^{n}_{\mu_2^{\#}}(x_1) = Ext\text{-}\int_{X_1}[f_n]_{x_1} d^{\#}\mu_2^{\#}, f^{n}_{\mu_1^{\#}}(x_2) = Ext\text{-}\int_{X_2}[f_n]^{x_2} d^{\#}\mu_1^{\#}. \tag{5.3.14}$$

This proves (5.3.11) and the lemma.

Proof of Theorem 5.3.1. Since an ${}^*\mathbb{R}_c^{\#}$-valued function f is Lebesgue #-integrable iff its positive f^+ and negative f^- parts are #-integrable, it is sufficient to prove the theorem only for nonnegative function $f \in L_1^{\#}(X_1\times X_2,\Sigma,\mu_1^{\#}\otimes\mu_2^{\#})$. But this was exactly done in Lemma 5.3.3.

§ 5.4.Lebesgue #-measure and integral in ${}^*\mathbb{R}_c^{\#n}$.

In this section, we study ${}^*\mathbb{R}_c^{\#n}, n \in {}^*\mathbb{N}$ and functions from ${}^*\mathbb{R}_c^{\#n}$to ${}^*\mathbb{R}_c^{\#}$ from the point of view of the Lebesgue #-measure and Lebesgue integration. All results presented below possess obvious ${}^*\mathbb{C}_c^{\#n}$-valued analogs. Then we define and study generalized Cantor sets which are interesting from the point of view of the set topology and the #-measure theory. Cantor sets are #-closed #-Borel nowhere #-dense subsets of the interval $[0,1]$ or, more generally, of a Hausdorff #-space.

Definition 5.4.1. The Lebesgue #-measure $\mu^{\#n}$ on ${}^*\mathbb{R}_c^{\#n}$ is the #-completion of the product of the Lebesgue #-measure on ${}^*\mathbb{R}_c^{\#n}$ according to Definition 5.3.1.The domain Σ^n of $\mu^{\#}$ (of course, $B^{\#}({}^*\mathbb{R}_c^{\#n}) \subseteq \Sigma^n$) is the class of Lebesgue #-measurable sets in ${}^*\mathbb{R}_c^{\#n}$. We write $d^{\#}x^n$ for $d^{\#}\mu^{\#n}$ and

$$Ext\text{-}\int f(x)d^{\#}x^n = Ext\text{-}\int f\, d^{\#}\mu^{\#n}.$$

We extend some of the results of previous section to the n-dimensional case with $n \in {}^*\mathbb{N}$. If $E = Ext\text{-}\prod_{j=1}^{n} E_j$ is a block in ${}^*\mathbb{R}_c^{\#n}$, we call sets $E_j \subseteq {}^*\mathbb{R}_c^{\#n}$ the sides of the block E.

Theorem **5.4.1**. Let $E \in \Sigma^n$. Then

(a) $\mu^{\#n}(E) = \inf\{\mu^{\#n}(U) : E \subseteq U, U\text{ \#-open}\} = \sup\{\mu^{\#n}(K) : K \subseteq E, K\text{ \#-compact}\}$;

(b) $E = A_1 \cup N_1 = A_2 \backslash N_2$, where A_1 is an $F_{\sigma^{\#}}$ set, A_2 is a $G_{\delta^{\#}}$ set, and $\mu^{\#n}(N_1) = \mu^{\#n}(N_2) = 0$;

(c) If $\mu^{\#n}(E) < {}^*\infty$ then, for any $\varepsilon \approx 0, \varepsilon > 0$, there is a hyperfinite family $\{R_j\}_{j=1}^{N}$ of disjoint blocks, whose sides are intervals such that $\mu^{\#n}(E \Delta \cup_{j=1}^{N} R_j) < \varepsilon$.

Proof: By the definition of product #-measures, if $E \in \Sigma^n$ and $\varepsilon \approx 0, \varepsilon > 0$, there is a $*$-countable family $\{T_j\}_{j=1}^{*\infty}$ of blocks such that $E \subseteq \cup_{j=1}^{*\infty} T_j$ and

$$Ext\text{-}\sum_{j=1}^{*\infty} \mu^{\#n}(T_j) \leq \mu^{\#n}(E) + \varepsilon.$$

For each j, by applying Theorem 5.2.3 to the sides of R_j, we can find blocks $U_j \supseteq F_j$ whose sides are #-open sets such that $\mu^{\#n}(U_j) < \mu(T_j) + \varepsilon 2^{-j}$.
If $U = \cup_{j=1}^{*\infty} U_j$ then U is #-open and

$$\mu^{\#n}(U) \leq Ext\text{-}\sum_{j=1}^{*\infty} \mu^{\#n}(U_j) \leq \mu^{\#n}(E) + 2\varepsilon.$$

This proves the first equation in part (a). The second equation and part (b) follow as in the proofs of Theorems **2.1.6** and **2.1.7**.
Next, if $\mu^{\#n}(E) < {}^*\infty$ then $\mu^{\#n}(U_j) < {}^*\infty$ for all j. Since the sides of U_j are $*$-countable unions of #-open intervals, by taking suitable hyperfinite subunions, we obtain blocks $V_j \subseteq U_j$ whose sides are hyperfinite unions of intervals such that $\mu^{\#n}(V_j) \geq \mu^{\#n}(U_j) - \varepsilon 2^{-j}$.If $N \in {}^*\mathbb{N}$ is sufficiently hyperfinite large, we have

$$\mu^{\#n}\left(E \backslash \bigcup_{j=1}^{N} V_j\right) \leq \mu^{\#n}\left(\bigcup_{j=1}^{N} U_j \backslash V_j\right) + \mu^{\#n}\left(\bigcup_{j=N+1}^{*\infty} U_j\right) < 2\varepsilon$$

and

$$\mu^{\#n}\left(\bigcup_{j=1}^{N} V_j \backslash E\right) \leq \mu^{\#n}\left(\bigcup_{j=1}^{*\infty} U_j \backslash E\right) < \varepsilon,$$

so $\mu^{\#n}(E \Delta \cup_{j=1}^{N} V_j) < 3\varepsilon$. Since $\cup_{j=1}^{N} V_j$ can be expressed as a hyperfinite disjoint union of rectangles whose sides are intervals, we have proved (c).

§ 5.5.Lebesgue #-integrable functions on ${}^*\mathbb{R}_c^{\#n}$

Let $\mu^{\#n}$ be the Lebesgue #-measure in ${}^*\mathbb{R}_c^{\#n}$. The set $M({}^*\mathbb{R}_c^{\#n}, \mu^{\#n})$ of all ${}^*\mathbb{R}_c^{\#}$-valued $\mu^{\#n}$-measurable functions on ${}^*\mathbb{R}_c^{\#n}$ is a vector space (addition and scalar multiplication are pointwise). By $L_1^{\#}({}^*\mathbb{R}_c^{\#n}, \mu^{\#n})$ we denote its subspace of all Lebesgue #-integrable functions (with finite in ${}^*\mathbb{R}_c^{\#}$ #-integral). Now write $f \approx g$ for f and g in $M({}^*\mathbb{R}_c^{\#n}, \mu^{\#n})$, whenever f and g differ only on a $\mu^{\#n}$-null set (a set of $\mu^{\#n}$-measure zero). It is easily seen that $\approx$ is an equivalencerelation. Let $L_0 = L_0({}^*\mathbb{R}_c^{\#n}, \mu^{\#n})$ be the set of equivalence classes of functions in $M({}^*\mathbb{R}_c^{\#n}, \mu^{\#n})$. We denote the equivalence classes of $f, g, \ldots by [f], [g], \ldots$. The set L_0 becomes a vector space over field ${}^*\mathbb{R}_c^{\#}$ by defining $[f]+[g] = [f+g]$ and $\alpha[f] = [\alpha f]$ for a real $\alpha \in {}^*\mathbb{R}_c^{\#}$. Observe that these definitions do not depend on the choice of f and g in their equivalence classes. The same is true for the partial order in L_0, if we define $[f] \leq [g]$ to mean $f(x) \leq g(x)$ for all $x \in {}^*\mathbb{R}_c^{\#n}$ except a null set. In practice, the elements of $L_0 = L_0({}^*\mathbb{R}_c^{\#n}, \mu^{\#n})$ are usually denoted by $f, g, \ldots$ and treated as if they were functions instead of equivalence classes of functions.

Definition 5.5.1.

Theorem 5.5.1. If $f \in L_1^{\#}(\mu^{\#n})$ and $\varepsilon \approx 0, \varepsilon > 0$, there is a simple function $\varphi = Ext\text{-}\sum_{j=1}^{N} \alpha_j \chi_{R_j}$, where each R_j is a product of intervals such that $Ext\text{-}\int |f-\varphi| d^{\#}\mu^{\#n} < \varepsilon$, and there is a #-continuous function g vanishing outside of a bounded in ${}^*\mathbb{R}_c^{\#n}$ set such that $Ext\text{-}\int |f-g| d^{\#}\mu^{\#n} < \varepsilon < \varepsilon$.

Proof. By the definition of Lebesgue #-integrable functions, we can approximate f by simple functions in $L_1^{\#}$-#-norm. Then use Theorem 5.4.1 to approximate a simple function by a function φ of the desired form. Finally, use the generalized Urysohn Lemma to approximate such φ by a #-continuous function.

Theorem 5.5.2.The Lebesgue #-measure on ${}^*\mathbb{R}_c^{\#n}$ is translation-invariant. Namely, let $a \in {}^*\mathbb{R}_c^{\#n}$. Define the shift $\tau_a : {}^*\mathbb{R}_c^{\#n} \to {}^*\mathbb{R}_c^{\#n}$ by $\tau_a(x) = x+a$.

(a) If $E \in \mathcal{L}^{\#n}$ then $\tau_a(E) \in \mathcal{L}^{\#n}$ and $\mu^{\#n}(\tau_a(E)) = \mu^{\#n}(E)$;

(b) If f : ${}^*\mathbb{R}_c^{\#n} \to {}^*\mathbb{R}_c^{\#}$ is Lebesgue #-measurable then so is $f \circ \tau_a$. Moreover, if either $f \geq 0$ or $f \in L_1^{\#}(\mu^{\#n})$ then

$$Ext\text{-}\int (f \circ \tau_a) d^{\#}\mu^{\#n} = Ext\text{-}\int f d^{\#}\mu^{\#n}. \tag{5.5.1}$$

Proof. Since τa and its inverse τ_{-a} are #-continuous, they preserve the class of #-Borel sets. The formula $\mu^{\#n}(\tau_a(E)) = \mu^{\#n}(E)$ follows easily from the trivial one dimensional variant of this result if E is a block. For a general #-Borel set E, the formula $\mu^{\#n}(\tau_a(E)) = \mu^{\#n}(E)$ follows from the previous step, since $\mu^{\#n}$ is determined by its action on blocks. Assertion (a) now follows immediately.

If f is Lebesgue #-measurable and B is a #-Borel set in ${}^*\mathbb{R}_c^{\#}$, we have $f^{-1}(B) = E \cup N$, where E is #-Borel and $\mu^{\#}(N) = 0$. But $\tau_a^{-1}(E)$ is Borel and $\mu^{\#}(\tau_a^{-1}(N)) = 0$, so $(f \circ \tau_a)^{-1}(B) \in \Sigma^n$ and f is Lebesgue #-measurable. The equality (5.5.1) reduces to the equality $\mu^{\#n}(\tau_{-a}(E)) = \mu^{\#n}(E)$ when $f = \chi_E$. It is true for simple functions by linearity, and hence for nonnegative #-measurable functions by the definition of #-integral. Taking positive and negative parts of real and imaginary parts, we obtain the result for $f \in L_1^{\#}(\mu^{\#n})$.

Theorem 5.5.3. (Generalized Lusin's theorem) If f is a Lebesgue #-measurable function on ${}^*\mathbb{R}_c^{\#n}$ and $\varepsilon \approx 0, \varepsilon > 0$ then there exist a #-measurable set $A \subseteq {}^*\mathbb{R}_c^{\#n}$ such that $\mu^{\#n}({}^*\mathbb{R}_c^{\#n} \backslash A) \leq \varepsilon$ and the restriction of f onto A is #-continuous.

Chapter IV. ${}^*\mathbb{R}_c^{\#}$-valued distributions.

§1. ${}^*\mathbb{R}_c^{\#}$-valued test functions and distributions

Definitions and theorems appropriate to analysis on non-Archemedean field ${}^*\mathbb{R}_c^{\#}$ and on complex field ${}^*\mathbb{C}_c^{\#} = {}^*\mathbb{R}_c^{\#} + i\,{}^*\mathbb{R}_c^{\#}$ are given in [1]-[2].

Definition 1.1.[3].(i) Let U be a free ultrafilters on $\mathbb{N}$ and introduce an equivalence relation on sequences in $\mathbb{R}^{\mathbb{N}}$ as $f_1 \sim_U f_2$ iff $\{i \in \mathbb{N} | f_1(i) = f_2(i)\} \in U$.

(ii) $\mathbb{R}^{\mathbb{N}}$ divided out by the equivalence relation $\sim_U$ gives us the nonstandard extension ${}^*\mathbb{R}$, the hyperreals; in symbols, ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\sim_U$ and similarly $\mathbb{N}^{\mathbb{N}}$ divided out by the equivalence relation $\sim_U$ gives us the nonstandard extension ${}^*\mathbb{N}$, the hyperintegers; in symbols,${}^*\mathbb{N} = \mathbb{N}^{\mathbb{N}}/\sim_U$.

Abbreviation 1.1.If $f \in \mathbb{R}^{\mathbb{N}}$, we denote its image in ${}^*\mathbb{R}$ by $[f]$,i.e.,$[f] = \{g \in \mathbb{R}^{\mathbb{N}} | g \sim_U f\}$.

Remark 1.1.For any real number $r \in \mathbb{R}$ let $\mathbf{r}$ denote the constant function $\mathbf{r}:\mathbb{N} \to \mathbb{R}$ with value r,i.e.,$\mathbf{r}(n) = r$,for all $n \in \mathbb{N}$.We then have a natural embedding

$*(\cdot) : \mathbb{R} \hookrightarrow {}^*\mathbb{R}$

by setting ${}^*r = [\mathbf{r}(n)]$ for all $r \in \mathbb{R}$.We denote it image $*(\mathbb{R})$ in ${}^*\mathbb{R}$ by ${}^*\mathbb{R}_{\mathbf{st}}$.

Definition 1.2.[3]. An element $x \in {}^*\mathbb{R}$ is called finite if $|x| < r$ for some $r \in \mathbb{Q}, r > 0$.

Abbreviation 1.2.For $x \in {}^*\mathbb{R}$ we abbreviate $x \in {}^*\mathbb{R}_{\mathbf{fin}}$ if x is finite.

Remark 1.2.[3]. Let $x \in {}^*\mathbb{R}_{\mathbf{fin}}$ be finite. Let D_1, be the set of $r \in \mathbb{Q}$ such that $r < x$ and D_2 the set of $r' \in \mathbb{Q}$ such that $x < r'$. The pair (D_1, D_2) forms a Dedekind cut in $\mathbb{R}$, hence determines a unique $r_0 \in \mathbb{R}$. A simple argument shows that $|x - r_0|$ is infinitesimal,i.e., $|x - r_0| \approx 0$.

Definition 1.3.[1].This unique r_0 is called the standard part of x and is denoted by ${}^\circ x$ or $st(x)$.

The following notation will be used throughout this paper.

$n \in \mathbb{N}^\#$ is a fixed positive integer and $U \subset {}^*\mathbb{R}_c^{\#n}$ is a fixed non-empty #-open subset of lnear space ${}^*\mathbb{R}_c^{\#n}$ over non Archemedan field${}^*\mathbb{R}_c^\#$.

$\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the standard natural numbers.

k will denote a non-negative integer or $\infty^\#$.

If f is a function then $\mathbf{Dom}(f)$ will denote its domain and the support of f, denoted by $supp(f)$,is defined to be the closure of the set $\{x \in \mathbf{Dom}(f) : f(x) \neq 0\}$ in $\mathbf{Dom}(f)$.

For two functions $f, g : U \to {}^*\mathbb{C}_c^\#$, the following notation defines external canonical pairing:

$$\langle f, g \rangle = Ext\text{-}\int_U f(x)g(x)d^\# x. \tag{1.1}$$

A multi-index of size $n \in \mathbb{N}^\#$ is an element in $\mathbb{N}^{\#n}$, if the size of multi-indices is omitted then the size should be assumed to be n. The length of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{\#n}$ is defined as $Ext\text{-}\sum_{i=1}^n \alpha_i$ and denoted by $|\alpha|$.Multi-indices are particularly useful when

dealing with functions of several variables, in particular we introduce the following canonical notations for a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{\#n}$,

$$\begin{aligned} x^\alpha &= x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \\ \partial^{\#\alpha} &= \frac{\partial^{\#|\alpha|}}{\partial^\# x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \end{aligned} \tag{1.2}$$

We also introduce a partial order of all multi-indices by $\beta \geq \alpha$ if and only if $\beta_i \geq \alpha_i$ for all $1 \leq i \leq n$. When $\beta \geq \alpha$ we define their multi-index binomial coefficient as:

$\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}$.

1.Let $k \in \mathbb{N}^\# \cup \infty^\#$.

2.Let $C^{\#k}(U)$ denote the vector space of all k-times #-continuously #-differentiable ${}^*\mathbb{R}_c^\#$-valued or ${}^*\mathbb{C}_c^\#$-valued functions on U.

For any #-compact subset $K \subseteq U$, let $C^{\#k}(K)$ and $C^{\#k}(K;U)$ both denote the vector space of all those functions $f \in C^{\#k}(U)$ such that $\mathbf{supp}(f) \subseteq K$.
Note that $C^{\#k}(K)$ depends on both K and U but we will only indicate K, where in particular, if $f \in C^{\#k}(K)$ then the domain of f is U rather than K. We will use the notation
$C^{\#k}(K;U)$ only when the notation $C^{\#k}(K)$ risks being ambiguous.
Every $C^{\#k}(K)$ contains the constant 0 map, even if $K = \varnothing$.
Let $C_c^{\#k}(U)$ denote the set of all $f \in C^{\#k}(U)$ such that $f \in C^{\#k}(K)$ for some #-compact subset K of U.
Equivalently, $C_c^{\#k}(U)$ is the set of all $f \in C^{\#k}(U)$ such that f has #-compact support.
$C_c^{\#k}(U)$ is equal to the union of all $C^{\#k}(K)$ as $K \subseteq U$ ranges over all #-compact subsets of U. If f is a ${}^*\mathbb{R}_c^{\#}$-valued function on U, then f is an element of $C_c^{\#k}(U)$ if and only if f is a $C^{\#k}$ bump function. Every ${}^*\mathbb{R}_c^{\#}$-valued test function on U is always also a ${}^*\mathbb{C}_c^{\#}$-valued test function on U.
For all $j,k \in \mathbb{N}$ and any #-compact subsets K and L of U, we have:
$C^{\#k}(K) \subseteq C_c^{\#k}(U) \subseteq C^{\#k}(U)$;
$C^{\#k}(K) \subseteq C^{\#k}(L)$ if $K \subseteq L C^{\#k}(K) \subseteq C^{\#j}(K)$ if $j \leq k$;
$C_c^{\#k}(U) \subseteq C_c^{\#j}(U)$ if $j \leq k$;
$C^{\#k}(U) \subseteq C^{\#j}(U)$ if $j \leq k$.
Definition1.1. Elements of $C_c^{\#\infty^{\#}}(U)$ are called ${}^*\mathbb{R}_c^{\#}$-valued test functions on U and $C_c^{\#\infty^{\#}}(U)$ is
called the space of ${}^*\mathbb{R}_c^{\#}$-valued test functions on U. We will use both $D(U)$ and $C_c^{\#\infty^{\#}}(U)$ to denote this space.
Definition1.2. Distributions on U are #-continuous ${}^*\mathbb{R}_c^{\#}$-valued linear functionals on $C_c^{\#\infty^{\#}}(U)$ when this vector space is endowed with a particular topology called the canonical
LF-topology.
The following proposition states two necessary and sufficient conditions for the #-continuity of a linear functional on $C_c^{\#\infty^{\#}}(U)$ that are often straightforward to verify.
Proposition1.1. A linear functional T on $C_c^{\#\infty^{\#}}(U)$ is #-continuous, and therefore a distribution, if and only if either of the following equivalent conditions are satisfied:
1.For every #-compact subset $K \subseteq U$ there exist constants $C > 0$ and $N \in \mathbb{N}$ dependent on K such that for all $f \in C_c^{\#\infty^{\#}}(U)$ with support contained in K
$|T(f)| \leq C \sup\{|\partial^{\#\alpha} f(x)|: x \in U, |\alpha| \leq N\}$.
2.For every #-compact subset $K \subseteq U$ and every sequence $\{f_i\}_{i=1}^{\infty^{\#}}$ in $C_c^{\#\infty^{\#}}(U)$ whose supports are contained in K, if $\{\partial^{\#\alpha} f_i\}_{i=1}^{\infty}$ #-converges uniformly to zero on U for every multi-index α, then $\#\text{-}\lim_{i\to\infty^{\#}} T(f_i) = 0$.

§ 2.The non-Archimedian external ${}^*\mathbb{R}_c^{\#}$-Valued Schwartz distributions.

Defined below are the tempered distributions, which form a subspace of $\mathcal{D}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$, the space of distributions on ${}^*\mathbb{R}_c^{\#n}$. This is a proper subspace: while every tempered distribution is a distribution and an element of $\mathcal{D}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ the converse is not true. Tempered distributions are useful if one studies the Fourier transform since all tempered distributions have a Fourier transform, which is not true for an arbitrary distribution in $\mathcal{D}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$.

§ 2.1.Schwartz space $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$.

Definition 2.1. A function $f : X \to {}^{*}\mathbb{R}_c^{\#}$ defined on some set X is called finitely bounded (or bounded) if the set of its values is finitely bounded, i.e., $f(X) \subset [a,b]$ where $a,b \in {}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}$. In other words, there exists a finite hyperreal number $M \in {}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}$ such that

$$|f(X)| \leq M. \tag{2.1}$$

Definition 2.2.A function $f : X \to {}^{*}\mathbb{R}_c^{\#}$ defined on some set X is called hyper finitely bounded (or hyper bounded) if the set of its values is hyper finitely bounded, i.e., $f(X) \subset [a,b]$ where $a,b \in {}^{*}\mathbb{R}_c^{\#}\backslash{}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}$. In other words, there exists a hyperfinite hyperreal number $M \in {}^{*}\mathbb{R}_c^{\#}\backslash{}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}$ such that $|f(X)| \leq M$.

Definition 2.3.For $n \in \mathbb{N}^{\#}$, an #-integrable function $\phi : {}^{*}\mathbb{R}_c^{\#n} \to {}^{*}\mathbb{R}_c^{\#}$is called #-rapidly decreasing if for all $\alpha \in \mathbb{N}^{\#n}$ the product function $x \mapsto x^{\alpha}\phi(x)$ is a finitely bounded or hyper finitely bounded function.

Remark 2.1.If ϕ is a #-rapidly decreasing function, then its integral exists

$$Ext\text{-}\int\limits_{{}^{*}\mathbb{R}_c^{\#n}} \phi(x)d^{\#n}x < \infty^{\#} \tag{2.2}$$

In fact for all $\alpha \in \mathbb{N}^{\#n}$ the integral of $x \mapsto x^{\alpha}\phi(x)$ exists

$$Ext\text{-}\int\limits_{{}^{*}\mathbb{R}_c^{\#n}} x^{\alpha}\phi(x)d^{\#n}x < \infty^{\#}. \tag{2.3}$$

Definition 2.4.The Schwartz space, $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$, is the space of all #-smooth functions in $C^{\#\infty^{\#}}({}^{*}\mathbb{R}_c^{\#n})$ that are rapidly decreasing at #-infinity along with all partial #-derivatives. Thus
$\phi : {}^{*}\mathbb{R}_c^{\#n} \to {}^{*}\mathbb{R}_c^{\#}$ is in the Schwartz space provided that any #-derivative of $\phi,$ multiplied with any power of $|x|,$ #-converges to 0 as $|x|\to \infty^{\#}.$ These functions form a #-complete TVS with a suitably defined family of seminorms. More precisely, for any multi-indices α and β define:

$$p_{\alpha,\beta}(\phi) = \sup_{x\in{}^{*}\mathbb{R}_c^{\#n}}|x^{\alpha}\partial^{\#\beta}\phi(x)|. \tag{2.1}$$

Then ϕ is in the Schwartz space $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$ if all the values satisfy: $p_{\alpha,\beta}(\phi) < \infty^{\#}$.
Thus

$$\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n},{}^{*}\mathbb{R}_c^{\#}) \triangleq \left\{\phi \in C^{\infty^{\#}}({}^{*}\mathbb{R}_c^{\#n},{}^{*}\mathbb{R}_c^{\#})|\forall\alpha,\beta \in \mathbb{N}^{\#n}(p_{\alpha,\beta}(\phi) < \infty^{\#})\right\}.$$

Similarly

$$\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n},{}^{*}\mathbb{C}_c^{\#}) \triangleq \left\{\phi \in C^{\infty^{\#}}({}^{*}\mathbb{R}_c^{\#n},{}^{*}\mathbb{C}_c^{\#})|\forall\alpha,\beta \in \mathbb{N}^{\#n}(p_{\alpha,\beta}(\phi) < \infty^{\#})\right\}$$

The family of seminorms $p_{\alpha,\beta}(\cdot)$ defines a locally convex topology on the Schwartz space $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$.
For $n = 1,$ the seminorms are norms on the Schwartz space $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#}).$ One can also use the following family of seminorms to define the topology:

$$|f|_{m,k} = \sup_{|p|\leq m}\left(\sup_{x\in{}^{*}\mathbb{R}_c^{\#n}}\{(1+|x|)^{k}|(\partial^{\#\alpha}f)(x)|\}\right), k,m \in \mathbb{N}^{\#}. \tag{2.2}$$

Otherwise, one can define a norm on $\mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ by

$$\|\phi\|_{k} = \max_{|\alpha|+|\beta|\leq k} \sup_{x\in{}^{*}\mathbb{R}_{c}^{\#n}} |x^{\alpha}\partial^{\beta}\phi(x)|, k \geq 1. \tag{2.3}$$

The Schwartz space $\mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ is a Fréchet space (that is, a #-complete metrizable locally convex space). Because the Fourier transform changes $\partial^{\#\alpha}$ into multiplication by x^{α} and vice versa, this symmetry implies that the Fourier transform of a Schwartz function is also a Schwartz function.

Definition 2.5. A sequence $\{f_i\}_{i=1}^{\infty^{\#}}$ #-converges to 0 in $\mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ if and only if the functions $(1+|x|)^{k}(\partial^{\#p}f_i)(x)$ #-converge to 0 uniformly in the whole of ${}^{*}\mathbb{R}_{c}^{\#n}$, which implies that such a sequence must converge to zero in $C^{\infty^{\#}}({}^{*}\mathbb{R}_{c}^{\#n})$.

The subset of all #-analytic Schwartz functions is #-dense in $\mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$

The Schwartz space is nuclear and the tensor product of two maps induces a canonical surjective TVS-isomorphisms $\mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#m})\ \widehat{\otimes}\ \mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n}) \to \mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#m+n})$,

where $\widehat{\otimes}$ represents the #-completion of the injective tensor product

§ 2.2.Schwartz space $\mathcal{S}_{\mathbf{fin}}^{\#}({}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#n})$

Definition 2.6.For $n \in \mathbb{N}$, an ${}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#n}$-valued and #-integrable function $\phi : {}^{*}\mathbb{R}_{c}^{\#n} \to {}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}$ is called #-rapidly decreasing if for all $\alpha \in \mathbb{N}^{n}$ the product function $x \mapsto x^{\alpha}\phi(x)$ is a finitely bounded function.

Remark 2.2.If ϕ is a #-rapidly decreasing ${}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#n}$-valued function, then its integral exists and finite,i.e.,

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#n}} \phi(x)d^{\#n}x \in {}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}. \tag{2.2}$$

In fact for all $\alpha \in \mathbb{N}^{n}$ the integral of $x \mapsto x^{\alpha}\phi(x)$ exists and finite,i.e.,

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#n}} x^{\alpha}\phi(x)d^{\#n}x \in {}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}. \tag{2.3}$$

It follows from () that for all $\alpha \in \mathbb{N}^{n}$ and for any $R \in {}^{*}\mathbb{R}_{c}^{\#}\backslash{}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}$

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#n}\backslash B(R)} x^{\alpha}\phi(x)d^{\#n}x \approx 0 \tag{2.3}$$

where $B(R) \triangleq \{x \in {}^{*}\mathbb{R}_{c}^{\#} \| |x| \leq R\}$

Definition 2.7.The Schwartz space, $\mathcal{S}_{\mathbf{fin}}^{\#}\left({}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#n}\right)$, is the space of all ${}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#n}$- valued #-smooth functions that are rapidly decreasing at #-infinity along with all partial #-derivatives any finite order $1 \leq m < \infty$.

Thus

$\phi : {}^{*}\mathbb{R}_{c}^{\#n} \to \mathbb{R}_{c}^{\#}$ is in the Schwartz space provided that any #-derivative of ϕ, multiplied with any power of $|x|$, #-converges to 0 as $|x| \to \infty^{\#}$. These functions form a #-complete TVS with a suitably defined family of seminorms. More precisely, for any multi-indices α and β define:

$$p_{\alpha,\beta}(\phi) = \sup_{x\in{}^{*}\mathbb{R}_{c}^{\#n}} |x^{\alpha}\partial^{\#\beta}\phi(x)|. \tag{2.1}$$

§ 2.3.Non-Archimedian tempered distributions $\mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_{c}^{\#n})$.

A non-Archimedean tempered distribution is a distribution $u \in \mathcal{D}'({}^*\mathbb{R}_c^{\#n})$ that does not "grow too fast" – at most polynomial (or tempered) growth – at #-infinity in all directions; in particular it is only defined on ${}^*\mathbb{R}_c^{\#n}$, not on any #-open subset. Formally, a tempered distribution is a #-continuous linear functional on the Schwartz space $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ of smooth functions with #-rapidly decreasing #-derivatives. The space of tempered distributions (with its natural topology) is denoted $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$. Every #-compactly supported distribution is a tempered distribution , yielding an inclusion $\mathcal{E}^{\#\prime}({}^*\mathbb{R}_c^{\#n}) \hookrightarrow \mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$.

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§ 3. The Fourier transform on $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n}), \mathcal{S}^{\#}_{\mathbf{fin}}({}^*\mathbb{R}_c^{\#n})$

We begin by defining the Fourier transform, and the inverse transform, on $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$,

$n \in \mathbb{N}^{\#}$, the Schwartz space of $C^{\infty^{\#}}$ functions of rapid decrease.

Definition 3.1. Suppose $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. The Fourier transform of $f(x)$ is the function $\widehat{f}(\lambda)$
given by

$$\widehat{f}(\lambda) = \frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int\limits_{{}^*\mathbb{R}_c^{\#n}} f(x)[Ext\text{-}\exp(-ix\cdot\lambda)]d^{\#n}x\right), \tag{3.1}$$

where $\mathbf{x}\cdot\boldsymbol{\lambda} = Ext\text{-}\sum_{i=1}^{n} x_i\lambda_i$. The inverse Fourier transform of f, denoted by $\check{f}$, is the function

$$\check{f}(\lambda) = \frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int\limits_{{}^*\mathbb{R}_c^{\#n}} f(x)[Ext\text{-}\exp(ix\cdot\lambda)]d^{\#n}x\right). \tag{3.2}$$

We will usualy write $\hat{f} = \mathcal{F}[f]$ and $\check{f} = \mathcal{F}^{-1}[f]$.
Since every function in Schwartz space is in $\mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#n})$, the above integrals (1.1) and (1.2) make sense.
We will use the standard multi-index notation. A multi-index $\alpha = \langle\alpha_1,\ldots,\alpha_n\rangle, n \in \mathbb{N}^{\#}$ is an n-tuple of nonnegative integers. The collection of all multi-indices will be denoted by I_+^n. The symbols $|\alpha|, x^\alpha, D^{\#\alpha}$, and x^2 are defined as follows:

$$\begin{gathered}
|\alpha| = Ext\text{-}\sum_{i=1}^{n}\alpha_i \\
x^\alpha = Ext\text{-}\prod_{i=1}^{n} x_i^{\alpha_i} \text{ or } Ext\text{-}(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}) \text{ or simbolically } x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \\
D^{\#\alpha}f(x) = Ext\text{-}\prod_{i=1}^{n}\frac{\partial^{\#\alpha_i}}{\partial^{\#}x^{\alpha_i}}f(x) \text{ or simbolically } D^{\#\alpha}f(x) = \frac{\partial^{\#|\alpha|}f(x)}{\partial^{\#}x^{\alpha_1}\partial^{\#}x^{\alpha_2}\cdots\partial^{\#}x^{\alpha_n}} \\
x^2 = Ext\text{-}\sum_{i=1}^{n}x_i^2.
\end{gathered} \tag{3.3}$$

Lemma 1.1.The maps $f \mapsto \hat{f}$ and $f \mapsto \check{f}$ are #-continuous linear transformations of $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ into $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$.Furthermore, if α and β are multi-indices, then

$$\left((i\lambda)^\alpha D^{\#\beta}\hat{f}\right)(\lambda) = \widehat{D^{\#\alpha}\left((-ix)^\beta f(x)\right)}(\lambda). \tag{3.4}$$

Proof The map $f \mapsto \hat{f}$ is clearly linear. Since

$$\left((i\lambda)^{\alpha} D^{\#\beta}\hat{f}\right)(\lambda) =$$

$$\frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} (\lambda^{\alpha})(-ix)^{\beta} f(x)[Ext\text{-}\exp(-ix\bullet\lambda)]f(x)d^{\#n}x\right) =$$

$$\frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} \frac{1}{(-i)^{\alpha}}(D_x^{\#\alpha}[Ext\text{-}\exp(-ix\bullet\lambda)])(-ix)^{\beta} f(x)d^{\#n}x\right) = \qquad (3.5)$$

$$\frac{(-i)^{\alpha}}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}x}^{\#\beta} [Ext\text{-}\exp(-ix\bullet\lambda)]D_x^{\#\alpha}\left((-ix)^{\beta} f(x)\right)d^{\#n}x\right).$$

We conclude that

$$\left\|\hat{f}\right\|_{\alpha,\beta} = \sup_{\lambda\in{}^*\mathbb{R}_c^{\#n}} \left|\lambda^{\alpha}\left(D^{\#\beta}\hat{f}\right)(\lambda)\right| \leq \frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |D_x^{\#\alpha}(x^{\beta} f(x))|d^{\#n}x\right) < \infty^{\#} \quad (3.6)$$

so $f \mapsto \hat{f}$ takes $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ into $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, and we have also proven (1.4).Furthermore, if k is large enough, $\int(1+x^2)^{-k}d^{\#n}x < \infty^{\#}$ so that

$$\left\|\hat{f}\right\|_{\alpha,\beta} \leq \frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} \frac{(1+x^2)^{-k}}{(1+x^2)^{-k}}\left|D_x^{\#\alpha}\left((-ix)^{\beta} f(x)\right)\right|d^{\#n}x\right) \leq$$

$$\frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} (1+x^2)^{-k}d^{\#n}x\right) \sup_{x\in{}^*\mathbb{R}_c^{\#n}} \left\{(1+x^2)^{+k}\left|D_x^{\#\alpha}\left((-ix)^{\beta} f(x)\right)\right|\right\}. \qquad (3.7)$$

Using generalized Leibnitz's rule we easily conclude that there exist multi-indices α_j, β_j and constants c_j so that

$$\left\|\hat{f}\right\|_{\alpha,\beta} \leq \sum_{j=1}^{M} c_j \|f\|_{\alpha_j,\beta_j}. \qquad (1.8)$$

Thus the map $f \mapsto \hat{f}$ is bounded and therefore #-continuous. The proof for $f \mapsto \check{f}$ is the same.

Theorem 1.1. (Generalized Fourier inversion theorem) The Fourier transform (3.1) is a linear bicontinuous bijection from $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ onto $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. Its inverse map is the inverse Fourier transform, i.e.,$\mathcal{F}^{-1}(\mathcal{F}[f]) = f$ and $\mathcal{F}(\mathcal{F}^{-1}[f]) = f$.

Proof. We will prove that $\mathcal{F}^{-1}(\mathcal{F}[f]) = f$. The proof that $\mathcal{F}(\mathcal{F}^{-1}[f]) = f$ is similar. $\mathcal{F}(\mathcal{F}^{-1}[f]) = f$ implies that $\mathcal{F}[f]$ is surjective and $\mathcal{F}^{-1}(\mathcal{F}[f]) = f$ implies that $\mathcal{F}[f]$ is injective. Since $\mathcal{F}[f]$ and $\mathcal{F}^{-1}[f]$ are #-continuous maps of $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ onto $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, it is sufficient to prove that $\mathcal{F}^{-1}(\mathcal{F}[f]) = f$ for f contained in the dense set $C_0^{\infty^{\#}}({}^*\mathbb{R}_c^{\#n})$. Let $C_{\varepsilon}, \varepsilon \approx 0$ be the cube of volume $(2/\varepsilon)^n$ centered at the origin in ${}^*\mathbb{R}_c^{\#n}$.Choose $\varepsilon \approx 0$ infinite small enough so that the support of f is contained in C_{ε}. Let $\mathbf{K}_{\varepsilon} = \{\mathbf{k} \in {}^*\mathbb{R}_c^{\#n} |$ each $k_i/\varepsilon\pi_{\#} \in \mathbf{k}$ is an integer $\}$

$$f(x)=Ext\text{-}\sum_{\mathbf{k}\in\mathbf{K}_\varepsilon}\left(\left(\tfrac{1}{2}\varepsilon\right)^{n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)],f\right)\left(\tfrac{1}{2}\varepsilon\right)^{n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)] \quad (3.9)$$

is just the hyper infinite Fourier series of f which #-converges uniformly in C_ε to f since f is #-continuously #-differentiable. Thus

$$f(x)=Ext\text{-}\sum_{\mathbf{k}\in\mathbf{K}_\varepsilon}\frac{\widehat{f}(k)[Ext\text{-}\exp(i\mathbf{k}\cdot x)]}{(2\pi_\#)^{n/2}}(\varepsilon\pi_\#)^n. \quad (3.10)$$

Since ${}^*\mathbb{R}_c^{\#n}$ is the disjoint union of the cubes of volume $(\varepsilon\pi_\#)^n$ centered about the points in $\mathbf{K}_\varepsilon$, the right-hand side of (1.10) is just a hyper finite Riemann sum for the integral of the function $\widehat{f}(k)[Ext\text{-}\exp(i\mathbf{k}\cdot x)]$. By the lemma 3.1, $\widehat{f}(k)[Ext\text{-}\exp(i\mathbf{k}\cdot x)]\in\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$, so the hyperfinite Riemann sums (1.10) #-converge to the integral. Thus $\mathcal{F}^{-1}(\mathcal{F}[f])=f$.

Corollary 3.1.Suppose $f\in\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$. Then

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}|f(x)|^2d^{\#n}x=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}|f(k)|^2d^{\#n}k. \quad (3.11)$$

Proof. This is really a corollary of the proof rather than the statement of Theorem 1.1. If f has #-compact support, then for $\varepsilon\approx 0$ small enough,

$$f(x)=Ext\text{-}\sum_{\mathbf{k}\in\mathbf{K}_\varepsilon}\left(\left(\tfrac{1}{2}\varepsilon\right)^{n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)],f\right)\left(\tfrac{1}{2}\varepsilon\right)^{n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)] \quad (3.12)$$

Since $\left\{\left(\tfrac{1}{2}\varepsilon\right)^{n/2n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)]\right\}_{\mathbf{k}\in\mathbf{K}_\varepsilon}$ is an orthonormal basis for $\mathcal{L}_2^\#(C_\varepsilon)$,

$$\begin{gathered}Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}|f(x)|^2d^{\#n}x=Ext\text{-}\int_{C_\varepsilon}|f(x)|^2d^{\#n}x=\sum_{\mathbf{k}\in\mathbf{K}_\varepsilon}\left|\left(\tfrac{1}{2}\varepsilon\right)^{n/2}([Ext\text{-}\exp(i\mathbf{k}\cdot x)],f(x))\right|^2=\\ \sum_{\mathbf{k}\in\mathbf{K}_\varepsilon}\left|\widehat{f}(k)\right|^2(\varepsilon\pi_\#)^n\underset{\varepsilon\to_\# 0}{\to_\#}Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}|f(k)|^2d^{\#n}k.\end{gathered} \quad (3.13)$$

This proves the corollary for $f\in C_0^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$. Since $f\mapsto\widehat{f}$ and $\|\cdot\|_2$ are #-continuous on $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$ and $C_0^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$ is #-dense, the result holds for all of $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$.

Definition 3.2. Let $T\in\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$the Fourier transform of T,denoted by $\hat{T}$ or $\mathcal{F}[T]$, is the tempered distribution defined by $\hat{T}(\varphi)=T\left(\widehat{\varphi}\right)$.

Suppose that $h,\varphi\in\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$,then by the polarization identity and the corollary to Theorem 1.1 we have $(h,\varphi)=\left(\widehat{h},\widehat{\varphi}\right)$. Substituting $\overline{\mathcal{F}[g]}=\mathcal{F}^{-1}[\overline{g}]$ for h, we obtain

$$T_{\widehat{g}}(\varphi)=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}\widehat{g}(x)\varphi(x)d^{\#n}x=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}g(x)\widehat{\varphi}(x)d^{\#n}x=T_g\left(\widehat{\varphi}\right)=\hat{T}_g(\varphi).$$

where $T_{\widehat{g}}$ and T_g are the distributions corresponding to the functions $\widehat{g}$ and g respectively. This shows that the Fourier transform on $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ extends the transform we previously defined on $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$.

Theorem 3.2. The Fourier transform is a one-to-one linear bijection from $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ to $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ which is the unique weakly #-continuous extension of the Fourier transform on $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$.

Proof. If hyper infinite sequence $\{\varphi_n\}_{n\in\mathbb{N}^\#}$ #-convergence to $\varphi\in\mathcal{S}^\#$, then by Theorem 1.1, hyper infinite sequence $\left\{\widehat{\varphi}_n\right\}_{n\in\mathbb{N}^\#}$ #-convergence to $\widehat{\varphi}\in\mathcal{S}^\#$,so

$T\left(\widehat{\varphi}_n\right) \to_{\#} T\left(\widehat{\varphi}\right)$ for each $T \in \mathcal{S}^{\#\prime}$. Thus $\#\text{-}\lim_{n\to\infty^{\#}} T\left(\widehat{\varphi}_n\right) = T\left(\widehat{\varphi}\right)$, which shows that T is a #-continuous linear functional on $\mathcal{S}^{\#}$. Furthermore, if $T_n \overset{w}{\to}_w T$, then $\widehat{T}_n \overset{w}{\to}_w \widehat{T}$ because $T\left(\widehat{\varphi}_n\right) \to_{\#} T\left(\widehat{\varphi}\right)$ implies $\widehat{T}(\varphi_n) \to_{\#} \widehat{T}(\varphi)$. Thus $T \mapsto \widehat{T}$ is weakly #-continuous.

Definition 3.3. Suppose that $f,g \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. Then the convolution of f and g, denoted by $f * g$, is the function

$$(f*g)(y) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} f(y-x)g(x)d^{\#n}x. \tag{3.14}$$

Convolutions frequently occur when one uses the Fourier transform because the Fourier transform takes products into convolutions.

Theorem 3.3.(a) For each $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, $g \mapsto f*g$ is a #-continuous map of $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ into $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$.

(b) $\widehat{fg} = (2\pi_{\#})^{-n/2}\widehat{f} * \widehat{g}$ and $\widehat{f*g} = (2\pi_{\#})^{n/2}\widehat{f}\widehat{g}$.

(c) For $f,g,h \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$,$f*g = g*f$ and $f*(g*h) = (f*g)*h$.

Definition 3.4. Suppose that $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n}), T \in \mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ and let $\widetilde{f}(x)$ denote the function, $f(-x)$. Then, the convolution of T and f denoted $T*f$ is the distribution in , $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ given by $(T*f)(\varphi) = T\left(\widetilde{f}*\varphi\right)$ for all $\varphi \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$.

The fact that g→ $\widetilde{f}*g$ is a #-continuous transformation guarantees that $T*f \in \mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$.

Abbreviation 3.1.Let f_y denote the function $f_y(x) = f(x-y)$ and $\widetilde{f}_y$ the function $f(y-x)$. When f is given by a longe expression $(\bullet\bullet\bullet)$, we will sometimes write $(\bullet\bullet\bullet)^{\sim}$ rather than $(\widetilde{\bullet\bullet\bullet})$.

Theorem 3.4. For each $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ the map $T \to T*f$ is a weakly #-continuous map of $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ into $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ which extends the convolution on $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. Furthermore,

(a) $T*f$ is a polynomially bounded $C^{\infty^{\#}}$ function. In fact, $(T*f)(y) = T\left(\widetilde{f}_y\right)$ and $D^{\#\beta}(T*f) = (D^{\#\beta}T)*f = T*D^{\#\beta}f$;

(b) $(T*f)*g = T*(f*g)$;

(c) $\widehat{T*f} = (2\pi_{\#})^{n/2}\widehat{f}\widehat{T}$.

Theorem 3.5. Let $T \in \mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ and $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. Then $\widehat{fT} \in O_M^n$ and $\widehat{fT}\,(k) = (2\pi_{\#})^{n/2}T(f[Ext\text{-}\exp(-ik\bullet x)])$. In particular, if T has #-compact support and $\psi \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ is identically one on a #-neighborhood of the support of T, then

$$\widehat{T}(k) = (2\pi_{\#})^{n/2}T(\psi[Ext\text{-}\exp(-ik\bullet x)]). \tag{3.15}$$

Proof By Theorem 3.4.c and the Fourier inversion formula we have $\widehat{fT} = (2\pi_{\#})^{n/2}\widehat{f}*\widehat{T}$. Thus $\widehat{fT} \in O_M^n$ and $\widehat{fT}\,(k) = (2\pi_{\#})^{n/2}\widehat{T}\left(\widetilde{\widehat{f}}_k\right) =$ $(2\pi_{\#})^{n/2}T(f[Ext\text{-}\exp(-ik\bullet x)])$.

Remark 3.1.We remark that one can also define the convolution of a distribution $T \in \mathcal{D}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ with an $f \in \mathcal{D}^{\#}({}^*\mathbb{R}_c^{\#n})$ by $(T*f)(y) = T\left(\widetilde{f}_y\right)$.

Definition 3.5. Let $j(x)$ be a positive $C^{\infty^{\#}}$ function whose support lies in the sphere of radius one about the origin in ${}^*\mathbb{R}_c^{\#n}$ and which satisfies $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} j(x)d^{\#n}x = 1$. The function $j_\varepsilon(x) = \varepsilon^{-n}j(x/\varepsilon), \varepsilon \approx 0$ is called an approximate identity.

Proposition 3.1. Suppose $T \in \mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ and let $j_\varepsilon(x)$ be an approximate identity. Then

$T * j_{\varepsilon}(x) \to_{\#} T$ weakly as $\varepsilon \to_{\#} 0$.
Proof. If $\varphi \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$, then $(T * j_{\varepsilon})(\varphi) = T\left(\widetilde{j}_{\varepsilon} * \varphi\right)$, so it is sufficient to show that $\widetilde{j}_{\varepsilon} * \varphi \to_{\#} \varphi$ in $\mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$. To do this it is sufficient to show that $(2\pi_{\#})^{n/2}\widehat{j}_{\varepsilon}\widehat{\varphi} \to_{\#} \widehat{\varphi}$ in $\mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$. Since $\widehat{j}_{\varepsilon}(\lambda) = j(\varepsilon\lambda)$ and $j(0) = (2\pi_{\#})^{n/2}$, it follows that $(2\pi_{\#})^{n/2}\widehat{j}_{\varepsilon}(x)$ #-converges to 1 uniformly on #-compact sets and is uniformly bounded. Similarly, $D^{\#\alpha}\widehat{j}_{\varepsilon}$ #-converges uniformly to zero. We conclude that $(2\pi_{\#})^{n/2}\widehat{j}_{\varepsilon}\widehat{\varphi} \to_{\#} \widehat{\varphi}$.
Theorem 3.6 (The generalized Plancherel theorem) The Fourier transform extends uniquely to a unitary map of $\mathcal{L}_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ onto $\mathcal{L}_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$. The inverse transform extends uniquely to its adjoint.
Proof The corollary to Theorem 3.1 states that if $f \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$, then $\|f\|_{2} = \left\|\widehat{f}\right\|_{2}$. Since $\mathcal{F}[\mathcal{S}^{\#}] = \mathcal{S}^{\#}$ is a surjective isometry on $\mathcal{L}_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$.
Theorem 3.7 (The generalized Riemann-Lebesgue lemma) The Fourier transform extends uniquely to a bounded map from $\mathcal{L}_{1}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ into $C^{\infty^{\#}}({}^{*}\mathbb{R}_{c}^{\#n})$, the #-continuous functions vanishing at $\infty^{\#}$.
Proof For $f \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$, we know that $\widehat{f} \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ and thus $\widehat{f} \in C^{\infty^{\#}}({}^{*}\mathbb{R}_{c}^{\#n})$. The estimate is trivial. The Fourier transform is thus a bounded linear map from a #-dense set of $\mathcal{L}_{1}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ into $C^{\infty^{\#}}({}^{*}\mathbb{R}_{c}^{\#n})$. By the generalized B.L.T. theorem, extends uniquely to a bounded linear transformation of $C^{\infty^{\#}}({}^{*}\mathbb{R}_{c}^{\#n})$ into $C^{\infty^{\#}}({}^{*}\mathbb{R}_{c}^{\#n})$.
Remark 3.2.We remark that the Fourier transform takes $\mathcal{L}_{1}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ into, but not onto $C^{\infty^{\#}}({}^{*}\mathbb{R}_{c}^{\#n})$.
A simple argument with test functions shows that the extended transform on $\mathcal{L}_{1}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ and $\mathcal{L}_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ is the restriction of the transform on $\mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_{c}^{\#n})$, but it is useful to have an explicit integral representation. For $f \in \mathcal{L}_{1}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$, this is easy since we can find $f_{m} \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ so that $\#\text{-}\lim_{m\to\infty^{\#}}\|f - f_{m}\|_{1} = 0$. Then, for each λ,

$$f(\lambda) = \#\text{-}\lim_{m\to\infty^{\#}}\left(\widehat{f}_{m}(\lambda)\right) =$$
$$\#\text{-}\lim_{m\to\infty^{\#}}\left\{\frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#n}}[Ext\text{-}\exp(-ik\cdot x)]f_{m}(x)d^{\#}x\right)\right\} = \tag{3.16}$$
$$\frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#n}}[Ext\text{-}\exp(-ik\cdot x)]f(x)d^{\#}x\right).$$

So, the Fourier transform of a function in $\mathcal{L}_{1}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ is given by the usual formula.
Next, suppose $f \in \mathcal{L}_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ and let

$$\chi_{R}(x) = \begin{cases} 1 & \text{if} \quad |x| \leq R \\ 0 & \text{if} \quad |x| > R \end{cases} \tag{3.17}$$

Then $\chi_{R}f \in \mathcal{L}_{1}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ and $\#\text{-}\lim_{R\to\infty^{\#}}\chi_{R}f = f$ in $\mathcal{L}_{2}^{\#}$, so by the generalized Plancherel theorem $\#\text{-}\lim_{R\to\infty^{\#}}\widehat{\chi_{R}f} = \widehat{f}$ in $\mathcal{L}_{2}^{\#}$. Thus

$$f(\lambda) = \#\text{-}\lim_{R\to\infty^{\#}}\frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int_{|x|\leq R}[Ext\text{-}\exp(-ik\cdot x)]f(x)d^{\#}x\right) \tag{3.18}$$

where by $\#\text{-}\lim_{R\to\infty^{\#}}$ we mean the #-limit in the $\mathcal{L}_{2}^{\#}$-norm. Sometimes we will dispense with $|x| \leq R$ and just write

$$f(\lambda) = \#\text{-}\lim_{R\to\infty^{\#}} \frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int[Ext\text{-}\exp(-ik\cdot x)]f(x)d^{\#}x\right) \quad (3.19)$$

for functions $f \in \mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#n})$.

We have proven above that $\mathcal{F} : \mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#n}) \to \mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#n})$ and $\mathcal{F} : \mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#n}) \to \mathcal{L}_{\infty^{\#}}^{\#}({}^*\mathbb{R}_c^{\#n})$ and in both cases is a bounded operator.

Theorem 3.8 (Generalized Hausdorff-Young inequality) Suppose $1 \leq q \leq 2$, and $p^{-1} + q^{-1} = 1$. Then the Fourier transform is a bounded map of $\mathcal{L}_p^{\#}({}^*\mathbb{R}_c^{\#n})$ to $\mathcal{L}_q^{\#}({}^*\mathbb{R}_c^{\#n})$ and its norm is less than or equal to $(2\pi_{\#})^{n(1/2-1/q)}$.

Chapter V.Non-Archiedean Hilbert Spaces over field $\widetilde{{}^*\mathbb{C}_c^{\#}}$.

§ 1. Non-Archiedean Hilbert Spaces over field $\widetilde{{}^*\mathbb{C}_c^{\#}}$ Basics.

Definition 1.1.(i) Let H be external hyper infinite dimensional vector space over field $\widetilde{{}^*\mathbb{C}_c^{\#}} = \widetilde{{}^*\mathbb{R}_c} + i\widetilde{{}^*\mathbb{R}_c}$.An inner #-product(or non-Archiedean inner product) on H is a $\widetilde{{}^*\mathbb{C}_c^{\#}}$-valued function, $\langle\cdot,\cdot\rangle_{\#} : H \times H \to {}^*\mathbb{C}_c$, such that

(1) $\langle \alpha x + \beta y, z\rangle_{\#} = \alpha\langle x,z\rangle + \beta\langle y,z\rangle_{\#}$, i.e. $x \to \langle x,z\rangle_{\#}$ is linear.

(2) $\overline{\langle x,y\rangle_{\#}} = \langle y,x\rangle_{\#}$.

(3) $\|x\|_{\#}^2 \equiv \langle x,x\rangle_{\#} \geq 0$ with equality $\|x\|_{\#}^2 = 0$ iff $x = 0$.

Notice that combining properties (1) and (2) that $x \to \langle z,x\rangle$ is anti-linear for fixed $z \in H$, i.e. $\langle z, ax + by\rangle_{\#} = \bar{a}\langle z,x\rangle_{\#} + \bar{b}\langle z,y\rangle_{\#}$.

(ii) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $H, \{a_n\}_{n=0}^{k} \subset H$.

We define external hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{k}} \subset H$ by

$$\begin{gathered}\{A_n;k\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{k}} = \\ = \left(a_0, a_1, \ldots, a_m, \ldots, a_{k-1}, \widehat{a_k}\right). \\ \widehat{a} = (a, a, \ldots.), a \in H.\end{gathered} \quad (0.1)$$

(iii) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $H : \{a_n\}_{n=0}^{\infty} \subset H$.

We define hyper infinite sequence $\{A_n\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{\infty}} \subset H$ by

$$\begin{gathered}\{A_n';\infty\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{\infty}} = \\ = \left(\widehat{a}_0, a_1, \ldots, a_k, \ldots \{a_n\}_{n=0}^{\infty}, \widehat{\{a_n\}_{n=0}^{\infty}}\right).\end{gathered} \quad (0.2)$$

(iv) Let $\{a_n\}_{n=0}^{N}, N \in {}^*\mathbb{N}\backslash\mathbb{N}$ be external hyperfinite sequence in $H : \{a_n\}_{n=0}^{N} \subset H$.

We define hyper infinite sequence $\overbrace{\{a_n\}_{n=0}^{N}} \subset H$ by

$$\begin{gathered}\{A_n;N\}_{n=0}^{*\infty} = \overbrace{\{a_n\}_{n=0}^{N}} = \\ = \left(a_0, a_1, , \ldots, a_m, \ldots, a_{N-1}a_N, \widehat{a_N}\right).\end{gathered} \quad (0.3)$$

(v) Let $\{a_n\}_{n=0}^{k}, k \in \mathbb{N}$ be finite sequence in $H, \{a_n\}_{n=0}^{N} \subset H$.

We define external finite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=k} a_n = \overbrace{\{c_n\}_{n=0}^{k}} = (c_0, c_1, \ldots, c_k, \ldots, \widehat{c}_k) \in [[\widehat{c}_k]] \tag{0.4}$$

where $c_0 = a_0, c_j = Ext\text{-}\sum_{n=0}^{n=j} a_n, 0 \leq j \leq k.$

(vi) Let $\{a_n\}_{n=0}^{\infty}$ be countable sequence in $H : \{a_n\}_{n=0}^{\infty} \subset H.$ We define external countable sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n$ by

$$\begin{gathered} Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n = \overbrace{\{c_n\}_{n=0}^{\infty}} = \\ = \left(c_0, c_1, \ldots, c_k, \ldots \{c_n\}_{n=0}^{\infty}, \widehat{\{c_n\}_{n=0}^{\infty}}\right) \in \left[\left[\widehat{\{c_n\}_{n=0}^{\infty}}\right]\right] \end{gathered} \tag{0.5}$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n, k \in \mathbb{N}.$

(vii) Let $\{a_n\}_{n=0}^{n=N}, N \in {}^*\mathbb{N}\backslash\mathbb{N}$ be external hyperfinite sequence in $H : \{a_n\}_{n=0}^{N} \subset H.$ We define external hyperfinite sum $Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n$ by

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = \overbrace{\{c_n\}_{n=0}^{n=N}} = (c_0, c_1, \ldots, c_k, \ldots, c_N, \widehat{c}_N) \in [[\widehat{c}_N]] \tag{0.6}$$

where $c_0 = a_0, c_k = Ext\text{-}\sum_{n=0}^{n=k} a_n,\ 0 \leq k \leq N, c_N = Ext\text{-}\sum_{n=0}^{n=N} a_n.$

(viii) Let $\{a_n\}_{n=0}^{n=N}, N \in {}^*\mathbb{N}\backslash\mathbb{N}$ be external hyperfinite sequence in $H : \{a_n\}_{n=0}^{N} \subset H$ such that $a_n \equiv 0$ for all $n \in {}^*\mathbb{N}\backslash\mathbb{N}.$ We assume that

$$Ext\text{-}\widehat{\sum}_{n=0}^{n=N} a_n = Ext\text{-}\widehat{\sum}_{n=0}^{n=\infty} a_n. \tag{0.7}$$

Remark 1.1. (i) Let $\{x_i\}_{i=1}^{N} \subset H$ and $\{y_i\}_{i=1}^{N} \subset H, N \in {}^*\mathbb{N}$ by external hyperfinite sequences; let $\{\alpha_i\}_{i=1}^{N} \subset \widetilde{{}^*\mathbb{C}_c^{\#}}$ and $\{\beta_i\}_{i=1}^{N} \subset \widetilde{{}^*\mathbb{C}_c^{\#}}.$ Then the equality holds

$$Ext\text{-}\widehat{\sum}_{i=1}^{N} \langle \alpha_i x_i + \beta_i y_i, z\rangle_{\#} = Ext\text{-}\widehat{\sum}_{i=1}^{N} \alpha_i \langle x_i, z\rangle_{\#} + Ext\text{-}\widehat{\sum}_{i=1}^{N} \beta_i \langle y_i, z\rangle_{\#} \tag{0.8}$$

(ii) Let $\{x_{ij}\} \subset H, N, K \in {}^*\mathbb{N}, K \leq j \leq i \leq N,$ by external hyperfinite sequences; let $\{\alpha_{ij}\}_{i=1}^{N} \subset \widetilde{{}^*\mathbb{C}_c^{\#}}$. Then the equality holds

$$\left\langle Ext\text{-}\widehat{\sum}_{K \leq j \leq i \leq N}^{N} \alpha_{ij} x_{ij}, z \right\rangle_{\#} = Ext\text{-}\widehat{\sum}_{i=K}^{N} \left(Ext\text{-}\widehat{\sum}_{i=K}^{N} \alpha_{ij} \langle x_{ij}, z\rangle_{\#} \right). \tag{0.9}$$

(iii) Let $\{x_i\}_{i=1}^{*\infty} \subset H$ by external hyperinfinite sequence in H. We call $\{x_i\}_{i=1}^{*\infty}$ a Cauchy hyperinfinite sequence if for any $\varepsilon \approx 0, \varepsilon > 0$ there is $N \in {}^*\mathbb{N}\backslash\mathbb{N}$such that for any $m, n > N, \|x_n - x_m\|_{\#} < \varepsilon.$

(iv) We now stand ready to give construction of $H^{\#}$. The members of $H^{\#}$ will be constructed as equivalence classes of Cauchy hyperinfinite sequences in H. Let $C(H)$ denote the set of all Cauchy hyperinfinite sequences in H. We must define an equivalence relation on $C(H)$.For $s \in C(H)$, denote by $[s]$ the set of all elements in $C(H)$
that are related to s. Then for any $s, t \in C(H)$, either $[s] = [t]$ or $[s]$ and $[t]$ are disjoint. Let $\{x_i\}_{i=1}^{*\infty}$ and $\{y_i\}_{i=1}^{*\infty}$ be in $C(H)$. Say they are equivalent (i.e. related $\{x_i\}_{i=1}^{*\infty} \sim \{y_i\}_{i=1}^{*\infty}$) if $\|x_n - y_n\|_{\#} \to_{\#} 0_{\widetilde{{}^*\mathbb{R}_c}}$; i.e. if the hyperinfinite sequence $\|x_n - y_n\|_{\#}$ #-tends to $0_{\widetilde{{}^*\mathbb{R}_c}}$ as $n \to {}^*\infty.$

(vi) Definition (iv) yields an equivalence relation $(\cdot \sim \cdot)$ on $C(H)$.

Proof. We need to show that this relation is reflexive, symmetric, and transitive.
• **Reflexive**: $x_n - x_n = 0_{\widetilde{^*\mathbb{R}_c}}, n \in {}^*\mathbb{N}$ and the sequence all of whose terms are $0_{\widetilde{^*\mathbb{R}_c}}$ clearly #-converges to $0_{\widetilde{^*\mathbb{R}_c}}$.So $\{x_i\}_{i=1}^{^*\infty}$ is related to $\{x_i\}_{i=1}^{^*\infty}$.
• **Symmetric**: Suppose $\{x_i\}_{i=1}^{^*\infty}$ is related to $\{y_i\}_{i=1}^{^*\infty}$, so $\|x_n - y_n\|_\# \to_\# 0_{\widetilde{^*\mathbb{R}_c}}$. But $y_n - x_n = -(x_n - y_n)$, and since only the absolute value $\|x_n - y_n\|_\# = \|y_n - x_n\|_\#$ comes into play in Definition (iv), it follows that $\|y_n - x_n\|_\# \to_\# 0_{\widetilde{^*\mathbb{R}_c}}$ as well. Hence, $\{y_i\}_{i=1}^{^*\infty}$ is related to $\{x_i\}_{i=1}^{^*\infty}$.
• **Transitive**: Suppose $\{x_i\}_{i=1}^{^*\infty}$ is related to $\{y_i\}_{i=1}^{^*\infty}$, and $\{y_i\}_{i=1}^{^*\infty}$ is related to $\{z_i\}_{i=1}^{^*\infty}$. This means that $\|x_n - y_n\|_\# \to_\# 0_{\widetilde{^*\mathbb{R}_c}}$ and $\|y_n - z_n\|_\# \to_\# 0_{\widetilde{^*\mathbb{R}_c}}$.
To be fully precise, let us fix $\varepsilon \approx 0, \varepsilon > 0$; then there exists an $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that for all $n > N, \|x_n - y_n\|_\# < \varepsilon/2$; also, there exists an M such that for all $n > M$, $\|y_n - z_n\|_\# < \varepsilon/2$. Well, then, as long as n is bigger than both N and M, we have that $\|x_n - z_n\|_\# = \|(x_n - y_n) + (y_n + z_n)\|_\# \leq \|x_n - y_n\|_\# + \|y_n - z_n\|_\# < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
So, choosing L equal to the $\max(N,M)$, we see that given $\varepsilon \approx 0, \varepsilon > 0$ we can always choose L so that for $n > L$, $\|x_n - z_n\|_\# < \varepsilon$. This means that $\|x_n - z_n\| \to_\# 0_{\widetilde{^*\mathbb{R}_c}}$ – i.e. $\{x_i\}_{i=1}^{^*\infty}$ is related to $\{z_i\}_{i=1}^{^*\infty}$.
So, we really have an equivalence relation ($\bullet \sim \bullet$), and therefore the set $C(H)$ is partitioned into disjoint subsets (equivalence classes).We will denote that partition $C(H)/\sim$ by $H^\#$

$$C(H)/\sim \;\triangleq H^\#. \tag{0.10}$$

(vii) assume that $[s] = \left[\{x_i\}_{i=1}^{^*\infty}\right] \in H^\#$and $[t] = \left[\{y_j\}_{i=1}^{^*\infty}\right] \in H^\#$, we define inner #-product $\langle[s],[t]\rangle_\#$ on $H^\#$ by

$$\left\langle\left[\{x_i\}_{i=1}^{^*\infty}\right],\left[\{y_j\}_{i=1}^{^*\infty}\right]\right\rangle_\# = Ext\text{-}\widehat{\sum}_{i,j=1}^{i,j={}^*\infty} \langle x_i,y_j\rangle_\#. \tag{0.11}$$

In particular if for all $i \neq j$ $\langle x_i,y_j\rangle_\# = 0_{\widetilde{^*\mathbb{R}_c}}$

$$\left\langle\left[\{x_i\}_{i=1}^{^*\infty}\right],\left[\{y_j\}_{i=1}^{^*\infty}\right]\right\rangle_\# = Ext\text{-}\widehat{\sum}_{i=1}^{i={}^*\infty} \langle x_i,y_i\rangle_\#. \tag{0.12}$$

Remark 1.2.The following formula useful:

$$\|x+y\|_\#^2 = \langle x+y,x+y\rangle_\# = \|x\|_\#^2 + \|y\|_\#^2 + \langle x,y\rangle_\# + \langle y,x\rangle_\# = \|x\|_\#^2 + \|y\|_\#^2 + 2\operatorname{Re}\langle x,y\rangle_\# \tag{1.1}$$

Theorem 1.1. (Generalized Schwarz Inequality). Let $(H,\langle\bullet,\bullet\rangle_\#)$ be an inner #-product space and $x,y \in H$. Assume that:
(1) at least one of hyperreals $\|x\|_\#, \|y\|_\#$ is invertible
in $\widetilde{^*\mathbb{R}_c^\#}$ then

$$|\langle x,y\rangle_\#| \leq \|x\|_\# \times \|y\|_\# \tag{1.2}$$

and equality holds iff x and y are linearly dependent.
(2) both of hyperreals $\|x\|_\#, \|y\|_\#$ is not invertible in $\widetilde{^*\mathbb{R}_c^\#}$ then

$$|\langle x,y\rangle_\#| \times \check{1}_{\widetilde{^*\mathbb{R}_c^\#}} \leq \|x\|_\# \times \|y\|_\#. \tag{1.2'}$$

Proof. (1) If $y = 0$, the result holds trivially. So assume that $y \neq 0$ and $\|y\|_\#$ is invertible
in $\widetilde{^*\mathbb{R}_c^\#}$. First off notice that if $x = \alpha y$ for some $\alpha \in \widetilde{^*\mathbb{C}_c^\#}$, then $\langle x,y\rangle = \alpha\|y\|_\#^2$ and hence

$|\langle x,y\rangle_\#| = |\alpha|\|y\|_\#^2 = \|x\|_\#\|y\|_\#$

Note that in this case $\alpha = \langle x,y\rangle\|y\|^2$. Now suppose that $x \in H$ is arbitrary, let $z \equiv x - \|y\|_\#^{-2}\langle x,y\rangle_\# y$. So z is the orthogonal projection of x onto y. Then

$$0 \leq \|z\|_\#^2 = \left\| x - \frac{\langle x,y\rangle_\#}{\|y\|_\#^2} y \right\|_\#^2 = \|x\|_\#^2 + \frac{|\langle x,y\rangle_\#|^2}{\|y\|_\#^4}\|y\|_\#^2 - 2\operatorname{Re}\left\langle x, \frac{\langle x,y\rangle_\#}{\|y\|_\#^2} y \right\rangle = \\ = \|x\|_\#^2 - \frac{|\langle x,y\rangle_\#|^2}{\|y\|_\#^2}. \tag{1.3}$$

from (1.3) it follows that $0 \leq \|y\|_\#^2\|x\|_\#^2 - |\langle x,y\rangle_\#|^2$ with equality iff $z = 0$ or equivalently iff $x = \|y\|_\#^{-2}\langle x,y\rangle_\# y$.

(2) Let $z \equiv x - \left(\|y\|_\#^{-1_*}\right)^2\langle x,y\rangle_\# y$. So z is the orthogonal projection of x onto y. Then

$$0 \leq \|z\|_\#^2 = \left\| x - \langle x,y\rangle_\#\left(\|y\|_\#^{-1_*}\right)^2 y \right\|_\#^2 = \\ \|x\|_\#^2 + |\langle x,y\rangle_\#|^2\left(\|y\|_\#^{-1_*}\right)^4\|y\|_\#^2 - 2\operatorname{Re}\left\langle x, \langle x,y\rangle_\#\left(\|y\|_\#^{-1_*}\right)^2 y\right\rangle = \\ = \|x\|_\#^2 - |\langle x,y\rangle_\#|^2\left(\|y\|_\#^{-1_*}\right)^2. \tag{1.3$'$}$$

From (1.3$'$) it follows that $0 \leq \|y\|_\#^2 \times \|x\|_\#^2 - |\langle x,y\rangle_\#|^2 \times \check{1}_{\widetilde{{}^*\mathbb{R}_c^\#}}$.

Corollary 1.1. Let $(H^\#, \langle\cdot,\cdot\rangle)$ be an inner #-product space and $\|x\|_\# := \sqrt{\langle x,x\rangle_\#}$. Then $\|\cdot\|_\#$ is a ${}^*\mathbb{R}_c$-valued #-norm on $H^\#$. Moreover $\langle\cdot,\cdot\rangle_\#$ is #-continuous on $H^\# \times H^\#$, where H is viewed as the #-normed space $(H^\#, \|\cdot\|_\#)$.

Proof. The only non-trivial thing to verify that $\|\cdot\|_\#$ is a #-norm is the triangle inequality:

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x,y\rangle_\# \leq \|x\|^2 + \|y\|^2 + 2\|x\|_\#\|y\|_\# = (\|x\|_\# + \|y\|_\#)^2$

where we have made use of Schwarz's inequality. Taking the square root of this inequality shows $\|x+y\| \leq \|x\| + \|y\|$. For the #-continuity assertion:

$|\langle x,y\rangle_\# - \langle x',y'\rangle_\#| = |\langle x-x',y\rangle_\# + \langle x',y-y'\rangle_\#| \leq \|y\|_\#\|x-x'\|_\# + \|x'\|_\#\|y-y'\|_\#$

$\leq \|y\|_\#\|x-x'\|_\# + (\|x\|_\# + \|x-x'\|_\#)\|y-y'\|_\# = \|y\|_\#\|x-x'\|_\# + \|x\|_\#\|y-y'\|_\#$

$+ \|x-x'\|_\#\|y-y'\|_\#$ from which it follows that $\langle\cdot,\cdot\rangle$ is #-continuous.

Definition 1.2. Let $(H, \langle\cdot,\cdot\rangle_\#)$ be an inner #-product space, we say $x,y \in H$ are orthogonal and write $x \perp y$ iff $\langle x,y\rangle_\# = 0$. More generally if $A \subset H$ is a set, $x \in H$ is orthogonal to A and write $x \perp A$ iff $\langle x,y\rangle = 0$ for all $y \in A$. Let $A_\perp = \{x \in H : x \perp A\}$ be the set of vectors orthogonal to A. We also say that a set $S \subset H$ is orthogonal if $x \perp y$ for all $x,y \in S$ such that $x \neq y$. If S further satisfies, $\|x\|_\# = 1$ for all $x \in S$, then S is said to be orthonormal.

Proposition 1.1. Let $(H, \langle\cdot,\cdot\rangle_\#)$ be an inner product space then

(1) (Parallelogram Law)

$$\|x+y\|_\#^2 + \|x-y\|_\#^2 = 2\|x\|_\#^2 + 2\|y\|_\#^2 \tag{1.4}$$

for all $x,y \in H$.

(2) (Pythagorean Theorem) If $S \subset H$ is a hyperfinite orthonormal set, then

$$\left\| Ext\text{-}\widehat{\sum}_{x \in S} x \right\|_\#^2 = Ext\text{-}\widehat{\sum}_{x \in S}\|x\|_\#^2 \tag{1.5}$$

(3) If $A \subset H^\#$ is a set, then $A_\perp$ is a #-closed linear subspace of $H^\#$.

Proof. We will assume that $H^{\#}$ is a complex Hilbert space with $\widetilde{{}^{*}\mathbb{C}_c^{\#}}$-valued inner product, the real case being easier. Statements (1) and (2) are proved by the following elementary computations:

$$\|x+y\|_{\#}^{2} = \|x\|_{\#}^{2} + \|y\|_{\#}^{2} + 2\operatorname{Re}\langle x,y\rangle_{\#} + \|x\|_{\#}^{2} + \|y\|_{\#}^{2} - 2\operatorname{Re}\langle x,y\rangle_{\#} = 2\|x\|_{\#}^{2} + 2\|y\|_{\#}^{2} \quad (1.6)$$

and

$$\left\| Ext\text{-}\widehat{\sum_{x\in S}} x \right\|_{\#}^{2} = \left\langle Ext\text{-}\widehat{\sum_{x\in S}} x, Ext\text{-}\widehat{\sum_{y\in S}} y \right\rangle_{\#} = Ext\text{-}\widehat{\sum_{x,y\in S}} \langle x,y\rangle_{\#} =$$
$$= Ext\text{-}\widehat{\sum_{x\in S}} \langle x,x\rangle_{\#} = Ext\text{-}\widehat{\sum_{x\in S}} \|x\|_{\#}^{2}. \quad (1.7)$$

Item 3. is a consequence of the #-continuity of $\langle\bullet,\bullet\rangle_{\#}$ and the fact that $A^{\perp} = \cap_{x\in A}\mathbf{Ker}(\langle\bullet,x\rangle)$ where $\mathbf{Ker}(\langle\bullet,x\rangle) = \{y \in H | \langle y,x\rangle_{\#} = 0\}$ is a #-closed subspace of H.

Definition 1.3. A non-Archiedean Hilbert space $H^{\#}$ is an inner #-product space $(H,\langle\bullet,\bullet\rangle_{\#})$ such that the induced Hilbertian #-norm is #-complete.

Example 1.3. Let $(X,M,\mu^{\#})$ be a #-measure space then $H^{\#} = L_2^{\#}(X,M,\mu^{\#})$ with inner #-product $\langle f,g\rangle_{\#} = Ext\text{-}\int_X f\bar{g}d^{\#}\mu^{\#}$ is a non-Archiedean Hilbert space. Note that every non-Archiedean Hilbert space $H^{\#}$ is “equivalent” to a Hilbert space of this form.

Definition 1.4. A subset C of a non-Archiedean vector space X is said to be convex if for all $x,y \in C$ the line segment $[x,y] = \{tx+(1-t)y : 0 \leq t \leq 1\}$ joining x to y is contained in C as well. (Notice that any vector subspace of X is convex.)

Definition 1.5.$M \subset H^{\#}$ **is** essentially #-closed if $\forall x(x \in H^{\#})\exists[\inf_{z\in M}\|x-z\|_{\#}]$.

Theorem 1.2. Suppose that $H^{\#}$ is a non-Archiedean Hilbert space and $M \subset H^{\#}$ be a essentially #-closed convex subset of $H^{\#}$. Then for any $x \in H^{\#}$ there exists a unique $y \in M$ such that $\|x-y\|_{\#} = \mathbf{dist}(x,M) = \inf_{z\in M}\|x-z\|_{\#}$ Moreover, if M is a vector subspace of $H^{\#}$, then the point y may also be characterized as the unique point in M such that $(x-y) \perp M$.

Proof. (1) Uniquiness: By replacing M by $M - x = \{m - x | m \in M\}$ we may assume $x = 0$. Let $\delta = \mathbf{dist}(0,M) = \inf_{m\in M}\|m\|_{\#}$ and $y,z \in M$. By the parallelogram law and the convexity of M one obtains

$$2\|y\|_{\#}^{2} + 2\|z\|_{\#}^{2} = \|y+z\|_{\#}^{2} + \|y-z\|_{\#}^{2} = 4\left\|\frac{y+z}{2}\right\|_{\#}^{2} + \|y-z\|_{\#}^{2} \geq 4\delta^{2} + \|y-z\|_{\#}^{2}. \quad (1.8)$$

Hence if $\|y\|_{\#} = \|z\|_{\#} = \delta$, then $2\delta^{2} + 2\delta^{2} \geq 4\delta^{2} + \|y-z\|_{\#}^{2}$, so that $\|y-z\|_{\#}^{2} = 0$. Therefore, if a minimizer for $\mathbf{dist}(0,\bullet)|_M$ exists, it is unique.

(2) Existence: Let $y_n \in M$ be chosen such that $\|y_n\|_{\#} = \delta_n \to_{\#} \delta \equiv \mathbf{dist}(0,M)$. Taking $y = y_m$ and $z = y_n$ in Eq.(1.8) shows $2\delta_m^{2} + 2\delta_n^{2} \geq 4\delta^{2} + \|y_n - y_m\|_{\#}^{2}$. Passing to the #-limit $m,n \to {}^{*}\infty$ in this equation implies, $2\delta^{2} + 2\delta^{2} \geq 4\delta^{2} +$ $+ \#\text{-}\lim\sup_{m,n\to{}^{*}\infty}\|y_n - y_m\|_{\#}^{2}$. Therefore $\{y_n\}_{n=1}^{{}^{*}\infty}$ is hyper infinite Cauchy sequence and hence #-convergent.Because M is #-closed, $y = \#\text{-}\lim\sup_{n\to{}^{*}\infty} y_n \in M$ and because $\|\bullet\|_{\#}$ is #-continuous, $\|y\|_{\#} = \#\text{-}\lim\sup_{n\to{}^{*}\infty}\|y_n\|_{\#} = \delta = \mathbf{dist}(0,M)$. So y is the desired point in M which is closest to 0.

Now for the second assertion we further assume that M is a #-closed subspace of H and $x \in H^{\#}$. Let $y \in M$ be the closest point in M to x. Then for $w \in M$, the

function $g(t) = \|x-(y+tw)\|_{\#}^2 = \|x-y\|_{\#}^2 - 2t\,\mathrm{Re}\langle x-y,w\rangle_{\#} + t^2\|w\|_{\#}^2$ has a minimum at $t = 0$. Therefore $0 = g'^{\#}(0) = -2\,\mathrm{Re}\langle x-y,w\rangle_{\#}$. Since $w \in M$ is arbitrary, this implies that $(x-y) \perp M$. Finally suppose $y \in M$ is any point such that $(x-y) \perp M$. Then for $z \in M$, by Pythagorean's theorem,
$\|x-z\|_{\#}^2 = \|x-y+y-z\|_{\#}^2 = \|x-y\|_{\#}^2 + \|y-z\|_{\#}^2 \geq \|x-y\|_{\#}^2$ which shows $[\mathbf{dist}(0,M)]^2 \geq \|x-y\|_{\#}^2$. That is to say y is the point in M closest to x.

Definition 1.6. $A : H^{\#} \to H^{\#}$ is a bounded in ${}^*\mathbb{R}_c^{\#}$ operator if and only if there exists some $M \in {}^*\mathbb{R}_c^{\#}, M > 0$ such that for all $x \in H^{\#}$, $\|Ax\|_{\#} \leq M\|x\|_{\#}$. The smallest such M if exists is called the operator #-norm of A and denoted by $\|A\|_{\#OP}$ or $\|A\|_{\#}$. Thus

$$\|A\|_{\#} = \sup_{\|x\|_{\#}=1}(\|Ax\|_{\#}) < {}^*\infty \tag{1.9}$$

if supremum in RHS of (1.9) exists and $\sup_{\|x\|_{\#}=1}(\|Ax\|_{\#}) < {}^*\infty$. Conversely if (1.9) holds

Proposition 1.2.A linear operator $A : H_1^{\#} \to H_2^{\#}$ between #-normed spaces is bounded in${}^*\mathbb{R}_c^{\#}$if and only if it is #-continuous.

Proof.Suppose that A is bounded in ${}^*\mathbb{R}_c^{\#}$ Then,for all vectors $x,h \in H_1^{\#}$ with $h \approx 0$ non zero we have $\|A(x+h)-A(x)\|_{\#} = \|A(h)\|_{\#} \leq M\|h\|_{\#}, M \in {}^*\mathbb{R}_c^{\#}, M > 0$. Letting h go to zero shows that A is #-continuous at x .Moreover, since the constant M does not depend on x, this shows that in fact A is uniformly #-continuous, and even Lipschitz #-continuous.
Conversely, it follows from the #-continuity at the zero vector that there exists a $\varepsilon \approx 0$, $\varepsilon > 0$ such that $\|A(h)\|_{\#} = \|A(h)-A(0)\|_{\#} \leq 1$ for all vectors $h \in H_1^{\#}$ with $\|h\|_{\#} \leq \varepsilon$. Thus, for all non-zero $x \in H_1^{\#}$, one has
$\|Ax\|_{\#} = \left\| \frac{\|x\|_{\#}}{\varepsilon} A\left(\varepsilon \frac{\|x\|_{\#}}{x}\right) \right\|_{\#} = \frac{\|x\|_{\#}}{\varepsilon} \left\| A\left(\varepsilon \frac{\|x\|_{\#}}{x}\right) \right\|_{\#} \leq \frac{\|x\|_{\#}}{\varepsilon}.$
This proves that A is bounded in${}^*\mathbb{R}_c^{\#}$.

Definition 1.7. Suppose that $A : H^{\#} \to H^{\#}$ is a bounded in ${}^*\mathbb{R}_c^{\#}$ operator. The #-adjoint of A, denote A^*, is the unique operator $A^* : H^{\#} \to H^{\#}$ such that $\langle Ax,y\rangle_{\#} = \langle x,A^*y\rangle_{\#}$. (The proof that A^* exists and is unique will be given in Proposition below.)
A bounded in ${}^*\mathbb{R}_c^{\#}$ operator $A : H^{\#} \to H^{\#}$ is self #-adjoint or Hermitian if $A = A^*$.

Definition 1.8. Let $H^{\#}$ be a non-Archiedean Hilbert space and $M \subset H$ be a #-closed subspace.The orthogonal projection of $H^{\#}$ onto M is the function $P_M : H^{\#} \to H^{\#}$ such that for $x \in H^{\#}, P_M(x)$ is the unique element in M such that $(x-P_M(x)) \perp M$.

Proposition 1.3. Let $H^{\#}$ be a non-Archiedean Hilbert space and $M \subset H^{\#}$ be a #-closed subspace.The orthogonal projection P_M satisfies:
(1) P_M is linear (and hence we will write P_Mx rather than $P_M(x)$.
(2) $P_M^2 = PM$ (P_M is a projection).
(3) $P_M^* = P_M$, (P_M is self-#-adjoint).
(4) $\mathbf{Ran}(P_M) = M$ and $\mathbf{ker}(P_M) = M^{\perp}$.

Proof. (1) Let $x_1,x_2 \in H^{\#}$ and $\alpha \in {}^*\mathbb{R}_c^{\#}$, then $P_Mx_1 + \alpha P_Mx_2 \in M$ and
$P_Mx_1 + \alpha P_Mx_2 - (x_1+\alpha x_2) = [P_Mx_1 - x_1 + \alpha(P_Mx_2 - x_2)] \in M^{\perp}$
showing $P_Mx_1 + \alpha P_Mx_2 = P_M(x_1+\alpha x_2)$, i.e. P_M is linear.
(2) Obviously $\mathbf{Ran}(P_M) = M$ and $P_Mx = x$ for all $x \in M$. Therefore $P_M^2 = P_M$.
(3) Let $x,y \in H^{\#}$, then since $(x-P_Mx) \in M^{\perp}$ and $(y-P_My) \in M^{\perp}$,
$\langle P_Mx,y\rangle_{\#} = \langle P_Mx, P_My + y - P_My\rangle_{\#} = \langle P_Mx,P_My\rangle_{\#} = \langle P_Mx + (x-P_M), P_My\rangle_{\#} = \langle x,PMy\rangle_{\#}$.
(4) It is clear that $\mathbf{Ran}(P_M) \subset M$. Moreover, if $x \in M$, then $P_Mx = x$ implies

that $\mathbf{Ran}(P_M) = M$. Now $x \in \mathbf{ker}(P_M)$ iff $P_M x = 0$ iff $x = x - 0 \in M^\perp$.
Corollary 1.2. Suppose that $M \subset H^\#$ is a proper closed subspace of a non-Archiedean Hilbert space $H^\#$, then $H^\# = M \oplus M^\perp$.
Proof. Given $x \in H^\#$, let $y = P_M x$ so that $x - y \in M^\perp$. Then $x = y + (x - y) \in M \oplus M^\perp$. If $x \in M \cap M\perp$, then $x \perp x$, i.e. $\|x\|_\#^2 = \langle x,x\rangle_\# = 0$. So $M \cap M^\perp = \{0\}$.
Proposition 1.4. (Generalized Riesz Theorem). Let $H^{\#*}$ be the dual space of $H^\#$. The map

$$z \in H^\# \overset{j}{\to} \langle \bullet, z\rangle_\# \in H^{\#*} \qquad (1.9')$$

is a conjugate linear #-isometric isomorphism.
Proof. The map j is conjugate linear by the axioms of the non-Archiedean inner products. Moreover, for $x,z \in H^\#, |\langle x,z\rangle_\#| \leq \|x\|_\# \ \|z\|_\#$ for all $x \in H^\#$ with equality when $x = z$. This implies that $\|jz\|_{H^{\#*}} = \langle \bullet, z\rangle_{H^{\#*}} = \|z\|_\#$. Therefore j is #-isometric and this shows that j is injective. To finish the proof we must show that j is surjective. So let $f \in H^{\#*}$ which we assume with out loss of generality is non-zero. Then $M = \mathbf{ker}(f)$ is a #-closed proper subspace of $H^\#$. Since, by Corollary 1.1, $H^\# = M \oplus M^\perp$, $f : H^\#/M \approx M^\perp \to {}^*\mathbb{C}_c^\#$ is a linear isomorphism. This shows that $\dim(M^\perp) = 1$ and hence $H^\# = M \oplus {}^*\mathbb{C}_c^\# x_0$ where $x_0 \in M^\perp \backslash \{0\}$. Alternatively, choose $x_0 \in M^\perp \backslash \{0\}$ such that $f(x_0) = 1$. For $x \in M^\perp$ we have $f(x - \lambda x_0) = 0$ provided that $\lambda = f(x)$. Therefore $x - \lambda x_0 \in M \cap M^\perp = \{0\}$, i.e. $x = \lambda x_0$. This again shows that $M^\perp$ is spanned by x_0. Choose $z = \lambda x_0 \in M^\perp$ such that $f(x_0) = \langle x_0, z\rangle$. (So $\lambda = \overline{f(x_0)}/\|x_0\|^2$.) Then for $x = m + \lambda x_0$ with $m \in M$ and $\lambda \in {}^*\mathbb{C}_c^\#, f(x) = \lambda f(x_0) = \lambda\langle x_0, z\rangle_\# = \langle \lambda x_0, z\rangle_\# =$
$= \langle m + \lambda x_0, z\rangle_\# = \langle x,z\rangle_\#$ which shows that $f = jz$.
Proposition 1.5. (Adjoints). Let $H^\#$ and $K^\#$ be a non-Archiedean Hilbert spaces and $A : H^\# \to K^\#$ be a bounded in ${}^*\mathbb{R}_c^\#$ operator. Then there exists a unique bounded operator $A^* : K^\# \to H^\#$ such that

$$\langle Ax, y\rangle_{K^\#} = \langle x, A^* y\rangle_{H^\#} \qquad (1.10)$$

for all $x \in H^\#$ and $y \in K^\#$. Moreover $(A + \lambda B)^* = A * + \bar{\lambda} B^*, A^{**} \doteq (A^*)^* = A$, $\|A^*\|_\# = \|A\|_\#$ and $\|A^*A\|_\# = \|A\|_\#^2$ for all $A,B \in L(H^\#, K^\#)$ and $\lambda \in {}^*\mathbb{C}_c^\#$.
Proof. For each $y \in K^\#$, then map $x \to \langle Ax, y\rangle_{K^\#}$ is in $H^{\#*}$ and therefore there exists by Proposition 12.15 a unique vector $z \in H^\#$ such that $\langle Ax, y\rangle_{K^\#} = \langle x, z\rangle_{H^\#}$ for all $x \in H^\#$. This shows there is a unique map $A^* : K^\# \to H^\#$ such that $\langle Ax, y\rangle_{K^\#} =$
$= \langle x, A^*(y)\rangle_{H^\#}$ for all $x \in H^\#$ and $y \in K^\#$. To finish the proof, we need only show A^* is linear and bounded in ${}^*\mathbb{R}_c^\#$ operator. To see A^* is linear, let $y_1, y_2 \in K^\#$ and $\lambda \in {}^*\mathbb{C}_c^\#$, then for any $x \in H^\#, \langle Ax, y_1 + \lambda y_2\rangle_{K^\#} = \langle Ax, y_1\rangle_{K^\#} + \bar{\lambda}\langle Ax, y_2\rangle_{K^\#}$
$= \langle x, A^*(y_1)\rangle_{K^\#} + \bar{\lambda}\langle x, A^*(y_2)\rangle_{K^\#} = \langle x, A^*(y_1) + \lambda A^*(y_2)\rangle_{K^\#}$ and by the uniqueness of $A^*(y_1 + \lambda y_2)$ we find $A^*(y_1 + \lambda y_2) = A^*(y_1) + \lambda A^*(y_2)$.
This shows A^* is linear and so we will now write $A^* y$ instead of $A^*(y)$. Since $\langle A^* y, x\rangle_{H^\#} = \langle x, A^* y\rangle_{H^\#} = \langle Ax, y\rangle_{K^\#} = \langle y, Ax\rangle_{K^\#}$ it follows that $A^{**} = A$. The assertion that $(A + \lambda B)^* = A^* + \bar{\lambda} B^*$ is left to the reader.
The following arguments prove the assertions about #-norms of A and A^* :
$\|A^*\|_\# = \sup_{k \in K^\#, \|k\|_\# = 1} \|A^* k\|_\# = \sup_{k \in K^\#, \|k\|_\# = 1} \sup_{h \in H^\#, \|h\|_\# = 1} |\langle A^* k, h\rangle_\#| =$
$= \sup_{h \in H^\#, \|h\|^\# = 1} \sup_{k \in K^\#, \|k\|_\# = 1} |\langle k, Ah\rangle_\#| = \sup_{h \in H^\#, \|h\|_\# = 1} \|Ah\|_\# = \|A\|_\#$,
$\|A^*A\|_\# \leq \|A^*\|_\# \|A\|_\# = \|A\|_\#^2$ and

$\|A\|_{\#}^{2} = \sup_{h\in H^{\#},\|h\|_{\#}=1}|\langle Ah,Ah\rangle_{\#}| = \sup_{h\in H^{\#},\|h\|_{\#}=1}|\langle h,A^{*}Ah\rangle_{\#}| \leq \sup_{h\in H^{\#},\|h\|_{\#}=1}\|A^{*}Ah\| =$
$= \|A^{*}A\|_{\#}$.

Corollary 1.3. Let $H^{\#},K^{\#},M^{\#}$ bea non-Archiedean Hilbert space, $A,B \in L(H^{\#},K^{\#})$, $C \in L(K^{\#},M^{\#})$ and $\lambda \in \widetilde{{}^{*}\mathbb{C}_{c}^{\#}}$. Then $(A+\lambda B)^{*} = A^{*}+\bar{\lambda}B^{*}$ and $(CA)^{*} =$
$= A^{*}C^{*} \in L(M^{\#},H^{\#})$.

Corollary 1.4. Let $H^{\#} = \widetilde{{}^{*}\mathbb{C}_{c}^{\#}}^{n}$ and $K^{\#} = \widetilde{{}^{*}\mathbb{C}_{c}^{\#}}^{n}$ equipped with the canonical inner products, i.e. $\langle z,w\rangle_{H^{\#}} = Ext\text{-}\widehat{\sum}_{1\leq i\leq n} z_{i}\cdot\overline{w}_{i}$ for $z,w \in H^{\#}$. Let A be an $m\times n$ external hyperfinite matrix thought of as a linear operator from $H^{\#}$ to $K^{\#}$. Then the hyperfinite matrix associated to $A^{*} : K^{\#} \to H^{\#}$ is the conjugate transpose of A.

Corollary 1.5. Let $K : L_{2}^{\#}(\nu^{\#}) \to L_{2}^{\#}(\mu^{\#})$ be the operator defined in Corollary 1.3. Then $K^{*} : L_{2}^{\#}(X,\mu^{\#}) \to L_{2}^{\#}(X,\nu^{\#})$ is the operator given by
$K^{*}g(y) = Ext\text{-}\int_{X}\overline{k(x,y)}g(x)d^{\#}\mu^{\#}(x)$.

Definition 1.9. $\{u_{\alpha}\}_{\alpha\in A} \subset H^{\#}$ is an orthonormal set if $u_{\alpha}\perp u_{\beta}$ for all $\alpha\neq\beta$ and $\|u_{\alpha}\|_{\#} = 1$.

Proposition 1.6 (Generalized Bessel's Inequality). Let $\{u_{\alpha}\}_{\alpha\in A}$ be an orthonormal set, then

$$Ext\text{-}\widehat{\sum}_{\alpha\in A}|\langle x,u_{\alpha}\rangle_{\#}|^{2} \leq \|x\|_{\#}^{2} \tag{1.11}$$

for all $x \in H^{\#}$. In particular the set $\{\alpha\in A : \langle x,u_{\alpha}\rangle_{\#}\neq 0\}$ is at most $*$-countable, i.e. $\mathbf{card}(A) = \mathbf{card}({}^{*}\mathbb{N})$ for all $x \in H^{\#}$.

Proof. Let $\Gamma \subset A$ be any hyperfinite set. Then

$$\begin{gathered} 0 \leq \left\| x - Ext\text{-}\widehat{\sum}_{\alpha\in\Gamma}\langle x,u_{\alpha}\rangle_{\#}u_{\alpha}\right\|_{\#}^{2} = \|x\|_{\#}^{2} - 2\operatorname{Re}\left(Ext\text{-}\widehat{\sum}_{\alpha\in\Gamma}\langle x,u_{\alpha}\rangle_{\#}\langle u_{\alpha},x\rangle_{\#}\right) + \\ +Ext\text{-}\widehat{\sum}_{\alpha\in\Gamma}|\langle x,u_{\alpha}\rangle_{\#}|^{2} = \|x\|_{\#}^{2} - Ext\text{-}\widehat{\sum}_{\alpha\in\Gamma}|\langle x,u_{\alpha}\rangle_{\#}|^{2} \end{gathered} \tag{1.12}$$

and (1.12) gives that

$$Ext\text{-}\widehat{\sum}_{\alpha\in\Gamma}|\langle x,u_{\alpha}\rangle_{\#}|^{2} \leq \|x\|_{\#}^{2}. \tag{1.13}$$

Taking the supremum of the inequality (1.13) of $\Gamma \subset\subset A$ then proves (1.11).

Proposition 1.7. Suppose $A \subset H^{\#}$ is an orthogonal set. Then $s = Ext\text{-}\sum_{v\in A} v$ exists in $H^{\#}$ iff $Ext\text{-}\widehat{\sum}_{v\in A}\|v\|_{\#}^{2} < {}^{*}\infty$. In particular A must be at most a $*$-countable set.

Moreover, $Ext\text{-}\widehat{\sum}_{v\in A}\|v\|_{\#}^{2} < {}^{*}\infty$, then

(1) $\|s\|_{\#}^{2} = Ext\text{-}\widehat{\sum}_{v\in A}\|v\|_{\#}^{2}$ and

(2) $\langle s,x\rangle_{\#} = Ext\text{-}\widehat{\sum}_{v\in A}\langle v,x\rangle_{\#}$ for all $x \in H^{\#}$.

Similarly if $\{v_{n}\}_{n=1}^{*\infty}$ is an orthogonal set, then $s = Ext\text{-}\widehat{\sum}_{n=1}^{*\infty}v_{n}$ exists in $H^{\#}$ iff $Ext\text{-}\widehat{\sum}_{n=1}^{*\infty}v_{n} < {}^{*}\infty$. In particular if $Ext\text{-}\widehat{\sum}_{n=1}^{*\infty}v_{n}$ exists, then it is independent of rearrange ments of $\{v_{n}\}_{n=1}^{*\infty}$.

Proof. Suppose $s = Ext\text{-}\widehat{\sum}_{v\in A} v$ exists. Then there exists $\Gamma \subset\subset A$ such that $Ext\text{-}\widehat{\sum}_{v\in\Lambda} \|v\|_{\#}^2 = \left\| Ext\text{-}\widehat{\sum}_{v\in\Lambda} v \right\|_{\#}^2 \leq 1$ for all $\Lambda \subset\subset A\backslash\Gamma$,wherein the first inequality we have used Pythagorean's theorem.

Taking the supremum over such Λ shows that $Ext\text{-}\widehat{\sum}_{v\in A\backslash\Gamma} \|v\|_{\#}^2 \leq 1$ and therefore $Ext\text{-}\widehat{\sum}_{v\in A} \|v\|_{\#}^2 \leq 1 + Ext\text{-}\widehat{\sum}_{v\in\Gamma} \|v\|_{\#}^2 < {}^*\infty$.

Conversely, suppose that $Ext\text{-}\widehat{\sum}_{v\in A} \|v\|_{\#}^2 < {}^*\infty$. Then for all $\varepsilon \approx 0, \varepsilon > 0$ there exists $\Gamma_\varepsilon \subset\subset A$ such that if $\Lambda \subset\subset A\backslash\Gamma_\varepsilon$,

$$\left\| Ext\text{-}\widehat{\sum}_{v\in\Lambda} v \right\|_{\#}^2 = Ext\text{-}\widehat{\sum}_{v\in\Lambda} \|v\|_{\#}^2 < \varepsilon. \tag{1.14}$$

Hence by $Ext\text{-}\widehat{\sum}_{v\in A} v$ exists.

For item 1, let Γ_ε be as above and set $s_\varepsilon = Ext\text{-}\widehat{\sum}_{v\in\Gamma_\varepsilon} v$. Then $|\|s\|_{\#} - \|s_\varepsilon\|_{\#}| \leq \|s - s_\varepsilon\|_{\#} < \varepsilon$ and by Eq.(1.14), $0 \leq \left(Ext\text{-}\widehat{\sum}_{v\in A} \|v\|_{\#}^2 \right) - \|s_\varepsilon\|_{\#}^2 \leq \varepsilon^2$

Letting $\varepsilon \to_{\#} 0$ we deduce from the previous two equations that $\|s_\varepsilon\|_{\#} \to_{\#} \|s\|_{\#}$ and $\|s_\varepsilon\|_{\#}^2 \to_{\#} Ext\text{-}\widehat{\sum}_{v\in A} \|v\|_{\#}^2$ as $\varepsilon \to_{\#} 0$ and therefore $\|s\|_{\#}^2 = Ext\text{-}\widehat{\sum}_{v\in A} \|v\|_{\#}^2$.

For the final assertion, let $s_N = Ext\text{-}\sum_{n=1}^{N} v_n$ and suppose that $\#\text{-}\lim_{N\to{}^*\infty} s_N = s$ exists in $H^{\#}$ and in particular $\{s_N\}_{N=1}^{{}^*\infty}$ is Cauchy. So for $N > M$: $Ext\text{-}\widehat{\sum}_{n=M+1}^{N} \|v_n\|_{\#}^2 =$

$= \|s_N - s_M\|_{\#}^2 \to_{\#} 0$ as $M, N \to {}^*\infty$which shows that $Ext\text{-}\widehat{\sum}_{n=1}^{{}^*\infty} v_n$ is #-convergent, i.e.$Ext\text{-}\widehat{\sum}_{n=1}^{{}^*\infty} v_n < {}^*\infty$.

Corollary 1.6.Suppose $H^{\#}$ is a non-Archiedean Hilbert space, $\beta \subset H^{\#}$ is an orthonormal set and $M = \mathbf{span}(\beta)$. Then

$$P_M x = Ext\text{-}\widehat{\sum}_{u\in\beta} \langle x, u\rangle_{\#} u, \tag{1.15}$$

$$\|P_M x\|_{\#}^2 = Ext\text{-}\widehat{\sum}_{u\in\beta} |\langle x, u\rangle_{\#}|^2, \tag{1.16}$$

and

$$\langle P_M x, y\rangle_{\#} = Ext\text{-}\widehat{\sum}_{u\in\beta} \langle x, u\rangle_{\#} \langle u, y\rangle_{\#}, \tag{1.17}$$

for all $x, y \in H^{\#}$.

Proof. By Bessel's inequality, $Ext\text{-}\widehat{\sum}_{u\in\beta} |\langle x, u\rangle_{\#}|^2 \leq \|x\|_{\#}^2$ for all $x \in H^{\#}$ and therefore by Proposition 12.18, $Px = Ext\text{-}\widehat{\sum}_{u\in\beta} \langle x, u\rangle_{\#} u$ exists in $H^{\#}$ and for all $x, y \in H^{\#}$,

$$\langle Px, y\rangle_{\#} = Ext\text{-}\widehat{\sum}_{u\in\beta} \langle\langle x, u\rangle_{\#} u, y\rangle_{\#} = Ext\text{-}\widehat{\sum}_{u\in\beta} \langle x, u\rangle_{\#} \langle u, y\rangle_{\#}. \tag{1.18}$$

Taking $y \in \beta$ in Eq. (1.18) gives $\langle Px, y\rangle = \langle x, y\rangle_{\#}$, i.e. that $\langle x - Px, y\rangle_{\#} = 0$ for all $y \in \beta$.

So $(x - Px) \perp \mathbf{span}(\beta)$ and by continuity we also have $(x - Px) \perp M = \#\text{-}\overline{\mathbf{span}(\beta)}$. Since Px is also in M, it follows from the definition of P_M that $Px = P_M x$ proving Eq. (1.15). Equations (1.16) and (1.17) now follow from (1.18), Proposition 1.7 and the fact that $\langle P_M x, y\rangle_\# = \langle P_M^2 x, y\rangle_\# = \langle P_M x, P_M y\rangle_\#$ for all $x, y \in H^\#$.

§2.Non-Archimedean Hilbert Space Basis.

Definition 2.1. (Basis). Let $H^\#$ be a non-Archiedean Hilbert space. A basis β of $H^\#$ is a maximal orthonormal subset $\beta \subset H^\#$.

Proposition 2.1. Every non-Archiedean Hilbert space $H^\#$ has an orthonormal basis.

Proof. Let $\mathcal{F}$ be the collection of all orthonormal subsets of $H^\#$ ordered by inclusion. If $\Phi \subset \mathcal{F}$ is linearly ordered then $\cup\Phi$ is an upper bound. By Zorn's Lemma there exists a maximal element $\beta \in \mathcal{F}$.

An orthonormal set $\beta \subset H^\#$ is said to be complete if $\beta^\perp = \{0\}$. That is to say if $\langle x, u\rangle_\# = 0$ for all $u \in \beta$ then $x = 0$.

Lemma 2.1. Let β be an orthonormal subset of $H^\#$ then the following are equivalent:

(1) β is a basis,

(2) β is #-complete and

(3) $\mathbf{span}(\beta) = H^\#$.

Proof. If β is not #-complete, then there exists a unit vector $x \in \beta^\perp \setminus \{0\}$. The set $\beta \cup \{x\}$ is an orthonormal set properly containing β, so β is not maximal. Conversely, if β is not maximal, there exists an orthonormal set $\beta_1 \subset H^\#$ such that $\beta \subsetneq \beta_1$. Then if $x \in \beta_1 \setminus \beta$, we have $\langle x, u\rangle_\# = 0$ for all $u \in \beta$ showing β is not #-complete. This proves the equivalence of (1) and (2). If β is not complete and $x \in \beta^\perp \setminus \{0\}$, then $\#\text{-}\overline{\mathbf{span}(\beta)} \subset x^\perp$ which is a proper subspace of $H^\#$. Conversely if $\mathbf{span}(\beta)$ is a proper subspace of $H^\#, \beta^\perp = \#\text{-}\overline{\mathbf{span}(\beta)}^\perp$ is a non-trivial subspace by Corollary 1.2 and β is not #-complete. This shows that (2) and (3) are equivalent.

Theorem 2.1. Let $\beta \subset H^\#$ be an orthonormal set. Then the following are equivalent:

(1) β is #-complete or equivalently a basis.

(2) $x = Ext\text{-}\widehat{\sum}_{u \in \beta} \langle x, u\rangle_\# u$ for all $x \in H^\#$.

(3) $\langle x, y\rangle_\# = Ext\text{-}\widehat{\sum}_{u \in \beta} \langle x, u\rangle_\# \langle u, y\rangle_\#$ for all $x, y \in H^\#$.

(4) $\|x\|_\#^2 = Ext\text{-}\widehat{\sum}_{u \in \beta} |\langle x, u\rangle_\#|^2$ for all $x \in H^\#$.

Proof. Let $M = \#\text{-}\overline{\mathbf{span}(\beta)}$ and $P = P_M$.

(1) ⇒ (2) By Corollary 1.6, $Ext\text{-}\widehat{\sum}_{u \in \beta} \langle x, u\rangle_\# u = P_M x$. Therefore

x- $Ext\text{-}\widehat{\sum}_{u \in \beta} \langle x, u\rangle_\# u = x - P_M x \in M^\perp = \beta^\perp = \{0\}$.

(2) ⇒ (3) is a consequence of Proposition 1.6.

(3) ⇒ (4) is obvious, just take $y = x$.

(4) ⇒ (1) If $x \in \beta^\perp$, then by 4), $\|x\|_\# = 0$, i.e. $x = 0$. This shows that β is #-complete.

Proposition 2.2. A non-Archimedean Hilbert space $H^\#$ is $*$-separable iff $H^\#$ has a $*$-countable orthonormal basis $\beta \subset H^\#$. Moreover, if $H^\#$ is $*$-separable, all orthonormal bases of $H^\#$ are $*$-countable.

Proof. Let $D \subset H^\#$ be a $*$-countable dense set $D = \{u_n\}_{n=1}^{*\infty}$. By Gram-Schmidt process there exists $\beta = \{v_n\}_{n=1}^{*\infty}$ an orthonormal set such that $\mathbf{span}(\{v_n|1 \leq n \leq N\}) \supseteq \mathbf{span}(\{u_n|1 \leq n \leq N\})$. So if $\langle x, v_n \rangle_\# = 0$ for all $n \in {}^*\mathbb{N}$ then $\langle x, u_n \rangle_\# = 0$ for all $n \in {}^*\mathbb{N}$. Since $D \subset H^\#$ is #-dense we may choose $\{w_k\} \subset D$ such that $x = \#\text{-}\lim_{k \to {}^*\infty} w_k$ and therefore $\langle x, x \rangle_\# = \#\text{-}\lim_{k \to {}^*\infty} \langle x, w_k \rangle = 0.$ That is to say $x = 0$ and β is #-complete.

Conversely if $\beta \subset H^\#$ is a $*$-countable orthonormal basis, then the $*$-countable set $D = \left\{ Ext\text{-}\widehat{\sum}_{u \in \beta} a_u u | a_u \in Q + iQ : \#\{u : a \neq 0\} < {}^*\infty \right\}$ is #-dense in $H^\#$.

Finally let $\beta = \{u_n\}_{n=1}^{*\infty}$ be an orthonormal basis and $\beta_1 \subset H^\#$ be another orthonormal basis. Then the sets $B_n = \{v \in \beta_1 | v, u_n \neq 0\}$ are $*$-countable for each $n \in {}^*\mathbb{N}$ and hence $B := \bigcup_{n=1}^{*\infty} B_n$ is a countable subset of β_1.

Suppose there exists $v \in \beta_1 \backslash B$, then $\langle v, u_n \rangle_\# = 0$ for all $n \in {}^*\mathbb{N}$ and since $\beta = \{u_n\}_{n=1}^{*\infty}$ is an orthonormal basis, this implies $v = 0$ which is impossible since $\|v\|_\# = 1$.

Therefore $\beta_1 \backslash B = \varnothing$ and hence $\beta_1 = B$ is $*$-countable.

Definition 2.2.A linear map $U : H^\# \to K^\#$ is an isometry if $\|Ux\|_{\#K^\#} = \|x\|_{\#H^\#}$ for all $x \in H^\#$ and U is unitary if U is also surjective.

Proposition 2.3. Let $U : H^\# \to K^\#$ be a linear map, show the following are equivalent:

(1) $U : H^\# \to K^\#$ is an isometry,

(2) $\langle Ux, Ux' \rangle_{\#K^\#} = \langle x, x' \rangle_{\#H^\#}$for all $x, x' \in H^\#$,

(3) $U^* U = \mathbf{id}_{H^\#}$.

Proposition 2.4. Let $U : H^\# \to K^\#$ be a linear map, show the following are equivalent:

(1) $U : H^\# \to K^\#$ is unitary

(2) $U^* U = \mathbf{id}_{H^\#}$ and $UU^* = \mathbf{id}_{K^\#}$.

(3) U is invertible and $U^{-1} = U^*$.

Proposition 2.5.Let $H^\#$ be a non-Archimedean Hilbert space. Then there exists a set X and a unitary map $U : H^\# \to l_2^\#(X)$. Moreover, if $H^\#$ is $*$-separable and $\dim(H^\#) = {}^*\infty$, then X can be taken to be ${}^*\mathbb{N}$ so that $H^\#$ is unitarily equivalent to $l_2^\#({}^*\mathbb{N})$.

Remark 2.1. Suppose that $\{u_n\}_{n=1}^{*\infty}$ is a #-total subset of $H^\#$, i.e. $\#\text{-}\overline{\mathbf{span}\{u_n\}} = H^\#$. Let $\{v_n\}_{n=1}^{*\infty}$be the vectors found by performing Gram-Schmidt on the set $\{u_n\}_{n=1}^{*\infty}$. Then $\{v_n\}_{n=1}^{*\infty}$is an orthonormal basis for $H^\#$.

§3.1.Weak #-Convergence.

Suppose $H^\#$ is an hyper infinite dimensional non-Archimedean Hilbert space and $\{x_n\}_{n=1}^{*\infty}$is an orthonormal subset of $H^\#$. Then, by Eq. (1.1), $\|x_n - x_m\|_\#^2 = 2$ for all $m \neq n$ and in particular, $\{x_n\}_{n=1}^{*\infty}$ has no #-convergent subsequences. From this we conclude that $C := \{x \in H^\# : \|x\|_\# \leq 1\}$, the #-closed unit ball in $H^\#$, is not #-compact. To overcome this problems it is sometimes useful to introduce a weaker topology on X having the property that C is #-compact.

Definition 3.1. Let $(X, \|\cdot\|_\#)$ be a non-Archimedean Banach space and X^* be its #-continuous dual. The weak topology, τ_w, on X is the topology generated by X^*. If $\{x_n\}_{n=1}^{*\infty} \subset X$ is a hyper infinite sequence we will write $x_n \overset{w}{\to}_\# x$ as $n \to {}^*\infty$ to mean that $x_n \to_\# x$ in the weak topology.

Because $\tau_w = \tau(X^*) \subset \tau\|\cdot\|_\# \triangleq \tau(\{\|x - \cdot\|_\# : x \in X\},$ it is harder for a function $f : X \to F$ to be #-continuous in the τ_w - topology than in the #-norm topology, $\tau\|\cdot\|_\#$. In particular if $\varphi : X \to F$ is a linear functional which is τ_w -continuous, then φ is

τ_w -continuous and hence $\varphi \in X^*$.

Proposition 3.1. Let $\{x_n\}_{n=1}^{*\infty} \subset X$ be a hyper infinite sequence, then $x_n \overset{w}{\to}_{\#} x \in X$ as $n \to {}^*\infty$iff $\varphi(x) = \#\text{-}\lim_{n\to{}^*\infty} \varphi(x_n)$ for all $\varphi \in X^*$.

Proof.By definition of τ_w,we have $x_n \overset{w}{\to}_{\#} x \in X$ iff for all $\Gamma \subset\subset X^*$ and $\varepsilon \approx 0, \varepsilon > 0$ there exists an $N \in {}^*\mathbb{N}$ such that $|\varphi(x) - \varphi(x_n)| < \varepsilon$ for all $n \geq N$ and $\varphi \in \Gamma$.

This later condition is easily seen to be equivalent to $\varphi(x) = \#\text{-}\lim_{n\to{}^*\infty} \varphi(x_n)$ for all $\varphi \in X^*$.

The topological space (X, τ_w) is still Hausdorff, however to prove this one needs to make use of the generalized Hahn Banach T**heorem 18.16** below. For the moment we will concentrate on the special case where $X = H^\#$ is a non-Archimedean Hilbert space in which case $H^{\#*} = \{\varphi_z := \langle \bullet, z\rangle_\# : z \in H^\#\}$, see

Propositions 3.2. If $x, y \in H^\#$ and $z = y - x \neq 0$,then
$0 < \varepsilon := \|z\|_\#^2 = \varphi_z(z) = \varphi_z(y) - \varphi_z(x)$.

Thus $V_x \triangleq \{w \in H^\# : |\varphi_z(x) - \varphi_z(w)| < \varepsilon/2\}$ and $V_y \triangleq \{w \in H^\# : |\varphi_z(y) - \varphi_z(w)| < \varepsilon/2\}$ are disjoint sets from τ_w which contain x and y respectively. This shows that $(H^\#, \tau_w)$ is a Hausdorff space. In particular, this shows that weak #-limits are unique if they exist.

Remark 3.1. Suppose that $H^\#$ is an $*$-infinite dimensional non-Archimedean Hilbert space and $\{x_n\}_{n=1}^{*\infty}$ an orthonormal subset of $H^\#$. Then generalized Bessel's inequality (Proposition 1.6) implies $x_n \overset{w}{\to}_{\#} 0 \in H^\#$ as $n \to {}^*\infty$. This points out the fact that if $x_n \overset{w}{\to}_{\#} x \in H^\#$ as $n \to {}^*\infty$, it is no longer necessarily true that $\|x\|_\# = \#\text{-}\lim_{n\to{}^*\infty} \|x_n\|_\#$. However we do always have $\|x\|_\# \leq \#\text{-}\liminf_{n\to{}^*\infty} \|x_n\|_\#$ because, $\|x\|_\#^2 = \#\text{-}\lim_{n\to{}^*\infty} \langle x_n, x\rangle_\# \leq \#\text{-}\liminf_{n\to{}^*\infty} [\|x_n\|_\# \|x\|_\#] =$
$= \|x\|_\# \#\text{-}\liminf_{n\to{}^*\infty} \|x_n\|_\#$.

Proposition 3.3. Let $H^\#$ be a non-Archimedean Hilbert space, $\beta \subset H^\#$ be an orthonormal basis for $H^\#$ and $\{x_n\}_{n=1}^{*\infty} \subset H^\#$ be a bounded in ${}^*\mathbb{R}_c^\#$ hyper infinite sequence, then the following properties are equivalent:

(1) $x_n \overset{w}{\to}_{\#} x \in H^\#$ as $n \to {}^*\infty$.

(2) $\langle x, y\rangle_\# = \#\text{-}\lim_{n\to{}^*\infty} \langle x_n, y\rangle_\#$ for all $y \in H^\#$.

(3) $\langle x, y\rangle_\# = \#\text{-}\lim_{n\to{}^*\infty} \langle x_n, y\rangle_\#$ for all $y \in \beta$.

Moreover, if $c_y \triangleq \#\text{-}\lim_{n\to{}^*\infty} \langle x_n, y\rangle_\#$ exists for all $y \in \beta$, then
$Ext\text{-}\widehat{\sum}_{y\in\beta} |c_y|^2 < {}^*\infty$ and $x_n \overset{w}{\to}_{\#} x \triangleq Ext\text{-}\widehat{\sum}_{y\in\beta} c_y y \in H^\#$ as $n \to {}^*\infty$.

Proof. 1. $\Rightarrow$ 2. This is a consequence of Propositions 1.4 (Generalized Riesz Theorem) and Proposition 3.2. $\Rightarrow$3. is trivial.

3. $\Rightarrow$ 1. Let $M \triangleq \sup_n \|x_n\|_\#$ and H_0 denote the #-algebraic span of β. Then for $y \in H^\#$ and $z \in H_0$,
$|\langle x - x_n, y\rangle_\#| \leq |\langle x - x_n, z\rangle_\#| + |\langle x - x_n, y - z|\rangle_\# \leq |\langle x - x_n, z\rangle_\#| + 2M\|y - z\|_\#$.

Passing to the #-limit in this equation implies
$\#\text{-}\limsup_{n\to{}^*\infty} |\langle x - x_n, y\rangle_\#| \leq 2M \|y - z\|_\#$
which shows $\#\text{-}\limsup_{n\to{}^*\infty} |\langle x - x_n, y\rangle_\#| = 0$ since H_0 is #-dense in $H^\#$.

To prove the last assertion, let $\Gamma \subset\subset \beta$. Then by Bessel's inequality (Proposition 1.6),$Ext\text{-}\widehat{\sum}_{y\in\Gamma} |c_y|^2 = \#\text{-}\lim_{n\to{}^*\infty} Ext\text{-}\widehat{\sum}_{y\in\Gamma} |\langle x - x_n, y\rangle_\#|^2 \leq \#\text{-}\liminf_{n\to{}^*\infty} \|x_n\|_\#^2 \leq M^2$.

Since $\Gamma \subset\subset \beta$ was arbitrary, we conclude that $Ext\text{-}\widehat{\sum}_{y\in\beta} |c_y|^2 \leq M < {}^*\infty$ and hence we may define $x \triangleq Ext\text{-}\widehat{\sum}_{y\in\beta} c_y y$. By construction we have $\langle x,y\rangle_\# = c_y = \#\text{-}\lim_{n\to{}^*\infty}\langle x_n,y\rangle_\#$ for all $y \in \beta$ and hence $x_n \overset{w}{\to}_\# x \in H^\#$ as $n \to {}^*\infty$ by what we have just proved.

Theorem 3.1. Suppose that $\{x_n\}_{n=1}^{^*\infty} \subset H^\#$ is a bounded in ${}^*\mathbb{R}_c^\#$ hyper infinite sequence. Then there exists a subsequence $y_k = x_{n_k}$ of $\{x_n\}_{n=1}^{^*\infty}$ and $x \in X$ such that $y_k \overset{w}{\to}_\# x$ as $k \to {}^*\infty$.

Proof. This is a consequence of Proposition 3.3.Let $H_0^\# = \#\text{-}\overline{\mathbf{span}\{x_n : n \in {}^*\mathbb{N}\}}$ is a $*$-separable non-Archimedean Hilbert subspace of $H^\#$. Let $\{\lambda_m\}_{m=1}^{^*\infty} \subset H_0^\#$ be an orthonormal basis and use hyper infinite Cantor's diagonalization argument to find a hyper infinite subsequence $y_k = x_{n_k}$ such that $c_m = \#\text{-}\lim_{k\to{}^*\infty}\langle y_k,\lambda_m\rangle_\#$ exists for all $m \in {}^*\mathbb{N}$. Finish the proof by appealing to Proposition 3.3.

Theorem 3.2. (Alaoglu's Theorem for a non-Archimedean Hilbert Spaces). Suppose that $H^\#$ is a $*$-separable non-Archimedean Hilbert space, $\mathbf{C} \triangleq \{x \in H^\# | \|x\|_\# \leq 1\}$ is the #-closed unit ball in $H^\#$ and $\{e_n\}_{n=1}^{^*\infty}$ is an orthonormal basis for $H^\#$. Then

$$\rho(x,y) = Ext\text{-}\sum_{n=1}^{^*\infty}(1/2^n)|\langle x-y,e_n\rangle_\#| \tag{3.1}$$

defines a non-Archimedean metric on $\mathbf{C}$ which is compatible with the weak topology on $\mathbf{C}$, $\tau_{\mathbf{C}} = (\tau_w)_{\mathbf{C}} = \{V\cap\mathbf{C}|V \in \tau_w\}$. Moreover $(\mathbf{C},\rho)$ is a #-compact non-Archimedean metric space.

Proof. That is simple to check that ρ is a #-${}^*\mathbb{R}_c^\#$- valued metric . Let τ_ρ be the topology on $\mathbf{C}$ induced by ρ. For any $y \in H^\#$ and $n \in {}^*\mathbb{N}$, the map $x \in H^\# \to \langle x-y,e_n\rangle_\# = \langle x,e_n\rangle_\# - \langle y,e_n\rangle_\#$ is τ_w continuous and since the sum in Eq. (3.1) is uniformly #-convergent for $x,y \in \mathbf{C}$, it follows that $x \to \rho(x,y)$ is $\tau_{\mathbf{C}}$ - continuous. This implies the #-open balls relative to ρ are contained in $\tau_{\mathbf{C}}$ and therefore $\tau_\rho \subset \tau_{\mathbf{C}}$. For the converse inclusion, let $z \in H^\#, x \to \varphi_z(x) = \langle z,x\rangle_\#$ be an element of $H^{\#*}$, and for $n \in {}^*\mathbb{N}$ let $z_N = Ext\text{-}\widehat{\sum}_{n=1}^{N}\langle z,e_n\rangle_\# e_n$. Then $\varphi_{z_N} = Ext\text{-}\widehat{\sum}_{n=1}^{N}\langle z,e_n\rangle_\#\varphi_{e_n}$ is ρ-#-continuous, being a hyperfinite linear combination of the φ_{e_n} which are easily seen to be ρ- #-continuous. Because $z_N \to_\# z$ as $N \to {}^*\infty$. it follows that $\sup_{x\in\mathbf{C}}|\varphi_z(x) - \varphi_{z_N}(x)| = \|z-z_N\|_\# \to_\# 0$ as $N \to {}^*\infty$.
Therefore $\varphi_z \upharpoonright \mathbf{C}$ is ρ- #-continuous as well and hence $\tau_{\mathbf{C}} = \tau(\varphi_z \upharpoonright \mathbf{C}|z \in H^\#) \subset \tau_\rho$. The last assertion follows directly from Theorem 3.1 and the fact that sequential #-compactness is equivalent to #-compactness for a non-Archimedean metric spaces.

Theorem 3.3. (Weak and Strong #-Differentiability). Suppose that $f \in L_2^\#\left(\widetilde{{}^*\mathbb{R}_c^\#}^n\right)$ and $v \in \widetilde{{}^*\mathbb{R}_c^\#}^n\backslash\{0\}$. Then the following are equivalent:

(1) There exists $\{t_n\}_{n=1}^{^*\infty} \subset \widetilde{{}^*\mathbb{R}_c^\#}\backslash\{0\}$ such that $\#\text{-}\lim_{n\to{}^*\infty} t_n = 0$ and

$$\sup_{n\in{}^*\mathbb{N}}\left\|\frac{f(\cdot\ + t_n v) - f(\cdot)}{t_n}\right\|_{\#2} < {}^*\infty.$$

(2) There exists $g \in L_2(\widetilde{{}^*\mathbb{R}_c^\#}^n)$ such that $\langle f,\partial_v^\#\varphi\rangle_\# = -\langle g,\varphi\rangle_\#$ for all $\varphi \in C_c^{^*\infty}\left(\widetilde{{}^*\mathbb{R}_c^\#}^n\right)$.

(3) There exists $g \in L_2^\#\left(\widetilde{{}^*\mathbb{R}_c^\#}^n\right)$ and $f_n \in C_c^{^*\infty}\left(\widetilde{{}^*\mathbb{R}_c^\#}^n\right)$ such that $f_n \overset{L_2^\#}{\to}_\# f$ and

$\partial_v^{\#} f_n \overset{L_2^{\#}}{\to}_{\#} g$ as $\to {}^*\infty$.

(4) There exists $g \in L_2^{\#}\left(\widetilde{{}^*\mathbb{R}_c^{\#}}^n\right)$ such that

$$\frac{f(\cdot\ + tv) - f(\cdot)}{t} \overset{L_2^{\#}}{\to}_{\#} g$$

as $t \to_{\#} 0$.

Proof. 1. $\Rightarrow$ 2. We may assume, using Theorem 3.1 and passing to a subsequence if necessary, that

$$\frac{f(\cdot\ + t_n v) - f(\cdot)}{t_n} \overset{w}{\to}_{\#} g$$

for some $g \in L_2^{\#}({}^*\mathbb{R}_c^{\#n})$. Now for $\varphi \in C_c^{{}^*\infty}({}^*\mathbb{R}_c^{\#n})$,

$$\langle g, \varphi \rangle_{\#} = \#\text{-}\lim_{n \to {}^*\infty} \left\langle \frac{f(\cdot\ + t_n v) - f(\cdot)}{t_n}, \varphi \right\rangle_{\#} = \#\text{-}\lim_{n \to {}^*\infty} \left\langle f, \frac{\varphi(\cdot\ + t_n v) - \varphi(\cdot)}{t_n} \right\rangle_{\#} =$$

$$= \left\langle f, \#\text{-}\lim_{n \to {}^*\infty} \frac{\varphi(\cdot\ + t_n v) - \varphi(\cdot)}{t_n} \right\rangle_{\#} = -\langle f, \partial_v^{\#} \varphi \rangle_{\#},$$

wherein we have used the translation invariance of Lebesgue #-measure and the dominated #-convergence theorem.

2. $\Rightarrow$ 3. Let $\varphi \in C_c^{{}^*\infty}\left(\widetilde{{}^*\mathbb{R}_c^{\#}}^n, \widetilde{{}^*\mathbb{R}_c^{\#}}\right)$ such that $Ext\text{-}\int_{\widetilde{{}^*\mathbb{R}_c^{\#}}^n} \varphi(x) d^{\#}x = 1$ and let $\varphi_m(x) = m^n \varphi(mx)$, then by **Proposition 11.24**, $h_m = \varphi_m * f \in C^{{}^*\infty}\left(\widetilde{{}^*\mathbb{R}_c^{\#}}^n\right)$ for all $m \in {}^*\mathbb{N}$ and $\partial_v^{\#} h_m(x) = \partial_v^{\#} \varphi_m * f(x) = Ext\text{-}\int_{\widetilde{{}^*\mathbb{R}_c^{\#}}^n} \partial_v^{\#} \varphi_m(x-y) f(y) d^{\#}y = \langle f, -\partial_v^{\#}[\varphi m(x - \cdot)] \rangle_{\#} = \langle g, \varphi_m(x - \cdot) \rangle_{\#} = \varphi_m * g(x)$.

By **Theorem11**.**21**, $h_m \to_{\#} f \in L_2^{\#}\left(\widetilde{{}^*\mathbb{R}_c^{\#}}^n\right)$ and $\partial_v^{\#} h_m = \varphi_m * g \to_{\#} g$ in $L_2^{\#}\left(\widetilde{{}^*\mathbb{R}_c^{\#}}^n\right)$ as $m \to {}^*\infty$. This shows 3. holds except for the fact that h_m need not have #-compact support. To fix this let $\psi \in C_c^{{}^*\infty}\left(\widetilde{{}^*\mathbb{R}_c^{\#}}^n, [0,1]\right)$ such that $\psi = 1$ in a neighborhood of 0 and let $\psi_\varepsilon(x) = \psi(\varepsilon x)$ and $(\partial_v^{\#}\psi)_\varepsilon(x) = (\partial_v^{\#}\psi)(\varepsilon x)$. Then

$\partial_v^{\#}\ (\psi_\varepsilon h_m) = \partial_v^{\#}\psi_\varepsilon h_m + \psi_\varepsilon \partial_v^{\#} h_m = \varepsilon(\partial_v^{\#}\psi)_\varepsilon h_m + \psi_\varepsilon \partial_v^{\#} h_m$

so that $\psi_\varepsilon h_m \to_{\#} h_m$ in $L_2^{\#}$ and $\partial_v^{\#}\ (\psi_\varepsilon h_m) \to_{\#} \partial_v^{\#} h_m$ in $L_2^{\#}$ as $\varepsilon \to_{\#} 0$. Let $f_m = \psi_{\varepsilon_m} h_m$ where ε_m is chosen to be greater than zero but small enough so that

$\|\psi_{\varepsilon_m} h_m - h_m\|_{\#2} + \|\partial_v^{\#}(\psi_{\varepsilon_m} h_m) - \partial_v^{\#} h_m\|_{\#2} < 1/m$.

Then $f_m \in C_c^{{}^*\infty}({}^*\mathbb{R}_c^{\#n})$, $f_m \to_{\#} f$ and $\partial_v^{\#} f_m \to_{\#} g$ in $L_2^{\#}$ as $m \to {}^*\infty$.

3. $\Rightarrow$ 4. By the fundamental theorem of calculus

$$\begin{aligned} \frac{\tau_{-tv} f_m(x) - f_m(x)}{t} &= \frac{f_m(x+tv) - f_m(x)}{t} = \frac{1}{t}\left(\int_0^1 \frac{d^{\#}}{d^{\#}s} f_m(x+stv) d^{\#}s\right) \\ &= \int_0^1 (\partial_v^{\#} f_m)(x+stv) d^{\#}s. \end{aligned} \tag{3.2}$$

Let

$$G_t(x) = \int_0^1 \tau_{-stv} g(x) d^{\#}s = \int_0^1 g(x+stv) d^{\#}s$$

which is defined for #-almost every x and is in $L_2^{\#}({}^*\mathbb{R}_c^{\#n})$ by generalized Minkowski's inequality for integrals. Therefore

$$\frac{\tau_{-tv}f_m(x) - f_m(x)}{t} - G_t(x) = \int_0^1 [(\partial_v^{\#} f_m)(x+stv) - g(x+stv)]d^{\#}s$$

and hence again

$$\left\| \frac{\tau_{-tv}f_m - f_m}{t} - G_t \right\|_{\#2} \leq \int_0^1 \|\tau_{-stv}(\partial v f_m) - \tau_{-stv}g\|_{\#2}\, d^{\#}s = \int_0^1 \|\partial v f_m - g\|_{\#2} d^{\#}s.$$

Letting $m \to {}^*\infty$ in this equation implies $(\tau_{-tv}f - f)/t = G_t$ #-a.e. Finally one more application of Minkowski's inequality for integrals implies,

$$\left\| \frac{\tau_{-tv}f - f}{t} - g \right\|_{\#2} = \|G_t - g\|_{\#2} = \left\| \int_0^1 (\tau_{-stv}g - g)d^{\#}s \right\|_{\#2} \leq \int_0^1 \|(\tau_{-stv}g - g)\|_{\#2} d^{\#}s$$

By the dominated convergence theorem and Proposition 11.13, the latter term tends to 0 as $t \to_{\#} 0$ and this proves 4. The proof is now complete since 4. $\Rightarrow$ 1. is trivial

Proposition 3.3. Let $(H^{\#}, \langle\cdot,\cdot\rangle_{\#})$ be a not necessarily #-complete inner product space and $\beta \subset H^{\#}$ be an orthonormal set. Then the following two conditions are equivalent:

(1) $x = Ext\text{-}\widehat{\sum_{u\in\beta}}\langle x,u\rangle_{\#}u$ for all $x \in H^{\#}$.

(2) $\|x\|_{\#}^2 = Ext\text{-}\widehat{\sum_{u\in\beta}}|\langle x,u\rangle_{\#}|^2$ for all $x \in H^{\#}$.

Moreover, either of these two conditions implies that $\beta \subset H^{\#}$ is a maximal orthonormal set. However $\beta \subset H^{\#}$ being a maximal orthonormal set is not sufficient to conditions for 1) and 2) hold.

Proof. As in the proof of **Theorem 12.24**, (1) implies (2). For (2) implies (1) let $\Lambda \subset\subset \beta$ and consider

$$\begin{aligned}\left\| x - \left(Ext\text{-}\widehat{\sum_{u\in\Lambda}}\langle x,u\rangle_{\#}u \right) \right\|_{\#}^2 &= \|x\|_{\#}^2 - 2\left(Ext\text{-}\widehat{\sum_{u\in\Lambda}}|\langle x,u\rangle_{\#}|^2 \right) + Ext\text{-}\widehat{\sum_{u\in\Lambda}}|\langle x,u\rangle_{\#}|^2 \\ &= \|x\|_{\#}^2 - \left(Ext\text{-}\widehat{\sum_{u\in\Lambda}}|\langle x,u\rangle_{\#}|^2 \right).\end{aligned} \tag{3.3}$$

Since $\|x\|_{\#}^2 = Ext\text{-}\widehat{\sum_{u\in\beta}}|\langle x,u\rangle_{\#}|^2$, it follows that for every $\varepsilon > 0, \varepsilon \approx 0,$ there exists $\Lambda_\varepsilon \subset\subset \beta$ such that for all $\Lambda \subset\subset \beta$ such that $\Lambda_\varepsilon \subset \Lambda$,

$$\left\| x - Ext\text{-}\widehat{\sum_{u\in\Lambda}}\langle x,u\rangle_{\#}u \right\|_{\#}^2 = \|x\|_{\#}^2 - Ext - \widehat{\sum_{u\in\Lambda}}|\langle x,u\rangle_{\#}|^2 < \varepsilon \tag{3.4}$$

showing that $x = Ext\text{-}\widehat{\sum_{u\in\beta}}\langle x,u\rangle_{\#}u.$

Suppose $x = (x_1,x_2,\ldots,x_n,\ldots) \in \beta^{\perp}.$ If (2) is valid then $\|x\|_{\#}^2 = 0,$ i.e. $x = 0.$ So β is maximal.

Let us now construct a counter example to prove the last assertion.

Take $H^{\#} = \mathbf{Span}(\{e_i\}_{n=1}^{*\infty}) \subset l_2^{\#}$ and let $\widetilde{u}_n = e_1 - (n+1)e_{n+1}$ for $n = 1,2.\ldots$ Applying Gramn-Schmidt to $\{\widetilde{u}_n\}_{n=1}^{*\infty}$ we construct an orthonormal set $\beta = \{u_n\}_{n=1}^{*\infty} \subset H^{\#}.$

I now claim that $\beta \subset H^{\#}$ is maximal. Indeed if $x = (x_1, x_2, \ldots, x_n, \ldots) \in \beta^{\perp}$ then $x \perp u_n$ for all n, i.e. $0 = \langle x, u_{n} \rangle_{\#} = x_1 - (n+1)x_{n+1}$.
Therefore $x_{n+1} = (n+1)^{-1}x_1$ for all n. Since $x \in \mathbf{Span}(\{e_i\}_{n=1}^{*\infty}), x_N = 0$ for some N sufficiently large and therefore $x_1 = 0$ which in turn implies that $x_n = 0$ for all n. So $x = 0$ and hence β is maximal in $H^{\#}$. On the other hand, β is not maximal in $l_2^{\#}$. In fact the above argument shows that $\beta^{\perp}$ in $l_2^{\#}$ is given by the span of $v = (1, 1/2, 1/3, 1/4, 1/5, \ldots)$. Let P be the orthogonal projection of $l_2^{\#}$ onto the $\mathbf{Span}(\beta) = v^{\perp}$. Then $Ext\text{-}\widehat{\sum}_{i=1}^{*\infty} \langle x, u_n \rangle_{\#} u_n = Px = x - \frac{\langle x, v \rangle_{\#}}{\|v\|_{\#}^2} v$ so that

$Ext\text{-}\widehat{\sum}_{i=1}^{*\infty} \langle x, u_n \rangle_{\#} u_n = x$ iff $x \in \mathbf{Span}(\beta) = v^{\perp} \subset l_2^{\#}$. For example if $x = (1, 0, 0, \ldots) \in H^{\#}$ (or more generally for $x = e_i$ for any i), $x \neq v^{\perp}$ and hence $Ext\text{-}\widehat{\sum}_{i=1}^{*\infty} \langle x, u_n \rangle_{\#} u_n \neq x$.

Proposition 3.4. (Parallelogram Law Converse). If $(X, \|\cdot\|_{\#})$ is a #-normed space such that Eq.(11.4) holds for all $x, y \in X$, then there exists a unique inner product on $\langle \cdot, \cdot \rangle_{\#}$ such that $\|x\|_{\#} = \sqrt{\langle x, x \rangle_{\#}}$ for all $x \in X$. In this case we say that $\|\cdot\|_{\#}$ is a Hilbertian #-norm.

Proof. If $\|\cdot\|_{\#}$ is going to come from an inner product $\langle \cdot, \cdot \rangle_{\#}$, it follows from Eq.(**12.1**) that $2\,\mathrm{Re}\langle x, x \rangle_{\#} = \|x + y\|_{\#}^2 - \|x\|_{\#}^2 - \|y\|_{\#}^2$ and $-2\,\mathrm{Re}\langle x, x \rangle_{\#} = \|x - y\|_{\#}^2 - \|x\|_{\#}^2 - \|y\|_{\#}^2$. Subtracting these two equations gives the "polarization identity,"

$$4\,\mathrm{Re}\langle x, x \rangle_{\#} = \|x + y\|_{\#}^2 - \|x - y\|_{\#}^2. \tag{3.5}$$

Replacing y by iy in this equation then implies that

$$4\,\mathrm{Im}\langle x, x \rangle_{\#} = \|x + iy\|_{\#}^2 - \|x - iy\|_{\#}^2. \tag{3.6}$$

from which we get

$$\langle x, y \rangle_{\#} = 1/4\left(Ext\text{-}\widehat{\sum_{\epsilon \in G}} \epsilon \|x + \epsilon y\|_{\#}^2 \right) \tag{3.7}$$

where $G = \{\pm 1, \pm i\}$ - a cyclic subgroup of ${}^*S^1 \subset {}^*\mathbb{C}_c^{\#}$. Hence if $\langle \cdot, \cdot \rangle_{\#}$ is going to exists we must define it by Eq. (3.7). Notice that

$$\begin{gathered}\langle x, x \rangle_{\#} = 1/4\left(Ext\text{-}\widehat{\sum_{\epsilon \in G}} \epsilon \|x + \epsilon x\|_{\#}^2 \right) = \|x\|_{\#}^2 + i\|x + ix\|_{\#}^2 - i\|x - ix\|_{\#}^2 = \\ \|x\|_{\#}^2 + i|1 + i|^2\|x\|_{\#}^2 - i|1 - i|^2\|x\|_{\#}^2 = \|x\|_{\#}^2.\end{gathered} \tag{3.8}$$

So to finish the proof of (4) we must show that $\langle x, y \rangle_{\#}$ in Eq. (3.7) is an inner product. Since

$$\begin{gathered}4\langle y, x \rangle_{\#} = Ext\text{-}\widehat{\sum_{\epsilon \in G}} \epsilon \|y + \epsilon x\|_{\#}^2 = Ext\text{-}\widehat{\sum_{\epsilon \in G}} \epsilon \|\epsilon(y + \epsilon x)\|_{\#}^2 = Ext\text{-}\widehat{\sum_{\epsilon \in G}} \epsilon \|(\epsilon y + \epsilon^2 x)\|_{\#}^2 \\ = \|y + x\|_{\#}^2 + \|-y + x\|_{\#}^2 + i\|y + ix\|_{\#}^2 - i\|-iy + x\|_{\#}^2 = \\ = \|x + y\|_{\#}^2 + \|x - y\|_{\#}^2 + i\|x - iy\|_{\#}^2 - i\|x + iy\|_{\#}^2 = 4\langle x, y \rangle_{\#}\end{gathered} \tag{3.9}$$

it suffices to show $x \to \langle x, y \rangle_{\#}$ is linear for all $y \in H^{\#}$. We will need to derive an identity from Eq. (1.4). To do this we make use of Eq. (1.4) three times to find

$$\begin{gathered}\|x+y+z\|_{\#}^{2} = -\|x+y-z\|_{\#}^{2}+2\|x+y\|_{\#}^{2}+2\|z\|_{\#}^{2} = \\ = \|x-y-z\|_{\#}^{2}-2\|x-z\|_{\#}^{2}-2\|y\|_{\#}^{2}+2\|x+y\|_{\#}^{2}+2\|x+y\|_{\#}^{2}+2\|z\|_{\#}^{2} = \\ \|y+z-x\|_{\#}^{2}-2\|x-z\|_{\#}^{2}-2\|y\|_{\#}^{2}+2\|x+y\|_{\#}^{2}+2\|z\|_{\#}^{2} = \\ -\|y+z+x\|_{\#}^{2}+2\|y+z\|_{\#}^{2}+2\|x\|_{\#}^{2}-2\|x-z\|_{\#}^{2}-2\|y\|_{\#}^{2}+2\|x+y\|_{\#}^{2}+2\|z\|_{\#}^{2}.\end{gathered} \tag{3.10}$$

Solving this equation for $\|x+y+z\|_{\#}^{2}$ gives

$$\|x+y+z\|_{\#}^{2} = \|x+z\|_{\#}^{2}+\|x+y\|_{\#}^{2}-\|x-z\|_{\#}^{2}+\|x\|_{\#}^{2}+\|z\|_{\#}^{2}-\|y\|_{\#}^{2}. \tag{3.11}$$

Using Eq. (3.11), for $x,y,z \in H^{\#}$,

$$\begin{gathered}4\operatorname{Re}\langle x+z,y\rangle_{\#} = \|x+z+y\|_{\#}^{2}-\|x+z-y\|_{\#}^{2} = \\ = \|y+z\|_{\#}^{2}+\|x+y\|_{\#}^{2}-\|x-z\|_{\#}^{2}+\|x\|_{\#}^{2}+\|z\|_{\#}^{2}-\|y\|_{\#}^{2}- \\ -\left(\|z-y\|_{\#}^{2}+\|x-y\|_{\#}^{2}-\|x-z\|_{\#}^{2}+\|x\|_{\#}^{2}+\|z\|_{\#}^{2}-\|y\|_{\#}^{2}\right) = \\ = \|z+y\|_{\#}^{2}-\|z-y\|_{\#}^{2}+\|x+y\|_{\#}^{2}-\|x-y\|_{\#}^{2} = \\ 4\operatorname{Re}\langle x,y\rangle_{\#}+4\operatorname{Re}\langle z,y\rangle_{\#}.\end{gathered} \tag{3.12}$$

Now suppose that $\delta \in G,$ then since $|\delta| = 1,$

$$\begin{gathered}4\langle\delta x,y\rangle_{\#} = 1/4\left(Ext\text{-}\widehat{\sum_{\epsilon\in G}}\epsilon\|\delta x+\epsilon y\|_{\#}^{2}\right) = 1/4\left(Ext\text{-}\widehat{\sum_{\epsilon\in G}}\epsilon\|\delta x+\delta^{-1}\epsilon y\|_{\#}^{2}\right) = \\ = 1/4\left(Ext\text{-}\widehat{\sum_{\epsilon\in G}}\delta\epsilon\|\delta x+\delta\epsilon y\|_{\#}^{2}\right) = 4\delta\langle x,y\rangle_{\#}.\end{gathered} \tag{3.13}$$

where in the third inequality, the substitution $\to \delta$ was made in the sum. So Eq.(3.13) says $\langle\pm ix,y\rangle_{\#} = \pm i\langle\pm ix,y\rangle_{\#},$ and $-\langle x,y\rangle_{\#} = \langle -x,y\rangle_{\#}$. Therefore

$$\operatorname{Im}\langle x,y\rangle_{\#} = \operatorname{Re}(-i\langle x,y\rangle_{\#}) = \operatorname{Re}(\langle -ix,y\rangle_{\#}) \tag{3.14}$$

which combined with Eq. (3.12.) shows

$\operatorname{Im}\langle x+z,y\rangle_{\#} = \operatorname{Re}\langle -ix-iz,y\rangle_{\#} = \operatorname{Re}\langle -ix,y\rangle_{\#}+\operatorname{Re}\langle -iz,y\rangle_{\#} = \operatorname{Im}\langle x,y\rangle_{\#}+\operatorname{Im}\langle z,y\rangle_{\#}$

and therefore (again in combination with Eq. (3.12)),

$\langle x+z,y\rangle_{\#} = \langle x,y\rangle_{\#}+\langle z,y\rangle_{\#}$ for all $x,y \in H^{\#}$.

Because of this equation and Eq. (3.13) to finish the proof that $x \to \langle x,y\rangle_{\#}$ is linear, it suffices to show $\lambda\langle x,y\rangle_{\#} = \langle\lambda x,y\rangle_{\#}$ for all $\lambda > 0.$ Now if $\lambda = m \in {}^{*}\mathbb{N},$ then

$\langle mx,y\rangle_{\#} = \langle x+(m-1)x,y\rangle_{\#} = \langle x,y\rangle_{\#}+\langle(m-1)x,y\rangle_{\#}$

so that by hyper infinite induction $\langle mx,y\rangle_{\#} = m\langle x,y\rangle_{\#}$. Replacing x by x/m then shows that $\langle x,y\rangle_{\#} = m\langle m^{-1}x,y\rangle_{\#},$ so that $\langle m^{-1}x,y\rangle_{\#} = m^{-1}\langle x,y\rangle_{\#}$ and so if $m,n \in {}^{*}\mathbb{N}$, we find $\left\langle\frac{n}{m}x,y\right\rangle_{\#} = n\left\langle\frac{1}{m}x,y\right\rangle_{\#} = \frac{n}{m}\langle x,y\rangle_{\#}$ so that $\lambda\langle x,y\rangle_{\#} = \langle\lambda x,y\rangle_{\#}$ for all $\lambda > 0$ and $\lambda \in \widetilde{{}^{*}\mathbb{Q}}$.

By #-continuity, it now follows that $\lambda\langle x,y\rangle_{\#} = \langle\lambda x,y\rangle_{\#}$ for all $\lambda \in \widetilde{{}^{*}\mathbb{R}_{c}^{\#}}, \lambda > 0.$

Proposition 3.5. Let $(H^{\#},\langle\cdot,\cdot\rangle_{\#})$ be a not necessarily #-complete inner product space and $\beta \subset H^{\#}$ be an orthonormal set. Then the following two conditions are equivalent:

(1) $x = Ext\text{-}\widehat{\sum_{u\in\beta}}\langle x,u\rangle_{\#}u$ for all $x \in H^{\#}$.

(2) $\|x\|_{\#}^{2} = Ext\text{-}\widehat{\sum_{u\in\beta}}|\langle x,u\rangle_{\#}|^{2}$ for all $x \in H^{\#}$.

Moreover, either of these two conditions implies that $\beta \subset H^{\#}$ is a maximal orthonormal set. However $\beta \subset H^{\#}$ being a maximal orthonormal set is not sufficient to conditions for 1) and 2) hold.

Proof. As in the proof of Theorem 2.1, (1) implies (2). For (2) implies (1) let $\Lambda \subset\subset \beta$ and consider

$$\left\| x - \left(Ext\text{-}\widehat{\sum_{u\in\Lambda}} \langle x,u\rangle_{\#} u \right) \right\|_{\#}^{2} = \|x\|_{\#}^{2} - 2\left(Ext\text{-}\widehat{\sum_{u\in\Lambda}} |\langle x,u\rangle_{\#}|^{2} \right) + Ext\text{-}\widehat{\sum_{u\in\Lambda}} |\langle x,u\rangle_{\#}|^{2} = \|x\|_{\#}^{2} - \left(Ext\text{-}\widehat{\sum_{u\in\Lambda}} |\langle x,u\rangle_{\#}|^{2} \right). \tag{3.15}$$

Since $\|x\|_{\#}^{2} = Ext\text{-}\widehat{\sum_{u\in\beta}} |\langle x,u\rangle_{\#}|^{2}$, it follows that for every $\varepsilon \approx 0_{{}^*\mathbb{R}_c^{\#}}, \varepsilon > 0_{{}^*\mathbb{R}_c^{\#}}$ there exists $\Lambda_{\varepsilon} \subset\subset \beta$ such that for all $\Lambda \subset\subset \beta$ such that $\Lambda_{\varepsilon} \subset \Lambda$,

$$\left\| x - \left(Ext\text{-}\widehat{\sum_{u\in\Lambda}} \langle x,u\rangle_{\#} u \right) \right\|_{\#}^{2} = \|x\|_{\#}^{2} - \left(Ext\text{-}\widehat{\sum_{u\in\Lambda}} |\langle x,u\rangle_{\#}|^{2} \right) < \varepsilon \tag{3.16}$$

showing that $x = Ext\text{-}\widehat{\sum_{u\in\beta}} \langle x,u\rangle_{\#}$. Suppose $x = (x_1, x_2, \ldots, x_n, \ldots) \in \beta^{\perp}$. If (2) is valid then $\|x\|_{\#}^{2} = 0_{\widetilde{{}^*\mathbb{R}_c^{\#}}}$, i.e. $x = 0$. So β is maximal. Let us now construct a counter example to prove the last assertion. Take $H^{\#} = \mathbf{Span}\{e_i\}_{i=1}^{{}^*\infty} \subset l_2^{\#}$ and let $\hat{u}_n = e_1 - (n+1)e_{n+1}$ for $n \in {}^*\mathbb{N}$. Applying Gramn-Schmidt to $\{\hat{u}_n\}_{n=1}^{{}^*\infty}$ we construct an orthonormal set $\beta = \{u_n\}_{n=1}^{{}^*\infty} \subset H^{\#}$.

We now claim that $\beta \subset H^{\#}$ is maximal. Indeed if $x = (x_1, x_2, \ldots, x_n, \ldots) \in \beta^{\perp}$ then $x \perp u_n$ for all $n \in {}^*\mathbb{N}$, i.e. $0_{\widetilde{{}^*\mathbb{R}_c^{\#}}} = \langle x, \hat{u}_n\rangle_{\#} = x_1 - (n+1)x_{n+1}$.

Therefore $x_{n+1} = (n+1)^{-1}x_1$ for all $n \in {}^*\mathbb{N}$. Since $x \in \mathbf{Span}\{e_i\}_{i=1}^{{}^*\infty}$, $x_N = 0$ for some N sufficiently large and therefore $x_1 = 0$ which in turn implies that $x_n = 0_{\widetilde{{}^*\mathbb{R}_c^{\#}}}$ for all $n \in {}^*\mathbb{N}$. So $x = 0_{\widetilde{{}^*\mathbb{R}_c^{\#}}}$ and hence β is maximal in $H^{\#}$. On the other hand, β is not maximal in $l_2^{\#}$. In fact the above argument shows that $\beta^{\perp}$ in $l_2^{\#}$ is given by the span of $v = (1_{\widetilde{{}^*\mathbb{R}_c^{\#}}}, 1_{\widetilde{{}^*\mathbb{R}_c^{\#}}}/2_{\widetilde{{}^*\mathbb{R}_c^{\#}}}, 1_{\widetilde{{}^*\mathbb{R}_c^{\#}}}/3_{\widetilde{{}^*\mathbb{R}_c^{\#}}}, 1_{\widetilde{{}^*\mathbb{R}_c^{\#}}}/4_{\widetilde{{}^*\mathbb{R}_c^{\#}}}, 1_{\widetilde{{}^*\mathbb{R}_c^{\#}}}/5_{\widetilde{{}^*\mathbb{R}_c^{\#}}}, \ldots)$. Let P be the orthogonal projection of $l_2^{\#}$ onto the $\mathbf{Span}(\beta) = v^{\perp}$.

$Ext\text{-}\widehat{\sum_{u\in\Lambda}} \langle x,u_n\rangle_{\#} u_n = Px = x - \dfrac{\langle x,v\rangle_{\#}}{\|v\|_{\#}^{2}} v$, so that $Ext\text{-}\widehat{\sum_{u\in\Lambda}} \langle x,u_n\rangle_{\#} u_n = x$ iff $x \in \mathbf{Span}(\beta) = v^{\perp} \subset l_2^{\#}$. For example if $x = (1_{\widetilde{{}^*\mathbb{R}_c^{\#}}}, 0_{\widetilde{{}^*\mathbb{R}_c^{\#}}}, 0_{\widetilde{{}^*\mathbb{R}_c^{\#}}}, \ldots) \in H^{\#}$(or more generally for $x = e_i$ for any $i \in {}^*\mathbb{N}$), $x \notin v^{\perp}$ and hence $Ext\text{-}\widehat{\sum_{u\in\Lambda}} \langle x,u_n\rangle_{\#} u_n \neq x$.

§ 3.2.#-Analytic vectors.Generalized Nelson's #-analytic vector theorem.

Let $\mathbf{H}^{\#}$ be a #-complex Hilbert space over field $\widetilde{{}^*\mathbb{C}_c^{\#}}$. The most natural way to construct a #-continuous one-parameter unitary group $U(t) : \mathbf{H}^{\#} \to \mathbf{H}^{\#}$ is to try to make sense of the power series $Ext\text{-}\widehat{\sum}_{n=0}^{\infty^{\#}}(itA)^n$ on a #-dense set of vectors. Notice that this can certainly be done if A is self-adjoint. For let E_{Ω} be the family of spectral projections for $A.$ Then on each of the spaces $E_{[-M,M]}$, A is a bounded operator and $Ext\text{-}\widehat{\sum}_{n=0}^{\infty^{\#}}(itA)^n/n!$ #-converges to $Ext\text{-}\exp(itA)$ in #-norm. In particular, for any $\varphi \in \bigcup_{M\geq 0} E_{[-M,M]}$,

$$\#\text{-}\lim_{N\to\infty^{\#}}\left(Ext\text{-}\widehat{\sum}_{n=0}^{N}\frac{(itA)^n}{n!}\right) = Ext\text{-}\exp(itA). \tag{3.1}$$

Since $\bigcup_{M\geq 0} E_{[-M,M]}$ is #-dense in $\mathbf{H}^{\#}$, we see that the group generated by a self-adjoint operator A is completely determined by the well-defined action of the hyper infinite series $Ext\text{-}\widehat{\sum}_{n=0}^{\infty^{\#}}(itA)^n/n!$ on a #-dense set. We will prove the #-converse: namely, if A is symmetric and has a #-dense set of vectors to which $Ext\text{-}\widehat{\sum}_{n=0}^{\infty^{\#}}(itA)^n/n!$ can be applied, then A is essentially self-#-adjoint. We need several definitions.

Definition1.1. Let A be an operator on a non-Archimedean Hilbert space $\mathbf{H}^{\#}$. The set $\mathbf{C}^{\infty^{\#}}(A) = \bigcap_{n=0}^{\infty^{\#}} D(A^n)$ is called the $\mathbf{C}^{\infty^{\#}}$-vectors for $A.$ A vector $\varphi \in \mathbf{C}^{\infty^{\#}}(A)$ is called an #-analytic vector for A if

$$Ext\text{-}\widehat{\sum}_{n=0}^{\infty^{\#}}\frac{\|A^n\varphi\|t^n}{n!} < {}^{*}\infty \tag{3.2}$$

for some $t > 0.$ If A is self-adjoint, then $\mathbf{C}^{\infty^{\#}}(A)$ will be #-dense in $D(A).$ However, in general, a symmetric operator may have no $\mathbf{C}^{\infty^{\#}}$-vectors at all even if A is essentially self-#-adjoint. We caution the reader to remember that #-analytic vectors and vectors of uniqueness (defined below) must be $\mathbf{C}^{\infty^{\#}}$- vectors for $A.$ A vector $\varphi \in D(A)$ can be an #-analytic vector for an extension of A but fail to be an #-analytic vector for A because it is not in $\mathbf{C}^{\infty^{\#}}(A)$.

Definition1.2.Suppose that A is symmetric. For each $\varphi \in \mathbf{C}^{\infty^{\#}}(A),$ define

$$D_{\varphi} = \left\{Ext\text{-}\widehat{\sum}_{n=0}^{N}\alpha_n A^n\varphi \,\middle|\, N \in {}^{*}\mathbb{N}, \alpha_n \in \widetilde{{}^{*}\mathbb{C}_c^{\#}}\right\}. \tag{3.3}$$

Let $\mathbf{H}^{\#}_{\varphi} = \#\text{-}\overline{D_{\varphi}}$ and define $A_{\varphi} : D_{\varphi} \to D_{\varphi}$ by $A_{\varphi}\left(Ext\text{-}\widehat{\sum}_{n=0}^{N}\alpha_n A^n\varphi\right) = Ext\text{-}\widehat{\sum}_{n=0}^{N}\alpha_n A^{n+1}\varphi.$

φ is called a vector of #-uniqueness if and only if A_{φ} is essentially self-#-adjoint on D_{φ} as an operator on $\mathbf{H}^{\#}_{\varphi}$.

Finally, a subset $S \subset \mathbf{H}^{\#}$ is called #-total if the set of hyperfinite linear combinations of elements of S is #-dense in $\mathbf{H}^{\#}$.

Lemma (Generalized Nussbaum's lemma) Let A be a symmetric operator and suppose that $D(A)$ contains a #-total set of vectors of #-uniqueness. Then A is essentially self-#-adjoint.

Proof We will show that $\mathbf{Ran}(A \pm\ i)$ are #-dense in $\mathbf{H}^{\#}$. By the fundamental criterion this will show that A is essentially self-#-adjoint. Suppose $\psi \in \mathbf{H}^{\#}$ and $\varepsilon > 0$ are given and let S denote the set of vectors of #-uniqueness. Since S is #-total we can find $(\alpha_n)_{n=1}^{N}$ and $(\psi_n)_{n=1}^{N}$ with $\psi_n \in S$ so that

$$\left\| \psi - Ext\text{-}\widehat{\sum}_{n=1}^{N} \alpha_n \psi_n \right\|_{\#} \leq \varepsilon/2. \quad (3.4)$$

Since ψ_n is a vector of #-uniqueness, there is a $\varphi_n \in D_{\psi_n}$ so that

$$\|\psi_n - (A+i)\varphi_n\|_{\#} \leq \frac{\varepsilon}{2}\left(Ext\text{-}\widehat{\sum}_{n=1}^{N} |\alpha_n| \right)^{-1}. \quad (3.5)$$

Setting $\varphi = Ext\text{-}\sum_{n=1}^{N} \alpha_n \varphi_n$ we have $\varphi \in D(A)$ and $\|\psi - (A+i)\varphi\|_{\#} < \varepsilon$.
Thus $\mathbf{Ran}(A+i)$ is #-dense. The proof for $(A-i)$ is the same.

Theorem 3.1. (Generalized Nelson's #-analytic vector theorem) Let A be a symmetric operator on a non-Archimedean Hilbert space $\mathbf{H}^{\#}$. If $D(A)$ contains a #-total set of #-analytic vectors, then A is essentially self-#-adjoint.

Proof By Generalized Nussbaum's lemma, it is enough to show that each #-analytic vector ψ is a vector of #-uniqueness. First notice that A_ψ always has self-#-adjoint extensions, since the operator

$$C : Ext\text{-}\widehat{\sum}_{n=0}^{N} \alpha_n A^n \psi \quad (3.6)$$

extends to a conjugation on $\mathbf{H}_\psi^{\#}$ which commutes with A_ψ. Suppose that B is a self-#-adjoint extension of A_ψ on $\mathbf{H}_\psi^{\#}$ and let $\mu^{\#}$ be the spectral #-measure for B associated to ψ. Since ψ is an #-analytic vector for A,

$$Ext\text{-}\widehat{\sum}_{n=0}^{N} \|A^n\psi\|_{\#}/n! < {}^*\infty \quad (3.7)$$

for some $t > 0$. Let $0 < s < t$. Then

$$\begin{gathered} Ext\text{-}\widehat{\sum}_{n=0}^{\infty^{\#}} \frac{s^n}{n!}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} |x|^n d^{\#}\mu^{\#} \right) \leq \\ \leq Ext\text{-}\widehat{\sum}_{n=0}^{\infty^{\#}} \frac{s^n}{n!}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} x^{2n} d^{\#}\mu^{\#} \right)^{1/2}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} d^{\#}\mu^{\#} \right)^{1/2} = \\ \|\psi\|_{\#} Ext\text{-}\widehat{\sum}_{n=0}^{\infty^{\#}} \frac{s^n}{n!} \|A^n\psi\|_{\#} < {}^*\infty. \end{gathered} \quad (3.8)$$

Therefore by generalized Fibini's theorem

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \left(Ext\text{-}\sum_{n=0}^{{}^*\infty} \frac{s^n}{n!}|x|^n \right) d^{\#}\mu^{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} Ext\text{-}(s|x|) d^{\#}\mu^{\#} < {}^*\infty. \quad (3.9)$$

As a result, the function

$$\langle \psi, [Ext\text{-}\exp(itB)]\psi \rangle_{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} [Ext\text{-}\exp(itx)] d^{\#}\mu^{\#} \quad (3.10)$$

has an #-analytic continuation

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} [Ext\text{-}\exp(izx)] d^{\#}\mu^{\#} \quad (3.11)$$

to the region $|\mathrm{Im}\, z| < t.$ Since

$$\left[\left(\frac{d^{\#}}{d^{\#}z}\right)^{k}\left(Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}}[Ext\text{-}\exp(izx)]d^{\#}\mu^{\#}\right)\right]_{z=0} = \tag{3.12}$$
$$= Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}}\left[Ext\text{-}\exp(ix)^{k}\right]d^{\#}\mu^{\#} = \left\langle\psi,(iA)^{k}\psi\right\rangle_{\#},$$

we obtain

$$\langle\psi,[Ext\text{-}\exp(isB)]\psi\rangle_{\#} = Ext\text{-}\widehat{\sum_{n=0}^{\infty^{\#}}}\frac{(is)^{n}}{n!} = \left\langle\psi,(iA)^{k}\psi\right\rangle_{\#} \tag{3.13}$$

for $|s| < t.$ Thus, for $|s| < t$ (and therefore for all s), the function $\langle\psi_1,[Ext\text{-}\exp(isB)]\psi_2\rangle_{\#}$ is completely determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_{\#}, n \in {}^{*}\mathbb{N}.$
Similar proof shows that $\langle\psi_1,[Ext\text{-}\exp(isB)]\psi_2\rangle_{\#}$ is determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_{\#}, n \in {}^{*}\mathbb{N}$for any $\psi_1,\psi_2 \in D_{\psi}.$ Since D_{ψ} is #-dense in $\mathbf{H}_{\psi}^{\#}$ and $Ext\text{-}\exp(isB)$ is unitary, $Ext\text{-}\exp(isB)$ is completely determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_{\#}, n \in {}^{*}\mathbb{N}$ for any $\psi_1,\psi_2 \in D_{\psi}.$Thus, all self-#-adjoint extensions of A_{ψ} generate the same unitary group, so by generalized Stone's theorem A_{ψ} has at most one self-#-adjoint extension. As we have already remarked, A_{ψ} has at least one self-#-adjoint extension. Thus A_{ψ} is essentially self-#-adjoint and ψ is a vector of uniqueness.
Corollary 3.1 A #-closed symmetric operator A is self-#-adjoint if and only if $D(A)$ contains a #-dense set of #-analytic vectors.
The statement of Corollary 1 is not true if "self-#-adjoint" is replaced by "essentially self-#-adjoint." A self-#-adjoint operator A may be essentially self-#-adjoint on a domain $D \subset D(A)$ and D may not even contain any #-vectors.
Corollary 3.2 Suppose that A is a symmetric operator and let D be a #-dense linear set contained in $D(A).$ Then, if D contains a #-dense set of #-analytic vectors and if D is invariant under $A,$ then A is essentially self-#-adjoint on $D.$
Proof Since D is invariant under A, each #-analytic vector for A in D is also an #-analytic vector for $A \upharpoonright D.$ Thus, by Theorem 3.1 $A \upharpoonright D$ is essentially self-#-aadjoint.
The reason that one needs the invariance condition in Corollary 2 is that for a vector $\psi \in D$ to be #-analytic for $A \upharpoonright D,$ it must first be $C^{*\infty}$ for $A \upharpoonright D.$ This requires that $A^n \in D$ for all $n \in {}^{*}\mathbb{N}.$

§4.The generalized Spectral Theorem

§ 4.1.The #-continuous functional calculus

In this section, we will discuss the generalized spectral theorem in its many guises. This structure theorem is a concrete description of all self-#-adjoint operators. There are several apparently distinct formulations of the spectral theorem. In some sense they are all equivalent.
The form we prefer says that every bounded self-#-adjoint operator is a multiplication operator. (We emphasize the word bounded since we will deal extensively with unbounded self-#-adjoint operators in the next chapter; there is a spectral theorem for unbounded operators which we discuss in Section § 4.3)
This means that given a bounded self-#-adjoint operator A on a non-Archimedean

Hilbert space $\mathbf{H}^{\#}$ over field $\widetilde{{}^{*}\mathbb{R}_{c}^{\#}}$ or $\widetilde{{}^{*}\mathbb{C}_{c}^{\#}}$, we can always find a #-measure $\mu^{\#}$ on a #-measure space M and a unitary operator $U : \mathbf{H}^{\#} \to L_{2}^{\#}(M, d^{\#}\mu^{\#})$ so that

$$(UAU^{-1}f)(x) = F(x)f(x) \tag{4.1.1}$$

for some bounded real-valued #-measurable function F on M.

In practice, M will be a union of copies of ${}^{*}\mathbb{R}_{c}^{\#}$ and F will be x so the core of the proof of the theorem will be the construction of certain #-measures. This will be done in Section

§ 4.2 by using the generalized Riesz-Markov theorem. Our goal in this section will be to

make sense out of $f(A)$, for f a #-continuous function.

In the next section, we will consider the #-measures defined by the functionals

$$f \mapsto \langle \psi, f(A)\psi \rangle_{\#} \tag{4.1.2}$$

for fixed $\psi \in \mathbf{H}^{\#}$.

Given a fixed operator A, for which f can we define $f(A)$? First, suppose that A is an arbitrary bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ operator. If $f(x) = Ext\text{-}\sum_{n=1}^{N} c_{n}x^{n}$, $N \in {}^{*}\mathbb{N}$ is a polynomial, we let $f(A) = Ext\text{-}\sum_{n=1}^{N} c_{n}A^{n}$. Suppose that $f(x) = Ext\text{-}\sum_{n=1}^{*\infty} c_{n}x^{n}$ is a hyper infinite power series with radius of #-convergence R. If $\|A\|_{\#} < R$ then hyper infinite power series $Ext\text{-}\sum_{n=1}^{*\infty} c_{n}A^{n}$ #-converges in $\mathcal{L}(H^{\#})$ so it is natural to set

$$f(A) = Ext\text{-}\sum_{n=1}^{*\infty} c_{n}A^{n} \tag{4.1.3}$$

In this last case, f was a function #-analytic in a domain including all of $\sigma(A)$.

The functional calculus we have talked about thus far works for any operator in any Banach space. The special property of self-adjoint operators or more generally normal operators is that $\|P(A)\|_{\#} = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$ for any polynomial P, so that one can use the B.L.T. theorem to extend the functional calculus to #-continuous functions. Our major goal in this section is the proof of:

Theorem 4.1.1. (#-continuous functional calculus) Let A be a self-#-adjoint operator on a Hilbert space $H^{\#}$. Then there is a unique map $\phi : C^{\#}(\sigma(A)) \to \mathcal{L}(H^{\#})$ with the following properties:

(a) ϕ is an algebraic $*$-homomorphism, that is,
$\phi(fg) = \phi(f)\phi(g), \phi(\lambda f) = \lambda\phi(f), \phi(1) = I, \phi(\bar{f}) = \phi(f)^{*}$.

(b) ϕ is #-continuous, that is, $\|\phi(f)\|_{\mathcal{L}(H^{\#})} \leq C\|f\|_{*\infty}$.

(c) Let f be the function $f(x) = x$; then $\phi(f) = A$.

Moreover, ϕ has the additional properties:

(d) If $A\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$.

(e) $\sigma[\phi(f)] = \{f(\lambda) | \lambda \in \sigma(A)\}$ [spectral mapping theorem].

(f) If $f \geq 0$, then $\phi(f) \geq 0$.

(g) $\|\phi(f)\|_{\#} = \|f\|_{*\infty}$. [this strengthens (b)].

The proof which we give below is quite simple, (a) and (c) uniquely determine $\phi(P)$ for any hyperfinite polynomial $P(x)$. By the generalized Weierstrass theorem, the set of polynomials is #-dense in $C^{\#}(\sigma(A))$ so the main part of the proof is showing that

$$\|P(A)\|_{\#op} = \|P(x)\|_{C^{\#}(\sigma(A))} = \sup_{\lambda \in \sigma(A)} |P(\lambda)|. \tag{4.1.4}$$

The existence and uniqueness of ϕ then follow from the generalized B.L.T. theorem. To prove the crucial equality, we first prove a special case of (e) (which holds for arbitrary bounded operators):

Lemma 4.1.1.Let $P(x) = Ext\text{-}\sum_{n=1}^{N} c_n x^n,\ N \in {}^*\mathbb{N}$. Let $P(A) = Ext\text{-}\sum_{n=1}^{N} c_n A^n$. Then

$$\sigma(P(A)) = \{P(\lambda)|\lambda \in \sigma(A)\}. \qquad (4.1.5)$$

Proof Let $\lambda \in \sigma(A)$. Since $x = \lambda$ is a root of $P(x) - P(\lambda)$, we have $P(x) - P(\lambda) = (x - \lambda)Q(x)$, so $P(A) - P(\lambda) = (A - \lambda)Q(A)$. Since $(A - \lambda)$ has no inverse neither does $P(A) - P(\lambda)$ that is, $P(\lambda) \in \sigma(P(A))$.

Conversely, let $\mu \in \sigma(P(A))$ and let $\lambda_1,\ldots,\lambda_n$ be the roots of $P(x) - \mu$, that is, $P(x) - \mu = a\left(Ext\text{-}\prod_{i=1}^{n}(x - \lambda_i)\right)$. If $\lambda_1,\ldots,\lambda_n \notin \sigma(A)$, then

$$(P(A) - \mu)^{-1} = a^{-1}\left(Ext\text{-}\prod_{i=1}^{n}(A - \lambda_i)^{-1}\right) \qquad (4.1.6)$$

so we conclude that some $\lambda_i \in \sigma(A)$ that is, $\mu = P(\lambda)$ for some $\lambda \in \sigma(A)$.

Definition Let $r(A) = \sup_{\lambda\in\sigma(A)} |\lambda|$. Then $r(A)$ is called the spectral radius of A.

Theorem 4.1.2. Let X be a Banach space, $A \in \mathcal{L}(X)$ Then $\lim_{n\to{}^*\infty} \sqrt[n]{\|A^n\|_{\#op}}$ exists and is equal to $r(A)$. If X is a Hilbert space and A is self-#-adjoint, then $r(A) = \|A\|_{\#op}$.

Lemma 4.1.2. Let A be a bounded in ${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator. Then

$$\|P(A)\|_{\#} = \sup_{\lambda\in\sigma(A)}|P(\lambda)|. \qquad (4.1.7)$$

Proof By Theorem 4.1.2 and by Lemma 4.1.1 we obtain

$$\begin{gathered}\|P(A)\|_{\#}^{2} = \|P(A)^{*}P(A)\|_{\#} = \|(\overline{P}P)(A)\|_{\#} = \\ = \sup_{\lambda\in\sigma\left((\overline{P}P)(A)\right)}|\lambda| = \sup_{\lambda\in\sigma(A)}|\overline{P}P(\lambda)| = \left(\sup_{\lambda\in\sigma(A)}|P(\lambda)|\right)^{2}.\end{gathered} \qquad (4.1.8)$$

Proof of **Theorem 4.1.1**. Let $\phi(P) = P(A)$. Then $\|\phi(P)\|_{\mathcal{L}(H^{\#})} = \|P\|_{C^{\#}(\sigma(A))}$ so ϕ has a unique linear extension to the #-closure of the polynomials in $C^{\#}(\sigma(A))$. Since the polynomials are an algebra containing $\mathbf{I}$, containing complex conjugates, and separating points, this #-closure is all of $C^{\#}(\sigma(A))$. Properties (a), (b), (c), (g) are obvious and if $\widetilde{\phi}$ obeys (a), (b), (c) it agrees with ϕ on polynomials and thus by #-continuity on $C^{\#}(\sigma(A))$ To prove (d), note that $\phi(P)\psi = P(\lambda)\psi$ and apply #-continuity. To prove (f), notice that if $f \geq 0$, then $f = g^2$ with g ${}^*\mathbb{R}_c^{\#}$-valued and $g \in C^{\#}(\sigma(A))$. Thus $\phi(f) = \phi(g)^2$ with $\phi(g)$ self-#-adjoint, so $\phi(f) \geq 0$.

Remark 4.1.1. In addition:

(1) $\phi(f) \geq 0$ if and only if $f \geq 0$.

(2) Since $fg = gf$ for all f,g, $\{f(A)|f \in C^{\#}(\sigma(A))\}$ forms an abelian algebra closed under adjoints. Since $\|\phi(f)\|_{\#} = \|f\|_{{}^*\infty}$ and $C^{\#}(\sigma(A))$ is #-complete, $\{f(A)|f \in C^{\#}(\sigma(A))\}$ is #-norm-#-closed. It is thus an non-Archimedean abelian $\mathbf{C}^*$ algebra of operators.

(3) $\mathbf{Ran}(\phi)$ is actually the non-Archimedean $\mathbf{C}^*$ algebra generated by A that is, the smallest $\mathbf{C}^*$-algebra containing A.

(4) This result, that $C^{\#}(\sigma(A))$ and the non-Archimedean $\mathbf{C}^*$-algebra generated by A are #-isometrically isomorphic

(5) (b) actually follows from (a) and Proposition 4.1.1. Thus (a) and (c) alone determine ϕ uniquely.

Proposition 4.1.1. Suppose that $\phi\colon C^{\#}(X) \to \mathcal{L}(H^{\#})$ is an algebraic $*$-homomorphism, X a #-compact metric space. Then

(a) If $f \geq 0$, then $\phi(f) \geq 0$.

(b) $\|\phi(f)\|_{\#} \leq \|f\|_{{}^*\infty}$.

Definition 4.1.1 if $n,k \in {}^*\mathbb{N}$ with $k \leq n$, then we define

$$\binom{n}{k} = \frac{n!^{\#}}{k!^{\#}(n-k)!^{\#}} \tag{4.1.8}$$

where $n!^{\#} = Ext\text{-}\prod_{1 \leq m \leq n} m,$ see ref [7].

Lemma 4.1.3. Whenever $n,k \in {}^*\mathbb{N}$ are such that $k \leq n$, then

$$\binom{n}{k} = \binom{n}{n-k}. \tag{4.1.9}$$

Proof. Directly from the formula (4.1.8)

$$\binom{n}{n-k} = \frac{n!^{\#}}{(n-k)!^{\#}[n-(n-k)]!^{\#}} = \frac{n!^{\#}}{(n-k)!^{\#}k!^{\#}} = \frac{n!^{\#}}{k!^{\#}(n-k)!^{\#}} = \binom{n}{k}. \tag{4.1.10}$$

Lemma 4.1.4. Let $n,k \in {}^*\mathbb{N}$ with $0 < k < n$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{4.1.11}$$

Proof. Directly by hyper infinite induction [7].

Proposition 4.1.2. (Generalized binomial theorem) Let $x,y \in {}^*\mathbb{R}_c^{\#}$ and let $n \in {}^*\mathbb{N}$, then we have

$$(x \dotplus y)^n = Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}. \tag{4.1.8}$$

Proof.We prove the result by hyper infinite induction. When $n = 1$, we trivially have

$$(x \dotplus y)^1 = x \dotplus y = \binom{1}{0}x + \binom{1}{1}y.$$

Suppose that there is an $n \in {}^*\mathbb{N}$for which the statement (4.1.8) is true. We then have

$$\begin{gathered}
(x \dotplus y)^{n+1} = (x \dotplus y)^n(x \dotplus y) = \\
\left[Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right](x \dotplus y) = \\
\left[Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right]x + \left[Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right]y = \\
\left[Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^{n+1-k}y^k\right] + \left[Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1}\right] = \\
x^{n+1} + Ext\text{-}\sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]x^{n-k}y^k + y^{n+1} = \\
\binom{n+1}{0}x^{n+1} + Ext\text{-}\sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]x^{n-k}y^k + \binom{n+1}{n+1}y^{n+1} = \\
= Ext\text{-}\sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k
\end{gathered}$$

where we have used Lemma 4.1.4.

Definition 4.1.2 (Hyperfinite Bernstein Polynomials). For each $n \in {}^*\mathbb{N}$, the n-th Bernstein Polynomial $B_n^{\#}(x,f)$ of a function $f \in C^{\#}([a,b],{}^*\mathbb{R}_c^{\#})$ is defined as

$$B_n^{\#}(x,f) = Ext\text{-}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}. \tag{4.1.9}$$

Lemma 4.1.3.For any $n \in {}^*\mathbb{N}$:

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = 1, \quad (4.1.10)$$

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(k-nx) = 0, \quad (4.1.11)$$

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(k-nx)^2 = nx(1-x). \quad (4.1.12)$$

Proof. To prove these identities, first, from the generalized binomial theorem, for any $n \in {}^*\mathbb{N}$ one obtains that

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = [x \dotplus (1-x)]^n = 1. \quad (4.1.13)$$

By the generalized binomial theorem we have

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}y^k z^{n-k} = (y \dotplus z)^n. \quad (4.1.14)$$

By the #-differentiating with respect to y of the identity (4.1.14) we obtain

$$\begin{gathered}\frac{d^{\#}}{d^{\#}y}\left[Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}y^k z^{n-k}\right] = Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}ky^{k-1}z^{n-k} = \\ = \frac{d^{\#}}{d^{\#}y}(y \dotplus z)^n = n(y \dotplus z)^{n-1}.\end{gathered} \quad (4.1.15)$$

Thus

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}ky^{k-1}z^{n-k} = n(y \dotplus z)^{n-1} \quad (4.1.16)$$

and therefore

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}ky^k z^{n-k} = ny(y \dotplus z)^{n-1}. \quad (4.1.17)$$

Replacing y by x and z by $1-x$ in the above expression, we have identity

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k} = nx. \quad (4.1.18)$$

From (4.1.18) we obtain

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}\frac{k}{n}x^k(1-x)^{n-k} = x. \quad (4.1.19)$$

From (4.1.19) and (4.1.13) we obtain the identity (4.1.11).By the #-differentiating with respect to y of the identity (4.1.17) we obtain

$$\begin{gathered}\frac{d^{\#}}{d^{\#}y}\left[Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}ky^k z^{n-k}\right] = Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}k^2y^{k-1}z^{n-k} = \\ = n\frac{d^{\#}}{d^{\#}y}y(y \dotplus z)^{n-1} = n(y \dotplus z)^{n-1} + n(n-1)y(y \dotplus z)^{n-2}.\end{gathered} \quad (4.1.20)$$

Thus

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}k^2y^{k-1}z^{n-k} = n(y \dotplus z)^{n-1} + n(n-1)y(y \dotplus z)^{n-2}. \quad (4.1.21)$$

and therefore

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}k^2y^k z^{n-k} = ny(y \dotplus z)^{n-1} + n(n-1)y^2(y \dotplus z)^{n-2}. \quad (4.1.22)$$

Replacing y by x and z by $1-x$ in the expression (4.1.22) we have identity

$$Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}k^2x^k(1-x)^{n-k} = nx + n(n-1)x^2. \qquad (4.1.23)$$

From (4.1.23) and (4.1.10)-(4.1.11) one obtains (4.1.12).

Theorem 4.1.2. (Generalized Weierstrass Approximation Theorem). Let $f \in C^{\#}([a,b], {}^*\mathbb{R}_c^{\#})$. Then there is a hyper infinite sequence of polynomials $p_n(x), n \in {}^*\mathbb{N}$ that #-converges uniformly to $f(x)$ on $[a,b]$.

Proof. For a #-continuous function f defined on $[0,1]$ by (4.1.9) and (4.1.10) we obtain

$$f(x) - B_n^{\#}(x) = Ext\text{-}\sum_{k=0}^{n}\left[f(x) - f\left(\frac{k}{n}\right)\right]\binom{n}{k}x^k(1-x)^k. \qquad (4.1.24)$$

From the identity (4.1.24) one obtains that

$$\left| f(x) - B_n^{\#}(x)\right| \leq Ext\text{-}\sum_{k=0}^{n}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^k. \qquad (4.1.25)$$

Since f is #-continuous on [0,1], it is bounded in ${}^*\mathbb{R}_c^{\#}$on $[0,1]$, i.e., there exists a number $M \in {}^*\mathbb{R}_c^{\#}$ such that $|f(x)| \leq M, x \in [0,1]$. Moreover f is uniformly #-continuous on $[0,1]$. Choose $\varepsilon \approx 0, \varepsilon > 0$, then there exists $\delta \approx 0, \delta > 0$ such that $x, y \in [0,1]$ with $|x-y| \leq \delta$ implies that $|f(x) - f(y)| \leq \varepsilon$. For $x \in [0,1]$, split the sum in the righthand side of (4.1.1) into two parts:

$$Ext\text{-}\sum_{|x-k/n|\leq\delta}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^k \qquad (4.1.12)$$

and

$$Ext\text{-}\sum_{|x-k/n|>\delta}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^k \qquad (4.1.13)$$

From (4.1.9) we obtain

$$Ext\text{-}\sum_{|x-k/n|\leq\delta}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^k \leq \varepsilon. \qquad (4.1.14)$$

From (4.1.9) we obtain

$$\begin{aligned} Ext\text{-}\sum_{|x-k/n|>\delta}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^k &\leq 2M\left[Ext\text{-}\sum_{|x-k/n|>\delta}\binom{n}{k}x^k(1-x)^k\right] \\ &\leq \frac{2M}{\delta^2}\left[Ext\text{-}\sum_{|x-k/n|>\delta}\left(x-\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^k\right] \leq \qquad (4.1.15) \\ &\leq \frac{2M}{\delta^2}\left[Ext\text{-}\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^k\right] \leq \frac{2M}{n\delta^2}. \end{aligned}$$

Finally we obtain

$$|f(x) - B_n^{\#}(x)| \leq \varepsilon \dotplus \frac{2M}{n\delta^2}. \qquad (4.1.16)$$

By choosing $n \in {}^*\mathbb{N}$ large enough the righthand side can be made less than 2ε. This estimate is independent of $x \in [0,1]$. Hence, for $\varepsilon > 0$ there exists a number $N \in {}^*\mathbb{N}$ such that $n \geq N$ and $x \in [0,1]$ imply $|f(x) - B_n^{\#}(x)| \leq 2\varepsilon$. Therefore f is the uniform #-limit of the polynomials $B_n^{\#}$.

Theorem 4.1.3.(Generalized B.L.T.theorem) Suppose that Z is a #-normed space, Y is a non-Archimedean Banach space, and $S \subset Z$ is a #-dense linear subspace of Z. If $T : S \to Y$ is a bounded linear transformation (i.e. there exists $C < {}^*\infty$ such that

$\|Tz\|_{\#} \leq C\ \|z\|_{\#}$ for all $z \in S$), then T has a unique extension to an element of $\mathcal{L}(Z,Y)$.

§ 4.2.The spectral #-measures

Theorem 4.2.1.(Generalized Riesz-Markov theorem) Let X be a locally #-compact non-Archimedean metric space endowed with ${}^*\mathbb{R}_c^{\#}$-valued metric.Let $C_c^{\#}(X)$ be the space of #-continuous #-compactly supported ${}^*\mathbb{C}_c^{\#}$-valued functions on X.
For any positive linear functional Φ on $C_c^{\#}(X)$, there is a unique #-measure $\mu^{\#}$ on X such that

$$\forall f \in C_c^{\#}(X) : \Phi(f) = Ext\text{-}\int_X f(x) d^{\#}\mu^{\#}(x).$$

Theorem 4.2.2.(Generalized Riesz lemma) Let Y be a #-closed proper vector subspace of a normed space $(X, \|\cdot\|_{\#})$ and let $\alpha \in {}^*\mathbb{R}_c^{\#}$ be any real number satisfying $0 < \alpha < 1$.Then there exists a vector $u \in X$ of unit #-norm $\|u\|_{\#} = 1$ such that $\|u - y\|_{\#} \geq \alpha$ for all $y \in Y$.
We are now introduce the #-measures corresponding to bounded in${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operators. Let A be an bounded in${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator. Let $\psi \in \mathbf{H}^{\#}$. Then

$$f \mapsto \langle \psi, f(A)\psi \rangle_{\#} \tag{4.2.1}$$

is a positive linear functional on $C^{\#}(\sigma(A))$. Thus, by the generalized Riesz-Markov theorem, there is a unique #-measure $\mu_{\psi}^{\#}(\cdot)$ on the #-compact set $\sigma(A)$ with the property

$$\langle \psi, f(A)\psi \rangle_{\#} = Ext\text{-}\int_{\sigma(A)} f(\lambda) d^{\#}\mu_{\psi}^{\#}. \tag{4.2.2}$$

Definition 4.2.1. The #-measure $\mu_{\psi}^{\#}(\cdot)$ is called the spectral #-measure associated with the vector $\psi \in \mathbf{H}^{\#}$.
The first and simplest application of the $\mu_{\psi}^{\#}(\cdot)$ is to allow us to extend the functional calculus to $B^{\#}({}^*\mathbb{R}_c^{\#})$, the bounded in${}^*\mathbb{R}_c^{\#}$ #-Borel functions on ${}^*\mathbb{R}_c^{\#}$. Let $g \in B^{\#}({}^*\mathbb{R}_c^{\#})$.
It is natural to define $g(A)$ so that $\langle \psi, g(A)\psi \rangle_{\#} = Ext\text{-}\int_{\sigma(A)} g(\lambda) d^{\#}\mu_{\psi}^{\#}$. The polarization identity lets us recover $\langle \psi, g(A)\varphi \rangle_{\#}$ from the proposed $\langle \psi, g(A)\psi \rangle_{\#}$ and then the Generalized Riesz lemma lets us construct $g(A)$.
Theorem 4.2.1.(spectral theorem-functional calculus form) Let A be a bounded in${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator on $\mathbf{H}^{\#}$. There is a unique map $\widehat{\phi} : B^{\#}({}^*\mathbb{R}_c^{\#}) \to \mathcal{L}(\mathbf{H}^{\#})$ so that
(a) $\widehat{\phi}$ is an algebraic $*$-homomorphism.
(b) $\widehat{\phi}$ is #-norm #-continuous: $\left\| \widehat{\phi}(f) \right\|_{\mathcal{L}(\mathbf{H}^{\#})} \leq \|f\|_{{}^*\infty}$.
(c) Let f be the function $f(x) = x$; then $\widehat{\phi}(f) = A$.
(d) Suppose $f_n(x) \to_{\#} f(x)$ for each x as $n \to {}^*\infty$and hyper infinite sequence $\|f_n\|_{{}^*\infty}, n \in {}^*\mathbb{N}$ is bounded in${}^*\mathbb{R}_c^{\#}$. Then $\widehat{\phi}(f_n) \to_{\#} \widehat{\phi}(f)$ as $n \to {}^*\infty$strongly.
Moreover $\widehat{\phi}(\cdot)$ has the properties:
(e) If $A\psi = \lambda\psi$, then $\widehat{\phi}(f) = f(\lambda)\psi$.
(f) If $f \geq 0$,then $\widehat{\phi}(f) \geq 0$.
(g) If $BA = AB$ then $\widehat{\phi}(f)B = B\widehat{\phi}(f)$.
Remark 4.2.1. Note that: (i) Theorem 4.2.1 can be proven directly by extending Theorem 4.1.1, part (d) requires the dominated #-convergence theorem. Or,

Theorem 4.2.1 can be proven by an easy corollary of Theorem 4.2.3 below.
The proof of Theorem 4.2.3 uses only the #-continuous functional calculus, $\widehat{\phi}$ extends ϕ and as before we write $\widehat{\phi}(f) = f(A)$. As in the #-continuous functional calculus, one has $f(A)g(A) = g(A)f(A)$.
(ii) Since $B^{\#}({}^*\mathbb{R}_c^{\#})$ is the smallest family closed under #-limits of form (d) containing all of $C^{\#}({}^*\mathbb{R}_c^{\#})$, we know that any $\widehat{\phi}(f)$ is in the Smallest non Archimedean C^*-algebra containing A which is also strongly #-closed; such an algebra is called a von Neumann #-algebra or non Archimedean W^*-algebra. When we study von Neumann #-algebras we will see that this follows from (g).
(iii) The #-norm equality of Theorem 4.2.1 carries over if we define $\|f\|'_{{}^*\infty}$ to be the $L^{\#}_{{}^*\infty}$ #-norm with respect to a suitable notion of "#-almost everywhere." Namely, pick an orthonormal basis $\{\psi_n\}_{n=1}^{{}^*\infty}$ and say that a property is true #-a.e. if it is true #-a.e. with respect to each $\mu^{\#}_{\psi_n}$ Then $\left\|\widehat{\phi}(f)\right\|_{\mathcal{L}(\mathbf{H}^{\#})} = \|f\|'_{{}^*\infty}$.

Definition 4.2.2. A vector $\psi \in \mathbf{H}^{\#}$ is called a cyclic vector for A if gyperfinite linear combinations of the elements $\{A^n\psi\}_{n=0}^{{}^*\infty}$ are #-dense in $\mathbf{H}^{\#}$.
Not all operators have cyclic vectors, but if they do.

Lemma 4.2.1. Let A be a bounded in${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator with cyclic vector ψ. Then, there is a unitary operator $U : \mathbf{H}^{\#} \to L_2^{\#}(\sigma(A), d^{\#}\mu_{\psi}^{\#})$, with $(UAU^{-1}f)(\lambda) = \lambda f(\lambda)$ where equality holds is in the sense of elements of $L_2^{\#}(\sigma(A), d^{\#}\mu_{\psi}^{\#})$.

Proof Define U by $U\phi(f) = f$ where f is #-continuous. U is essentially the inverse of the map ϕ of Theorem 4.1.1. To show that U is well defined operator we compute $\|\phi(f)\psi\|_{\#}^2 = \langle\psi, \phi^*(f)\phi(f)\psi\rangle_{\#} = \left\langle\psi, \phi\left(\overline{f} \times f\right)\psi\right\rangle_{\#} = Ext\text{-}\int|f(\lambda)|^2 d^{\#}\mu_{\psi}^{\#}$.
Therefore, if $f = g$ a.e. with respect to $\mu_{\psi}^{\#}$, then $\phi(f)\psi = \phi(g)\psi$. Thus U is well defined on $\{\phi(f)\psi | f \in C^{\#}(\sigma(A))\}$ and is #-norm preserving. Since ψ is cyclic it #-closure #-$\overline{\{\phi(f)\psi | f \in C^{\#}(\sigma(A))\}} = \mathbf{H}^{\#}$ so by the generalized B.L.T. theorem U extends to an #-isometric map of $\mathbf{H}^{\#}$ into $L_2^{\#}(\sigma(A), d^{\#}\mu_{\psi}^{\#})$. Since $C^{\#}(\sigma(A))$ is #-dense in $L_2^{\#}$, $\mathbf{Ran}\ U = L_2^{\#}(\sigma(A), d^{\#}\mu_{\psi}^{\#})$. Finally, if $f \in C^{\#}(\sigma(A))$ one obtains
$(UAU^{-1}f)(\lambda) = [UA\phi(f)](\lambda) = [U\phi(xf)](\lambda) = \lambda f(\lambda)$.
By #-continuity, this extends from $C^{\#}(\sigma(A))$ to $L_2^{\#}$.
To extend this lemma to arbitrary Ay we need to know that A has a family of invariant subspaces spanning $\mathbf{H}^{\#}$ so that A is cyclic on each subspace:

Lemma 4.2.2. Let A be a self-adjoint operator on a $*$-separable Hilbert space $\mathbf{H}^{\#}$.
Then there is a direct sum decomposition $\mathbf{H}^{\#} = Ext\text{-}\bigoplus_{n=1}^{N}\mathbf{H}_n^{\#}$ with $N \in {}^*\mathbb{N}$ or

$\mathbf{H}^{\#} = Ext\text{-}\bigoplus_{n=1}^{{}^*\infty}\mathbf{H}_n^{\#}$ such that:

(a) A leaves each $\mathbf{H}_n^{\#}$ invariant, that is, $\psi \in \mathbf{H}_n^{\#}$ implies $A\psi \in \mathbf{H}_n^{\#} = Ext\text{-}\bigoplus_{n=1}^{{}^*\infty}\mathbf{H}_n^{\#}$

so that:
(b) For each $n \in {}^*\mathbb{N}$, there is a $\phi_n \in \mathbf{H}_n^{\#}$ which is cyclic for $A \restriction \mathbf{H}_n^{\#}$, i.e.
$\mathbf{H}_n^{\#} = \#\text{-}\overline{\{f(A)\phi_n | f \in C^{\#}(\sigma(A))\}}$

Theorem 4.2.3 (spectral theorem-multiplication operator form) Let A be a bounded in${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator on $\mathbf{H}^{\#}$, a $*$-separable Hilbert space. Then, there exist #-measures $\{\mu_n^{\#}\}_{n=1}^{N}$ with $N \in {}^*\mathbb{N}$ or $\{\mu_n^{\#}\}_{n=1}^{{}^*\infty}$ on $\sigma(A)$ and a

unitary operator $U : \mathbf{H}^{\#} \to \bigoplus_{n=1}^{N} L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu_n^{\#})$ or $U : \mathbf{H}^{\#} \to \bigoplus_{n=1}^{{}^*\infty} L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu_n^{\#})$
so that $(UAU^{-1}\psi)_n(\lambda) = \lambda\psi_n(\lambda)$
where we write an element $\psi \in \bigoplus_{n=1}^{N} L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu_n^{\#})$ as an N-tuple $\langle\psi_1(\lambda),\ldots,\psi_N(\lambda)\rangle$
or $*$-tuple
This realization of A is called a spectral representation.
Proof. Use Lemma 4.2.2 to find the decomposition and then use Lemma 4.2.1 on each component.
This theorem tells us that every bounded self-#-adjoint operator is a multiplication operator on a suitable #-measure space; what changes as the operator changes are the underlying #-measures. Explicitly:
Corolarly 4.2.1. Let A be a bounded in${}^*\mathbb{R}_c^{\#}$ self-adjoint operator on a $*$-separable Hilbert space $\mathbf{H}^{\#}$. Then there exists a finite in${}^*\mathbb{R}_c^{\#}$ measure space $\langle M, \mu^{\#}\rangle$, a bounded in${}^*\mathbb{R}_c^{\#}$ function F on M, and a unitary map, $U : \mathbf{H}^{\#} \to L_2^{\#}(M, d^{\#}\mu^{\#})$ so that $(UAU^{-1}f)(m) = F(m)f(m)$.
Proof Choose the cyclic vectors ϕ_n so that $\|\phi_n\|_{\#} = 2^{-n}$. Let $M = \bigcup_{n=1}^{N*} \mathbb{R}_c^{\#}$
i.e. the union of $N \in {}^*\mathbb{N}$ copies of ${}^*\mathbb{R}_c^{\#}$. Define μ by requiring that its restriction to the n-th copy of ${}^*\mathbb{R}_c^{\#}$ be μ_n. Since $\mu(M) = Ext\text{-}\sum_{n=1}^{N} \mu_n^{\#}({}^*\mathbb{R}_c^{\#}) < {}^*\infty$, μ_n is finite in${}^*\mathbb{R}_c^{\#}$. We also notice that this last theorem is essentially a rigorous form of the formaal Dirac notation. If we write $\phi_n = \phi(x;n)$, we see that in the “new representation defined by U” one has

$$\langle\psi,\phi\rangle_{\#} = Ext\text{-}\sum_n Ext\text{-}\int d^{\#}\mu_n^{\#}\overline{\psi(\lambda;n)}\phi(\lambda;n)$$

and

$$\langle\psi,A\phi\rangle_{\#} = Ext\text{-}\sum_n Ext\text{-}\int d^{\#}\mu_n^{\#}\overline{\psi(\lambda;n)}\lambda\phi(\lambda;n).$$

These are the Dirac type formulas familiar to physicists except that the formal sums of Dirac are replaced with integrals over spectral measures, where we define:
Definition 4.2.3. The #-measures $d^{\#}\mu_n$ are called spectral measures; they are just $d^{\#}\mu_{\psi}$ for suitable ψ.
Remark 4.2.2. Notice these #-measures are not uniquely determined.
We now investigate the connection between spectral measures and the spectrum.
Definition 4.2.3. If $\{\mu_n^{\#}\}_{n=1}^{N}, N \in {}^*\mathbb{N}$ is a family of #-measures, the support of $\{\mu_n^{\#}\}_{n=1}^{N}$ is the complement of the largest #-open set B with $\mu_n^{\#}(B) = 0$ for all $n \in {}^*\mathbb{N}$ so

$$\mathbf{supp}\left(\{\mu_n^{\#}\}_{n=1}^{N}\right) = \#\text{-}\overline{\bigcup_{n=1}^{N} \mathbf{supp}(\mu_n^{\#})}. \qquad (4.2.1)$$

Proposition 4.2.1. Let A be a self-#-adjoint operator and $\{\mu_n^{\#}\}_{n=1}^{N}, N \in {}^*\mathbb{N}$ a family of spectral #-measures. Then
$\sigma(A) = \mathbf{supp}\left(\{\mu_n^{\#}\}_{n=1}^{N}\right)$.
Definition 4.2.4. Let F be a ${}^*\mathbb{R}_c^{\#}$-valued function on a #-measure space $\langle M, \mu^{\#}\rangle$.
We say λ is in the **essential range** of F if and only if
$\mu^{\#}\{m|\lambda - \varepsilon < F(m) < \lambda \dotplus \varepsilon\} > 0$.
for all $\varepsilon \approx 0, \varepsilon > 0$.
Proposition 4.2.2. Let F be a bounded in ${}^*\mathbb{R}_c^{\#}$ ${}^*\mathbb{R}_c^{\#}$-valued function on a #-measure space $\langle M, \mu^{\#}\rangle$. Let T_f be the operator on $L_2^{\#}(M, d^{\#}\mu^{\#})$ given by $(T_F g)(m) = F(m)g(m)$

Then $\sigma(T_F)$ is the essential range of F.
Definition 4.2.5. Let A be a bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operator on $H^\#$
Let $H_{\mathbf{pp}}^\# = \{\psi|\mu_\psi^\#$is pure point$\}$, $H_{\mathbf{ac}}^\# = \{\psi|\mu_\psi^\#$ is absolutely #-continuous},
$H_{\mathbf{sing}}^\# = \{\psi|\mu_\psi^\#$ is #-continuous singular$\}$.
We have thus proven.
Theorem 4.2.4.$H^\# = H_{\mathbf{pp}}^\# \oplus H_{\mathbf{ac}}^\# \oplus H_{\mathbf{sing}}^\#$. Each of these subspaces is invariant under A. $A \upharpoonright H_{\mathbf{pp}}^\#$ has a #-complete set of eigenvectors, $A \upharpoonright H_{\mathbf{ac}}^\#$ has only absolutely #-continuous spectral #-measures and $A \upharpoonright H_{\mathbf{sing}}^\#$ has only #-continuous singular spectral #-measures.
Definition 4.2.6. $\sigma_{\mathbf{pp}}(A) = \{\lambda|\lambda$ is an eigenvalue of $A\}$,
$\sigma_{\#\mathbf{cont}}(A) = \sigma(A \upharpoonright H_{\#\mathbf{cont}}^\# = H_{\mathbf{sing}}^\# \oplus H_{\mathbf{ac}}^\#)$,
$\sigma_{\mathbf{ac}}(A) = \sigma(A \upharpoonright H_{\mathbf{ac}}^\#)$,
$\sigma_{\mathbf{sing}}(A) = \sigma(A \upharpoonright H_{\mathbf{sing}}^\#)$.
These sets are called the **pure point**, #-**continuous**, **absolutely** #-**continuous**, and **singular** (or #-**continuous singular**) **spectrum** respectively.
Remark 4.2.2. While it may happen that $\sigma_{\mathbf{ac}}(A) \cup \sigma_{\mathbf{sing}}(A) \cup \sigma_{\mathbf{pp}}(A) \neq \sigma(A)$ this is only true because we did not define $\sigma_{\mathbf{pp}}(A)$ as $\sigma(A \upharpoonright H_{\mathbf{pp}}^\#)$ but rather as the actual set of eigenvalues.
Proposition 4.2.3. $\sigma_{\#\mathbf{cont}}(A) = \sigma_{\mathbf{ac}}(A) \cup \sigma_{\mathbf{sing}}(A)$,
$\sigma(A) = \#\text{-}\overline{\sigma_{\mathbf{pp}}(A)} \cup \sigma_{\#\mathbf{cont}}(A)$.
The sets need not be disjoint, however. The reader should be warned that $\sigma_{\mathbf{sing}}(A)$ may have nonzero #-Lebesgue measure. For many purposes, breaking up the spectrum in this way gives useful information.
Finally, we turn to the question of making canonical choices for the spectral #-measures, a subject which goes under the title of “multiplicity theory.” We will describe the basic results without proof:

§ 4.2.1. Multiplicity free operators

We must first ask when A is unitariiy equivalent to multiplication by x on $L_2^\#({}^*\mathbb{R}_c^\#, d^\#\mu^\#)$, that is, when only one spectral #-measure is needed. An symple examples tells us this happens in the finite-dimensional case only when A has no repeated eigenvalues, so we define:
Definition 4.2.7. A bounded in ${}^*\mathbb{R}_c^\#$self-#-adjoint operator A is called **multiplicity free** if and only if A is unitarily equivalent to multiplication by A on $L_2^\#({}^*\mathbb{R}_c^\#, d^\#\mu^\#)$ for some #-measure $\mu^\#$.
One is interested in intrinsic characterizations of “multiplicity free” and there are several:
Theorem 4.2.5. The following are equivalent:
(a) A is multiplicity free.
(b) A has a cyclic vector.
(c) $\{B|AB = BA\}$ is an abelian algebra.
#-**Measure classes**
Next we must ask about the nonuniqueness of the #-measure in the multiplicity free case. Suppose $d^\#\mu^\#$ on ${}^*\mathbb{R}_c^\#$ is given and let F be a #-measurable function which is positive and nonzero #-a.e. with respect to $\mu^\#$ and locally $L_1^\#({}^*\mathbb{R}_c^\#, d^\#\mu^\#)$, that is, $\int_\Sigma |F|d^\#\mu^\# < {}^*\infty$ for every compact set $\Sigma \subset {}^*\mathbb{R}_c^\#$. Then $d^\#\nu = Fd^\#\mu^\#$ is a #-Borel

#-measure and the map, $U : L_1^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\nu) \to L_1^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu^{\#})$ given by $(Uf)(\lambda) = U(\lambda f)$ is unitary (onto since $F \neq 0$ #-a.e.) and $\lambda(Uf) = U(\lambda f)$, Thus, an operator A with a spectral representation in terms of pi could just as well be represented in terms of ν. By the generalized Radon-Nikodym theorem, $d^{\#}\nu = Fd^{\#}\mu^{\#}$ with F #-a.e. nonzero if and only if $\nu^{\#}$ and $\mu^{\#}$ have the same sets of #-measure zero. This suggests the definition:

Definition 4.2.8. Two #-Borel #-measures $\mu^{\#}$ and $\nu^{\#}$ are called equivalent if and only if they have the same sets of #-measure zero. An equivalence class $\langle\mu^{\#}\rangle$ is called a #-**measure class**.

Then, the nonuniqueness question is answered by:

Proposition 4.2.7. Let $\mu^{\#}$ and $\nu^{\#}$ be #-Borel #-measures on ${}^*\mathbb{R}_c^{\#}$ with bounded in ${}^*\mathbb{R}_c^{\#}$ support. Let $A_{\mu^{\#}}$ be the operator on $L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu^{\#})$ given by $(A_{\mu^{\#}}f)(\lambda) = \lambda f(\lambda)$ and similarly for $A_{\nu^{\#}}$ on $L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\nu^{\#})$. Then $A_{\mu^{\#}}$ and $A_{\nu^{\#}}$ are unitarily equivalent if and only if $\mu^{\#}$ and $\nu^{\#}$ are equivalent #-measures.

§ 4.2.2. Operators of uniform multiplicity

If one wants a canonical listing of the eigenvalues of a matrix, it is natural to list all eigenvalues of multiplicity one, all eigenvalues of multiplicity two, etc. We thus need a way of saying that A is an operator of uniform multiplicity two, three, etc. It is natural to take:

Definition 4.2.9. A bounded self-adjoint operator A is said to be of uniform multiplicity $m \in {}^*\mathbb{N}$ if A is unitarily equivalent to multiplication by λ on $Ext\text{-}\bigoplus L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu^{\#})$ where there are m terms in the external sum and $\mu^{\#}$ is a fixed #-Borel #-measure.

That this is a good definition is shown by

Proposition 4.2.8.If A is unitarily equivalent to multiplication by λ on $Ext\text{-}\bigoplus L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu^{\#})$
(m times) and on $Ext\text{-}\bigoplus L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\nu)$ (n times), then $m - n$ and $\mu^{\#}$ and $\nu^{\#}$ are equivalent #-measures.

§ 4.2.3.Disjoint #-measure classes.The multiplicity theorem

In listing eigenvalues of multiplicity one, two, three, etc. in the finitedimensional case, we must add a requirement that prevents us from counting an eigenvalue of multiplicity three once as an eigenvalue of multiplicity one and once as an eigenvalue of multiplicity two. In the hyperfinite-dimensional case, we avoid this “error” by requiring the lists to be disjoint. The analogous notion for #-measures is:

Definition 4.2.10. Two #-measure classes $\langle\mu^{\#}\rangle$ and $\langle\nu^{\#}\rangle$ are called disjoint if any $\mu_1^{\#} \in \langle\mu^{\#}\rangle$ and $\nu_1^{\#} \in \langle\nu^{\#}\rangle$ are mutually singular.

We can now state the basic theorem:

Theorem 4.2.6.(commutative multiplicity theorem) Let A be abounded in ${}^*\mathbb{R}_c^{\#}$self-#-adjoint operator on a Hilbert space $H^{\#}$. Then there is a decomposition $Ext\text{-}\bigoplus_{m=1}^{{}^*\infty} H_m^{\#}$ so that

(a) A leaves each $H_i^{\#}$ invariant.
(b) $A \upharpoonright H_m^{\#}$ has uniform multiplicity $m \in {}^*\mathbb{N}$.
(c) The #-measure classes $\langle \mu_m^{\#} \rangle$ associated with the spectral representation of $A \upharpoonright H_m^{\#}$ are mutually disjoint.

Remark 4.2.3. Moreover, the subspaces $\{H_m^{\#}\}_{m=1}^{*\infty}$ (some of which may be zero) and the #-measure classes $\{\langle \mu_m^{\#} \rangle\}_{m=1}^{*\infty}$ are uniquely determined by (a)-(c).

The spectral theorem with the multiplicity theory just described is thus one of those gems of mathematics: a structure theorem, that is, a theorem that describes all objects of a certain sort up to a natural equivalence. Each bounded in ${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator A is described by a family of mutually disjoint #-measure classes on $[-\|A\|_{\#}, \|A\|_{\#}]$; two operators are unitarily equivalent if and only if their spectral multiplicity #-measure classes are identical.

§ 4.3. Spectral projections.

In the last section, we constructed a functional calculus, $f \mapsto f(A)$ for any #-Borel function and any bounded in ${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator A. The most important functions gained in passing from the continuous functional calculus to the #-Borel functional calculus are the characteristic functions of sets.

Definition 4.3.1. Let A be a bounded self-#-adjoint operator and Ω a #-Borel set of ${}^*\mathbb{R}_c^{\#}$. $P_\Omega = \chi_\Omega(A)$ is called a spectral projection of A.

As the definition suggests, P_Ω is an orthogonal projection since $\chi_\Omega = \chi_\Omega^2 = 1$ pointwise. The properties of the family of projections $\{P_\Omega | \Omega \text{ an arbitrary #-Borel set}\}$ is given by the following elementary translation of the functional calculus.

Proposition 4.3.1. The family $\{P_\Omega\}$ of spectral projections of a bounded self-#-adjoint operator A, has the following properties:
(a) Each P_Ω is an orthogonal projection.
(b) $P_\varnothing = 0$; $P_{(-a,a)} = I$ for some $a \in {}^*\mathbb{R}_c^{\#}$.
(c) If $\Omega = Ext\text{-}\bigcup_{n=1}^{*\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \varnothing$ for all $n \neq m$ then

$$P_\Omega = s\text{-}\#\text{-}\lim_{N \to {}^*\infty} \left(Ext\text{-}\sum_{n=1}^{N} P_{\Omega_n} \right). \quad (4.3.1)$$

(d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Definition 4.3.2. A family of projections obeying (a)-(c) is called a projection-valued #-measure (p.v.#-m.).

We remark that (d) follows from (a) and (c) by abstract considerations.

As one might guess, one can integrate with respect to a p.v.#-m. If P_Ω is a p.v.#-m., then $\langle \phi, P_\Omega \phi \rangle_{\#}$ is an ordinary #-measure for any ϕ. We will use the symbol $d^{\#}\langle \phi, P_\lambda \phi \rangle_{\#}$ to mean integration with respect to this #-measure. By generalized Riesz lemma methods, there is a unique operator B with $\langle \phi, B\phi \rangle_{\#} = Ext\text{-}\int f(\lambda) d^{\#}\langle \phi, P_\lambda \phi \rangle_{\#}$.

Theorem 4.3.1. If P_Ω is a p.v.#-m. and f a bounded in ${}^*\mathbb{R}_c^{\#}$ #-Borel function on $\mathbf{supp}(P_\Omega)$, then there is a unique operator B which we denote $Ext\text{-}\int f(\lambda) d^{\#} P_\lambda$ so that

$$\langle \phi, B\phi \rangle_{\#} = Ext\text{-}\int f(\lambda) d^{\#}\langle \phi, P_\lambda \phi \rangle_{\#}. \quad (4.3.2)$$

Theorem 4.3.2.(spectral theorem-p.v.#-m. form) There is a one-one correspondence between (bounded) self-#-adjoint operators A and (bounded) projection valued #-measures $\{P_\Omega\}$ given by

$$A \mapsto \{P_\Omega\} = \{\chi_\Omega(A)\} \tag{4.3.3}$$

and

$$\{P_\Omega\} \mapsto A = Ext\text{-}\int \lambda d^\# P_\lambda. \tag{4.3.4}$$

Spectral projections can be used to investigate the spectrum of A.

Proposition 4.3.1. $\lambda \in \sigma(A)$ if and only if $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ for any $\varepsilon > 0$.

The essential element of the proof is that $\|(A-\lambda)^{-1}\|_\# = [dist(\lambda, \sigma(A))]^{-1}$.

This suggests that we distinguish between two types of spectrum.

Definition 4.3.3. We say that (i) $\lambda \in \sigma_{\mathbf{ess}}(A)$, the essential spectrum of A if and only if $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ is hyper infinite dimensional for all $\varepsilon > 0$.

(ii) If $\lambda \in \sigma(A)$ but $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ is hyperfinite dimensional for some $\varepsilon > 0$, we say $\lambda \in \sigma_{\mathbf{disc}}(A)$, the discrete spectrum of $A.P$ is hyper infinite dimensional means $\mathbf{Ran}(P)$ is hyper infinite dimensional.

Thus, we have a second decomposition of $\sigma(A)$. Unlike the first, it is a decomposition into two necessarily disjoint subsets. We note that $\sigma_{\mathbf{disc}}$ is not necessarily #-closed, but notice that.

Theorem 4.3.3 $\sigma_{\mathbf{ess}}(A)$ is always #-closed.

Proof Let $\lambda_n \to_\# \lambda$ with each $\lambda_n \in \sigma_{\mathbf{ess}}(A)$. Since any #-open interval I about λ contains an interval about some $\lambda_n, P_I(A)$ is hyper infinite dimensional.

The following three theorems give alternative descriptions of $\sigma_{\mathbf{disc}}$ and $\sigma_{\mathbf{ess}}$;

Theorem 4.3.4 $\lambda \in \sigma_{\mathbf{disc}}$ if and only if both the following hold:

(a) λ is an #-isolated point of $\sigma(A)$ that is, for some $\varepsilon \approx 0$, $(\lambda-\varepsilon, \lambda+\varepsilon) \cap \sigma(A) = \{\lambda\}$.

(b) λ is an eigenvalue of hyperfinite multiplicity, i.e., $\{\psi | A\psi = \lambda\psi\}$ is hyperfinite dimensional.

Theorem 4.3.5 $\lambda \in \sigma_{\mathbf{ess}}$ if and only if one or more of the following holds:

(a) $\lambda \in \sigma_{\#\mathbf{cont}}(A) \leftrightarrow \sigma_{\mathbf{ac}}(A) \cup \sigma_{\mathbf{sing}}(A)$.

(b) λ is a #-limit point of $\sigma_{\mathbf{pp}}(A)$.

(c) λ is an eigenvalue of hyper infinite multiplicity.

Theorem 4.3.6 (Generalized Weyl's criterion) Let A be a bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operator. Then (i) $\lambda \in \sigma(A)$ if and only if there exists $\{\psi_n\}_{n=1}^{^*\infty}$ with $\|\psi_n\|_\# = 1$ and $\#\text{-}\lim_{n\to^*\infty} \|(A-\lambda)\psi_n\|_\# = 0$.

(ii)$\lambda \in \sigma_{\mathbf{ess}}(A)$ if and only if the above $\{\psi_n\}$ can be chosen to be orthogonal.

As one might guess, the essential spectrum cannot be removed by essentially hyperfinite dimensional perturbations. In Section 4.4, we will prove a general theorem which implies that $\sigma_{\mathbf{ess}}(A) = \sigma_{\mathbf{ess}}(B)$ if $A \backslash B$ is #-compact.

Finally, we discuss one useful formula relating the resolvent and spectral projections. It is a matter of computation to see that

$$f_\varepsilon(x) \to_\# \begin{cases} 0 & \text{if} \quad x \notin [a,b] \\ 1/2 & \text{if} \quad x = a \vee x = b \\ 1 & \text{if} \quad x \in (a,b) \end{cases}$$

if $\varepsilon \to_\# 0$, where

$$f_\varepsilon(x) = (2\pi_\# i)^{-1}\left(Ext\text{-}\int_a^b\left[(x-\lambda-i\varepsilon)^{-1}-(x-\lambda+i\varepsilon)^{-1}\right]d^\#\lambda\right). \quad (4.3.5)$$

Moreover, $|f_\varepsilon(x)|$ is bounded in $x \in^* \mathbb{R}_c^\#$ uniformly in $\varepsilon \approx 0$, so by the functional calculus, one obtains that.

Theorem 4.3.7 (Generalized Stone's formula) Let A be a bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operator. Then

$$\begin{gathered}\mathbf{s}\text{-}\lim_{\varepsilon\to_\# 0}(2\pi_\# i)^{-1}\left(Ext\text{-}\int_a^b\left[(A-\lambda-i\varepsilon)^{-1}-(A-\lambda+i\varepsilon)^{-1}\right]d^\#\lambda\right) = \\ = \frac{1}{2}\left[P_{[a,b]}+P_{(a,b)}\right].\end{gathered} \quad (4.3.6)$$

§ 4.4.The #-continuous functional calculus related to unbounded in${}^*\mathbb{R}_c^\#$ self-#-adjoint operators

In this section we will show how the spectral theorem for bounded in${}^*\mathbb{R}_c^\#$ self-#-adjoint operators which we developed in § 4.3 can be extended to unbounded in${}^*\mathbb{R}_c^\#$ self-#-adjoint operators. To indicate what we are aiming for, we first prove the following:

Proposition 4.4.1. Let $\langle M,\mu^\#\rangle$ be a #-measure space with $\mu^\#$ a hyperfinite #-measure. Suppose that *f* is a #-measurable, ${}^*\mathbb{R}_c^\#$-valued function on M which is finite or hyperfinite a.e.$\mu^\#$. Then the operator $T_f : \varphi \to f\varphi$ on $L_2^\#(M,d^\#\mu^\#)$ with domain

$$D(T_f) = \{\varphi | f\varphi \in L_2^\#(M,d^\#\mu^\#)\} \quad (4.4.1)$$

is self-#-adjoint and $\sigma(T_f)$ is the essential range of T_f.

Proof T_f is clearly symmetric. Suppose that $\psi \in D(T_f^*)$ and let

$$\chi_N = \begin{cases} 1 & \text{if} \quad |f(m)| \leq N \\ 0 & \text{otherwise}\end{cases}$$

Then, using the generalized monotone #-convergence theorem,

$$\|T_f^*\psi\|_\# = \#\text{-}\lim_{N\to^*\infty}\|\chi_N T_f^*\psi\|_\# = \#\text{-}\lim_{N\to^*\infty}\left(\sup_{\|\varphi\|_\#=1}\left|\left\langle\varphi,\chi_N T_f^*\psi\right\rangle_\#\right|\right) =$$

$$\#\text{-}\lim_{N\to^*\infty}\left(\sup_{\|\varphi\|_\#=1}|\langle\chi_N T_f\varphi,\psi\rangle_\#|\right) = \#\text{-}\lim_{N\to^*\infty}\left(\sup_{\|\varphi\|_\#=1}|\langle\varphi,\chi_N f\psi\rangle_\#|\right) =$$

$$\#\text{-}\lim_{N\to^*\infty}\|\chi_N f\psi\|_\#$$

Thus, $f\psi \in L_2^\#(M,d^\#\mu^\#)$, so $\psi \in D(T_f)$ and therefore T_f is self-#-adjoint. That $\sigma(T_f)$ is the essential range of *f* follows as in the bounded case.

With more information about *f*, one can say something about the domains on which T_f is essentially self-#-adjoint:

Proposition 4.4.2. Let *f* and T_f obey the conditions in Proposition 4.4.1. Suppose in addition that $f \in L_p^\#(M,d^\#\mu^\#)$ for $2 < p < {}^*\infty.$ Let D be any #-dense set in $L_q^\#(M,d^\#\mu^\#)$ where $q^{-1}+p^{-1} = 1/2.$ Then D is a #-core for T_f.

Proof Let us first show that $L_q^\#$ is a #-core for $T_f.$ By the generalized Holder's inequality $\|g\|_{\#2} \leq \|1\|_{\#p}\cdot\|g\|_{\#q}$, and $\|fg\|_{\#2} \leq \|f\|_{\#p}\cdot\|g\|_{\#q}$ so $L_p^\# \subset D(T_f)$.

Moreover, if $g \in D(T_f)$ let $g_n, n \in {}^*\mathbb{N}$ be that function which is zero where

$|g(m)| > n$ and equal to g otherwise. By the generalized dominated convergence theorem, $g_n \to_\# g$ and $fg_n \to_\# fg$ in $L_2^\#$. Since each g_n is in $L_q^\#$, we conclude that $L_q^\#$ is a #-core for T_f. Now let D be #-dense in $L_q^\#$ and let $g \in L_q^\#$. Find $g_n \in D$ with $g_n \to_\# g$ in $L_q^\#$. Since $\|g_n - g\|_{\#2} \leq \|1\|_{\#p} \cdot \|g_n - g\|_{\#q}$ and
$\|T_f(g_n - g)\|_{\#2} \leq \|f\|_{\#p} \cdot \|g_n - g\|_{\#q}$, $g \in$ #- $\overline{D(T_f \restriction D)}$.
Thus $L_q^\# \subset D(T_f \restriction D)$ so D is a #-core. Unless f$\in L_{^*\infty}^\#(M, d^\#\mu^\#)$ the operator T_f described in Propositions 4.4.1 and 4.4.2 will be unbounded.
Thus, we have found a large class of unbounded self-#-adjoint operators. In fact, we have found them all.
Theorem 4.4.1. (spectral theorem-multiplication operator form) Let A be a self-adjoint operator on a $^*\infty$-dimensional a non-Archimedean Hilbert space $\mathbf{H}^\#$ with domain $D(A)$. Then there is a #-measure space $\langle M, \mu^\# \rangle$ with $\mu^\#$ a hyperfinite #-measure, a unitary operator $U : \mathbf{H}^\# \to L_2^\#(M, d^\#\mu^\#)$, and a $^*\mathbb{R}_c^\#$-valued function f on M which is finite or hyperfinite $\mu^\#$-a.e. so that
(a) $\psi \in D(A)$ if and only if $f(\cdot)(U\psi)(\cdot) \in L_2^\#(M, d^\#\mu^\#)$.
(b) If $\varphi \in U[D(A)]$, then $(UAU^{-1}\varphi)(m) = f(m)\varphi(m)$.
Proof It easily verify that $A + i$ and $A - i$ are one to one correspondence and $\mathbf{Ran}(A \pm i) = \mathbf{H}^\#$. Since $A \pm i$ are #-closed, $(A \pm i)^{-1}$ are #-closed and therefore bounded in $^*\mathbb{R}_c^\#$. Note that the operators $(A + i)^{-1}$ and $(A - i)^{-1}$ commute. The equality $\langle (A - i)\psi, (A + i)^{-1}(A + i)\varphi \rangle_\# = \langle (A + i)^{-1}(A - i)\psi, (A + i)\varphi \rangle_\#$ and the fact that $\mathbf{Ran}(A \pm i) = \mathbf{H}^\#$ shows that $((A + i)^{-1})^* = (A - i)^{-1}$. Thus the operator $(A + i)^{-1}$ is normal.
We now use the easy extension of the spectral theorem for bounded in $^*\mathbb{R}_c^\#$ self-#-adjoint operators to bounded in $^*\mathbb{R}_c^\#$ normal operators. The proof of this extension is a straightforward. We conclude that there is a #-measure space $\langle M, \mu^\# \rangle$ with $\mu^\#$ a hyperfinite #-measure, a unitary operator $U : \mathbf{H}^\# \to L_2^\#(M, d^\#\mu^\#)$, and a #-measurable, bounded, in $^*\mathbb{R}_c^\#$ $^*\mathbb{C}_c^\#$-valued function $g(m)$ so that $U(A + i)^{-1}U^{-1}\varphi(m) = g(m)\varphi(m)$ for all $\varphi \in L_2^\#(M, d^\#\mu^\#)$. Since $\mathbf{Ker}((A + i)^{-1})$ is empty, $g(m) \neq 0$ a.e.$\mu^\#$, so the function $f(m) = g^{-1}(m) - i$ is hyperfinite a.e.$\mu^\#$. Now, suppose $\psi \in D(A)$. Then $\psi = (A + i)^{-1}\varphi$ for some $\varphi \in \mathbf{H}^\#$ and $U\psi = gU\varphi$. Since fg is bounded in $^*\mathbb{R}_c^\#$, we conclude that $f(U\psi) \in L_2^\#(M, d^\#\mu^\#)$. Conversely, if $f(U\psi) \in L_2^\#(M, d^\#\mu^\#)$, then there is a $\varphi \in \mathbf{H}^\#$ so that $U\varphi = (f + i)U\psi$. Thus, $gU\varphi = g(f + i)U\psi = U\psi$, so $\psi = (A + i)^{-1}\varphi$ which shows that $\psi \in D(A)$. This proves (a).
To prove (b) notice that if $\psi \in D(A)$ then $\psi = (A + i)^{-1}\varphi$ for some $\varphi \in \mathbf{H}^\#$ and $A\psi = \varphi - i\psi$. Therefore, $(UA\psi)(m) = (U\varphi)(m) - i(U\psi)(m) = (g^{-1}(m) - i)(U\psi)(m) = f(m)(U\psi)(m)$. Finally, if $\mathrm{Im}(f) > 0$ on a set of nonzero Lebesgue #-measure, there is a bounded in $^*\mathbb{R}_c^\#$ set B in the upper half plane so that $S = \{x | f(x) \in B\}$ has nonzero Lebesgue #-measure. If $\chi(x)$ is the characteristic function of S then $f\chi \in L_2^\#(M, d^\#\mu^\#)$ and $\mathrm{Im}\langle \chi, f\chi \rangle > 0$. This contradicts the fact that multiplication by /is self-adjoint (since it is unitarily equivalent to A). Thus f is $^*\mathbb{R}_c^\#$-valued function.
There is a natural way to define functions of a self-#-adjoint operator by using the above theorem. Given a bounded in $^*\mathbb{R}_c^\#$ #-Borel function h on $^*\mathbb{R}_c^\#$ we define
$$h(A) = UT_{h(f)}U^{-1} \tag{4.4.2}$$
where $T_{h(f)}$ is the operator on $L_2^\#(M, d^\#\mu^\#)$ which acts by multiplication by the function

$h(f(m))$. Using this definition the following theorem follows easily from Theorem 4.4.1.
Theorem 4.4.2. (spectral theorem-functional calculus form) Let A be a self-#-adjoint operator on $\mathbf{H}^{\#}$. Then there is a unique map $\widehat{\phi}$ from the bounded #-Borel functions on ${}^{*}\mathbb{R}_{c}^{\#}$ into $\mathcal{L}(\mathbf{H}^{\#})$ so that
(a) $\widehat{\phi}$ is an algebraic $*$-homomorphism.
(b) $\widehat{\phi}$ is #-norm #-continuous, that is, $\left\|\widehat{\phi}(h)\right\|_{\mathcal{L}(\mathbf{H}^{\#})} \leq \|h\|_{{}^{*}\infty}$
(c) Let $h_n(x), n \in {}^{*}\mathbb{N}$ be a hyper infinite sequence of bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ #-Borel functions with $\#\text{-}\lim_{n\to{}^{*}\infty} h_n(x) = x$
for each x and $|h_n(x)| \leq |x|$ for all x and $n \in {}^{*}\mathbb{N}$. Then, for any $\psi \in D(A)$, $\#\text{-}\lim_{n\to{}^{*}\infty} \widehat{\phi}(h_n)\psi = A\psi$.
(d) If $h_n(x) \to_{\#} h(x)$ pointwise and if the hyper infinite sequence $\|h_n\|_{{}^{*}\infty}, n \in {}^{*}\mathbb{N}$ is bounded in ${}^{*}\mathbb{R}_{c}^{\#}$, then $\widehat{\phi}(h_n) \to_{\#} \widehat{\phi}(h)$ strongly.
In addition:
(e) If $A\psi = \lambda\psi$ then $\widehat{\phi}(h) = h(\lambda)\psi$.
(f) If $h \geq 0$, then $\widehat{\phi}(h) \geq 0$.
The functional calculus is very useful. For example, it allows us to define the exponential $Ext\text{-}\exp(itA)$ and prove easily many of its properties as a function of t (see the next section). In the case where A is bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ we do not need the functional calculus to define the exponential since we can define $Ext\text{-}\exp(itA)$ by the power series which #-converges in #-norm.
The functional calculus is also used to construct spectral #-measures and can be used to develop a multiplicity theory similar to that for bounded self-#-adjoint operators.
A vector $\psi \in \mathbf{H}^{\#}$ is said to be cyclic for A if $\{g(A)\psi | g \in C^{{}^{*}\infty}({}^{*}\mathbb{R}_{c}^{\#})\}$ is #-dense in $\mathbf{H}^{\#}$. If ψ is a cyclic vector, then it is possible to represent $\mathbf{H}^{\#}$ as $L_2^{\#}({}^{*}\mathbb{R}_{c}^{\#}, d^{\#}\mu_{\psi}^{\#})$ where $\mu_{\psi}^{\#}$ is the measure satisfying $Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}} g(x)d^{\#}\mu_{\psi}^{\#}(x) = \langle\psi, g(A)\psi\rangle_{\#}$in such a way that A becomes multiplication by x. In general, $\mathbf{H}^{\#}$ decomposes into a direct sum of cyclic subspaces so the #-measure space, M in Theorem 4.4.1 can be realized as a union of copies of ${}^{*}\mathbb{R}_{c}^{\#}$. As in the case of bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ operators we can define $\sigma_{\mathbf{ac}}(A), \sigma_{\mathbf{pp}}(A), \sigma_{\mathbf{sing}}(A)$ and decompose $\mathbf{H}^{\#}$ accordingly.
Finally, the spectral theorem in its projection-valued #-measure form follows easily from the functional calculus. Let P_{Ω} be the operator $\chi_{\Omega}(A)$ where χ_{Ω} is the characteristic function of the measurable set $\Omega \subset {}^{*}\mathbb{R}_{c}^{\#}$. The family of operators $\{P_{\Omega}\}$ has the following properties:
(a) Each P_{Ω} is an orthogonal projection.
(b) $P_{\varnothing} = 0$; $P_{(-{}^{*}\infty,{}^{*}\infty)} = I$.
(c) If $\Omega = Ext\text{-}\bigcup_{n=1}^{{}^{*}\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \varnothing$ for all $n \neq m$ then
$$P_{\Omega} = s\text{-}\#\text{-}\lim_{N\to{}^{*}\infty}\left(Ext\text{-}\sum_{n=1}^{N} P_{\Omega_n}\right). \tag{4.4.3}$$
(d) $P_{\Omega_1}P_{\Omega_2} = P_{\Omega_1\cap\Omega_2}$.
Definition 4.4.1. Such a family is called a projection-valued #-measure (p.v.#-m.).
Remark 4.4.1. This is a generalization of the notion of bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ projection-valued #-measure introduced in § 4.3.In that we only require $P_{(-{}^{*}\infty,{}^{*}\infty)} = I$ rather than $P_{(-a,a)} = I$ for some $a \in {}^{*}\mathbb{R}_{c}^{\#}$. For $\varphi \in \mathbf{H}^{\#}, \langle\varphi, P_{\Omega}\varphi\rangle_{\#}$ is a well-defined Borel

#-measure on ${}^*\mathbb{R}_c^\#$ which we denote by $d^\#\langle\varphi,P_\lambda\varphi\rangle_\#$ as in § 4.3.
The complex ${}^*\mathbb{C}_c^\#$-valued #-measure $d^\#\langle\varphi,P_\lambda\psi\rangle_\#$ is defined by polarization. Thus, given a bounded in ${}^*\mathbb{R}_c^\#$ #-Borel function g we can define $g(A)$ by

$$\langle\varphi,g(A)\varphi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi,P_\lambda\varphi\rangle_\# \qquad (4.4.4)$$

It is not difficult to show that this map $g \mapsto g(A)$ has the properties (a)-(d) of Theorem 4.4.1, so $g(A)$ as defined by (4.4.4) coincides with the definition of $g(A)$ given by Theorem 4.4.1. Now, suppose g is an unbounded ${}^*\mathbb{C}_c^\#$-valued #-Borel function and let

$$D_g = \left\{\varphi|Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi,P_\lambda\varphi\rangle_\# < {}^*\infty\right\}. \qquad (4.4.5)$$

Then, D_g is #-dense in $H^\#$ and an operator $g(A)$ is defined on D_g by

$$\langle\varphi,g(A)\varphi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi,P_\lambda\varphi\rangle_\#. \qquad (4.4.6)$$

As in § 4.3, we write symbolically

$$g(A) = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#P_\lambda. \qquad (4.4.7)$$

In particular, for $\varphi,\psi \in D(A)$,

$$\langle\varphi,A)\psi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi,P_\lambda\varphi\rangle_\#. \qquad (4.4.8)$$

if g is ${}^*\mathbb{R}_c^\#$-valued, then $g(A)$ is self-#-adjoint on D_g. We summarize:
Theorem 4.4.3. (spectral theorem-projection valued measure form) There is a one-to-one correspondence between self-#-adjoint operators A and projection-valued #-measures $\{P_\Omega\}$ on $\mathbf{H}^\#$ the correspondence being given by

$$A = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} \lambda d^\#P_\lambda. \qquad (4.4.9)$$

We use the functional calculus developed above to define $Ext\text{-}\exp(itA)$.
Theorem 4.4.4. Let A be a self-#-adjoint operator and define $U(t) = Ext\text{-}\exp(itA)$. Then
(a) For each $t \in {}^*\mathbb{R}_c^\#, U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s,t \in {}^*\mathbb{R}_c^\#$.
(b) If $\varphi \in \mathbf{H}^\#$ and $t \to_\# t_0$, then $U(t)\varphi \to_\# U(t_0)\varphi$.
(c) For any $\psi \in D(A)$: $\dfrac{U(t)\psi-\psi}{t} \to_\# iA\psi$ as $t \to_\# 0$.
(d) If $\#\text{-}\lim_{t\to_\# 0}\dfrac{U(t)\psi-\psi}{t}$ exists, then $\psi \in D(A)$.
Proof (a) follows immediately from the functional calculus and the corresponding statements for the complex-valued function $Ext\text{-}\exp(it\lambda)$. To prove (b) observe that

$$\|Ext\text{-}\exp(itA)\varphi-\varphi\|_\#^2 = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}|Ext\text{-}\exp(it\lambda)-1|^2d^\#\langle P_\lambda\varphi,\varphi\rangle_\#. \qquad (4.4.10)$$

Since $|Ext\text{-}\exp(it\lambda)-1|^2$ is dominated by the #-integrable function $g(\lambda) = 2$ and since for each $\lambda \in {}^*\mathbb{R}_c^\#$: $|Ext\text{-}\exp(it\lambda)-1|^2 \to_\# 0$ as $t \to_\# 0$ we conclude that $\|U(t)\varphi-\varphi\|_\#^2 \to_\# 0$ as $t \to_\# 0$, by the generalized Lebesgue dominated-#-convergence theorem. Thus $t \mapsto U(t)$ is strongly #-continuous at $t = 0$, which by the group property proves $t \mapsto U(t)$ is strongly #-continuous everywhere. The proof of (c), which again uses the dominated #-convergence theorem and the estimate $|Ext\text{-}\exp(ix)-1|^2 \leq |x|$. To prove (d), we define

$$D(B) = \left\{\psi \middle| \#\text{-}\lim_{t \to_\# 0} \frac{U(t)\psi - \psi}{t} \text{ exists}\right\} \quad (4.4.11)$$

and let

$$iB\psi = \#\text{-}\lim_{t \to_\# 0} \frac{U(t)\psi - \psi}{t}. \quad (4.4.12)$$

A simple computation shows that B is symmetric.By (c), $B \supset A$, so $B = A$.

Definition 4.4.2. An operator-valued function $U(t)$ satisfying (a) and (b) is called a strongly #-continuous one-parameter unitary group.

Definition 4.4.3. If $U(t)$ is a strongly #-continuous one-parameter unitary group, then the self-#-adjoint operator A with $U(t) = Ext\text{-}\exp(itA)$ is called the infinitesimal generator of $U(t)$.

Suppose that $U(t)$ is a weakly #-continuous one-parameter unitary group. Then $\|U(t)\varphi - \varphi\|_\#^2 = \|U(t)\varphi\|_\#^2 - \langle U(t)\varphi, \varphi\rangle_\# - \langle \varphi, U(t)\varphi\rangle_\# + \|\varphi\|_\#^2 \to_\# 0$ as $t \to_\# 0$. Thus $U(t)$ is actually strongly #-continuous. As a matter of fact, to conclude that $U(t)$ is strongly #-continuous one need only show that $U(t)$ is weakly #-measurable,that is, that $\langle U(t)\varphi, \psi\rangle_\#$ is #-measurable for each φ and ψ. This startling result sometimes useful since in applications one can often show that $\langle U(t)\varphi, \psi\rangle_\#$ is the #-limit of a hyper infinite sequence of #-continuous functions;$\langle U(t)\varphi, \psi\rangle_\#$ is therefore #-measurable and by generalized von Neumann's theorem $U(t)$ is then strongly #-continuous.

Theorem 4.4.5. Let $U(t)$ be a one-parameter group of unitary operators on a hy infinite dimensional Hilbert space $\mathbf{H}^\#$. Suppose that for all $\varphi, \psi \in \mathbf{H}^\#, \langle U(t)\psi, \varphi\rangle_\#$ is #-measurable. Then $U(t)$ is strongly #-continuous.

Proof. Let $\psi \in \mathbf{H}^\#$.Then for all $\varphi \in \mathbf{H}^\#$, $\langle U(t)\psi, \varphi\rangle_\#$ is a bounded in ${}^*\mathbb{R}_c^\#$ #-measurable function and $\varphi \mapsto \int_0^a \langle U(t)\psi, \varphi\rangle_\# d^\# t$ is a linear functional on $\mathbf{H}^\#$ of #-norm less than or equal to $a\|\varphi\|_\#$. Thus, by the generalized Riesz lemma there is a $\psi_a \in \mathbf{H}^\#$ so that

$$\langle \psi_a, \varphi\rangle_\# = \int_0^a \langle U(t)\psi, \varphi\rangle_\# d^\# t. \quad (4.4.13)$$

Note that

$$\begin{gathered}\langle U(b)\psi_a, \varphi\rangle_\# = \langle \psi_a, U(-b)\varphi\rangle_\# = \int_0^a \langle U(t)\psi, U(-b)\varphi\rangle_\# d^\# t = \\ \int_0^a \langle U(t+b)\psi, \varphi\rangle_\# d^\# t = \int_b^{a+b} \langle U(t)\psi, \varphi\rangle_\# d^\# t.\end{gathered} \quad (4.4.14)$$

From (4.1.14) we obtain

$$\begin{gathered}|\langle U(b)\psi_a, \varphi\rangle_\# - \langle \psi_a, \varphi\rangle_\#| = \\ = \left|\int_0^b \langle U(t)\psi, \varphi\rangle_\# d^\# t\right| + \left|\int_b^{a+b} \langle U(t)\psi, \varphi\rangle_\# d^\# t\right| \leq 2a\|\varphi\|_\#\|\psi\|_\#\end{gathered} \quad (4.4.15)$$

and therefore #-$\lim_{b\to_\# 0}\langle U(b)\psi_a,\varphi\rangle_\# = \langle\psi_a,\varphi\rangle_\#$ so that $U(b)$ is weakly and therefore strongly #-continuous on the set of vectors of the form $\{\psi_a|\psi\in\mathbf{H}^\#\}$. It remains only to show that this set is #-dense, since by by an $\varepsilon\approx 0,\varepsilon/3$ argument we can then conclude that $t\mapsto U(t)$ is strongly #-continuous on $\mathbf{H}^\#$. Suppose that $\varphi\in\{\psi_a|\psi\in\mathbf{H}^\#, a\in{}^*\mathbb{R}_c^\#\}^\intercal$ and let $\{\psi^{(n)}\}_{n\in{}^*\mathbb{N}}$ be an orthonormal basis for $\mathbf{H}^\#$. Then for each $n\in{}^*\mathbb{N}$

$$Ext\text{-}\int_0^a\langle U(t)\psi^{(n)},\varphi\rangle_\# d^\# t = \left\langle\psi_a^{(n)},\varphi\right\rangle_\# = 0 \tag{4.4.16}$$

for all $a\in{}^*\mathbb{R}_c^\#$ which implies that $\langle U(t)\psi^{(n)},\varphi\rangle_\# = 0$ except for $t\in S_n$, a set of Lebesgue #-measure zero. Choose $t_0\notin\cup_{n\in{}^*\mathbb{N}}S_n$. Then $\langle U(t_0)\psi^{(n)},\varphi\rangle_\# = 0$ for all $n\in{}^*\mathbb{N}$ which implies that $\varphi = 0$, since $U(t_0)$ is unitary.

Theorem 4.4.6.Suppose that $U(t)$ is a strongly continuous one-parameter unitary group. Let D be a #-dense domain which is invariant under $U(t)$ and on which $U(t)$ is strongly #-differentiable. Then i^{-1} times the strong #-derivative of $U(t)$ is essentially self-#-adjoint on D and its #-closure is the #-infinitesimal generator of $U(t)$.

This theorem has a reformulation which is sufficiently important that we state it as a theorem.

Theorem 4.4.7. Let A be a self-adjoint operator on $\mathbf{H}^\#$ and D be a #-dense linear set contained in $D(A)$. If for all t, $Ext\text{-}\exp(itA) : D\to D$ then D is a #-core for A.

Theorem 4.4.8.Let $U(t)$ be a strongly #-continuous one-parameter unitary group on a Hilbert space $\mathbf{H}^\#$. Then, there is a self-#-adjoint operator A on $\mathbf{H}^\#$ so that $U(t) = Ext\text{-}\exp(itA)$.

Proof Part (d) of Theorem 4.4.4 suggests that we obtain A by differentiating $U(t)$ at $t = 0$. We will show that this can be done on a #-dense set of especially nice vectors and then show that the #-limiting operator is essentially self-#-adjoint by using the basic criterion. Finally, we show that the exponential of this #-limiting operator is just $U(t)$. Let $f\in C_0^{*\infty}({}^*\mathbb{R}_c^\#)$ and for each $\varphi\in\mathbf{H}^\#$ define

$$\varphi_f = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} f(t)U(t)\varphi d^\# t. \tag{4.4.17}$$

Since $U(t)$ is strongly #-continuous the integral in (4.4.7) can be taken to be a Riemann integral. Let D be the set of hyperfinite linear combinations of all such φ_f with $\varphi\in\mathbf{H}^\#$ and $f\in C_0^{*\infty}({}^*\mathbb{R}_c^\#)$. If $j_\varepsilon(t)$ is the approximate identity then

$$\begin{gathered}\|\varphi_{j_\varepsilon}-\varphi\|_\# = \left\|Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} j_\varepsilon(t)[U(t)\varphi-\varphi]d^\# t\right\|_\# \leq \\ \leq\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} j_\varepsilon(t)d^\# t\right)\sup_{t\in[-\varepsilon,\varepsilon]}\|U(t)\varphi-\varphi\|_\#.\end{gathered} \tag{4.4.18}$$

Since $U(t)$ is strongly #-continuous, D is #-dense in $\mathbf{H}^\#$. We have used the inequality

$$\left\|Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} h(t)d^\# t\right\|_\# \leq Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}\|h(t)\|_\# d^\# t \tag{4.4.19}$$

for non-Archimedean Banach space-valued #-continuous functions on the real line ${}^*\mathbb{R}_c^\#$ (which can be proven using the approximate partial sums as in the ${}^*\mathbb{R}_c^\#$-valued case). For $\varphi_f \in D$ we obtain that

$$\left(\frac{U(s)-I}{s}\right)\varphi_f = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} f(t)\left(\frac{U(s+t)-U(t)}{s}\right)\varphi d^\# t =$$
$$Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} \frac{f(\tau-s)-f(\tau)}{s} U(\tau)\varphi d^\#\tau \to_\# -Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} f^{\#'}(\tau)U(\tau)\varphi d^\#\tau = \varphi_{-f^{\#'}} \quad (4.4.20)$$

since $[f(t-s)-f(t)]/s$ #-converges to $-f^{\#'}(t)$ uniformly. For $\varphi_f \in D$ we define $A\varphi_f = i^{-1}\varphi_{-f^{\#'}}$. Note that $U(t) : D \to D, A : D \to D$ and $U(t)A\varphi_f = AU(t)\varphi_f$ for $\varphi_f \in D$. Futhermore if $\varphi_f, \varphi_g \in D$ we obtain that

$$\langle A\varphi_f, \varphi_g\rangle_\# = \#\text{-}\lim_{s\to_\# 0}\left\langle\left(\frac{U(s)-I}{is}\right)\varphi_f, \varphi_g\right\rangle_\# =$$
$$= \#\text{-}\lim_{s\to_\# 0}\left\langle\varphi_f, \left(\frac{I-U(-s)}{is}\right)\varphi_g\right\rangle_\# = \frac{1}{i}\left\langle\varphi_f, \varphi_{-g^{\#'}}\right\rangle_\# = \langle\varphi_f, A\varphi_g\rangle_\# \quad (4.4.21)$$

so A is symmetric. Now we show that A is essentially self-#-adjoint. Suppose that there is a $u \in D(A^*)$ so that $A^*u = iu.$ Then for each $\varphi \in D(A) = D$

$$\frac{d^\#}{d^\# t}\langle U(t)\varphi, u\rangle_\# = \langle iAU(t)\varphi, u\rangle_\# = -i\langle U(t)\varphi, A^*u\rangle_\# = -i\langle U(t)\varphi, iu\rangle_\# = \langle U(t)\varphi, u\rangle_\# \quad (4.4.22)$$

Thus, the ${}^*\mathbb{C}_c^\#$-valued function $f(t) = \langle U(t)\varphi, u\rangle_\#$ satisfies the ordinary differential equation $f^{\#'} = f$ so $f(t) = f(0)[Ext\text{-}\exp(t)]$. Since $U(t)$ has #-norm one, $|f(t)|$ is bounded, in ${}^*\mathbb{R}_c^\#$ which implies that $f(0) = \langle\varphi, u\rangle_\# = 0.$ Since D is #-dense, $u = 0.$ A similar proof shows that $A^*u = -iu$ can have no nonzero solutions. Therefore A is essentially self-#-adjoint on D.

Let $V(t) = Ext\text{-}\exp(it(\#\text{-}\bar{A})).$ It remains to show that $U(t) = V(t).$ Let $\varphi \in D(A).$ Since $\varphi \in D((\#\text{-}\bar{A})), V(t)\varphi \in D((\#\text{-}\bar{A}))$ and $V^{\#'}(t)\varphi = iAV(t)\varphi$ by (c) of Theorem 4.4.4, We already know that $U(t)\varphi \in D \subset D(\#\text{-}\bar{A})$ for all$\in {}^*\mathbb{R}_c^\#$. Let $w(t) = U(t)\varphi - V(t)\varphi$. Then $w(t)$ is a strongly #-differentiable vector-valued function and

$$w^{\#'}(t) = iAU(t)\varphi - i(\#\text{-}\bar{A})V(t)\varphi = iAw(t). \quad (4.4.23)$$

Thus

$$\frac{d^\#}{d^\# t}\|w(t)\|_\#^2 = -i\langle(\#\text{-}\bar{A})w(t), w(t)\rangle_\# + i\langle w(t), (\#\text{-}\bar{A})w(t)\rangle_\#. \quad (4.4.24)$$

Therefore $w(t) = 0$ for all $t \in {}^*\mathbb{R}_c^\#$ since $w(t) = 0$. This implies that $U(t)\varphi = V(t)\varphi$ for all $t \in {}^*\mathbb{R}_c^\#, \varphi \in D.$ Since D is #-dense in $\mathbf{H}^\#, U(t) = V(t).$

Remark 4.4.2. Finally, we have the following generalization of Stone's theorem 4.4.8. If g is a ${}^*\mathbb{R}_c^\#$-valued #-Borel function on ${}^*\mathbb{R}_c^\#$, then

$$g(A) = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\# P_\lambda \quad (4.4.25)$$

defined on D_g (4.4.5) is self-#-adjoint. If g is bounded, $g(A)$ coincides with $\widehat{\phi}(g)$ in Theorem 4.4.2.

We conclude with several remarks. First, generalized Stone's formula, given in Theorem 4.3.7 relates the resolvent and the projection-valued measure associated with any self-#-adjoint operator. The proof is the same as in the bounded in ${}^*\mathbb{R}_c^\#$ case. The spectrum of an unbounded self-#-adjoint operator is an unbounded subset of the real axis ${}^*\mathbb{R}_c^\#$. One can define discrete and essential spectrum; Theorem 4.3.6

(Generalized Weyl's criterion) still holds if one adds the criterion that the vectors $\{\psi_n\}$ must be in the domain of A.

Finally, we note that the measure space of Theorem 4.4.1 can always be chosen so that

Proposition **4.4.2** is applicable.

The following theorem says that every strongly #-continuous unitary group arises as the exponential of a self-#-adjoint operator.

Theorem 4.4.9. Let $U(\mathbf{t}) = U(t_1,\ldots,t_n)$ be a strongly continuous map of ${}^*\mathbb{R}_c^{\#n}$ into the unitary operators on a hyper infinite dimensional Hilbert space $\mathbf{H}^{\#}$ satisfying $U(\mathbf{t}+\mathbf{s}) = U(\mathbf{t})U(\mathbf{s})$ Let D be the set of hyperfinite linear combinations of vectors of the form

$$\varphi_f = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} f(\mathbf{t})U(\mathbf{t})d^{\#n}t \qquad (4.4.26)$$

where $\varphi \in \mathbf{H}^{\#}, f \in C_0^{\#^*\infty}({}^*\mathbb{R}_c^{\#n})$. Then D is a domain of essential self-#-adjointness for each of the generators A_j of the one-parameter subgroups $U(0,0,\ldots,t_j,..,0)$, each $A_j : D \to D$ and the A_j commute, $j = 1,\ldots,n$. Furthermore, there is a projection-valued #-measure P_Ω on ${}^*\mathbb{R}_c^{\#n}$ so that

$$\langle\varphi, U(\mathbf{t})\psi\rangle_{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} Ext\text{-}\exp(i\langle\mathbf{t},\boldsymbol{\lambda}\rangle)d^{\#}\langle\varphi, P_{\boldsymbol{\lambda}}\psi\rangle_{\#} \qquad (4.4.27)$$

for all $\varphi,\psi \in \mathbf{H}^{\#}$.

Proof Let A_j be the infinitesimal generator of $U_j(t_j) = U(0,\ldots,t_j,..,,0)$. The procedure used in the proof of Theorem 4.4.8 shows that $D \subset D(A_j)$, $A_j : D \to D$, and $U_j(t_j) : D \to D$. Theorem 4.4.7 shows that A_j is essentially self-#-adjoint on D. Because of the relation $U(\mathbf{t}+\mathbf{s}) = U(\mathbf{t})U(\mathbf{s})$, $U_j(t_j)$ commutes with $U_i(t_i)$ for all $t_j, t_i \in {}^*\mathbb{R}_c^{\#}$.

Therefore, it follows from Theorem 4.5.1, that A_i and A_j commute in the sense that is, their spectral projections commute.Let P_Ω^j be the projection-valued #-measure on ${}^*\mathbb{R}_c^{\#}$ corresponding to A_j. Define a projection valued #-measure P_Ω on ${}^*\mathbb{R}_c^{\#n}$ by defining it first on rectangles $r_n = Ext\text{-}\prod_{i=1}^{n}(a_i,b_i)$ by $P_{r_n} = Ext\text{-}\prod_{i=1}^{n} P^i_{(a_i,b_i)}$ and then letting P_Ω be the unique extension to the smallest $\sigma^{\#}$-algebra containing the rectangles, namely the #-Borel sets. Notice that, by Theorem 4.5.1, the $P^j_{\Omega_j}$ commute since the groups U_j commute. For each $\varphi,\psi \in \mathbf{H}^{\#}$, $\langle\varphi, P_{\boldsymbol{\Omega}}\psi\rangle_{\#}$ is a ${}^*\mathbb{C}_c^{\#}$-valued #-measure of hyperfinite mass which we denote by $d^{\#}\langle\varphi, P_{\boldsymbol{\lambda}}\psi\rangle_{\#}$.

Applying generalized Fubini's theorem we conclude that

$$\langle\varphi, U(\mathbf{t})\psi\rangle_{\#} = \left\langle \varphi, Ext\text{-}\prod_{i=1}^{n} U(t_i)\psi \right\rangle_{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} Ext\text{-}\exp(i\langle\mathbf{t},\boldsymbol{\lambda}\rangle)d^{\#}\langle\varphi, P_{\boldsymbol{\lambda}}\psi\rangle_{\#}. \qquad (4.4.28)$$

§ 4.5.Nearstandard $C^*_{\#}$ algebras generated by spectral prodjections related to unbounded in ${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operators.

Suppose that A and B are two unbounded self-#-adjoint operators on a non-Archimedean Hilbert space $H^{\#}$. We would like to find a reasonable definition for the statement: "A and B commute."

This cannot be done in the straightforward way since $AB - BA$ may not make sense on any vector $\psi \in H^{\#}$ for example, one might have $(\mathbf{Ran}(A)) \cap D(B) = \varnothing$ in which case BA does not have a meaning. This suggests that we find an equivalent formulation of commutativity for bounded self-#-adjoint operators. The spectral theorem for bounded self-#-adjoint operators A and B shows that in that case $AB - BA = 0$ if and only if all their projections, $\{P_{\Omega}^{A}\}$ and $\{P_{\Omega}^{B}\}$, commute, We take this as our definition in the unbounded case.

Definition 4.5.1. Two possibly unbounded in${}^{*}\mathbb{R}_{c}^{\#}$ self-#-adjoint operators A and B are said to commute if and only if all the projections in their associated projection-valued #-measures commute.

Remark 4.5.1. The spectral theorem shows that if A and B commute, then all the bounded in${}^{*}\mathbb{R}_{c}^{\#}$ #-Borel functions of A and B also commute. In particular, the resolvents $R_{\lambda}(A)$ and $R_{\mu}(B)$ commute and the unitary groups $Ext\text{-}\exp(itA)$ and $Ext\text{-}\exp(isA)$ commute.

The converse statement is also true and this shows that the above definition of "commute" is reasonable:

Theorem 4.5.1. Let A and B be self-#-adjoint operators on a non-Archimedean Hilbert spaceHilbert space $H^{\#}$.

Then the following three statements are equivalent:

(a) Spectral projections $P_{(a,b)}^{A}$ and $P_{(c,d)}^{B}$, commute.

(b) If $\operatorname{Im}\lambda$ and $\operatorname{Im}\mu$ are nonzero, then $R_{\lambda}(A)R_{\mu}(B) - R_{\mu}(B)R_{\lambda}(A) = 0$.

(c) For all $s,t \in {}^{*}\mathbb{R}_{c}^{\#}$,$[Ext\text{-}\exp(itA)][Ext\text{-}\exp(isB)] = [Ext\text{-}\exp(isB)][Ext\text{-}\exp(itA)]$.

Proof The fact that (a) implies (b) and (c) follows from the functional calculus. The fact that (b) implies (a) easily follows from the formula which expresses the spectral projections of A and B as strong #-limits of the resolvents (generalized Stone's formula) together with the fact that

$$s\text{-}\#\text{-}\lim_{\varepsilon\to_{\#}0}[i\varepsilon R_{a+i\varepsilon}(A)] = P_{\{a\}}^{A}. \tag{4.5.1}$$

To prove that (c) implies (a), we use some simple facts about the Fourier transform. Let $f \in S^{\#}({}^{*}\mathbb{R}_{c}^{\#})$. Then, by generalized Fubini's theorem,

$$\begin{gathered} Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}} f(t)\langle[Ext\text{-}\exp(itA)]\varphi,\psi\rangle_{\#}d^{\#}t = \\ = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}} f(t)\Big(Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}}\big([Ext\text{-}\exp(-it\lambda)]d_{\lambda}^{\#}\langle P_{\lambda}^{A}\varphi,\psi\rangle_{\#}\big)\Big)d^{\#}t = \\ = \sqrt{2\pi_{\#}}\left(Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}}\widehat{f}(\lambda)d_{\lambda}^{\#}\langle P_{\lambda}^{A}\varphi,\psi\rangle_{\#}\right) = \sqrt{2\pi_{\#}}\left\langle\varphi,\widehat{f}(A)\psi\right\rangle_{\#}. \end{gathered} \tag{4.5.2}$$

Thus, using (c) and generalized Fubini's theorem again,

$$\begin{gathered} \left\langle\varphi,\widehat{f}(A)\widehat{g}(B)\psi\right\rangle_{\#} = \\ Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}} Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#}} f(t)g(s)\langle\varphi,[Ext\text{-}\exp(-itA)][Ext\text{-}\exp(-isB)]\psi\rangle_{\#}d^{\#}sd^{\#}t = \\ = \left\langle\varphi,\widehat{g}(B)\widehat{f}(A)\psi\right\rangle_{\#} \end{gathered} \tag{4.5.3}$$

so, for all $f,g \in S^{\#}({}^{*}\mathbb{R}_{c}^{\#})$, $\widehat{f}(A)\widehat{g}(B) - \widehat{g}(B)\widehat{f}(A) = 0$.

Since the Fourier transform maps $S^{\#}({}^{*}\mathbb{R}_{c}^{\#})$ onto $S^{\#}({}^{*}\mathbb{R}_{c}^{\#})$ we conclude that $f(A)g(B) = g(B)f(A)$ for all $f,g \in S^{\#}({}^{*}\mathbb{R}_{c}^{\#})$. But, the characteristic function, $\chi_{(a,b)}$ can be expressed as the pointwise #-limit of a hyperinfinite sequence $f_{n}, n \in {}^{*}\mathbb{N}$ of uniformly bounded functions in $S^{\#}({}^{*}\mathbb{R}_{c}^{\#})$. By the functional calculus,

$$s\text{-}\#\text{-}\lim_{n\to{}^*\infty} f_n(A) = P^A_{(a,b)}. \tag{4.5.4}$$

Similarly,we find uniformly bounded $g_n \in S^\#({}^*\mathbb{R}^\#_c)$ #-converging pointwise to $\chi_{(c,d)}$ and

$$s\text{-}\#\text{-}\lim_{n\to{}^*\infty} g_n(B) = P^B_{(c,d)}. \tag{4.5.5}$$

Since the f_n and g_n are uniformly bounded in${}^*\mathbb{R}^\#_c$ and

$$f_n(A)g_n(B) = g_n(B)f_n(A) \tag{4.5.6}$$

for each $n \in {}^*\mathbb{N}$, we conclude that $P^A_{(a,b)}$ and $P^B_{(c,d)}$, commute which proves (a).

Definition 4.5.2. Let $A : H^\# \to H^\#$ be bounded in${}^*\mathbb{R}^\#_c$ self-#-adjoint operator. The operator A is **essentially bounded** in${}^*\mathbb{R}^\#_c$ if there is $\mathbf{st}(\|A\|_\#) \in \mathbb{R}$ and $\mathbf{st}(\|A\|_\#) \neq \infty$.

Remark 4.5.2. Note that if A is essentially bounded in${}^*\mathbb{R}^\#_c$ operator then for any nearstandard vector $\psi \in H^\#$ vector $A\psi$ again nearstandard, i.e. $\mathbf{st}(\|A\psi\|_\#) \neq \infty$.

Definition 4.5.3.Let A and B be self-#-adjoint essentially bounded in${}^*\mathbb{R}^\#_c$operators on a non-Archimedean Hilbert spaceHilbert space $H^\#$..The operators A and B are $\approx$-commute if $\|AB\|_\# \approx \|BA\|_\#$

Remark 4.5.3. Note that the operators A and B are $\approx$-commute if for any nearstandard vector $\psi \in H^\# : A\psi \approx B\psi$.

Theorem 4.5.2. Let A and B be self-#-adjoint operators on a non-Archimedean Hilbert space $H^\#$ and essentially bounded in${}^*\mathbb{R}^\#_c$. Then the following three statements are equivalent:

(a) Spectral projections $P^A_{(a,b)}$ and $P^B_{(c,d)}$, $\approx$-commute.

(b) If $\operatorname{Im}\lambda$ and $\operatorname{Im}\mu$ are nonzero, then $R_\lambda(A)R_\mu(B)$ and $R_\mu(B)R_\lambda(A)$ $\approx$-commute.

(c) For all $s,t \in {}^*\mathbb{R}^\#_c$,$[Ext\text{-}\exp(itA)][Ext\text{-}\exp(isB)]$ and $[Ext\text{-}\exp(isB)][Ext\text{-}\exp(itA)]$ $\approx$-commute.

Theorem 4.5.3. Let A and B be self-#-adjoint operators on a non-Archimedean Hilbert space $H^\#$. Then the following three statements are equivalent:

(a) Spectral projections $P^A_{(a,b)}$ and $P^B_{(c,d)}$, $\approx$-commute.

(b) For all $s,t \in {}^*\mathbb{R}^\#_c$,$[Ext\text{-}\exp(itA)][Ext\text{-}\exp(isB)] = [Ext\text{-}\exp(isB)][Ext\text{-}\exp(itA)]$. $\approx$-commute.

§4.6. ${}^*\mathbb{C}^\#_c$-valued quadratic forms.

One consequence of the generalized Riesz lemma is that there is a one-to-one correspondence between bounded in ${}^*\mathbb{R}^\#_c$ quadratic forms and bounded in ${}^*\mathbb{R}^\#_c$ operators; that is, any sesquilinear map $q : H \times H \to {}^*\mathbb{C}^\#_c$ which satisfies $|q(\varphi,\psi)_\#| < M\|\varphi\|_\#\|\psi\|_\#$ is of the form $q(\varphi,\psi) = \langle\varphi,A\psi\rangle_\#$ for some bounded operator A. As one might expect, the situation is more complicated if one removes the boundedness restriction. It is the relationship between unbounded forms and unbounded operators which we study briefly in this section.

Definition 4.6.1. A quadratic form is a map $q : Q(q) \times Q(q) \to {}^*\mathbb{C}^\#_c$, where $Q(q)$ is a #-dense linear subset of H called the form domain, such that $q(\cdot,\psi)$ is conjugate linear and $q(\varphi,\cdot)$ is linear for $\varphi,\psi \in Q(q)$. If $q(\varphi,\psi) = \overline{q\langle\psi,\varphi\rangle)_\#}$ we say that q is symmetric. If $q(\varphi,\varphi) \geq 0$ for all $\varphi \in Q(q)$, q is called positive, and if $q(\varphi,\varphi) \geq -M\|\varphi\|^2_\#$ for some $M \in {}^*\mathbb{R}^\#_c$we say that q is semibounded in ${}^*\mathbb{R}^\#_c$.

Notice that if q is semibounded, then it is automatically symmetric if H is complex.

Example 4.6.1. Let $H = \mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#})$ and $Q(q) = C_0^{\infty^{\#}}({}^*\mathbb{R}_c^{\#})$ with $q(f,g) = f(0)g(0)$. Then q is a positive quadratic form. Since $q(f,g) = \delta^{\#}(fg)$ one could formally write
$q(fg) = (f,Ag)$
where $A : g \mapsto \delta^{\#}(x)g(x)$. Since multiplication by $\delta(x)$ is not an operator, q is an example
of a quadratic form not likely to be associated with an operator.

Example 4.6.2 Let A be a self-#-adjoint operator on $H^{\#}$. Let us pass to a spectral representation of A, so that A is multiplication by x on $\otimes_{n=1}^{N} \mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#}, \mu_n^{\#})$. Let

$$Q(q) = \left\{ (\psi_n)_{n=1}^{N} | Ext\text{-}\sum_{n=1}^{N} Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} |x| |\psi_n(x)|^2 d^{\#}\mu_n^{\#} \right\} < {}^*\infty \tag{4.6.1}$$

and for $\varphi, \psi \in Q(q)$ define

$$q(\varphi,\psi) = \sum_{n=1}^{N} \left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} x\overline{\varphi_n(x)}\psi_n(x) d^{\#}\mu_n^{\#} \right). \tag{4.6.2}$$

We call q the quadratic form associated with A and write $Q(q) = Q(A)$; $Q(A)$ is called the form domain of the operator A. For $\psi, \varphi \in Q(A)$, we will write $q(\varphi,\psi) = \langle \varphi, A\psi \rangle_{\#}$ although A does not make sense on all $\psi \in Q(A)$, then $Q(A)$ is in some sense the largest domain on which q can be defined.

To investigate the deep connection between self-#-adjointness and semi-bounded in ${}^*\mathbb{R}_c^{\#}$ quadratic forms we need to extend the notion of “#-closed” from operators to forms. An operator A is #-closed if and only if its graph is #-closed which is the same as saying that $D(A)$ is complete under the #-norm $\|\psi\|_A = \|A\psi\|_{\#} + \|\psi\|_{\#}$.

Analogously we define:

Definition 4.6.2. Let q be a semibounded in ${}^*\mathbb{R}_c^{\#}$ quadratic form, $q(\psi,\psi) \geq -M\|\psi\|_{\#}^2$ is called #-closed if $Q(q)$ is complete under the #-norm

$$\|\psi\|_{\#+1} = \sqrt{q(\psi,\psi) \dotplus (M \dotplus 1)\|\psi\|_{\#}^2}. \tag{4.6.3}$$

If q is #-closed and $D \subset Q(q)$ is #-dense in $Q(q)$ in the $\|\psi\|_{\#+1}$ #-norm, then D is called a form #-core for q.

Notice that $\|\psi\|_{\#+1}$ comes from the inner product

$$\langle \psi, \varphi \rangle_{\#+1} = q(\psi,\varphi) \dotplus (M \dotplus 1)\langle \psi, \varphi \rangle_{\#}. \tag{4.6.4}$$

It is not hard to see that q is #-closed if and only if whenever
$\varphi_n \in Q(q)$ $\varphi_n \overset{H^{\#}}{\to} \varphi$ and $q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to_{\#} 0$, as $n,m \to {}^*\infty$, then $\varphi \in Q(q)$ and $q(\varphi_n - \varphi, \varphi_n - \varphi) \to_{\#} 0$. This criterion and the dominated #-convergence theorem show that the form q associated with a semibounded self-#-adjoint operator (Example 4.6.2) is #-closed. Furthermore, any operator #-core for A is a form #-core for q.

Now, let $q(f,g) = f(0)g(0)$ as in Example 4.6.1 and $\varphi_n \in C_0^{\infty^{\#}}({}^*\mathbb{R}_c^{\#})$. Then $\varphi_n \to_{\#} 0$, and $q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to_{\#} 0$, but $q(\varphi_n, \varphi_n) \to_{\#} 1 \neq q(0,0)$ which proves that q has no #-closed extensions. Therefore, even though q is positive (and therefore symmetric) there is no semibounded self-#-adjoint operator A so that $q(f,g) = \langle f, Ag \rangle_{\#}$ for all $f,g \in C_0^{\infty^{\#}}({}^*\mathbb{R}_c^{\#})$.

The deep fact about semibounded quadratic forms is that unlike the case for operators,
they cannot be #-closed and symmetric, yet not self-#-adjoint.

Theorem 4.6.2. If q is a #-closed semibounded in ${}^*\mathbb{R}_c^{\#}$ quadratic form, then q is the

quadratic form of a unique self-#-adjoint operator.
Proof We may assume without loss of generality that q is positive. Then, since q is #-closed and symmetric, $Q(q)$ is a Hilbert space, which we denote by $H^{\#}_{+1}$, under the inner product $\langle\varphi,\psi\rangle_{\#+1} = q(\varphi,\psi) + \langle\varphi,\psi\rangle_{\#}$. We denote by $H^{\#}_{-1}$ the space of bounded in ${}^*\mathbb{R}^{\#}_c$ conjugate linear functionals on $H^{\#}_{+1}$. Let j, given by $\psi \overset{j}{\mapsto} \langle\bullet,\psi\rangle_{\#}$ be the linear imbedding of $H^{\#}$ into $H^{\#}_{-1}$ is bounded in ${}^*\mathbb{R}^{\#}_c$ because

$$|[j(\psi)(\varphi)]| \leq \|\varphi\|_{\#}\|\psi\|_{\#} \leq \|\varphi\|_{\#}\|\psi\|_{\#} \leq \|\varphi\|_{\#+1}\|\psi\|_{\#}. \tag{4.6.5}$$

Since the identity map i embeds $H^{\#}_{+1}$ in $H^{\#}$ we have a “scale of spaces”

$$H^{\#}_{+1} \overset{i}{\to} H^{\#} \overset{j}{\to} H^{\#}_{-1}. \tag{4.6.6}$$

We now exploit the generalized Riesz lemma. Given $\Phi \in H^{\#}_{+1}$, let $\widehat{B}\Phi$ be the element of $H^{\#}_{-1}$ which acts by $[\widehat{B}\Phi](\varphi) = q(\varphi,\Phi) + (\varphi,\Phi)_{\#}$. By the generalized Riesz lemma, $\widehat{B}$ is an isometric isomorphism of $H^{\#}_{+1}$ onto $H^{\#}_{-1}$. Let $D(B) = \left\{\psi \in H^{\#}_{+1} | \widehat{B}\psi \in \mathbf{Ran}(j)\right\}$.
Define now B on $D(B)$ by $B = j^{-1}\widehat{B}$. Notice that

$$H^{\#} \supset H^{\#}_{+1} \overset{B}{\to} H^{\#}_{-1} \overset{j}{\leftarrow} H^{\#}. \tag{4.6.7}$$

First, we prove that the range of j is #-dense in $H^{\#}_{-1}$. If it were not, there would be a $\lambda \in H^{\#*}_{-1}$ so that $\lambda \neq 0$, and $\lambda[j(\psi)] = 0$ for each $\psi \in H^{\#}$. By the generalized Riesz Lemma, there is a $\varphi_{\lambda} \neq 0$ in $H^{\#}_{+1}$ so that $0 = \lambda[j(\psi)] = [j(\psi)](\varphi_{\lambda}) = \langle\varphi_{\lambda},\psi\rangle_{\#}$ for all $\psi \in H^{\#}$. Since $\varphi_{\lambda} \neq 0$, this is impossible. Therefore $\mathbf{Ran}(j)$ is #-dense in $H^{\#}_{-1}$.
Since B is an isometric isomorphism we conclude that $D(B)$ is $\|\bullet\|_{\#+1}$ #-dense in $H^{\#}_{+1}$.
Further, since $\|\bullet\|_{\#} \leq \|\bullet\|_{\#+1}$ and $H^{\#}_{+1}$ is #-norm #-dense in $H^{\#}, D(B)$ is #-norm #-dense in $H^{\#}$. Suppose $\varphi,\psi \in D(B)$. Then one obtains that

$$\langle\varphi,B\psi\rangle_{\#} = q(\varphi,\psi) + \langle\varphi,\psi\rangle_{\#} = \overline{q(\psi,\varphi) + \langle\psi,\varphi\rangle_{\#}} = \overline{\langle\psi,B\varphi\rangle_{\#}} = \langle B\varphi,\psi\rangle_{\#}. \tag{4.6.8}$$

Thus, B is a #-densely defined symmetric operator.
We will prove now that B is self-#-adjoint. Let $C = \left(\widehat{B}\right)^{-1} j$. C takes $H^{\#}$ into $H^{\#}$ and is an everywhere defined symmetric operator. By the generalized Hellinger-Toeplitz theorem, C is a bounded in ${}^*\mathbb{R}^{\#}_c$ self-#-adjoint operator. Moreover, C is injective.
A simple application of the spectral theorem in multiplication operator form shows that $C^{-1} : \mathbf{Ran}\ (C) \to H^{\#}$ is a self-#-adjoint operator. But $C^{-1} = B$.
We now define $A = B - I$. Then A is also self-#-adjoint on $D(A) = D(B)$ and for $\varphi,\psi \in D(A)$, $\langle\varphi,A\psi\rangle_{\#} = q(\varphi,\psi)$. Since $D(A)$ is $\|\bullet\|_{\#+1}$ #-dense in $H^{\#}_{+1}$ is the quadratic form associated to A. Uniqueness is obvious.
Thus, there is an principal distinction between semi-bounded in ${}^*\mathbb{R}^{\#}_c$ symmetric operators and semi-bounded in ${}^*\mathbb{R}^{\#}_c$ quadratic forms. For symmetric operators, there is never any problem finding #-closed extensions.
Reark 4.6.1. Note that: (1) If A and B are self-#-adjoint operators and $D(A) \subset D(B)$ with $B \upharpoonright D(A) = A$ then $A = B$. But it can happen that a and b are #-closed semibounded in ${}^*\mathbb{R}^{\#}_c$ quadratic forms and $b \upharpoonright Q(a)xQ(a) = a$ without having $a = b$.
(2) Let A be a symmetric operator that is semibounded in ${}^*\mathbb{R}^{\#}_c$. Let q be the quadratic form $q(\varphi,\psi) = \langle\varphi,A\psi\rangle_{\#}$ with $Q(a) = D(A)$. Suppose that q has a #-closure , that is, a smallest #-closed form which extends it. Then the self-#-adjoint operator A which corresponds to $\widehat{q}$ (by Theorem 4.6.2) may be bigger than the operator #-closure of A.

(3) While a general quadratic form may have no #-closed extensions, forms that come directly from semibounded in ${}^*\mathbb{R}_c^\#$ operators always have #-closures and thus semibounded in ${}^*\mathbb{R}_c^\#$ operators always have self-#-adjoint extensions.

§ 4.7. #-Convergence of unbounded in ${}^*\mathbb{R}_c^\#$ operators

One of the main difficulties with unbounded in ${}^*\mathbb{R}_c^\#$ operators is that they are only #-densely defined. This difficulty is especially troublesome when one wants to find a notion of #-convergence for a hyper infinite sequence $A_n \to_\# A, n \in {}^*\mathbb{N}$ of unbounded in ${}^*\mathbb{R}_c^\#$ operators since the domains of the operators A_n may have no vector in common. For example, if $A_n = (1 - n^{-1})x$ on $L_2^\#({}^*\mathbb{R}_c^\#)$, it is clear that in some sense $A_n \to_\# A = x;$ yet we could have been given domains $D(A_n)$ and $D(A)$ of essential self-#-adjointness for these operators which have no nonzero vector in common. Of course, in this simple case the #-closures of A_n and A all have the same domain, but in general this will not be true, and in any case, one is often forced to deal with domains of essential self-adjointness since closures of operators are sometimes difficult to compute. It is very natural to say that self-#-adjoint operators are “close” if certain bounded in ${}^*\mathbb{R}_c^\#$ functions of them are “close.” Most of this section is devoted to this approach. However, we also introduce graph #-limits, a topic which will be explored further.

Definition 4.7.1. Let $(A_n)_{n\in{}^*\mathbb{N}}$ and A be self-#-adjoint operators. Then A_n is said to #-converge to A in the #-norm resolvent sense (or #-norm generalized sense) if $R_\lambda(A_n) \to_\# R_\lambda(A)$ in #-norm for all λ with $\operatorname{Im}\lambda \neq 0$. A_n is said to #-converge to A in the strong resolvent sense (or strong generalized sense) if $R_\lambda(A_n) \to_\# R_\lambda(A)$ strongly for all λ with $\operatorname{Im}\lambda \neq 0.$.

We have not introduced the notion of weak resolvent #-convergence since weak resolvent #-convergence implies strong resolvent #-convergence. The following theorem shows that #-norm resolvent #-convergence is the right generalization of #-norm convergence for bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operators. A similar result holds for strong resolvent #-convergence, but the analogue for weak #-convergence is not true.

Theorem 4.7.1. Let $(A_n)_{n=1}^{{}^*\infty}$ and A be a family of uniformly bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operators. Then $A_n \to_\# A$ as $n \to {}^*\infty$ in the #-norm resolvent sense if and only if $A_n \to_\# A$ as $n \to {}^*\infty$ in #-norm.

Proof. Let $A_n \to_\# A$ as $n \to {}^*\infty$ in #-norm. Then if $\operatorname{Im}\lambda \neq 0$, $(A_n - A)(A - \lambda)^{-1} \to_\# 0$ in #-norm. Thus, using the equality $(A_n - \lambda)^{-1} = (A - \lambda)^{-1}\left[I + (A_n - A)(A - \lambda)^{-1}\right]^{-1}$ we obtain that $(A_n - \lambda)^{-1} \to_\# (A - \lambda)^{-1}$ in #-norm as $n \to {}^*\infty$.

Conversely, suppose $A_n \to_\# A$ as $n \to {}^*\infty$ in the #-norm resolvent sense. Then, since $A_n - A = (A_n - i)(A_n - i)^{-1}[(A - i)^{-1} - (A_n - i)^{-1}](A - i),$ we conclude that $\|A_n - A\|_\# \leq (\sup_n\|A_n\|_\# + 1)\|(A - i)^{-1} - (A_n - i)^{-1}\|_\#(\|A\|_\# + 1) \to_\# 0$ as $n \to {}^*\infty$.

The following theorem shows that to prove generalized convergence one need only show #-convergence of the resolvents at one point off the hyperreal axis ${}^*\mathbb{R}_c^\#$.

Theorem 4.7.2. Let $(A_n)_{n=1}^{{}^*\infty}$ and A be self-#-adjoint operators, and let $\lambda_0 \in {}^*\mathbb{C}_c^\#$.

(a) If $\operatorname{Im}\lambda_0 \neq 0$ and $\|R_{\lambda_0}(A_n) - R_{\lambda_0}(A)\|_\# \to_\# 0$, then $A_n \to_\# A$ as $n \to {}^*\infty$ in the #-norm resolvent sense.

(b) If $\operatorname{Im}\lambda_0 \neq 0$ and if $R_{\lambda_0}(A_n)\varphi - R_{\lambda_0}(A)\varphi \to_\# 0$, for all $\varphi \in H^\#$ then $A_n \to_\# A$ as $n \to {}^*\infty$ in the strong resolvent sense.

Proof (a) Both $R_\lambda(A)$ and $R_\lambda(A_n)$ are analytic in the half-plane of ${}^*\mathbb{C}_c^\#$ containing λ_0 and have hyper infinite power series around λ_0,

$$\begin{aligned} R_\lambda(A) &= Ext\text{-}\sum_{m=0}^{^*\infty}(\lambda_0-\lambda)^m[R_{\lambda_0}(A)]^{m+1}, \\ R_\lambda(A_n) &= Ext\text{-}\sum_{m=0}^{^*\infty}(\lambda_0-\lambda)^m[R_{\lambda_0}(A_n)]^{m+1} \end{aligned} \tag{4.7.1}$$

which #-converge in #-norm in the circle $|\lambda-\lambda_0| < |\mathrm{Im}\,\lambda_0|^{-1}$. Since $R_{\lambda_0}(A_n) \to_\# R_{\lambda_0}(A)$ in #-norm, $R_\lambda(A_n) \to_\# R_\lambda(A)$ in #-norm for A in this circle. Therefore, by repeating this process, we get #-convergence for all A in the half-plane of ${}^*\mathbb{C}_c^\#$ containing λ_0.
Furthermore, since

$$\begin{gathered} \|R_{\overline{\lambda_0}}(A_n)-R_{\overline{\lambda_0}}(A)\|_\# = \|(R_{\lambda_0}(A_n)-R_{\lambda_0}(A))^*\|_\# = \\ \|R_{\lambda_0}(A_n)-R_{\lambda_0}(A)\|_\# \to_\# 0 \text{ as } n\to{}^*\infty \end{gathered} \tag{4.7.2}$$

the same argument shows that the resolvents converge in #-norm in the hal-fplane of ${}^*\mathbb{C}_c^\#$ containing λ_0.
(b) The proof is the same as the proof of (a) except for two things. First, we consider the vector-valued functions $R_\lambda(A_n)\varphi$ and $R_\lambda(A)\varphi$. Secondly, since the map $T\to T^*$ is not #-continuous in the strong topology, one needs a separate argument to get from one half-plane of ${}^*\mathbb{C}_c^\#$ to the other. Suppose that λ_0 is in the lower half-plane of ${}^*\mathbb{C}_c^\#$. Then, as in (a), we get #-convergence everywhere in the lower half-plane of ${}^*\mathbb{C}_c^\#$, in particular at $\lambda=-i$. The formula

$$\begin{gathered} (A_n-i)^{-1}-(A-i)^{-1} = \\ [(A_n+i)(A_n-i)^{-1}][(A_n+i)^{-1}-(A+i)^{-1}][(A_n+i)(A_n-i)^{-1}] \end{gathered} \tag{4.7.3}$$

which follows from elementary calculations, can then be used to prove that hyper infinite sequence $(A_n-i)^{-1}, n\in{}^*\mathbb{N}$ #-converges strongly to $(A-i)^{-1}$. The above argument then shows that hyper infinite sequence $R_\lambda(A_n)\,, n\in{}^*\mathbb{N}$ #-converges strongly to $R_\lambda(A)$ everywhere in the upper half-plane of ${}^*\mathbb{C}_c^\#$.
For alternative ways of proving that strong #-convergence, $R_\lambda(A_n) \overset{s}{\to}_\# R_\lambda(A)$ in one half-plane implies strong #-convergence in the other half-plane, see Theorem 4.7.9.
We will investigate several aspects of generalized #-convergence. First, we ask how resolvent #-convergence is related to the #-convergence of other bounded functions of A_n and A. Secondly, we investigate the relationship between the spectra of A_n and the spectrum of A if $A_n \to_\# A$ in a generalized sense. Finally, we give criteria on the operators A_n, A themselves which are sufficient to guarantee that $A_n \to_\# A$ as $n\to{}^*\infty$ in a generalized sense.
Theorem 4.7.3. Let A_n and A be self-#-adjoint operators.
(a) If $A_n \to_\# A$ as $n\to{}^*\infty$in the #-norm resolvent sense and f is a #-continuous function on${}^*\mathbb{R}_c^\#$ vanishing at ${}^*\infty$, then $\|f(A_n)-f(A)\|_\# \to_\# 0$ as $n\to{}^*\infty$
(b) If $A_n \to_\# A$ in the strong resolvent sense and f is a bounded in ${}^*\mathbb{R}_c^\#$ #-continuous function on ${}^*\mathbb{R}_c^\#$, then $f(A_n)\varphi \to_\# f(A)\varphi$ as $n\to{}^*\infty$, for all $\varphi\in H^\#$.
Proof By the generalized Stone-Weierstrass theorem, polynomials in $(x+i)^{-1}$ and $(x-i)^{-1}$ are #-dense in $C^{^*\infty}({}^*\mathbb{R}_c^\#)$, the #-continuous functions vanishing at hyper

infinity. Thus, given $\varepsilon \approx 0, \varepsilon > 0,$ we can find an hyperfinite polynomial $P(s,t)$ so that

$$\|f(x) - P\Big((x+i)^{-1},(x-i)^{-1}\Big)\|_{^*\infty} \leq \frac{\varepsilon}{3}. \quad (4.7.4)$$

Therefore,

$$\|f(A_n) - P\Big((A_n+i)^{-1},(A_n-i)^{-1}\Big)\|_{^*\infty} \leq \frac{\varepsilon}{3} \quad (4.7.5)$$

and

$$\|f(A) - P\Big((A+i)^{-1},(A-i)^{-1}\Big)\|_{^*\infty} \leq \frac{\varepsilon}{3}. \quad (4.7.6)$$

If $A_n \to_{\#} A$ as $n \to {}^*\infty$in the #-norm resolvent sense, then

$$P\Big((A_n+i)^{-1},(A_n-i)^{-1}\Big) \to_{\#} P\Big((A+i)^{-1},(A-i)^{-1}\Big) \quad (4.7.7)$$

in #-norm as $n \to {}^*\infty,$and thus for hyperfinite n large enough,$\|f(A_n) - f(A)\|_{\#} \leq \varepsilon.$
This proves (a).
To prove (b) we first note that the same proof as above shows that if $A_n \to_{\#} A$ in the strong resolvent sense and $h \in C^{^*\infty}({}^*\mathbb{R}_c^{\#})$, then $h(A_n)\varphi \to_{\#} h(A)\varphi$. Let $\psi \in H^{\#}$ and $\varepsilon \approx 0, \varepsilon > 0$ be given and define $g_m(x) = Ext\text{-}\exp(-x^2/m)$. Since $g_m(x) \to_{\#} 1$ pointwise, $g_m(A)\psi \to_{\#} \psi$ by spectral theorem , so we can find an m with $\|g_m(A)\psi - \psi\|_{\#} \leq \varepsilon(6\|f\|_{^*\infty})^{-1}$.Furthermore since $g_m \in C^{^*\infty}({}^*\mathbb{R}_c^{\#}), g_m(A_n)\psi \to_{\#} g_m(A)\psi$ by the remark above, so we can find an $N_0,$ so that $n > N_0$ implies $\|g_m(A_n)\psi - g_m(A)\psi\|_{\#} \leq \varepsilon(6\|f\|_{^*\infty})^{-1}$. Therefore, if $n \geq N_0,$

$$\|g_m(A_n)\psi - \psi\|_{\#} \leq \varepsilon(3\|f\|_{^*\infty})^{-1}. \quad (4.7.8)$$

Since fg_m is #-continuous and goes to zero at ${}^*\infty$, there is an N_1 so that $n \geq N_1$ implies

$$\|f(A_n)g_m(A_n)\psi - f(A)g_m(A)\psi\|_{\#} \leq \frac{\varepsilon}{3}. \quad (4.7.9)$$

Let $N = \max(N_0, N_1)$. Then for $n \geq N,$

$$\begin{aligned}\|f(A_n)\psi - f(A)\psi\|_{\#} &\leq \|f(A_n)g_m(A_n)\psi - f(A)g_m(A)\psi\|_{\#} + \\ &+ \|A_n\|_{\#}\|g_m(A_n)\psi - \psi\|_{\#} + \|A\|_{\#}\|g_m(A)\psi - \psi\|_{\#}.\end{aligned} \quad (4.7.10)$$

Since ψ and ε were arbitrary, this proves (b).
As an example of an application of part (a) let $(A_n)_{n=1}^{^*\infty}$ and A be positive self-#-adjoint operators. Then, if $A_n \to_{\#} A$ in the #-norm resolvent sense $Ext\text{-}\exp(-tA_n)$ #-converges in #-norm to $Ext\text{-}\exp(-tA)$ for each positive t. To see that part (a) does not extend to all of $C^{\#}({}^*\mathbb{R}_c^{\#})$, notice that on $L_2^{\#}({}^*\mathbb{R}_c^{\#})$ the operators $A_n = (1 - n^{-1})x$ #-converge to the operator $A = x$ in the #-norm resolvent sense but $\|Ext\text{-}\exp(iA_n) - Ext\text{-}\exp(iA_n)\|_{\#} = 1$ for all $n \in {}^*\mathbb{N}$.
An important application of part (b) is the following generalization of the classical Trotter theorem.
Theorem 4.7.4. Let $(A_n)_{n=1}^{^*\infty}$ and A be self-#-adjoint operators. Then $A_n \to_{\#} A$ in the strong resolvent sense if and only if $Ext\text{-}\exp(itA_n)$ #-converges strongly to $Ext\text{-}\exp(itA)$ for each t.
Proof Since $Ext\text{-}\exp(itx)$ is a bounded #-continuous function of $x,$ Theorem 4.7.3 implies that if $A_n \to_{\#} A$ in the strong resolvent sense, then $Ext\text{-}\exp(itA_n) \to_{\#} Ext\text{-}\exp(itA)$ as $n \to {}^*\infty,$strongly for each t.
To prove the theorem in the other direction, we first derive a formula for the resolvent

of a self-#-adjoint operator A. Suppose that $\operatorname{Im}\mu < 0$. Then, by the functional calculus

$$\begin{gathered}\langle\psi, R_\mu(A)\varphi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}\left(\frac{1}{\mu-\lambda}\right)d^\#\langle\psi, P_\lambda\varphi\rangle_\# = \\ Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}\left(Ext\text{-}\int_0^{{}^*\infty} i[Ext\text{-}\exp(-it\mu)][Ext\text{-}\exp(it\lambda)]d^\#t\right) = \\ Ext\text{-}\int_0^{{}^*\infty} i[Ext\text{-}\exp(-it\mu)]^\#\langle\psi, Ext\text{-}\exp(itA)\varphi\rangle_\# = \\ \left\langle\psi, Ext\text{-}\int_0^{{}^*\infty} i[Ext\text{-}\exp(-it\mu)][Ext\text{-}\exp(itA)]\varphi d^\#t\right\rangle_\#.\end{gathered} \tag{4.7.11}$$

Therefore,

$$R_\mu(A)\varphi = Ext\text{-}\int_0^{{}^*\infty} i[Ext\text{-}\exp(-it\mu)][Ext\text{-}\exp(itA)]\varphi d^\#t \tag{4.7.12}$$

where the #-integral is a Riemann #-integral. The third step in the computation uses generalized Fubini's theorem. Applying (4.7.12) to the operators A_n and A we obtain

$$\begin{gathered}\|R_\mu(A_n)\varphi - R_\mu(A)\varphi\|_\# \leq \\ Ext\text{-}\int_0^{{}^*\infty}[Ext\text{-}\exp(t\operatorname{Im}\mu)]\|[Ext\text{-}\exp(itA_n) - Ext\text{-}\exp(itA)]\varphi\|_\# d^\#t\end{gathered} \tag{4.7.13}$$

so if $Ext\text{-}\exp(itA_n) \to_\# Ext\text{-}\exp(itA)$ as $n \to {}^*\infty$ for each t, $\|R_\mu(A_n)\varphi - R_\mu(A)\varphi\|_\# \to_\# 0$ as $n \to {}^*\infty$ by the generalized Lebesgue dominated convergence theorem. Using a formula similar to (4.7.12) one concludes in the same way that
$\|R_\mu(A_n)\varphi - R_\mu(A)\varphi\|_\# \to_\# 0$ for $\operatorname{Im}\mu > 0$. We remark that it is possible to show that if $A_n \to_\# A$ in the strong resolvent sense, then $Ext\text{-}\exp(itA_n) \to_\# Ext\text{-}\exp(itA)$ for each φ uniformly for t in any gyperfinite interval.

Theorem 4.7.5. (Generalized Trotter-Kato theorem) Let $(A_n)_{n=1}^{{}^*\infty}$ be a sequence of self-#-adjoint operators. Suppose that there exist points, λ_0 in the upper half-plane of ${}^*\mathbb{C}_c^\#$ and μ_0 in the lower half-plane of ${}^*\mathbb{C}_c^\#$ so that $R_{\lambda_0}(A_n)\varphi$ and $R_{\mu_0}(A_n)\varphi$ #-converge as $n \to {}^*\infty$ for each $\varphi \in H^\#$. Suppose further that one of the limiting operators, T_{λ_0} or T_{μ_0}, has a #-dense range. Then there exists a self-#-adjoint operator A so that $A_n \to_\# A$ as $n \to {}^*\infty$ in the strong resolvent sense.

Proof Since $\|R_{\lambda_0}(A_n)\|_\# \leq |\operatorname{Im}\lambda_0|^{-1}$, $\|T_{\lambda_0}\| \leq |\operatorname{Im}\lambda_0|^{-1}$, and so

$$T_\lambda = Ext\text{-}\sum_{n=0}^{{}^*\infty}(\lambda_0-\lambda)^n(T_{\lambda_0})^{n+1}. \tag{4.7.14}$$

is well defined for $|\lambda_0 - \lambda| \leq |\operatorname{Im}\lambda_0|^{-1}$. Furthermore, since $R_{\lambda_0}(A_n)\varphi \to_\# T_{\lambda_0}\varphi$, $R_\lambda(A_n)\varphi \to_\# T_\lambda\varphi$ in the same circle.

Continuing in this way we can define an #-analytic operator valued function T_λ in the half-plane containing λ_0 which is the strong #-limit of $R_\lambda(A_n)$. Since the half-plane is simply #-connected, the determination of T_λ at a point λ is independent of the path taken from λ_0 to λ. The same argument for the half-plane containing shows that we can extend the definition of T_λ to that half-plane of ${}^*\mathbb{C}_c^\#$ so that for all λ with $\operatorname{Im}\lambda \neq 0$

$$T_\lambda\varphi = \#\text{-}\lim_{n\to{}^*\infty} R_\lambda(A_n)\varphi. \qquad (4.7.15)$$

The T_λ commute, satisfy the first resolvent equation, and $T_\lambda^* = T_\lambda$ since these statements are true for each $R_\lambda(A_n)$ It follows from the first resolvent formula and the commutativity that the ranges of all the T_λ are equal; we denote this common range by D. By hypothesis, D is #-dense and this implies that the kernel of T_λ is empty, since $\mathbf{Ker}(T_\lambda) = (\mathbf{Ran}(T_\lambda^*))^\top = (\mathbf{Ran}(T_\lambda))^\top = D^\top = \{0\}$. We can therefore define $A = \lambda I - T_\lambda^{-1}$ on D and a short calculation with the resolvent equation shows that this definition is independent of which λ with $\operatorname{Im}\lambda \neq 0$, is chosen. Since $\mathbf{Ran}(A \pm i) = \mathbf{Ran}(T_{\pm i}^{-1})$ operator A is self-#-adjoint. It is clear that the resolvent of A is T_λ.

Notice that in the Trotter-Kato theorem we need convergence at two points, one in the upper half-plane and one in the lower half-plane of ${}^*\mathbb{C}_c^\#$. For, we cannot use Theorem 4.7.3 until we know that the #-limiting operator is self-#-adjoint, and the self-#-adjointness proof depends on the #-convergence in both half-planes of ${}^*\mathbb{C}_c^\#$.

The Trotter-Kato theorem is important since its hypotheses do not assume the a priori existence of a #-limiting operator A. It can be used to assert the existence of a generalized #-limit of a sequence of self-#-adjoint operators. This can also be done with the one-parameter groups. To see why it is necessary to use the resolvents or groups rather than the operators themselves to prove such an existence theorem consider the following example: Let A be a closed symmetric operator which is not self-#-adjoint but which has a self-#-adjoint extension $\widetilde{A}$. Let P_n be the spectral projection of $\widetilde{A}$ corresponding to the interval $[-n,n]$. Then $P_n\widetilde{A}P_n$ are bounded self-#-adjoint operators (and therefore essentially self-#-adjoint on $D(A)$) such that for all $\varphi \in D(A) : P_n\widetilde{A}P_n\varphi \to_\# \widetilde{A}\varphi = A\varphi$. Thus the P_nAP_n are essentially self-#-adjoint on $D(A)$ and the strong #-limit exists but the #-limit is not essentially self-#-adjoint.

One of the most useful aspects of generalized #-convergence is that the spectra and projections of the A_n are related to the spectrum and projections of A.

Theorem 4.7.6. Let $(A_n)_{n=1}^{{}^*\infty}$and A be self-#-adjoint operators and suppose that $A_n \to_\# A$ in the #-norm resolvent sense. Then

(a) If $\mu \notin \sigma(A)$, then $\mu \notin \sigma(A_n)$ for $n \in {}^*\mathbb{N}$ sufficiently large and

$$\|R_\mu(A_n) - R_\mu(A)\|_\# \to_\# 0 \qquad (4.7.16)$$

(b) Let $a,b \in {}^*\mathbb{R}_c^\#$, $a < b$, and suppose that $a \in \rho(A)$, $b \in \rho(A)$. Then

$$\|P_{(a,b)}(A_n) - P_{(a,b)}(A)\|_\# \to_\# 0 \qquad (4.7.17)$$

Proof (a) We need only consider the case where $\mu \in {}^*\mathbb{R}_c^\#$. Since $\mu \in \rho(A)$, there is a $\delta \approx 0, \delta > 0$ so that $(\mu-\delta,\mu+\delta) \cap \sigma(A) = \varnothing$. Thus, by the functional calculus, $\|R_{\mu\dotplus i\delta/3}(A)\|_\# < 1/\delta$. Now, we can find N so that $\|R_{\mu\dotplus i\delta/3}(A_n)\|_\# < 2/\delta$ for $n \geq N$ which implies that the power series for $R_\lambda(A_n)$ about $\mu \dotplus i\delta/3$ has radius of #-convergence at least $5/2$. We already know that where the series #-converges it is an inverse for A_n. So, $\mu \in \rho(A_n)$ for $n \geq N$ and $\|R_\mu(A_n) - R_\mu(A)\|_\# \to_\# 0$ as $n \to {}^*\infty$.

To prove (b), we note that since $a,b \in \rho(A)$, there exists $\varepsilon < (1/2)(b-a)$ and an N, so that $\sup_{n\geq N}\left\{\|(A_n-a)^{-1}\|_\#, \|(A_n-b)^{-1}\|_\#\right\} \leq 1/\varepsilon$. Therefore, by the functional calculus, $\sigma(A_n) \cap (a,b) \subset (a \dotplus \varepsilon, b-\varepsilon)$ for $n \geq N$. Let f_ε be a #-continuous function which equals one on $(a+\varepsilon, b-\varepsilon)$ and is equal to zero outside (a,b). Then $P_{(a,b)}(A_n) = f_\varepsilon(A_n)$ and $P_{(a,b)}(A) = f_\varepsilon(A)$ and so by Theorem 4.7.3 one obtains $\|P_{(a,b)}(A_n) - P_{(a,b)}(A)\|_\# \to_\# 0$ as $n \to {}^*\infty$.

Theorem 4.7.7. Let $(A_n)_{n=1}^{*\infty}$ and A be self-#-adjoint operators and suppose that $A_n \to_\# A$ in the strong resolvent sense. Then

(a) If $a,b \in {}^*\mathbb{R}_c^\#$ $a < b$, and $(a,b) \cap \sigma(A_n) = \varnothing$ for all $n \in {}^*\mathbb{N}$, then $(a,b) \cap \sigma(A) = \varnothing$. That is, if $\lambda \in \sigma(A)$, then there exists $\lambda_n \in \sigma(A_n)$ so that $\lambda_n \to_\# \lambda$.

(b) If $a,b \in {}^*\mathbb{R}_c^\#$ $a < b$, and $a,b \notin \sigma_{\mathbf{pp}}(A)$ then $P_{(a,b)}(A_n)\varphi \to_\# P_{(a,b)}(A)\varphi$ for all $\varphi \in H^\#$.

Proof By the functional calculus, the statement that $(a,b) \cap \sigma(A_n) = \varnothing$ is equivalent to the statement that $\|(A-\lambda_0)^{-1}\|_\# \leq \sqrt{2}/(b-a)$ where $\lambda_0 = (a+b)/2 + i(b-a)/2$. But $(A_n - \lambda_0)^{-1}$ #-converges strongly to $(A-\lambda_0)^{-1}$ so we have $\|(A-\lambda_0)^{-1}\|_\# \leq$ #-$\overline{\lim}_{n\to{}^*\infty}\|(A_n-\lambda_0)^{-1}\|_\# \leq \sqrt{2}/(b-a)$. This proves (a).

To prove (b), we find uniformly bounded sequences of #-continuous functions $(f_n)_{n=1}^{*\infty}$ and $(g_n)_{n=1}^{*\infty}$ so that $0 \leq f_n \leq \chi_{(a,b)}, f_n(x) \to_\# \chi_{(a,b)}(x)$ pointwise and $\chi_{(a,b)} \leq g_n$, $g_n(x) \to_\# \chi_{[a,b]}(x)$ pointwise. Then $f_n(A) \to_\# P_{(a,b)}(A)$ and $g_n(A) \to_\# P_{[a,b]}(A)$ strongly. Since $a,b \notin \sigma_{\mathbf{pp}}(A)$, $P_{(a,b)}(A) = P_{[a,b]}(A)$ which means that given ψ and $\varepsilon \approx 0, \varepsilon > 0$, we can find #-continuous functions $f,g,$ with $f \leq \chi_{(a,b)} \leq \chi_{[a,b]} \leq g$ so that $\|f(A)\psi - f(A)\psi\|_\# \leq \varepsilon/5$ By Theorem 4.7.3(b) we can find $N \in {}^*\mathbb{N}$ so that $n > N$ implies $\|f(A_n)\psi - f(A)\psi\|_\# \leq \varepsilon/5$ and $\|g(A_n)\psi - g(A)\psi\|_\# \leq \varepsilon/5$ so by an $\varepsilon/3$ argument we get $\|g(A_n)\psi - g(A_n)\psi\|_\# \leq 3\varepsilon/5$. Since the functional calculus implies $\|f(A)\psi - P_{(a,b)}(A)\psi\|_\# \leq \|f(A)\psi - g(A)\psi\|_\#$ another $\varepsilon/3$ argument implies $\|P_{(a,b)}(A)\psi - P_{(a,b)}(A)\psi\|_\# \leq \varepsilon$.

Remark.Part (a) of Theorem 4.7.7 says that the spectrum of the #-limiting operator cannot suddenly expand. It can, however, contract rather spectacularly as the following example shows: Let $(A_n)_{n=1}^{*\infty} = (x/n)_{n=1}^{*\infty}$ on $L_2^\#({}^*\mathbb{R}_c^\#)$ then A_n #-converges to the zero operator in the strong resolvent sense. For each n, $\sigma(A_n) = {}^*\mathbb{R}_c^\#$, but the spectrum of the #-limiting operator contains only the origin. An easy application of part (a) is the statement that if the A_n are positive and $A_n \to_\# A$ in the strong resolvent sense, then A is positive.

If A_n #-converges to A in #-norm resolvent sense, Theorem 4.7.6 tells us that the spectrum of the #-limiting operator cannot suddenly contract in the sense that if $\lambda \in \sigma(A_n)$ for all sufficiently infinitely large n, then $\lambda \in \sigma(A)$. Notice that in the example $A_n = x/n$ above, A_n does not #-converge to A in the #-norm resolvent sense. The principle of noncontraction of the spectrum under #-norm resolvent #-convergence remains true even when A_n and A are not self-#-adjoint. But the principle of nonexpansion of the spectrum in the strong resolvent #-limit is not always valid for general not-necessarily-self-#-adjoint operators. In fact, there exists a #-norm #-convergent sequence of uniformly bounded operators $A_n \to_\# A$ with $\sigma(A_n)$ the unit circle in ${}^*\mathbb{C}_c^\#$ for each $n \in {}^*\mathbb{N}$ and $\sigma(A)$ the entire unit disc. Thus the reader should be careful to apply Theorem 4.7.7 only in the self-#-adjoint case.

In applications, one is usually given the operators $(A_n)_{n=1}^{*\infty}$ and A on domains of self-#-adjointness or essential self-#-adjointness and it may be very difficult to compute the resolvents. Thus, in order to use Theorem 4.7.6 and Theorem 4.7.7 one must have criteria on the operators $(A_n)_{n=1}^{*\infty}$ and A themselves which guarantee

#-norm or strong resolvent #-convergence.

Theorem 4.7.8. (a) Let $(A_n)_{n=1}^{*\infty}$ and A be self-#-adjoint operators and suppose that D is a common #-core for all A_n and A. If $A_n\varphi \to_\# A\varphi$ for each $\varphi \in D$ then $A_n \to_\# A$ as $n \to {}^*\infty$ in the strong resolvent sense.

(b) Let $(A_n)_{n=1}^{*\infty}$ and A be self-#-adjoint operators with a common domain, D. Norm D with $\|\varphi\|_{\#A} = \|A\varphi\|_\# \dot{+} \|\varphi\|_\#$. If $\sup_{\|\varphi\|_{\#A}=1}(\|(A_n - A)\varphi\|_\#) \to_\# 0$ as $n \to {}^*\infty$ then $A_n \to_\# A$ in the #-norm resolvent sense.

(c) Let $(A_n)_{n=1}^{*\infty}$ and A be positive self-#-adjoint operators with a common form domain $H_{+1}^\#$ which we norm with $\|\psi\|_{\#+1} = \langle\psi, A\psi\rangle_\# + \langle\psi,\psi\rangle_\#$. If $A_n \to_\# A$ in #-norm in the sense of maps from $H_{+1}^\#$ to $H_{-1}^\#$ that is, if

$$\sup_{\psi\neq 0,\varphi\in D} \frac{|\langle\varphi,(A - A_n)\psi\rangle_\#|}{\|\varphi\|_{\#+1}\|\psi\|_{\#+1}} = \sup_{\psi\neq 0,\psi\in D} \frac{|\langle\varphi,(A - A_n)\psi\rangle_\#|}{\langle\psi,(A + I)\psi\rangle_\#} \to_\# 0 \qquad (4.7.18)$$

then $A_n \to_\# A$ in the #-norm resolvent sense.

Proof (a) Let $\varphi \in D$, $\psi = (A + i)\varphi$, then $\left[(A_n + i)^{-1} - (A + i)^{-1}\right]\psi = (A_n + i)^{-1}(A - A_n)\varphi$ #-converges to zero as as $n \to {}^*\infty$, since $(A - A_n)\varphi \to_\# 0$ and the $(A_n + i)^{-1}$ are uniformly bounded. Since D is a #-core for A the set of such ψ is #-dense so for all $\varphi \in H^\# : (A_n + i)^{-1}\varphi \to_\# (A + i)^{-1}\varphi$. A similar proof holds for $(A_n - i)^{-1}$.

We sketch the proofs of (b) and (c). For (b), first one proves that the hypothesis is equivalent to $(A_n - A)(A + i)^{-1} \to_\# 0$ in the ordinary $H^\#$-operator #-norm. Thus $\left(I + (A_n - A)(A + i)^{-1}\right)^{-1}$ exists and #-converges to I in #-norm as as $n \to {}^*\infty$. As a result $(A_n + i)^{-1} = (A + i)^{-1}\left(I + (A_n - A)(A + i)^{-1}\right)^{-1} \to_\# (A + i)^{-1}$in #-norm. Similarity $(A_n - i)^{-1} \to_\# (A - i)^{-1}$. To prove (c), we first prove that the hypothesis is equivalent to $(A + I)^{-1/2}(A_n - A)(A + I)^{-1/2} \to_\# 0$ in the ordinary operator #-norm. Using the identity $(A_n + I)^{-1} = (A + I)^{-1/2}\left[I + (A + I)^{-1/2}(A_n - A)(A + I)^{-1/2}\right]^{-1}(A + I)^{-1/2}$ one then follows the proof of (b).

§ 4.8.Graph #-limits.

Definition 4.8.1.Let $(A_n)_{n=1}^{*\infty}$ be a hyper infinite sequence of operators on a non Archimedean Hilbert space $H^\#$, We say that a pair $\langle\psi,\varphi\rangle_\# \in H^\# \times H^\#$ is in the strong graph #-limit of A_n as as $n \to {}^*\infty$if we can find $\psi_n \in D(A_n)$ so that $\psi_n \to_\# \psi$, $A_n\psi_n \to_\# \varphi$. We denote the set of pairs in the strong graph #-limit by $\Gamma_{*\infty}^{\mathfrak{s}}$. If $\Gamma_{*\infty}^{\mathfrak{s}}$ is the graph of an operator A we say that A is the strong graph #-limit of A_n and write

$$A = \mathbf{st.gr.}\text{-\#-}\lim A_n. \qquad (4.8.1)$$

First, we consider the case where all the A_n are self-#-adjoint and A is also self-#-adjoint

Theorem 4.8.1. Suppose that $(A_n)_{n=1}^{*\infty}$ and A are self-#-adjoint operators. Then $A_n \to_\# A$ in the strong resolvent sense if and only if $A = \mathbf{st.gr.}\text{-\#-}\lim A_n$.

Proof Suppose first that $(A_n + i)^{-1} \to_\# (A + i)^{-1}$ strongly. Suppose $\varphi \in D(A)$. Then $\varphi_n = (A_n + i)^{-1}(A + i)\varphi \to_\# \varphi$ and $A_n\varphi_n = (A + i)\varphi - i\varphi$, so $\langle\varphi, A\varphi\rangle_\# \in \Gamma_{*\infty}^{\mathfrak{s}}$. Thus $\Gamma(A) \subset \Gamma_{*\infty}^{\mathfrak{s}}$. On the other hand, suppose $\varphi_n \in D(A_n), \varphi_n \to_\# \varphi$ and $A_n\varphi_n \to_\# \psi$. We let $\eta_n = (A + i)^{-1}(A_n + i)\varphi_n \in D(A)$, then

$$\begin{aligned}\eta_n - \varphi_n &= [(A+i)^{-1} - (A_n+i)^{-1}][(A_n+i)\varphi_n] = \\ &= [(A+i)^{-1} - (A_n+i)^{-1}][(A_n+i)\varphi_n - \psi - i\varphi] + \\ &\quad +[(A+i)^{-1} - (A_n+i)^{-1}][\psi + i\varphi] \to_{\#} 0\end{aligned} \tag{4.8.2}$$

as $n \to {}^*\infty$. Thus, $\eta_n \to_{\#} \varphi$ and $A\eta_n = (A_n+i)\varphi_n - i\eta_n \to_{\#} \psi$ so since A is #-closed $\langle\varphi,\psi\rangle_{\#} \in \Gamma(A)$. Thus, $\Gamma(A) = \Gamma^{\mathbf{s}}_{{}^*\infty}$.

Conversely, suppose that $A = \mathbf{st.gr.}\text{-\#-}\lim A_n$. Let $\varphi \in D(A)$. Then there exist $\varphi_n \in D(A_n)$ so that $\varphi_n \to_{\#} \varphi$ and $A_n\varphi_n \to_{\#} A\varphi$ as $n \to {}^*\infty$. Thus,

$[(A_n+i)^{-1} - (A+i)^{-1}][(A+i)\varphi] = (A_n+i)^{-1}[(A_n+i)\varphi - (A_n+i)\varphi_n] - (\varphi - \varphi_n) \to_{\#} 0$

as $n \to {}^*\infty$since $\|(A_n+i)^{-1}\|_{\#} \leq 1, (A_n+i)\varphi_n \to_{\#} (A+i)\varphi$, and $\varphi_n \to_{\#} \varphi$. Since $\mathbf{Ran}(A+i) = H.^{\#}$ the strong #-convergence of $(A_n+i)^{-1}$ to $(A+i)^{-1}$ follows.

Remark 4.8.1. Thus, we see that if the #-limit is self-#-adjoint, then strong graph and strong resolvent #-convergence are the same. It is in the case when we do not know a priori that the #-limit is self-#-adjoint that strong graph #-limits are particularly important. For example, the existence of graph #-limits can sometimes be combined with other information to prove that the #-limit is self-#-adjoint.

Theorem 4.8.2. Let $(A_n)_{n=1}^{*\infty}$ be a hyper infinite sequence of symmetric operators.

(a) Let $D^{\mathbf{s}}_{{}^*\infty} = \left\{\psi|\langle\psi,\varphi\rangle_{\#} \in \Gamma^{\mathbf{s}}_{{}^*\infty}\text{for some } \varphi\right\}$.If $D^{\mathbf{s}}_{{}^*\infty}$ is #-dense, then $\Gamma^{\mathbf{s}}_{{}^*\infty}$ is the graph of an operator.

(b) Suppose that $D^{\mathbf{s}}_{{}^*\infty}$ is #-dense and let $A = \mathbf{st.gr.}\text{-\#-}\lim A_n$.

Then A is symmetric and #-closed.

Proof We will prove (a); the proof of (b) is obvious. Suppose $\varphi_n, \varphi_n' \in D(A_n)$ and $\varphi_n \to_{\#} \varphi, \varphi_n' \to_{\#} \varphi$ and $A_n\varphi_n \to_{\#} \psi, A_n\varphi_n' \to_{\#} \psi'$. Let $\eta \in D^{\mathbf{s}}_{{}^*\infty}$. Then there is an $\eta_n \in D(A_n)$, so that $\eta_n \to_{\#} \eta$ and $A_n\eta_n \to_{\#} \rho$ as $n \to {}^*\infty$. Thus,

$\langle\psi - \psi, \eta\rangle_{\#} = \text{\#-}\lim_{n\to{}^*\infty}\langle A_n(\varphi_n - \varphi_n'), \eta_n\rangle_{\#} = \text{\#-}\lim_{n\to{}^*\infty}\langle\varphi_n - \varphi_n', A_n\eta_n\rangle_{\#} = 0$

so $\psi = \psi'$ since $D^{\mathbf{s}}_{{}^*\infty}$ is #-dense.

We also define weak graph #-limits. We give the definition and state one theorem.

Definition 4.8.2. Let $(A_n)_{n=1}^{*\infty}$ be a gyper infinite sequence of operators on $H^{\#}$. We say that $\langle\psi,\varphi\rangle_{\#} \in H^{\#} \times H^{\#}$ is in the weak graph #-limit $\Gamma^{\mathbf{w}}_{{}^*\infty}$ if we can find $\psi_n \in D(A_n)$ so that $\psi_n \overset{\|\cdot\|_{\#}}{\to_{\#}} \psi$ and $A_n\psi_n \to_{\#} \varphi$ weakly. If $\Gamma^{\mathbf{w}}_{{}^*\infty}$ is the graph of an operator, A we say that A is the weak graph #-limit of A_n and abbreviated as $A = \mathbf{w.gr.}\text{-\#-lim } A_n$.

Theorem 4.8.3. Let $(A_n)_{n=1}^{*\infty}$ be a gyper infinite sequence of symmetric operators. If $D^{\mathbf{s}}_{{}^*\infty} = \left\{\psi|\langle\psi,\varphi\rangle_{\#} \in \Gamma^{\mathbf{s}}_{{}^*\infty}\text{for some } \varphi\right\}$ is #-dense, then $\Gamma^{\mathbf{w}}_{{}^*\infty}$ is the graph of a symmetric operator.

Remark 4.8.2. Finally we note that if A_n is a uniformly bounded sequence of operators then $A = \mathbf{w.gr.}\text{-\#-lim } A_n$ if and only if $A_n \to_{\#} A$ as $n \to {}^*\infty$in the weak operator topology. This fact shows that the notions of weak graph #-limit and weak resolvent #-convergence are distinct. It is not true that weak graph #-limits are necessarily #-closed if each A_n is symmetric.

§ 4.9. Generalized Trotter product formula

Theorem 4.9.1. (Generalized Lie product formula) Let A and B be external hyperfinite-dimensional matrices.Then

$$Ext\text{-}\exp(A+B) = \text{\#-}\lim_{n\to{}^*\infty}\{[Ext\text{-}\exp(A/n)] \times [Ext\text{-}\exp(B/n)]\}^n. \tag{4.9.1}$$

Proof Let $S_n = Ext\text{-}\exp((A+B)/n)$ and $T_n = [Ext\text{-}\exp(A/n)] \times [Ext\text{-}\exp(B/n)]$. Then

$$S_n^n - T_n^n = Ext\text{-}\sum_{m=0}^{n-1} S_n^m (S_n - T_n) T_n^{n-m-1} \qquad (4.9.2)$$

so that

$$\begin{gathered}\|S_n^n - T_n^n\|_{\#} \leq n(\max(\|S_n\|_{\#}, \|T_n\|_{\#}))^{n-1} \|S_n - T_n\|_{\#} \leq \\ \leq n\|S_n - T_n\|_{\#}[Ext\text{-}\exp(\|A\|_{\#} + \|B\|_{\#})].\end{gathered} \qquad (4.9.3)$$

Since

$$\begin{gathered}\|S_n - T_n\|_{\#} = \\ \left\| Ext\text{-}\sum_{m=0}^{{}^*\infty} \frac{1}{m!^{\#}}\left(\frac{A+B}{n}\right)^m - \left(Ext\text{-}\sum_{m=0}^{{}^*\infty} \frac{1}{m!^{\#}}\left(\frac{A}{n}\right)^m\right)\left(Ext\text{-}\sum_{m=0}^{{}^*\infty} \frac{1}{m!^{\#}}\left(\frac{B}{n}\right)^m\right)\right\|_{\#} \qquad (4.9.4) \\ \leq C/n\end{gathered}$$

where constant C depends only on $\|A\|_{\#}$ and $\|B\|_{\#}$we conclude that
$\#\text{-}\lim_{n\to{}^*\infty} \|S_n^n - T_n^n\|_{\#} = 0.$

Remark 4.9.1. This theorem and its proof can be extended to the case where A and B are unbounded self-#-adjoint operators and $A+B$ is self-#-adjoint on $D(A) \cap D(B)$.

Theorem 4.9.2 Let A and B be self-#-adjoint operators on $H^{\#}$ and suppose that $A+B$ is self-#-adjoint on $D = D(A) \cap D(B)$. Then

$$Ext\text{-}\exp[it(A+B)] = \mathbf{s}\text{-}\#\text{-}\lim_{n\to{}^*\infty}\{[Ext\text{-}\exp(itA/n)] \times [Ext\text{-}\exp(itB/n)]\}^n. \quad (4.9.5)$$

Proof Let $\psi \in D$. Then

$$\begin{gathered}s^{-1}\{[Ext\text{-}\exp(isA)] \times [Ext\text{-}\exp(isB)] - I\}\psi = \\ s^{-1}\{[Ext\text{-}\exp(isA)] - I\}\psi + s^{-1}\{[Ext\text{-}\exp(isB)] - I\}\psi \to_{\#} i(A+B)\psi\end{gathered} \qquad (4.9.6)$$

and

$$s^{-1}\{[Ext\text{-}\exp(isA)] \times [Ext\text{-}\exp(isB)] - I\}\psi \to_{\#} i(A+B)\psi \qquad (4.9.7)$$

as $s \to_{\#} 0$. Letting

$$K(s) = s^{-1}\{[Ext\text{-}\exp(isA)] \times [Ext\text{-}\exp(isB)] - [Ext\text{-}\exp(is(A+B))]\} \qquad (4.9.8)$$

we see that $K(s)\psi \to_{\#} 0$ as $s \to_{\#} 0$, for each $\psi \in D$. Since $A+B$ is self-#-adjoint on D, D is a
Banach space under the #-norm

$$\|\psi\|_{\#A+B} = \|(A+B)\psi\|_{\#} \dotplus \|\psi\|_{\#}. \qquad (4.9.9)$$

Each of the maps $K(s) : D \to H^{\#}$ is bounded and $K(s)\psi \overset{H^{\#}}{\to_{\#}} 0$ as $s \to_{\#} 0$ or ${}^*\infty$ for each $\psi \in D$.
Thus, we conclude from the uniform boundedness theorem that the $K(s)$ are uniformly bounded, that is, there is a constant C so that $\|K(s)\psi\|_{\#} \leq C\|\psi\|_{\#A+B}$ for all $s \in {}^*\mathbb{R}_c^{\#}$ and $\psi \in D$.Therefore, an $\varepsilon/3$ argument shows that on $\|\cdot\|_{\#A+B}$ #-compact subsets of D $K(s)\psi \to_{\#} 0$ uniformly.Since $A \dotplus B$ is self-#-adjoint on D, $[Ext\text{-}\exp(is(A+B))]\psi \in D$ if $\psi \in D$. Moreover, $s \to [Ext\text{-}\exp(is(A+B))]\psi$ is a #-continuous map of ${}^*\mathbb{R}_c^{\#}$ into D when D is given the $\|\cdot\|_{\#A+B}$ #-norm topology. Thus $\{[Ext\text{-}\exp(is(A+B))]\psi | s \in [-1,1]\}$ is a $\|\cdot\|_{\#A+B}$ #-compact set in D for each fixed ψ.

We are now ready to mimic the proof of the generalized Lie product formula. We know that

$$t^{-1}\{[Ext\text{-}\exp(itA)] \times [Ext\text{-}\exp(itB)] - [Ext\text{-}\exp(it(A+B))]\} \times \times [Ext\text{-}\exp(is(A+B))]\psi \to_{\#} 0 \quad (4.9.10)$$

uniformly for $s \in [-1,1]$. Therefore, we write

$$(\{[Ext\text{-}\exp(itA/n)] \times [Ext\text{-}\exp(itB/n)]\}^n - [Ext\text{-}\exp(it(A+B)/n)]^n)\psi = Ext\text{-}\sum_{k=0}^{n}\{[Ext\text{-}\exp(itA/n)] \times [Ext\text{-}\exp(itB/n)]\}^k \times [(\{[Ext\text{-}\exp(itA/n)] \times [Ext\text{-}\exp(itB/n)]\} - [Ext\text{-}\exp(it(A+B)/n)])] \times [Ext\text{-}\exp(it(A+B)/n)]^{n-k-1}\psi \quad (4.9.11)$$

The #-norm of the RHS of (4.9.11)

$$|t| \times \max_{|s|<t} \left\| \left(\frac{t}{n}\right)^{-1}\{[Ext\text{-}\exp(it(A+B)/n)] - \{[Ext\text{-}\exp(itA/n)] \times [Ext\text{-}\exp(itB/n)]\}\} \right\|_{\#} \quad (4.9.12)$$

and so we conclude that

$$\{[Ext\text{-}\exp(itA/n)] \times [Ext\text{-}\exp(itB/n)]\}^n\psi \overset{H^{\#}}{\to_{\#}} Ext\text{-}\exp(it(A+B))\psi \quad (4.9.13)$$

as $n \to {}^*\infty$ if $\psi \in D$; Since D is #-dense and the operators are bounded by one, this statement holds on all of $H^{\#}$. The above proof shows that on a fixed vector the #-convergence is uniform for t in a #-compact subset of ${}^*\mathbb{R}_c^{\#}$.

Remark 4.9.2. The same argument can be used to show that

$$\mathbf{s}\text{-}\#\text{-}\lim_{n\to{}^*\infty}\{[Ext\text{-}\exp(itA/n)] \times [Ext\text{-}\exp(itB/n)]\}^n = Ext\text{-}\exp(t(A+B)) \quad (4.9.14)$$

if A and B satisfy the same hypotheses and in addition are semibounded. The following result is considerably stronger than Theorem 4.9.2 since it only requires essential self-#-adjoint ness of $A \dotplus B$ on $D(A) \cap D(B)$.

Theorem 4.9.3 (the generalized Trotter product formula) If A and B are self-#-adjoint operators and $A + B$ is essentially self-adjoint on $D(A) \cap D(B)$ then

$$\mathbf{s}\text{-}\#\text{-}\lim_{n\to{}^*\infty}\{[Ext\text{-}\exp(itA/n)] \times [Ext\text{-}\exp(itB/n)]\}^n = Ext\text{-}\exp(it(A+B)) \quad (4.9.15)$$

Moreover, if A and B are bounded from below, then

$$\mathbf{s}\text{-}\#\text{-}\lim_{n\to{}^*\infty}\{[Ext\text{-}\exp(-tA/n)] \times [Ext\text{-}\exp(-tB/n)]\}^n = Ext\text{-}\exp(-t(A+B)). \quad (4.9.16)$$

§ 4.10.The polar decomposition

Note that an arbitrary bounded operator T can be written $T = U|T|$ where $|T|$ is positive and self-#-adjoint and U is a partial isometry. Moreover, the conditions that $\mathbf{Ker}(|T|) = = \mathbf{Ker}(T)$ and that the initial space of U equals $(\mathbf{Ker}(T))^{\perp}$ uniquely determine $|T|$ and U. In this section we extend this result to closed #-unbounded operators. As in the bounded case, U is easy to construct once $|T|$ has been constructed and, as in the bounded case, we will let $|T| = \sqrt{T^*T}$. In the bounded case, the hard part was the construction of the square root. Now that we have the spectral theorem, it is easy to construct $\sqrt{T^*T}$ if we can prove that T^*T is a positive self-

#-adjoint operator. It is this fact that is hard in the unbounded case. A priori, it is not clear that $\{\psi|\psi \in D(T)$ and $T\psi \in D(T^{*})\}$ is different from $\{0\}$. In fact, this set is #-dense, but our approach using the theory of semi-bounded quadratic forms does not require us to prove this.

Theorem 4.10.1. (the polar decomposition) Let Γ be an arbitrary #-closed operator on a
non Archimedean Hilbert space $H^{\#}$. Then, there is a positive self-adjoint operator $|T|$, with $D(|T|) = D(T)$ and a partial isometry U with initial space, $(\mathbf{Ker}(T))^{\perp}$, and final space #-$\overline{\mathbf{Ran}(T)}$ so that $T = U|T|$ and U are uniquely determined by these properties together with the additional condition $\mathbf{Ker}(|T|) = \mathbf{Ker}(T)$.

Proof. Define the ${}^{*}\mathbb{C}_{c}^{\#}$-valued quadratic form $s(\psi,\varphi)$ on $D(T)$ by

$$s(\psi,\varphi) = \langle T\psi, T\varphi\rangle_{\#}. \qquad (4.10.1)$$

Quadratic form $s(\psi,\varphi)$ is clearly positive. Now suppose $\|\psi_n - \psi_m\|_{\#+1} \to_{\#} 0$. Then $\|\psi_n - \psi_m\|_{\#} \to_{\#} 0$ and $\|T(\psi_n - \psi_m)\|_{\#} \to_{\#} 0$. Since T is #-closed there is a $\psi \in D(T)$ with

$$\|\psi_n - \psi\|_{\#} \dot{+} \|T(\psi_n - \psi)\|_{\#} \to_{\#} 0, \qquad (4.10.2)$$

i.e. $\|\psi_n - \psi\|_{\#+1} \to_{\#} 0$.

Thus $s(\psi,\varphi)$ is a #-closed form. Therefore, by **Theorem VIII.15**, there is a unique, positive self-#-adjoint operator S with $Q(S) = D(T)$ and $s(\psi,\varphi) = \langle\psi, S\varphi\rangle_{\#}$ in the sense of ${}^{*}\mathbb{C}_{c}^{\#}$-valued quadratic forms. Let $|T| = S^{1/2}$. Then $D(|T|) = Q(S) = D(T)$ and by construction $\||T|\psi\|_{\#}^{2} = s(\psi,\psi) = \|T\psi\|_{\#}^{2}$ so $\mathbf{Ker}(|T|) = \mathbf{Ker}(T)$. Define the operator $U : \mathbf{Ran}(|T|) \to \mathbf{Ran}(T)$ by $U|T|\psi = T\psi$. Since $\||T|\psi\|_{\#} = \|T\psi\|_{\#}$, U is well defined and #-norm preserving. Thus U extends to a partial isometry from $\mathbf{Ran}(|T|)$ to $\mathbf{Ran}(T)$. Finally, since $|T|$ is self-#-adjoint, #-$\overline{\mathbf{Ran}(T)} = (\mathbf{Ker}(|T|))^{\perp} = (\mathbf{Ker}(T))^{\perp}$. Uniqueness is obvious.

§ 5.Tensor products and second quantization.

§ 5.1.Tensor products.

In this section we describe some aspects of the theory of tensor products of operators on non Archimedean Hilbert spaces. Let A and. B be #-densely defined operators on non Archimedean Hilbert spaces $H_1^{\#}$ and $H_2^{\#}$ respectively. We will denote by $D(A) \otimes D(B)$ the set of hyperfinite linear combinations of vectors of the form $\varphi \otimes \psi$ where $\varphi \in D(A)$ and $\psi \in D(B)$. $D(A) \otimes D(B)$ is dense in $H_1^{\#} \otimes H_2^{\#}$ We define $A \otimes B$ on $D(A) \otimes D(B)$ by

$$A \otimes B(\varphi \otimes \psi) = A\varphi \otimes B\psi \qquad (5.1.1)$$

and extend by linearity.

Proposition 5.1.1 The operator $A \otimes B$ is well defined. Further, if A and B are #-closable,
so is $A \otimes B$.

Proof Suppose that $Ext\text{-}\sum c_i\, \varphi_i \otimes \psi_i$ and $Ext\text{-}\sum d_j \varphi_j' \otimes \psi_j'$ are two representations of the same vector $f \in D(A) \otimes D(B)$. Using Gram-Schmidt orthogonalization we obtain bases $\{\eta_k\}$ and $\{\theta_l\}$ for the spaces spanned by $\{\varphi_i\} \cup \{\varphi_i'\}$ and $\{\psi_j\} \cup \{\psi_j'\}$ respectively so that $\eta_k \in D(A)$ and $\theta_l \in D(B)$. $\varphi_i \otimes \psi_i$ and $\varphi_j' \otimes \psi_j'$ can be expressed

$$\varphi_i \otimes \psi_i = Ext\text{-}\sum_{kl} \alpha^i_{kl}(\eta_k \otimes \theta_l) \quad (5.1.2)$$

and

$$\varphi'_j \otimes \psi'_j = Ext\text{-}\sum_{kl} \beta^j_{kl}(\eta_k \otimes \theta_l). \quad (5.1.3)$$

Since the two expressions for f give the same vector, $Ext\text{-}\sum_i c_i\alpha^i_{kl} = Ext\text{-}\sum_j d_j\beta^j_{kl}$ for each pair (k,l). Thus,

$$\begin{gathered}(A \otimes B)\left[Ext\text{-}\sum_i c_i(\varphi_i \otimes \psi_i)\right] = Ext\text{-}\sum_{kl}\left(Ext\text{-}\sum_i c_i\alpha^i_{kl}\right)(A\eta_k \otimes B\theta_l) = \\ Ext\text{-}\sum_{kl}\left(Ext\text{-}\sum_j d_j\beta^j_{kl}\right)(A\eta_k \otimes B\theta_l) = (A \otimes B)\left[Ext\text{-}\sum_j d_j(\varphi'_j \otimes \psi'_j)\right]\end{gathered} \quad (5.1.4)$$

so $A \otimes B$ is well defined. If g is any vector in $D(A^*) \otimes D(B^*)$ then $\langle (A \otimes B)f, g\rangle_\# = \langle f, (A^* \otimes B^*)g\rangle_\#$ so

$$D(A^*) \otimes D(B^*) \subset D((A \otimes B)^*). \quad (5.1.5)$$

If A and B are #-closable, $D(A*)$ and $D(B^*)$ are #-dense. Therefore, in that case $(A \otimes B)^*$ is #-densely defined which proves that $A \otimes B$ is #-closable.
Similarly, if A and B are #-closable then $A \otimes I + I \otimes B$ defined on $D(A) \otimes D(B)$ is #-closable.

Definition 5.1.1. Let A and B be #-closable operators on a non Archimedean Hilbert spaces $H^\#_1$ and $H^\#_2$. The tensor product of A and B is the #-closure of the operator $A \otimes B$ defined on $D(A) \otimes D(B)$. We will denote the #-closure by $A \otimes B$ also. Usually $A \dotplus B$ will denote the #-closure of $A \otimes I + I \otimes B$ on $D(A) \otimes D(B)$.

Proposition 5.1.1. Let A and B be bounded in ${}^*\mathbb{R}^\#_c$ operators on a non Archimedean Hilbert spaces $H^\#_1$ and $H^\#_2$. Then $\|A \otimes B\|_\# = \|A\|_\# \times \|B\|_\#$.

Proof Let $\{\varphi_k\}$ and $\{\psi_l\}$ be orthonormal bases for $H^\#_1$ and $H^\#_2$ and suppose $Ext\text{-}\sum_{kl} c_{kl}\varphi_k \otimes \psi_l$ is a gyperfinite sum. Then

$$\begin{gathered}\|(A \otimes I)\left[Ext\text{-}\sum_{kl} c_{kl}\varphi_k \otimes \psi_l\right]\|^2_\# = Ext\text{-}\sum_l \|Ext\text{-}\sum_k c_{kl}A\varphi_k\|^2_\# = \\ \leq Ext\text{-}\sum_l \|A\|^2_\#\left(Ext\text{-}\sum_k |c_{kl}|^2\right) = \|A\|^2_\#\|Ext\text{-}\sum_k c_{kl}\varphi_k \otimes \psi_l\|^2_\#.\end{gathered} \quad (5.1.6)$$

Since the set of such gyperfinite sums is #-dense in $H^\#_1 \otimes H^\#_2$, we conclude that $\|A \otimes I\|_\# \leq \|A\|_\#$. Thus $\|A \otimes B\|_\# \leq \|A \otimes I\|_\# \times \|B \otimes I\|_\# \leq \|A\|_\# \times \|B\|_\#$.
Conversely, given $\varepsilon \approx 0, \varepsilon > 0$, there exist unit vectors $\varphi \in H^\#_1$, $\psi \in H^\#_2$ so that

$$\|A\varphi\|_\# \geq \|A\|_\# - \varepsilon \quad (5.1.7)$$

and

$$\|B\psi\|_\# \geq \|B\|_\# - \varepsilon. \quad (5.1.8)$$

Then

$$\|A \otimes B(\varphi \otimes \psi)\|_\# = \|A\varphi\|_\# \times \|B\psi\|_\# \geq \|A\|_\# \times \|B\|_\# - \varepsilon\|A\|_\# - \varepsilon\|B\|_\# \dotplus \varepsilon^2. \quad (5.1.9)$$

Since $\varepsilon > 0$ is arbitrary $\|A \otimes B\|_\# \geq \|A\|_\# \times \|B\|_\#$. which concludes the proof.

Remark 5.1.1.We notice that both of the above propositions have natural generalizations to arbitrary hyperfinite tensor products of operators. This can be proven directly or by using the associativity of the hyperfinite tensor product of a non Archimedean Hilbert spaces.

Remark 5.1.2.We turn now to questions of self-adjointness and spectrum. Let $(A_k)^N_{k=1}$ be a hyperfinite family of operators, A_k self-#-adjoint on $H^\#_k$. We will denote the

#-closure of $I_1 \otimes \cdots \otimes A_k \otimes \cdots \otimes I_N$ on $D = Ext\text{-}\otimes_{k=1}^{N} D(A_k)$ by A_k also. Let $P(x_1,\ldots,x_N)$ be a polynomial with ${}^*\mathbb{R}_c^{\#}$-valued coefficients of degree n_k in x_k. Then, the operator $P(A_1,\ldots,A_N)$ makes sense on $Ext\text{-}\otimes_k D(A_{n_k})$ since $D(A_{n_k}) \subset D(A_l)$ for all $l \leq n_k$. In fact, $P(A_1,\ldots,A_N)$ is essentially self-#-adjoint on that domain.

Theorem 5.1.1. Let A_k be a self-#-adjoint operator on $H_k^{\#}$. Let $P(x_1,\ldots,x_N)$ be a polynomial with ${}^*\mathbb{R}_c^{\#}$-valued coefficients of degree n_k in the k-th variable and suppose that D_k^l is a domain of essential self-#-adjointness for $A_k^{n_k}$. Then,

(a) $P(A_1,\ldots,A_N)$ is essentially self-adjoint on $D^l = Ext\text{-}\otimes_{k=1}^{N} D_k^l$.

(b) The spectrum of #-$\overline{P(A_1,\ldots,A_N)}$ is the #-closure of the range of $P(A_1,\ldots,A_N)$ on the product of the spectra of the A_k. That is $\sigma\left(\#\text{-}\overline{P(A_1,\ldots,A_N)}\right) = \#\text{-}\overline{P(\sigma(A_1),\ldots,\sigma(A_N))}$.

Proof We will first prove that $P(A_1,\ldots,A_N)$ is essentially self-#-adjoint on $D = Ext\text{-}\otimes_{k=1}^{N} D(A_k^{n_k})$. By the spectral theorem, there is a #-measure space $\langle M_k, \mu_k^{\#}\rangle$ so that A_k is unitarily equivalent to multiplication by a ${}^*\mathbb{R}_c^{\#}$-valued #-measurable function f_k on $L_2^{\#}(M_k, d^{\#}\mu_k^{\#})$. Thus we may assume that $\mu_k^{\#}$ is hyperfinite and that $f_k \in \bigcap_{1 \leq p < {}^*\infty} L_p^{\#}(M_k, d^{\#}\mu_k^{\#})$. Furthermore $Ext\text{-}\otimes_{k=1}^{N} L_2^{\#}(M_k, d^{\#}\mu_k^{\#})$ is naturally isomorphic to $L_2^{\#}(Ext\text{-}\times_{k=1}^{N} M_k, Ext\text{-}\otimes_{k=1}^{N} d^{\#}\mu_k^{\#})$. Under this isomorphism $P(A_1,\ldots,A_N)$ corresponds to multiplication by $P(f_1,\ldots,f_N)$ and D corresponds to the set of hyperfinite linear combinations of hyperfinite linear combinations of functions $Ext\text{-}\prod_{i=1}^{N} \phi_i(m_i)$ such that $f_k^{n_k}\phi_k \in L_2^{\#}(M_k, d^{\#}\mu_k^{\#})$.

To prove essential self-#-adjointness we use result from functional calculus. First, since $\mu_k^{\#}$ is hyperfinite and $f_k^{n_k}\phi_k \in L_2^{\#}(M_k, d^{\#}\mu_k^{\#})$ we conclude that $f_k^{l} \in L_p^{\#}(M_k, d^{\#}\mu_k^{\#})$ for $1 \leq p < {}^*\infty$. From this it follows immediately that $P(f_1,\ldots,f_N)$ is in $L_p^{\#}$ for all such p; in particular $P(f_1,\ldots,f_N) \in L_4^{\#}(Ext\text{-}\times_{k=1}^{N} M_k, Ext\text{-}\otimes_{k=1}^{N} d^{\#}\mu_k^{\#})$. Since $f_k^{n_k}$ is self-#-adjoint on D_k, D_k contains the characteristic functions of #-measurable sets in M_k.

Thus D contains all hyperfinite linear combinations of the characteristic functions of rectangles. By the property on product #-measures we conclude that the characteristic function of any #-measurable set in M_k is equal to such a hyperfinite linear combination except on a set of arbitrarily small $Ext\text{-}\otimes_{k=1}^{N} d^{\#}\mu_k^{\#}$ #-measure. Thus the simple functions on $Ext\text{-}\times_{k=1}^{N} M_k$ can be approximated in the $L_p^{\#}$ sense with $1 \leq p < {}^*\infty$ by elements of D. In particular D is #-dense in $L_4^{\#}(Ext\text{-}\times_{k=1}^{N} M_k, Ext\text{-}\otimes_{k=1}^{N} d^{\#}\mu_k^{\#})$. Essential self-#-adjointness now follows from Proposition 5.1.2.

To show that $P(A_1,\ldots,A_N)$ is essentially self-adjoint on D^l we need only show that #-$\overline{P(A_1,\ldots,A_N) \restriction D^l}$ extends $P(A_1,\ldots,A_N) \restriction D$. Suppose $Ext\text{-}\otimes_{k=1}^{N} \phi_k \in D$. Then $\phi_k \in D(A_k^{n_k})$, so since D_k^l is a domain of essential self-#-adjointness of $A_k^{n_k}$ there is a hyper infinite sequence $(\phi_k^l)_{l=1}^{{}^*\infty}$ so that $\phi_k^l \to_{\#} \phi_k$ and $A_k^{n_k}\phi_k^l \to_{\#} A_k^{n_k}\phi_k$. An easy estimate shows that this implies that $A_k^m\phi_k^l \to_{\#} A_k^m\phi_k$ for all $1 \leq m \leq n_k$. Therefore $Ext\text{-}\otimes_{k=1}^{N} \phi_k^l \to_{\#} Ext\text{-}\otimes_{k=1}^{N} \phi_k$ and $P(A_1,\ldots,A_N)(Ext\text{-}\otimes_{k=1}^{N} \phi_k^l) \to_{\#} P(A_1,\ldots,A_N)(Ext\text{-}\otimes_{k=1}^{N} \phi_k)$

The same argument works for hyperfinite linear combinations of vectors of the form $Ext\text{-}\otimes_{k=1}^{N} \phi_k$ so #-$\overline{P(A_1,\ldots,A_N) \restriction D^l}$ extends $P(A_1,\ldots,A_N) \restriction D$. This completes the proof of (a).

To prove (b), suppose that $\lambda \in \sigma\left(\#\text{-}\overline{P(A_1,\ldots,A_N)}\right)$. If I is any #-open interval about λ then $P^{-1}(A_1,\ldots,A_N)(I)$ contains a product $Ext\text{-}\times_{k=1}^{N} I_k$ of open #-intervals so that $I_k \cap \sigma(A_k) \neq \varnothing$. Since $\sigma(A_k) = \#\text{-}\mathbf{ess\ range}(f_k^{n_k}), \mu_k^{\#}\left[\left(f_k^{n_k}\right)^{-1}(I_k)\right] \neq 0$ so $\mu[P(f_1,\ldots,f_N)(I)] \neq 0$. That is, $\lambda \in \#\text{-}\mathbf{ess\ range}(P(f_1,\ldots,f_N))$ which equals

$\sigma\left(\#\text{-}\overline{P(A_1,\ldots,A_N)}\right)$.

Conversely if $\lambda \notin \#\text{-}\overline{P(\sigma(A_1),\ldots,\sigma(A_N))}$ then $(\lambda - P(f_1,\ldots,f_N))^{-1}$ is bounded #-a.e. on $Ext\text{-}\times_{k=1}^{N} M_k$ so $\lambda \in \rho\left(\#\text{-}\overline{P(A_1,\ldots,A_N)}\right)$.

Remark 5.1.3. If $A_1\ldots,A_N, N \in {}^*\mathbb{N}$ are bounded in ${}^*\mathbb{R}_c^\#$, $P(A_1,\ldots,A_N)$ is #-closed, but in general it is not.

Corollary 5.1.1. Let $A_1\ldots,A_N, N \in {}^*\mathbb{N}$ be self-#-adjoint operators on $H_1^\#,\ldots,H_N^\#$ and suppose that, for each k, D_k is a domain of essential self-#-adjointness for A_k. Then,

(a) The operators $A_\pi = Ext\text{-}\otimes_{k=1}^{N} A_k$ and $A_\Sigma = Ext\text{-}\sum_{k=1}^{N} A_k$ are essentially self-#-adjoint on $D = Ext\text{-}\otimes_{k=1}^{N} D_k$.

(b) $\sigma(A_\pi) = \#\text{-}\overline{Ext\text{-}\prod_{k=1}^{N}\sigma(\bar{A}_k)}$ and $\sigma(A_\Sigma) = \#\text{-}\overline{Ext\text{-}\sum_{k=1}^{N}\sigma(\bar{A}_k)}$.

Example 5.1.1. Suppose that $V(x)$ is a potential so that $H_1 = -\nabla_x^\# + V(x)$ is essentially self-#-adjoint on $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#3})$. Then $H_2 = -\nabla_x^\# + V(x) - \nabla_y^\# + V(y)$ is essentially self-#-adjoint on the set of hyperfinite sums of products $\varphi(x)\psi(y)$, with $\varphi,\psi \in \mathcal{S}^\#({}^*\mathbb{R}_c^{\#3})$. Further $\sigma(H_2) = \#\text{-}\overline{\sigma(H_1)+\sigma(H_1)}$. is obvious.

§ 5.2.Non-Archimedean Fock spaces.

Let $H^\#$ be a non-Archimedean Hilbert space and denote by $H^{\#n}, n \in {}^*\mathbb{N}$ the n-fold tensor product $H^{\#n} = Ext\text{-}\otimes_{k=1}^{n} H^\#$ and define

$$\mathcal{F}^\#(H^\#) = Ext\text{-}\otimes_{n=0}^{*\infty} H^{\#n} \tag{5.2.1}$$

$\mathcal{F}^\#(H^\#)$ is called a non-Archimedean Fock space over $H^\#$; it will be $*$-separable if $H^\#$ is. For example, if $H^\# = L_2^\#({}^*\mathbb{R}_c^\#)$, then an element $\psi \in \mathcal{F}(H^\#)$ is a hyper infinite sequence of functions

$$\psi = \{\psi_0, \psi_1(x_1), \psi_1(x_1,x_2), \psi_1(x_1,x_2,x_3), \ldots, \} \tag{5.2.2}$$

so that

$$|\psi_0|^2 + Ext\text{-}\sum_{n=1}^{*\infty}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} \psi_n(x_1,\ldots,x_n)d^{\#n}x\right) < {}^*\infty, \tag{5.2.3}$$

where $\psi_0 \in {}^*\mathbb{C}_c^\#, d^{\#n}x = Ext\text{-}\prod_{i=1}^{n} d^{\#n}x_i$. Actually, it is not $\mathcal{F}^\#(H^\#)$ itself, but two of its subspaces which are used in quantum field theory. These two subspaces are constructed as follows: Let $\mathbf{P}_n, n \in {}^*\mathbb{N}$ be the permutation group on $n \in {}^*\mathbb{N}$ elements and let $\{\varphi_k\}_{k=1}^{*\infty}$ be a basis for $H^\#$. For each $\boldsymbol{\sigma} \in \mathbf{P}_n$ we define an operator (which we also denote by a) on basis elements of $H^{\#(n)}, n \in {}^*\mathbb{N}$ by

$$\boldsymbol{\sigma}(Ext\text{-}\otimes_{i=1}^{n}\varphi_{k_i}) = Ext\text{-}\otimes_{i=1}^{n}\varphi_{k_{\boldsymbol{\sigma}(i)}} \tag{5.2.4}$$

a extends by linearity to a bounded in ${}^*\mathbb{R}_c^\#$ operator (of #-norm one) on $H^\#$ so we can define

$$\mathbf{S}_n = \left(\frac{1}{n!^\#}\right) Ext\text{-}\sum_{\boldsymbol{\sigma}\in\mathbf{P}_n}\boldsymbol{\sigma}. \tag{5.2.5}$$

It is easy to show that $\mathbf{S}_n^2 = \mathbf{S}_n$ and $\mathbf{S}_n^* = \mathbf{S}_n$, so $\mathbf{S}_n$ is an orthogonal projection The range of $\mathbf{S}_n$ is called the n-fold symmetric tensor product of $H^\#$. In the case where $H^\# = L_2^\#({}^*\mathbb{R}_c^\#)$ and $H^{\#n} = Ext\text{-}\otimes_{k=1}^{n} L_2^\#({}^*\mathbb{R}_c^\#) = L_2^\#({}^*\mathbb{R}_c^{\#n}), \mathbf{S}_n H^{\#n}$ is just the subspace of $L_2^\#({}^*\mathbb{R}_c^{\#n})$ of all functions left invariant under any permutation of the variables. We now define

$$\mathcal{F}^{\#}_{\mathbf{s}}(H^{\#}) = Ext\text{-}\bigoplus_{n=0}^{^*\infty}\mathbf{S}_n H^{\#n} \qquad (5.2.6)$$

$\mathcal{F}^{\#}_{\mathbf{s}}(H^{\#})$ is called the symmetric non Archimedean Fock space over $H^{\#}$ or the non Archimedean Boson Fock space over $H^{\#}$.

§ 5.3.Second quantization of the free Hamiltonian.

Let $H^{\#}$ be a non Archimedean Hilbert space, $\mathcal{F}^{\#}(H^{\#})$ the associated non Archimedean Fock space over $H^{\#}$. Suppose that A is a self-#-adjoint operator on $H^{\#}$ with a domain of essential self-#-adjointness D.Corresponding to each such A we can define an operator $d\Gamma^{\#}(A)$ on $\mathcal{F}^{\#}(H^{\#})$ as follows.Let

$$A^{(n)} = A\otimes\ I\otimes\cdots\otimes I+I\otimes\ A\otimes\ \cdots\otimes I+I\otimes\cdots\otimes I\ \otimes A \qquad (5.3.1)$$

on $Ext\text{-}\otimes_{i=1}^{n}D$ as follows. Let $D_A\subset\mathcal{F}^{\#}(H^{\#})$ be the set of $\{\psi_0,\psi_1,\ldots\}$ such that $\psi_n = 0$ for n large enough and $\psi_n\in Ext\text{-}\otimes_{k=1}^{n}D$ for each n. D_A is #-dense in $\mathcal{F}^{\#}(H^{\#})$ since D is #-dense in $H^{\#}$. Define $A^{(0)} = 0$ and $d\Gamma^{\#}(A) = Ext\text{-}\sum_{n=0}^{^*\infty}A^{(n)}$. $d\Gamma^{\#}(A)$ makes sense on D_A and obviously to be symmetric. By Theorem 5.1.1, $A^{(n)}$ is essentially self-#-adjoint on $Ext\text{-}\otimes_{k=1}^{n}D$.Thus $A^{(n)}+\mu i$ has a #-dense range on $Ext\text{-}\otimes_{k=1}^{n}D$ whenever $\mu\in{}^*\mathbb{R}^{\#}_c$ and $\mu\neq 0$. From this it follows that $d\Gamma^{\#}(A)\pm i$ has a #-dense range on D_A.Thus $d\Gamma^{\#}(A)$ is essentially self-#-adjoint on D_A . If A is the quantum mechanical operator which corresponds to the free energy, $d\Gamma^{\#}(A)$ is called the second quantization of the free energy. $d\Gamma^{\#}(A)$ commutes with the projections onto the symmetric and antisymmetric non Archimedean Fock spaces and it follows that $d\Gamma^{\#}(A)\restriction\mathcal{F}^{\#}_{\mathbf{s}}(H^{\#})$ and $d\Gamma^{\#}(A)\restriction\mathcal{F}^{\#}_{\mathbf{a}}(H^{\#})$ are essentially self-#-adjoint on $D\cap\mathcal{F}^{\#}_{\mathbf{s}}(H^{\#})$ and $D\cap\mathcal{F}^{\#}_{\mathbf{a}}(H^{\#})$ respectively.

Chapter IV.Non-Archimedean Banach spaces endroved with ${}^*\mathbb{R}^{\#}_c$-valued norm.

1.Definitions and examples

A non-Archimedean normed space with ${}^*\mathbb{R}^{\#}_c$-valued norm (#-norm) is a pair $(X,\|\cdot\|_{\#})$ consisting of a vector space X over a non-Archimedean scalar field ${}^*\mathbb{R}^{\#}_c$ or complex field ${}^*\mathbb{C}^{\#}_c$ together with a distinguished norm $\|\cdot\|_{\#}: X\to{}^*\mathbb{R}^{\#}_c$. Like any norms, this #-norm induces a translation invariant distance function, called the canonical or (norm) induced non-Archimedean ${}^*\mathbb{R}^{\#}_c$-valued metric for all vectors $x,y\in X$, defined by

$$d^{\#}(x,y) = \|x-y\|_{\#} = \|y-x\|_{\#}. \qquad (1.1)$$

Thus (1.1) makes X into a metric space $(X,d^{\#})$. A hyper infinite sequence $(x_n)_{n=1}^{\infty^{\#}}$ is called $d^{\#}$-Cauchy or Cauchy in $(X,d^{\#})$ or $\|\cdot\|_{\#}$ -Cauchy if for every hyperreal $r\in{}^*\mathbb{R}^{\#}_c$, $r>0$, there exists some $N\in\mathbb{N}^{\#}$ such that

$$d^{\#}(x_n,x_m) = \|x_n-x_m\|_{\#} < r, \qquad (1.2)$$

where m and n are greater than N. The canonical metric $d^{\#}$ is called a #-complete metric if the pair $(X,d^{\#})$ is a #-complete metric space, which by definition means for every $d^{\#}$-Cauchy sequence $(x_n)_{n=1}^{\infty^{\#}}$ in $(X,d^{\#})$, there exists some $x\in X$ such that

$$\#\text{-}\lim_{n\to\infty^{\#}}\|x_n - x\|_{\#} = 0 \tag{1.3}$$

where because $\|x_n - x\|_{\#} = d^{\#}(x_n, x)$, this hyper infinite sequence's #-convergence to x can equivalently be expressed as: $\#\text{-}\lim_{n\to\infty^{\#}} x_n = x$ in $(X, d^{\#})$.

Definition 1.1. The normed space $(X, \|\cdot\|_{\#})$ is a non-Archimedean Banach space endroved with ${}^{*}\mathbb{R}_c^{\#}$-valued norm if the #-norm induced metric $d^{\#}$ is a #-complete metric, or said differently, if $(X, d^{\#})$ is a #-complete metric space. The #-norm $\|\cdot\|_{\#}$ of a #-normed space $(X, \|\cdot\|_{\#})$ is called a #-complete #-norm if $(X, \|\cdot\|_{\#})$ is a non-Archimedean Banach space endroved with ${}^{*}\mathbb{R}_c^{\#}$-valued #-norm.

Remark 1.1.For any #-normed space $(X, \|\cdot\|_{\#})$, there exists an L-semi-inner product $\langle\cdot,\cdot\rangle_{\#} : X \times X \to {}^{*}\mathbb{R}_c^{\#}$ such that $\|x\|_{\#} = \sqrt{\langle x, x\rangle_{\#}}$ for all $x \in X$; in general, there may be infinitely many L-semi-inner products that satisfy this condition. L-semi-inner products are a generalization of inner products, which are what fundamentally distinguish non-Archimedean Hilbert spaces from all other non-Archimedean Banach spaces. Characterization in terms of hyper infinite series,see ref. [1].

The vector space structure allows one to relate the behavior of hyper infinite Cauchy sequences to that of #-converging hyper infinite series of vectors.

Remark 1.2.A #-normed space X is a non-Archimedean Banach space if and only if each absolutely #-convergent hyper infinite series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} v_n$ in X #-converges in X,i.e., $Ext\text{-}\sum_{n=1}^{\infty^{\#}} \|v_n\| < \infty^{\#}$ implies that $Ext\text{-}\sum_{n=1}^{\infty^{\#}} v_n$ #-converges in X.

2.Linear operators,isomorphisms

If X and Y are #-normed spaces over the same ground field ${}^{*}\mathbb{R}_c^{\#}$, the set of all #-continuous ${}^{*}\mathbb{R}_c^{\#}$-linear maps $T : X \to Y$ is denoted by $B^{\#}(X, Y)$. In hyper infinite-dimensional spaces, not all linear maps are #-continuous. A linear mapping from a #-normed space X to another normed space is #-continuous if and only if it is bounded or hyper bounded on the #-closed unit ball of X. Thus, the vector space $B^{\#}(X, Y)$ can be endroved with the operator norm

$$\|T\| = \sup\{\|Tx\|_{\#Y} \mid x \in X, \|x\|_{\#X} \le 1\}. \tag{2.1}$$

For Y a non-Archimedean Banach space, the space $B^{\#}(X, Y)$ is a Banach space with respect to this #-norm.

If X is a non-Archimedean Banach space, the space $B^{\#}(X) = B^{\#}(X, X)$ forms a unital Banach algebra; the multiplication operation is given by the composition of linear maps.

Definition 2.1.If X and Y are #-normed spaces, they are #-isomorphic #-normed spaces if there exists a linear bijection $T : X \to Y$ such that T and its inverse T^{-1} are #-continuous. If one of the two spaces X or Y is #-complete then so is the other space. Two #-normed spaces X and Y are #-isometrically isomorphic if in addition, T is an #-isometry, that is, $\|T(x)\| = \|x\|$ for every $x \in X$.

Definition 2.2.Let $\{X, \|\cdot\|\}$ be standard Banach space.For $x \in {}^{*}X$ and $\varepsilon > 0, \varepsilon \approx 0$ we define the open $\approx$-ball about x of radius ε to be the set

$B_{\varepsilon}(x) = \{y \in {}^{*}X | {}^{*}\|x - y\| < \varepsilon\}$.

Definition 2.3.Let $\{X, \|\cdot\|\}$ be standard Banach space, $Y \subset X$ thus ${}^{*}Y \subseteq {}^{*}X$ and let

$x \in {}^*X$. Then x is an $*$-accumulation point of ${}^*X$ if for every $\varepsilon > 0, \varepsilon \approx 0, Y \cap (B_\varepsilon(x) \backslash \{x\}) \neq \varnothing$.

Definition 2.4.Let $\{X, \|\cdot\|\}$ be a standard Banach space, let $Y \subseteq {}^*X, Y$ is $*$-closed if every $*$-accumulation point of Y is an element of Y.

Definition 2.5.Let $\{X, \|\cdot\|\}$ be standard Banach space.We shall say that internal hyper infinite sequence $\{x_n\}_{n=1}^{n={}^*\infty}$ in ${}^*X$ $*$-converges to $x \in {}^*X$ as $n \to {}^*\infty$if for any $\varepsilon > 0, \varepsilon \approx 0$ there is $N \in {}^*\mathbb{N}$ such that for any $n > N : {}^*\|x_n - x\| < \varepsilon$.

Definition 2.6.Let $\{X, \|\cdot\|\}, \{Y, \|\cdot\|\}$ be a standard Banach spaces. A linear internal operator $A : D(A) \subseteq {}^*X \to {}^*Y$is $*$-closed if for every internal hyper infinite sequence $\{x_n\}_{n=1}^{n={}^*\infty}$ in $D(A)$ $*$-converging to $x \in {}^*X$ such that $Ax_n \to y \in {}^*Y$ as $n \to {}^*\infty$ one has $x \in D(A)$ and $Ax = y$. Equivalently, A is $*$-closed if its graph is $*$-closed
in the direct sum ${}^*X \oplus {}^*Y$.

Given a linear operator $A : {}^*X \to {}^*Y$, not necessarily $*$-closed, if the $*$-closure of its graph in ${}^*X \oplus {}^*Y$ happens to be the graph of some operator, that operator is called the $*$-closure of A, and we say that A is $*$-closable. Denote the $*$-closure of A by $*$-$\bar{A}$. It follows that A is the restriction of $*$-$\bar{A}$ to $D(A)$.

A $*$-core (or $*$-essential domain) of a $*$-$\bar{A}$closable operator is a subset $C \subset D(A)$ such that the $*$-closure of the restriction of A to C is $*$-$\bar{A}$.

Definition 2.7. The graph of the linear transformation $T : H \to H$ is the set of pairs $\{\langle\varphi, T\varphi\rangle | (\varphi \in D(T))\}$.

The graph of T, denoted by $\Gamma(T)$, is thus a subset of $H \times H$ which is a non-Archimedean
Hilbert space with inner product $(\langle\varphi_1, \psi_1\rangle, \langle\varphi_2, \psi_2\rangle)$.

T is called a #-closed operator if $\Gamma(T)$ is a #-closed subset of $H \times H$.

Definition 2.8. Let T_1 and T be operators on H. If $\Gamma(T_1) \supset \Gamma(T)$, then T_1 is said to be an
extension of T and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$ and $T_1\varphi = T\varphi$ for all $\varphi \in D(T)$.

Definition 2.9. An operator T is #-closable if it has a #-closed extension. Every #-closable
operator has a smallest #-closed extension, called its #-closure, which we denote by #-$\bar{T}$.

Theorem 2.1.If T is #-closable, then $\Gamma(\#\text{-}\bar{T}) = \#\text{-}\overline{\Gamma(T)}$.

Definition 2.10.Let T be a #-densely defined linear operator on a non-Archimedean Hilbert space H. Let $D(T^*)$ be the set of $\varphi \in H$ for which there is an $\xi \in H$ with $(T\psi, \varphi) = (\psi, \xi)$ for all $\psi \in D(T)$.

For each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$. T^* is called the #-adjoint of T. Note that $\varphi \in D(T^*)$ if and only if $|(T\psi, \varphi)| \leq C\|\psi\|$ for all $\psi \in D(T)$. We note that $S \subset T$ implies $T^* \subset S^*$.

Theorem 2.2. Let T be a #-densely defined operator on a non-Archimedean Hilbert space H.

Then:(i) T^* is #-closed.

(ii) T is #-closabie if and only if $D(T^*)$ is #-dense in which case $T = T^{**}$.

(iii) If T is #-closabie, then $(\#\text{-}\bar{T})^* = T^*$.

Definition 2.11. Let T be a #-closed operator on a Hilbert space H. A complex number

$\lambda \in {}^*\mathbb{C}_c^\#$ is in the resolvent set, $\rho(T)$, if $\lambda I - T$ is a bijection of $D(T)$ onto H with a a finitely or hyper finitely bounded inverse. If $\lambda \in \rho(T)$, $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.

The definitions of spectrum, point spectrum, and residual spectrum are the same for unbounded operators as they are for bounded operators. We will sometimes refer to the spectrum of nonclosed, but closabie operators. In this case we always mean the spectrum of the closure.

3. Symmetric and self-#-adjoint operators: the basic criterion for self-#-adjointness.

Definition 3.1. A #-densely defined operator T on a non-Archimedean Hilbert space is called symmetric (or Hermitian) if $T \subset T^*$, that is, if $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$.

Equivalently, T is symmetric if and only if $(T\varphi, \psi) = (\varphi, T\psi)$ for all $\varphi, \psi \in D(T)$

Definition 3.2. T is called self-adjoint if $T = T^*$, that is, if and only if T is symmetric and $D(T) = D(T^*)$.

A symmetric operator is always #-closable, since $D(T^*) \supset D(T)$ is #-dense in H. If T is symmetric, T^* is a closed extension of T so the smallest #-closed extension T^{**} of T must be contained in T^*. Thus for symmetric operators, we have $T \subset T^{**} \subset T^*$. For #-closed symmetric operators, $T = T^{**} \subset T^*$ and, for self-adjoint operators, $T = T^{**} = T^*$

From this one can easily see that a #-closed symmetric operator T is self-adjoint if and only if T^* is symmetric.

The distinction between #-closed symmetric operators and self-adjoint operators is very important. It is only for self-adjoint operators that the spectral theorem holds and it is only self-adjoint operators that may be #-exponentiated to give the one-parameter unitary groups which give the dynamics in QFT. Chapter X is mainly devoted to studying methods for proving that operators are self-adjoint. We content ourselves here with proving the basic criterion for selfadjointness.

First, we introduce the useful notion of essential self-adjointness.

Definition 3.3 A symmetric operator T is called essentially self- #-adjoint if its #-closure #-$\overline{T}$ is self- #-adjoint. If T is #-closed, a subset $D \subset D(T)$ is called a core for T if

$\overline{\# - T \restriction D} = T$.

If T is essentially self-#-adjoint, then it has one and only one self-#-adjoint extension. The importance of essential self-#-adjointness is that one is often given a nonclosed symmetric operator T. If T can be shown to be essentially self-#-adjoint, then there is uniquely associated to Ta self-adjoint operator $T = T^{**}$. Another way of saying this is that if A is a self-#-adjoint operator, then to specify A uniquely one need not give the exact domain of A (which is often difficult), but just some #-core for A

Chapter V. Semigroups of operators on a non-Archimedean Banach spaces.

§1.Semigroups on non-Archimedean Banach spaces and their generators.

A family of #-bounded operators $\{T(t)|0 < t < \infty^{\#}\}$ on external hyper infinite dimensional

non-Archimedean Banach space X endoved with ${}^{*}\mathbb{R}^{\#}_{c,+}$ - valued norm $\|\bullet\|_{\#}$ is called a strongly #-continuous semigroup if:

(a) $T(0) = I$

(b) $T(s)T(t) = T(s+t)$ for all $s,t \in {}^{*}\mathbb{R}^{\#}_{c,+}$

(c) For each $\varphi \in X, t \mapsto T(t)$ is #-continuous mapping.

We will see that strongly continuous semigroups are the “exponentials,” $T(t) = Ext\text{-}\exp(-tA)$, of a certain class of operators. .

We begin by studying a special class of semigroups:

Definition 1.1. A family $\{T(t)|0 < t < \infty^{\#}\}$ of bounded or hyper bounded operators on external hyper infinite dimensional Banach space X is called a contraction semigroup if it is a strongly #-continuous semigroup and moreover $\|T(t)\|_{\#} < 1$ for all $t \in [0,\infty^{\#})$. Note that the all theorems about general strongly #-continuous semigroups are easy generalizations of the corresponding theorems for #-contraction semigroups. Thus, we study the special case first. We then briefly discuss the general theory and conclude the section by studying another special class, #-holomorphic semigroups.

Proposition 1.1. Let $T(t)$ be a strongly #-continuous semigroup on a non-Archimedean Banach space X and set $A\varphi = \#\text{-}\lim_{r\to_{\#}0} A_r\varphi$ where $D(A) = \{\varphi|\ \#\text{-}\lim_{r\to_{\#}0} A_r\varphi$ exists$\}$. Then A is #-closed and #-densely defined. A is called the infinitesimal generator of $T(t)$. We will also say that A generates $T(t)$ and write $T(t) = Ext\text{-}\exp(-tA)$.

Proof.Let $T(t)$ be a contraction semigroup on a Banach space X. We obtain the generator of $T(t)$ by #-differentiation. Set $A_t = t^{-1}(I - T(t))$ and define

$$D(A) = \{\varphi|\ \#\text{-}\lim_{t\to_{\#}0} A_t\varphi \text{ exists}\}.$$

For $\varphi \in D(A)$, we define $A\varphi = \#\text{-}\lim_{t\to_{\#}0} A_t\varphi$. Our first goal is to show that $D(A)$ is #-dense. For $\varphi \in X$, we set

$$\varphi_s = Ext\text{-}\int_0^s T(t)\varphi d^{\#}t. \tag{2.1}$$

For any $r > 0$, we get

$$T(r)\varphi_s = Ext\text{-}\int_0^s T(t+r)\varphi d^{\#}t \tag{2.2}$$

thus

$$\begin{gathered} A_r\varphi_s = -\frac{1}{r}\left(Ext\text{-}\int_0^s [T(t+r)\varphi - T(t)\varphi]d^{\#}t\right) = \\ -\frac{1}{r}\left(Ext\text{-}\int_s^{r+s} T(t)\varphi d^{\#}t\right) + \frac{1}{r}\left(Ext\text{-}\int_s^{r} T(t)\varphi d^{\#}t\right). \end{gathered} \tag{2.3}$$

From Eq.(2.3) one obtains #-$\lim_{r\to_{\#}0} A_r\varphi_s = -T(s)\varphi + \varphi$. Therefore, for each $\varphi \in X$

and $s > 0$, $\varphi_s \in D(A)$. Since $s^{-1}\varphi_s \to_{\#} \varphi$ as $\to_{\#} 0$, A is #-densely defined. Furthermore, if $\varphi \in D(A)$, then $A_r T(t)\varphi = T(t)A_r\varphi$, so $T(t) : D(A) \to D(A)$ and

$$\frac{d^{\#}}{d^{\#}t}T(t)\varphi = -AT(t)\varphi = -T(t)A\varphi \tag{2.4}$$

A is also #-closed, for if $\varphi_n \in D(A)$, #-$\lim_{n\to\infty^{\#}}\varphi_n = \varphi$, and #-$\lim_{n\to\infty^{\#}} A\varphi_n = \psi$, then

$$\begin{gathered}\#\text{-}\lim_{r\to_{\#}0} A_r\varphi = \#\text{-}\lim_{r\to_{\#}0}\#\text{-}\lim_{n\to\infty^{\#}}\left[-\frac{1}{r}(T(r)\varphi_n - \varphi_n)\right] = \\ \#\text{-}\lim_{r\to_{\#}0}\#\text{-}\lim_{n\to\infty^{\#}}\frac{1}{r}\left(Ext\text{-}\int_s^r T(t)A\varphi_n d^{\#}t\right) = \\ \#\text{-}\lim_{r\to_{\#}0}\frac{1}{r}\left(Ext\text{-}\int_s^r T(t)\psi d^{\#}t\right)\end{gathered} \tag{2.5}$$

so $\varphi \in D(A)$ and $A\varphi = \psi$.
The formal Laplace transform

$$\frac{1}{\lambda + A} = -\left(Ext\text{-}\int_0^{\infty^{\#}}(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))d^{\#}t\right) \tag{2.6}$$

suggests that all $\mu \in {}^*\mathbb{C}_c^{\#}$ with $\operatorname{Re}\mu < 0$ are in $\rho(A)$. This is in fact true and the formula (2.6) holds in the strong sense. For suppose that $\operatorname{Re}\lambda > 0$. Then, since $\|Ext\text{-}\exp(-tA)\| < 1$, the formula (2.7)

$$R\varphi = Ext\text{-}\int_0^{\infty^{\#}}(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA)\varphi)d^{\#}t \tag{2.7}$$

defines a hyper bounded linear operator of #-norm less than or equal to $(\operatorname{Re}\lambda)^{-1}$. Moreover, for $r > 0$,

$$\begin{gathered}A_r R\varphi = -\frac{1}{r}\left(Ext\text{-}\int_0^{\infty^{\#}}(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-(t+r)A) - Ext\text{-}\exp(-tA))\varphi d^{\#}t\right) = \\ \frac{1 - Ext\text{-}\exp(\lambda r)}{r}\left(Ext\text{-}\int_0^{\infty^{\#}}(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^{\#}t\right) + \\ \frac{Ext\text{-}\exp(\lambda r)}{r}\left(Ext\text{-}\int_0^{r}(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^{\#}t\right)\end{gathered} \tag{2.8}$$

so as $r \to_{\#} 0, A_r R\varphi \to_{\#} (\varphi - \lambda R\varphi)$. Thus $R\varphi \in D(A)$ and $AR\varphi = \varphi - \lambda R\varphi$ which implies $(\lambda + A)R\varphi = \varphi$. In addition, for $\varphi \in D(A)$ we have $AR\varphi = RA\varphi$ since

$$A\left(Ext\text{-}\int_0^{\infty^\#}(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^\# t\right) =$$
$$Ext\text{-}\int_0^{\infty^\#}(Ext\text{-}\exp(-\lambda t))A(Ext\text{-}\exp(-tA))\varphi d^\# t = \quad (2.9)$$
$$Ext\text{-}\int_0^{\infty^\#}(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))A\varphi d^\# t.$$

The first equality follows by approximation with external hyperfinite Riemann sums (see [1]) from the facts that $(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi$ and $A(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))$ are #-integrable, A is #-closed. Thus, for $\varphi \in D(A)$, $R(\lambda + A)\varphi = \varphi = (\lambda + A)R\varphi$ which implies that

$$R = (\lambda + A)^{-1}. \quad (2.10)$$

The properties of A which we have derived are also sufficient to guarantee that A generates a contraction semigroup. In fact, we only need information about real positive A.

Theorem 1.1. (Generalized Hille-Yosida theorem) A necessary and sufficient condition that a #-closed
linear operator A on a Banach space X generate a contraction semigroup is that
(i) $(-\infty^\#, 0) \subset \rho(A)$
(ii) $\|(\lambda + A)^{-1}\|_\#$ for all $\lambda > 0$.
Furthermore, if A satisfies (i) and (ii), then the entire #-open left half-plane is contained in $\rho(A)$ and

$$(\lambda + A)^{-1}\varphi = -Ext\text{-}\int_0^{\infty^\#}(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))d^\# t \quad (2.11)$$

for all $\varphi \in X$ and λ with $\operatorname{Re}\lambda > 0$. Finally, if $T_1(t)$ and $T_2(t)$ are contraction semigroups generated by A_1 and A_2 respectively, then $T_2(t) \neq T_1(t)$ for some t implies that $A_1 \neq A_2$.

Proof. Since we showed above that conditions (i) and (ii) are necessary and that (2.11)
holds, we need only show sufficiency. So, suppose that A is a #-closed operator on X satisfying (i) and (ii). For $\lambda > 0$, define $A^{(\lambda)} = \lambda - \lambda^2(\lambda + A)^{-1}$. We will show that as $\lambda \to \infty^\#$, $A^{(\lambda)} \to_\# A$ strongly on $D(A)$ and then construct $Ext\text{-}\exp(-tA)$ as the strong #-limit of the semigroups $Ext\text{-}\exp(-tA^{(\lambda)})$. For $\varphi \in D(A)$, $A^{(\lambda)}\varphi = \lambda(\lambda + A)^{-1}A\varphi$. Moreover, by (ii),

$$\#\text{-}\lim_{\lambda\to\infty^\#}[\lambda(\lambda + A)^{-1}\varphi - \varphi] = \#\text{-}\lim_{\lambda\to\infty^\#}[-(\lambda + A)^{-1}A\varphi] = 0. \quad (2.12)$$

By condition (ii) the family $\{\lambda(\lambda + A)^{-1} | \lambda > 0\}$ is #-uniformly hyperfinitely bounded in #-norm, so since $D(A)$ is #-dense, $\#\text{-}\lim_{\lambda\to\infty^\#}[\lambda(\lambda + A)^{-1}\psi] = \psi$ for all $\psi \in X$. Thus $\#\text{-}\lim_{\lambda\to\infty^\#} A^{(\lambda)}\varphi = A\varphi$ for all $\varphi \in D(A)$. Since A is hyperfinitely bounded, the semigroups $Ext\text{-}\exp(-tA^{(\lambda)})$ can be defined by hyper infinite power series. Since

$$\|Ext\text{-}\exp(-tA^{(\lambda)})\|_{\#} = \|(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(t\lambda^{2}(\lambda+A)^{-1}))\|_{\#} \leq$$
$$\leq (Ext\text{-}\exp(-\lambda t))\left(Ext\text{-}\sum_{n=0}^{\infty^{\#}}\frac{t^{n}\lambda^{2n}}{n!}\|(\lambda+A)^{-1}\|_{\#}^{n}\right) \leq 1 \qquad (2.13)$$

they are contraction semigroups. For all $\mu,\lambda,t > 0,$ and all $\varphi \in D(A),$ we have

$$[Ext\text{-}\exp(-tA^{(\lambda)})]\varphi - [Ext\text{-}\exp(-tA^{(\mu)})]\varphi =$$
$$Ext\text{-}\int_{0}^{t}\frac{d^{\#}}{d^{\#}s}(Ext\text{-}\exp(-sA^{(\lambda)}))((Ext\text{-}\exp(-(t-s)A^{(\lambda)}))\varphi)d^{\#}s \qquad (2.14)$$

so,

$$\|[Ext\text{-}\exp(-tA^{(\lambda)})]\varphi - [Ext\text{-}\exp(-tA^{(\mu)})]\varphi\|_{\#} \leq$$
$$Ext\text{-}\int_{0}^{t}\|(Ext\text{-}\exp(-sA^{(\lambda)}))((Ext\text{-}\exp(-(t-s)A^{(\lambda)})))\|_{\#}\|A^{(\mu)}\varphi - A^{(\lambda)}\varphi\|_{\#}d^{\#}s \leq \qquad (2.15)$$
$$\leq t\|A^{(\mu)}\varphi - A^{(\lambda)}\varphi\|_{\#}.$$

We have used the fact that Ext-$\exp(-tA^{(\lambda)})$ and $[Ext\text{-}\exp(-(t-s)A^{(\mu)})]$ commute since $\{A^{(\lambda)}|\lambda > 0\}$ is a commuting family. Since we have proven above that #-$\lim_{\lambda\to\infty^{\#}}A^{(\lambda)}\varphi = A\varphi, \{Ext\text{-}\exp(-tA^{(\lambda)})\}$ is Cauchy as $\lambda \to \infty^{\#}$ for each $t > 0$ and $\varphi \in D(A)$. Since $D(A)$ is #-dense and the Ext-$\exp(-tA^{(\lambda)})$ are uniformly hyperfinitely bounded, the same statement holds for all $\varphi \in X.$ Now, define

$$T(t)\varphi = \#\text{-}\lim_{\lambda\to\infty^{\#}}[Ext\text{-}\exp(-tA^{(\lambda)})\varphi]. \qquad (2.16)$$

$T(t)$ is a semigroup of contraction operators since these properties are preserved under strong #-limits. The above inequality shows that the #-convergence in Eq.(2.16) is uniform for t restricted to a hyperfinite interval, so $T(t)$ is strongly #-continuous since Ext-$\exp(-tA^{(\lambda)})$ is. Thus, $T(t)$ is a contraction semigroup.It remains to show that the infinitesimal generator of $T(t),$ call it $\widetilde{A},$ is equal to $A.$ For all t
and $\varphi \in D(A)$,

$$[Ext\text{-}\exp(-tA^{(\lambda)})\varphi] - \varphi == -\left[Ext\text{-}\left[\int_{0}^{t}Ext\text{-}\exp(-sA^{(\lambda)})\right]A^{(\lambda)}\varphi d^{\#}s\right] \qquad (2.17)$$

so, since #-$\lim_{\lambda\to\infty^{\#}}A^{(\lambda)}\varphi = A\varphi$, we have

$$T(t)\varphi - \varphi = -\left[Ext\text{-}\int_{0}^{t}T(s)A\varphi d^{\#}s\right]. \qquad (2.18)$$

Thus, $\widetilde{A}_{t}\varphi \to_{\#} A\varphi$ as $t \to_{\#} 0.$ Therefore $D\left(\widetilde{A}\right) \supset D(A)$ and $\widetilde{A} \upharpoonright D(A) = A.$ For $\lambda > 0,$ $(\lambda + A)^{-1}$ exists by hypothesis and $\left(\lambda + \widetilde{A}\right)^{-1}$ exists by the necessity part of the theorem.

§2 Hypercontractive semigroups

In the previous section we discussed $\mathcal{L}_{\#}^{p}$-contractive semigroups. In this section we will prove a self-adjointness theorem for operators of the form $A + V$ where V is a multiplication operator and A generates an $\mathcal{L}_{\#}^{p}$-contractive semigroup that satisfies a strong additional property.

Definition 2.1. Let $\langle M,\mu^{\#}\rangle$ be a #-measure space with $\mu^{\#}(M) = 1$ and suppose that A is a positive self-adjoint operator on $\mathcal{L}_{\#}^{2}(M,d^{\#}\mu^{\#})$. We say that $Ext\text{-}\exp(-tA)$ is a hypercontractive semigroup if:

(i) $Ext\text{-}\exp(-tA)$ is $\mathcal{L}_{\#}^{p}$-contractive;

(ii) for some $b > 2$ and some constant C_b, there is a $T > 0$ so that $\|Ext\text{-}\exp(-tA)\varphi\|_b \leq C_b\|\varphi\|_2$ for all $\varphi \in \mathcal{L}_{\#}^{2}(M,d^{\#}\mu^{\#})$.

By Theorem X.55, condition (i) implies that $Ext\text{-}\exp(-tA)$ is a strongly #-continuous contraction semigroup for all $p < \infty^{\#}$. Holder's inequality shows that

$$\|\cdot\|_q \leq \|\cdot\|_p \tag{1}$$

if $p \geq q$. Thus the $\mathcal{L}_{\#}^{p}$-Spaces are a nested family of spaces which get smaller as p gets larger; this suggests that (ii) is a very strong condition. The following proposition shows that b plays no special role.

Proposition 2.1. Let $Ext\text{-}\exp(-tA)$ be a hypercontractive semigroup on $\mathcal{L}_{\#}^{2}(M,d^{\#}\mu^{\#})$. Then for all $p,q \in (1,\infty^{\#})$, there is a constant C_{pq} and a $t_{pq} > 0$ so that if $t > t_{pq}$ then $\|Ext\text{-}\exp(-tA)\varphi\|_p \leq C_{pq}\|\varphi\|_q$ for all $\varphi \in \mathcal{L}_{\#}^{q}$.

Proof. The case where $p < q$ follows immediately from (i) and (1). So suppose that $p > q$. Since $Ext\text{-}\exp(-tA) : \mathcal{L}_{\#}^{2} \to \mathcal{L}_{\#}^{b}$ and $Ext\text{-}\exp(-tA) : \mathcal{L}_{\#}^{\infty^{\#}} \to \mathcal{L}_{\#}^{\infty^{\#}}$, the generalized Riesz-Thorin theorem implies that there is a constant C so that for all $r \geq 2$, $\|Ext\text{-}\exp(-tA)\varphi\|_r \leq C\|\varphi\|_{br/2}$. We now consider two cases. First, if $q \geq 2$ we choose n large enough so that $2(b/2)^n > p$. Then $\|Ext\text{-}\exp(-nTA)\varphi\|_{2(b/2)^n} \leq C\|\varphi\|_2$ so the conclusion follows if $2 < q,p > 2(b/2)^n$, by using (1), and hypothesis (i). If $1 < q < 2$, then we choose n large enough so that $2(b/2)^n > p$ and $q > c$ where $c^{-1} + (2(b/2)^n)^{-1} = 1$. Since A is self-adjoint and $Ext\text{-}\exp(-nTA)\varphi$ is a bounded or hyper bounded map from $\mathcal{L}_{\#}^{2}$ to $\mathcal{L}_{\#}^{2(b/2)^n}$, $(Ext\text{-}\exp(-nTA))^* = Ext\text{-}\exp(-nTA)$ is a bounded or hyper bounded map from $\mathcal{L}_{\#}^{c}$ to $\mathcal{L}_{\#}^{2}$. Thus $Ext\text{-}\exp(-2nTA)$ is a bounded or hyper bounded map from $\mathcal{L}_{\#}^{c}$ to $\mathcal{L}_{\#}^{2(b/2)^n}$. Since $c < q < p < 2(b/2)^n$, (1) implies the proposition.

Theorem 2.1. The operator $-\frac{1}{2}d^{\#2}/d^{\#}x^2 + xd^{\#}/d^{\#}x$ on $\mathcal{L}_{\#}^{2}\left({}^{*}\mathbb{R}_{c}^{\#},\pi_{\#}^{-1/2}Ext\text{-}\exp(-x^2)d^{\#}x\right)$ is positive and essentially self-adjoint on the set of hyperfinite linear combinations of Hermite polynomials, and generates a hypercontractive semigroup.

As a preparation for our main theorem, we prove the following result.

Theorem 2.2 Let $\langle M,\mu\rangle$ be a #-measure space with $\mu(M) = 1$ and let H_0 be the generator of a hypercontractive semigroup on $\mathcal{L}_{\#}^{2}(M,d\mu)$. Let V be a real-valued measurable function on $\langle M,\mu^{\#}\rangle$ such that $V \in \mathcal{L}_{\#}^{p}(M,d^{\#}\mu^{\#})$ for all $p \in [1,\infty^{\#})$ and $Ext\text{-}e^{-tV} \in \mathcal{L}_{\#}^{1}(M,d^{\#}\mu^{\#})$ for all $t > 0$. Then $H_0 + V$ is essentially self-#-adjoint on $C^{\infty^{\#}}(H_0) \cap D(V)$ and is bounded below. $C^{\infty^{\#}}(H_0) = \bigcap_{p\in\mathbb{N}^{\#}} D(H_0^p)$

Chapter VI. Singular Perturbations of Selfadjoint Operators on a non-Archimedean Hilbert space.

§1. Introduction

We study the sum $A+B$ of two #-selfadjoint operators on a non-Archimedean Banach spaces, and we find sufficient conditions for $C = A+B$ to be #-selfadjoint. Our technique is to approximate B by a hyperinfinite sequence of bounded #-selfadjoint operators $B_n, n \in {}^*\mathbb{N}$ and so to approximate C by #-selfadjoint operators $C_n = A+B_n$.
We answer three questions separately:
1.When do the operators C_n have a #-$\lim C$? 2.When is C a #-selfadjoint operator? 3.When is $C = A+B$?
In Theorem 8 we give a set of estimates on the relative size of A and B which ensure a positive answer to all three questions. Hence these estimates show that $A+B = C$ is #-selfadjoint. In another paper [5], we use Theorem 2.8 to prove the existence of a self-interacting, causal quantum field in 4-dimensional space-time. Formally this field theory is Lorentz covariant and has non-trivial scattering; this application was the motivation for the present work.
In order to investigate the meaning of #-$\lim_{n\to{}^*\infty} C_n$, we give a new definition for the strong #-convergence of a hyperinfinite sequence of operators. Consequences of this definition are worked out in Section 2. In Section 3 we give estimates on operators C_n which are sufficient to ensure that the #-$\lim_{n\to{}^*\infty} C_n = C$ exists and that C is maximal symmetric or #-selfadjoint. This result is given in Theorem 5 and Corollary 6.
In Section 4 we investigate whether #-$\lim_{n\to{}^*\infty} C_n = C$ is equal to $A+B$.
We combine this work in Theorem 8, our second main theorem, where B is a singular, but nearly positive #-selfadjoint perturbation of a positive #-selfadjoint operator A. To illustrate this theorem, let $A \geq I$ and let B be essentially #-selfadjoint on

$$D^\# = \bigcap_{n\in{}^*\mathbb{N}} D(A^n). \tag{1.0}$$

Assume now that, for some $\beta > 0$ and some α,

$$A^{-(1-\beta)}BA^{-(1-\beta)} \text{ and } A^\beta BA^\alpha \tag{1.1}$$

are #-densely defined, bounded operators. Also, for some positive $a,\varepsilon \in {}^*\mathbb{R}^\#_{c+}$ satisfying $2a+\varepsilon < 1$, suppose that there is a constant $b \in {}^*\mathbb{R}^\#_c$ such that, as bilinear forms on $D \times D$,

$$0 \leq aA + B + b \tag{1.2}$$

and

$$0 \leq \varepsilon A^2 + [A^{1/2},[A^{1/2},B]] + b. \tag{1.3}$$

Then $A+B$ is #-selfadjoint.
We see from this example that neither the operator B nor the bilinear form B need be bounded relative to A.
While it may not appear evident, the conditions (1.1)-(1.3) are closely related to a more easily understandable estimate on $D^\# \times D^\#$,

$$A^2 + B^2 c(A+B)^2 + c. \tag{1.4}$$

In fact, estimates (1.1)-(1.3) are chosen because they allow us not only to prove (1.4), but also the similar inequality where B is replaced by B_n.
Let us now see that if $A+B$ is #-selfadjoint, then (1.4) must hold for every vector in $D(A+B) = D(A) \cap D(B)$.

Proposition 1.1. Let A and B be #-closed operators. Then $A+B$ is #-closed if and only if there is a constant $c \in {}^*\mathbb{R}_c^\#$ such that for all $\psi \in D(A+B)$

$$\|A\psi\|_\# + \|B\psi\|_\# \leq \|(A+B)\psi\|_\# + c\|\psi\|_\# \tag{1.5}$$

and (1.5) is equivalent to (1.4) on $D(A+B) \times D(A+B)$.

Proof: Certainly (1.5) implies that $A+B$ is #-closed. Conversely, assume that $A+B$ is #-closed and introduce the #-norms on $D(A+B) = D(A) \cap D(B)$,

$$\|\psi\|_{\#1} \triangleq \|\psi\|_\# + \|A\psi\|_\# + \|B\psi\|_\# \tag{1.6}$$

and

$$\|\psi\|_{\#2} \triangleq \|\psi\|_\# + \|(A+B)\psi\|_\# \tag{1.7}$$

Then $D(A+B), \|\bullet\|_{\#2}$ is a non-Archimedean Banach space because $A+B$ is #-closed. The identity map from $D(A+B), \|\bullet\|_{\#2}$ to $D(A+B), \|\bullet\|_{\#1}$ has a #-closed graph because $A,B,$ and $A+B$ are c#-losed. By the #-closed graph theorem, the identity map is #-continuous; hence

$$\|\psi\|_{\#1} \leq c\|\psi\|_{\#2}. \tag{1.7$'$}$$

Proposition 1.2.Let $A \geq I, B$ be #-selfadjoint operators with $D^\# \subset D$(B) and suppose (1.2) and (1.3) hold. Then (1.4) is valid on $D^\# \times D^\#$.

Proof The operators $A^2, B^2, AB, BA,$ and $A^{1/2}BA^{1/2}$ define bilinear forms on $D^\# \times D^\#$. Using (1.2) and (1.3), we have the inequality:

$A^2 + B^2 = (A+B)^2 - 2A^{1/2}BA^{1/2} - [A^{1/2},[A^{1/2},B]] \leq (A+B)^2 + (2a+\varepsilon)A^2 + 2Ab + b$

which establishes (1.4).

§2. Strong #-Convergence of Operators

Let $\mathcal{L}(C)$ be the graph of the operator C. For any hyperinfinite sequence $\{C_n\}, n \in {}^*\mathbb{N}$ of #-densely defined operators we define

$$\mathcal{L}_{^*\infty}(C) = \{\phi, \chi | \phi = \#\text{-}\lim_{n\to{}^*\infty} \phi_n, \phi_n \in D(C_n), \chi = \#\text{-}\lim_{n\to{}^*\infty} C_n\phi_n\}. \tag{8}$$

In general, $\mathcal{L}_{^*\infty}$ will not be the graph of an operator. If the hyperinfinite sequence $\{C_n^*\}, \quad n \in {}^*\mathbb{N}$#-converges strongly on a #-dense domain D to an operator C^*, namely,

$$C^*\psi = \#\text{-}\lim_{n\to{}^*\infty} C_n^*\psi, \psi \in D,$$

then $\mathcal{L}_{^*\infty}$ is the graph of some operator C^*. In particular, if each C_n is self #-adjoint, and if the C_n #-converge on a #-dense set D to an operator C defined on D, then $\mathcal{L}_{^*\infty} = \mathcal{L}_{^*\infty}(C_{^*\infty})$ and $C_{^*\infty}$ is a symmetric extension of C.

Definition 2.1. G #-CONVERGENCE. The hyperinfinite sequence of operators $C_n, n \in {}^*\mathbb{N}$ #-converge strongly to $C_{^*\infty}$ in the sense of graphs, written

$$C_n \to_{\#G} C_{^*\infty} \tag{8$'$}$$

if $\mathcal{L}_{^*\infty}$ is the graph of a #-densely defined operator $C_{^*\infty}$.

Remark 2.1.Note that for a hyperinfinite sequence of uniformly bounded operators $\{C_n^*\}_{n\in{}^*\mathbb{N}}$ such that $C_n \to_{\#G} C_{^*\infty}, C_{^*\infty}$ is the usual strong #-limit of the operators $C_n, n \in {}^*\mathbb{N}$ and is everywhere defined.

Definition 2.2.R #-CONVERGENCE. Let the resolvents $R_n(z) = (C_n - z)^{-1}, n \in {}^*\mathbb{N}$ exist for some $z \in {}^*\mathbb{C}_c^\#$, and be uniformly bounded in n. The operators C_n #-converge strongly to $C_{^*\infty}$ in the sense of resolvents, written

$$C_n \to_{\#R} C_{^*\infty} \tag{8''}$$

if the resolvents $R_n(z)$ #-converge strongly to an operator $R(z)$, which has a #-densely defined inverse.

Remark 2.2.Note thatIn that case, the operator $C_{^*\infty} = R^{-1}(z) + z$ exists for all $z \in {}^*\mathbb{C}_c^\#$ for which the strong #-limit of the $R_n(z)$ exists, and $R^{-1}(z) + z$ is independent of z.

Remark 2.3.Note that G #-convergence is weaker than R #-convergence, in the case $C_n = C_n^*$ at least, because, as we shall show, in this case $C_n \to_{\#R} C_{^*\infty}$ implies $C_n \to_{\#G} C_{^*\infty}$. It seems likely that G #-convergence is strictly weaker than R #-convergence; this could be established by giving an example for which $C_n^* = C_n \to_{\#G} C_{^*\infty}$ with $C_{^*\infty}$ not maximal symmetric. The importance of G #-convergence is that it is technically easier to verify-and gives less information about the #-limit-than R #-convergence, whiIe automatically selecting the correct domain in the case that R #-convergence also holds. The most familiar examples of G #-convergence occur where there is C_n strong #-convergence on a #-dense domain. A less trivial example occurs where there is $D(C_n)$ is independent of n, but apparently

$$D(C) \cap D(C_n) = \{0\}.$$

We have the following connection between G and R #-convergence for a hyperinfinite sequence of #-selfadjoint operators.

Proposition 3.Let $C_n, n \in {}^*\mathbb{N}$ be #-selfadjoint.

(a) The domain $D_{^*\infty} = \left\{\phi | \{\phi, \chi\} \in \mathcal{L}_{^*\infty} \text{ for some } \chi\right\}$ is #-dense in H and oniy if $C_n \to_{\#G} C_{^*\infty}$, and in this case $C_{^*\infty}$ is necessarily symmetric.

(b) If $R_n(z) = (C_n - z)^{-1}, n \in {}^*\mathbb{N}$ #-converges to a bounded operator $R(z)$ for an unbounded set of z's with $\|zR,(z)\|_\#$ bounded uniformly in $z \in {}^*\mathbb{C}_c^\#$ and $n \in {}^*\mathbb{N}$ and if $C_n \to_{\#G} C_{^*\infty}$, then each $R(z)$ is invertible.

(c) If $R_n(z)$ #-converges to an invertible $R(z)$, then $C_n \to_{\#R} C$.

(d) If $C_n \to_{\#R} C$, then $C_n \to_{\#G} C_{^*\infty}, \mathcal{L}_{^*\infty} = \mathcal{L}(C)$, and C is maximal symmetric.

(e) Conversely, if $C_n \to_{\#G} C$, where C is maximal symmetric, then $C_n \to_{\#R} C$.

In case the #-limit of the $C_n, n \in {}^*\mathbb{N}$ is actually selfadjoint, there are further connections between G and R #-convergence.

Theorem 4.

(a) $C_n \to_{\#G} C$, and $C = C^*$.

(b) $C_n \to_{\#R} C$, and $C = C^*$.

(c) The hyper infinite sequences $\{R_n(z)\}$ and $\{[R_n(z)]^*\}, n \in {}^*\mathbb{N}$ #-converge strongly and #-$\lim_{n \to {}^*\infty} R_n(z)$ is invertible for some z.

(d) Statement (c) holds for all non-real $z \in {}^*\mathbb{C}_c^\#$

§3.Estimates on a G #-convergent hyper infinite sequence

In this section we give estimates which are sufficient to assure that it G #-convergent sequence of operators is R #-convergent, and that the limit is maximal symmetric or selfadjoint. In order to measure the rate of #-convergence, we introduce a selfadjoint operator $N \geq I$ and the associated non-Archimedean Hilbert spaces H_λ with the scalar product

$$\langle\psi,\psi\rangle_{\#\lambda} = \langle N^{\lambda/2}\psi, N^{\lambda/2}\psi\rangle_{\#}. \tag{3.1}$$

By standard identifications we have for $\lambda \geq 0 : H_\lambda \subset H_0 \subset H_{-1}$ and $H_0 = H$.
If $D : H_\alpha \to H_\beta$ is a #-densely defined, bounded operator from H_α to H_β, we let $\|D\|_{\#\alpha,\beta}$ denote its #-norm. Setting $\|D\|_{\#} = \|D\|_{\#0,0}$ we obtain

$$\|D\|_{\#\alpha,\beta} = \|N^{\beta/2}DN^{-\alpha/2}\|. \tag{3.2}$$

Let $C_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of #-selfadjoint operators, and consider the following three conditions.
(i) Suppose that $C_n - C_m$ is a #-densely defined, bounded operator from $H_\lambda$to $H_{-\lambda},$for some $\lambda,$and that as $n,m \to {}^*\infty$

$$\|C_n - C_m\|_{\#\lambda,-\lambda} \to_{\#} 0. \tag{3.3}$$

(ii) Suppose that, for some p and for an unbounded set of $z = x + iy \in {}^*\mathbb{C}_c^{\#}$ in the sector $|x| \leq const \times |y|$,

$$\|R_n(z)\|_{\#\mu,\lambda} \leq M(z), \tag{3.4}$$

where the bound $M(z)$ is uniform in $n \in {}^*\mathbb{N}$.
(iii) Suppose that, for the above z's,

$$\|R_n(\bar{z})\|_{\#\mu,\lambda} \leq M(z). \tag{3.5}$$

Theorem 5. Let $C_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of #-selfadjoint operators with a common domain, such that

$$C_n \to_{\#G} C.$$

If conditions (i) and (ii) hold, then

$$C_n \to_{\#R} C$$

and C is maximal symmetric.
Corollary 6. If in addition to the hypothesis of Theorem 5, condition (iii) also holds, then C is #-selfajoint.
Remark 3.1.(1) If $\mu = 0$ in (ii), then the resolvents #-converge uniformly.
(2) If the C_n are uniformly semibounded from below, then we may choose the z in condition (ii) to be infinite large negative numbers. In that case the conclusion of Theorem 5 is that $C_n \to_{\#R} C = C^*$.

§ 4.Estimates for singular perturbations

In this section we consider a singular perturbation B of a #-selfadjoint operator A. We give estimates on B which ensure that the sum $A + B$ is #-selfadjoint.
Abbreviation 4.1.We abbreviate $A^{\#-}$ instead #-$\bar{A}$.
Definition 4.1. A #-core of an operator C is a domain D contained in $D(C)$ such that $C = (C \upharpoonright D)^{\#-}$.
Lemma 7. Let $A, A_n, n \in {}^*\mathbb{N}, B, B_n, n \in {}^*\mathbb{N}$ and $C_n = A, +B_n, n \in {}^*\mathbb{N}$ be #-selfadjoint operators with a common #-core D. Assume the hypotheses of Theorem 5 and Corollary 6 for $C_n, n \in {}^*\mathbb{N}$ and suppose also that, for $\theta \in D$,

$$\|(A - A_n)\theta\|_{\#} + \|(B - B_n)\theta\|_{\#} \to_{\#} 0 \text{ as } n \to {}^*\infty \tag{4.9}$$

and

$$\|A_n\theta\|_{\#}^2 + \|B_n\theta\|_{\#}^2 \leq const. \times \|\theta\|_{\#}^2 + const. \times \|C_n\theta\|_{\#}^2, \tag{4.10}$$

with constants independent of n. Then $A+B$ is #-selfadjoint and $C_n \to_{\#R} A+B$.
Remark 4.1.As hypothesis for our next theorem, our second main result, we assume that $N \leq A$ and that N and A commute. Let

$$D^{*\infty}(A) = \bigcap_{n\in{}^*\mathbb{N}} A(A^n) \quad (4.11)$$

the elements of $D^{*\infty}(A)$ are called $C^{*\infty}$ vectors for A. Assume that $D^{*\infty}(A)$ is a #-core for the #-selfadjoint operator B. Also assume that, as bilinear forms on $D^{*\infty} \times D^{*\infty}$, and for some α and ε in the indicated ranges,

$$0 \leq \alpha N + B + const., 0 \leq \alpha < 1/2 \quad (4.12)$$

and

$$0 \leq \varepsilon A^2 + const \times B + [A^{1/2},[A^{1/2},B]] + const., 2\alpha + \varepsilon < 1. \quad (4.13)$$

Let B be a bounded operator from H_ν to $H_{-\nu}$ and from H_α to H_β for some α, β and $\nu, \beta > 0$ (H_α is defined following Theorem 4.) If $\nu \geq 2$, assume that for all $\varepsilon > 0$

$$0 \leq \varepsilon N^{\mu+2} + [N^{(\mu+1)/2},[N^{(\mu+1)/2},B]] + const. \quad (4.14)$$

as bilinear forms on $D^{*\infty} \times D^{*\infty}$, for some $\mu > \nu - 2$.
Theorem 8. Under the above hypothesis, $A+B$ is #-selfadjoint.

Chapter IX.

§1.Free scalar field

Let $\mathbf{H}^\#$ be a #-complex Hilbert space over field $\mathbb{C}_c^\#$ and let $\mathcal{F}(\mathbf{H}^\#) = \bigoplus_{n=0}^{\infty^\#} \mathbf{H}_\#^{(n)}$

(where $\mathbf{H}_\#^{(n)} = \bigoplus_{k=1}^{n} \mathbf{H}^\#$) be the Fock space over $\mathbf{H}^\#$. Our goal is to define the abstract free field on $\mathcal{F}_{\mathbf{s}}(\mathbf{H}^\#)$, the Boson subspace of $\mathcal{F}(\mathbf{H}^\#)$; to do this we need to introduce several other families of operators and some terminology. Let $f \in \mathbf{H}^\#$ be fixed. For vectors in $\mathbf{H}_\#^{(n)}$ of the form $\eta = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n$ we define a map $b^-(f) : \mathbf{H}_\#^{(n)} \to \mathbf{H}_\#^{(n-1)}$ by

$$b^-(f)\eta = (f,\psi_1)(\psi_2 \otimes \cdots \otimes \psi_n) \quad (1)$$

$b^-(f)$ extends by linearity to finite linear combinations of such η, the extension is well defined, and $\|b^-(f)\eta\| \leq \|f\| \times \|\eta\|$. Thus $b^-(f)$ extends to a bounded map (of norm $\|f\|$) of $\mathbf{H}_\#^{(n)}$ into $\mathbf{H}_\#^{(n-1)}$. Since this is true for each n (except for $n=0$ in which case we define $b^-(f) : \mathbf{H}_\#^{(0)} \to 0$), $b^-(f)$ is in a natural way a bounded operator of norm $\|f\|$ from $\mathcal{F}(\mathbf{H}^\#)$ to $\mathcal{F}(\mathbf{H}^\#)$. It is easy to check that $b^+(f) = (b^-(f))^*$ takes each $\mathbf{H}_\#^{(n)}$ into $\mathbf{H}_\#^{(n+1)}$ with the action

$$b^+(f)\eta = f \otimes \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \quad (2)$$

on product vectors. Notice that the map $f \mapsto b^+(f)$ is linear, but $f \mapsto b^-(f)$ is antilinear.

Let $\mathbf{S}_n$ be the symmetrization operators introduced in Section II.4. Then $\mathbf{S} = \bigoplus_{n=0}^{\infty^\#} \mathbf{S}_n$ is the projection onto the symmetric Fock space $\mathcal{F}_s(\mathbf{H}^\#) = \bigoplus_{n=0}^{\infty^\#} \mathbf{S}_n \mathbf{H}^{\#(n)}$ We will write $\mathbf{S}_n \mathbf{H}^{\#(n)} = \mathbf{H}_s^{\#(n)}$ and call $\mathbf{H}_s^{\#(n)}$ the n-particle subspace of $\mathcal{F}_s(\mathbf{H}^\#)$. Notice that $b^-(f)$ takes $\mathcal{F}_s(\mathbf{H}^\#)$ into itself, but that $b^+(f)$ does not. A vector $\Psi = \{\psi^{(n)}\}_{n=1}^{\infty^\#}$ for which $\psi^{(n)} = 0$ for all except finitely many n is called a finite particle vector. We will denote the set of finite particle vectors by F_0. The vector $\mathbf{\Omega}_0 = \langle 1,0,0,\ldots\rangle$ plays a special role; it is called the vacuum.

Let A be any self-adjoint operator on $\mathbf{H}^\#$ with domain of essential selfadjointness D. Let $D_A = \left\{\Psi \in F_0 | \psi^{(n)} \in \otimes_{k=1}^n D \text{ for each } n \in \mathbb{N}^\#\right\}$ and define $d\Gamma^\#(A)$ on $D_A \cap \mathbf{H}_s^{\#(n)}$ as

$$d\Gamma^\#(A) = A \otimes I \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + \otimes I \cdots \otimes I \otimes A. \tag{3}$$

Note that $d\Gamma^\#(A)$ is essentially self-adjoint on D_A ; $d\Gamma^\#(A)$ is called the second quantization of A. For example, let $A = I$. Then its second quantization $N = d\Gamma^\#(I)$ is essentially self-adjoint on F_0 and for $\psi \in \mathbf{H}_s^{\#(n)}, N\psi = n\psi$. N is called the number operator. If U is a unitary operator on $\mathbf{H}^\#$, we define $d\Gamma^\#(U)$ to be the unitary operator on $\mathcal{F}_s(\mathbf{H}^\#)$ which equals $Ext\text{-}\otimes_{k=1}^n U$ when restricted to $\mathbf{H}_s^{\#(n)}$ for $n > 0$, and which equals
the identity on $\mathbf{H}_s^{\#(0)}$. If $Ext\text{-}\exp(itA)$ is a #-continuous unitary group on $\mathbf{H}^\#$, then $\Gamma^\#(Ext\text{-}\exp(itA))$ is the group generated by $d\Gamma^\#(A)$, i.e., $\Gamma^\#(Ext\text{-}\exp(itA)) = Ext\text{-}\exp[itd\Gamma^\#(A)]$.

Deinition1.1. We define the annihilation operator $a^-(f)$ on $\mathcal{F}_s(\mathbf{H}^\#)$ with domain F_0 by

$$a^-(f) = \sqrt{N+1}\, b^-(f) \tag{4}$$

$a^-(f)$ is called an annihilation operator because it takes each $(n+1)$-particle subspace into the n-particle subspace. For each ψ and η in F_0,

$$\left(\sqrt{N+1}\, b^-(f)\psi, \eta\right) = \left(\psi, Sb^+(f)\sqrt{N+1}\right). \tag{5}$$

Then Eq.(5) implies that

$$(a^-(f))^* \upharpoonright F_0 = Sb^+(f)\sqrt{N+1} \tag{6}$$

The operator $(a^-(f))^*$ is called a creation operator. Both $a^-(f)$ and $a^-(f)^* \upharpoonright F_0$ are #-closable; we denote their #-closures by $a^-(f)$ and $a^-(f)^*$ also.

Example 1.1. If $\mathbf{H}^\# = L_2^\#(M, d^\#\mu)$, then $\bigotimes_{i=1}^n L_2^\#(M, d^\#\mu) = L_2^\#(\times_{i=1}^n M, \otimes_{i=1}^n d^\#\mu)$ and that $S\bigotimes_{i=1}^n L_2^\#(M, d^\#\mu) = L_{2,s}^\#(\times_{i=1}^n M, \otimes_{i=1}^n d^\#\mu)$, where $L_{2,s}^\#$ is the set of functions in $L_2^\#$ which are invariant under permutations of the coordinates. The operators $a^-(f)$ and $a^-(f)^*$ are given by

$$\begin{aligned} a^-(f)\psi^{(n)}(m_1,\ldots,m_n) &= \sqrt{n+1}\left(Ext\text{-}\int_M \tilde{f}(m)\psi^{(n+1)}(m,m_1,\ldots,m_n)d^\#\mu\right) \\ a^-(f)^*\psi^{(n)}(m_1,\ldots,m_n) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n f(m_i)\psi^{(n-1)}(m_1,\ldots,\tilde{m}_i,\ldots,m_n) \end{aligned} \tag{7}$$

where $\tilde{m}_i$ means that m_i is omitted. If A operates on $L_2^\#(M, d^\#\mu)$ by multiplication by the ${}^*\mathbb{R}_c^\#$-valued function $\omega(m)$, then

$$(d\Gamma^\#(A)\psi)^{(n)}(m_1,\ldots,m_n) = \left(\sum_{i=1}^n \omega(m_i)\right)\psi^{(n)}(m_1,\ldots,m_n) \tag{8}$$

Eq.(6) implies that the Segal field operator $\mathbf{\Phi}_S^\#(f)$ on F_0 defined by

$$\mathbf{\Phi}_S^{\#}(f) = \frac{1}{\sqrt{2}}[a^{-}(f) + a^{-}(f)^{*}] \tag{9}$$

is symmetric and essentially self-#-adjoint. The mapping from $\mathbf{H}^{\#}$ to the self-#-adjoint operators on $\mathcal{F}_s(\mathbf{H}^{\#})$ given by

$$f \mapsto \mathbf{\Phi}_S^{\#}(f) \tag{10}$$

is called the Segal quantization over $\mathbf{H}^{\#}$. Notice that the Segal quantization is a real (but not complex) linear map since $f \mapsto a^{-}(f)$ is antilinear and $f \mapsto a^{-}(f)^{*}$ is linear. The following theorem gives the properties of the Segal quantization.

Theorem 1.1. Let $\mathbf{H}^{\#}$ be hyper infinite dimensional Hilbert space over field ${}^{*}\mathbb{C}_c = {}^{*}\mathbb{R}_c^{\#} + i{}^{*}\mathbb{R}_c^{\#}$and $\mathbf{\Phi}_S^{\#}(f)$ the corresponding Segal quantization. Then:

(a) (self-adjointness) For each $f \in \mathbf{H}^{\#}$ the operator $\mathbf{\Phi}_S^{\#}(f)$ is essentially self-adjoint on F_0,
the hyperfinite particle vectors.

(b) (cyclicity of the vacuum) $\mathbf{\Omega}_0$ is in the domain of all hyperfinite products $\prod_{i=1}^{n}\mathbf{\Phi}_S^{\#}(f_i), n \in \mathbb{N}^{\#}$
and the set $\left\{\prod_{i=1}^{n}\mathbf{\Phi}_S^{\#}(f_i)\mathbf{\Omega}_0 \middle| f_i \text{ and } n \text{ arbitrary}\right\}$ is #-total in $\mathcal{F}_s(\mathbf{H}^{\#})$.

(c) (commutation relations) For each $\psi \in F_0$ and $f,g \in \mathbf{H}^{\#}$

$$[\mathbf{\Phi}_S^{\#}(f)\mathbf{\Phi}_S^{\#}(g) - \mathbf{\Phi}_S^{\#}(g)\mathbf{\Phi}_S^{\#}(f)]\psi = i\operatorname{Im}(f,g)_{\mathbf{H}^{\#}}\psi. \tag{11}$$

Further, if $W(f)$ denotes the external unitary operator $Ext\text{-}\exp(i\mathbf{\Phi}_S^{\#}(f))$ then

$$W(f+g) = \left[Ext\text{-}\exp\left(\frac{-i\operatorname{Im}(f,g)_{\mathbf{H}^{\#}}}{2}\right)\right]W(f)W(g) \tag{12}$$

(d) (#-continuity) If $\{f_n\}_{n=1}^{\infty^{\#}}$is hyper infinite sequence such as $\#\text{-}\lim_{n\to\infty^{\#}} f_n = f$ in $\mathbf{H}^{\#}$, then: (i) $\#\text{-}\lim_{n\to\infty^{\#}} W(f_n)\psi$ exists for all $\psi \in \mathcal{F}_s(\mathbf{H}^{\#})$ and

$$\#\text{-}\lim_{n\to\infty^{\#}} W(f_n)\psi = W(f)\psi \tag{13}$$

(ii) $\#\text{-}\lim_{n\to\infty^{\#}} \mathbf{\Phi}_S^{\#}(f_n)\psi$ exists for all $\psi \in F_0$ and

$$\#\text{-}\lim_{n\to\infty^{\#}} \mathbf{\Phi}_S^{\#}(f_n)\psi = \mathbf{\Phi}_S^{\#}(f)\psi. \tag{14}$$

(e) For every unitary operator U on $\mathbf{H}^{\#}, \Gamma^{\#}(U) : D\left(\overline{\mathbf{\Phi}_S^{\#}(f)}\right) \to D\left(\overline{\mathbf{\Phi}_S^{\#}(Uf)}\right)$ and for $\psi \in D\left(\overline{\mathbf{\Phi}_S^{\#}(Uf)}\right)$

$$\Gamma^{\#}(U)\overline{\mathbf{\Phi}_S^{\#}(f)}\Gamma^{\#}(U)^{-1}\psi = \overline{\mathbf{\Phi}_S^{\#}(Uf)}\psi \tag{15}$$

for all $f \in \mathbf{H}^{\#}$.

Proof. Let $\psi \in \mathbf{H}_s^{\#(n)}$. Since $\mathbf{\Phi}_S^{\#}(f) : F_0 \to F_0$, ψ is in $C^{\infty^{\#}}(\mathbf{\Phi}_S^{\#}(f))$. Further, it follows from Eq.(5)-Eq.(6), and the fact that $\|b^{-}(f)\| = \|f\|$, that

$$\left\|(a^{\bigstar}(f))^k\psi\right\|_{\#} \leq \left(Ext\text{-}\prod_{i=1}^{k}\sqrt{p+i}\right)\|f\|_{\#}^{k}\|\psi\|_{\#} \tag{16}$$

where $a^{\bigstar}(f)$ represents either $a^{-}(f)$ or $a^{-}(f)^{*}$. Therefore,

$$\|\mathbf{\Phi}_S^{\#}(f)^k\psi\|_{\#} \leq 2^{k/2}((n+k)!)^{1/2}\|f\|_{\#}^{k}\|\psi\|_{\#} \tag{17}$$

Since $Ext\text{-}\sum_{k=0}^{{}^{*}\infty} t^k 2^{k/2}((n+k)!)^{1/2}\|f\|_{\#}^{k}\|\psi\|_{\#} < {}^{*}\infty$ for all t,ψ is an #-analytic vector for $\mathbf{\Phi}_S^{\#}(f)$. Since F_0 is #-dense in $\mathcal{F}_s(\mathbf{H}^{\#})$ and is left invariant by $\mathbf{\Phi}_S^{\#}(f)$ is essentially self-adjoint on F_0 by generalized Nelson’s analytic vector theorem (Theorem).

The proof of (b) is obviously.

To prove (c) one first computes that if $\psi \in F_0$, then

$$a^{-}(f)a^{-}(g)^{*}\psi - a^{-}(g)^{*}a^{-}(f)\psi = (f,g)\psi \tag{18}$$

Eq.(11) follows immediately. Although Eq.(11) and Eq.(12) are formally equivalent, Eq.(11) by itself does not imply Eq.(12) We sketch a proof of Eq.(12) which uses special properties of the vectors in F_0. Let $\psi \in \mathbf{H}_s^{\#(p)}$. Then

$$\|\mathbf{\Phi}_S^{\#}(f)^n\mathbf{\Phi}_S^{\#}(g)^m\psi\|_{\#} \leq 2^{(n+m)/2}\left(Ext\text{-}\prod_{i=1}^{n+m}\sqrt{p+i}\right)\|f\|_{\#}^{n}\|g\|_{\#}^{m}\|\psi\|_{\#} \tag{19}$$

which implies that hyper infinite series $Ext\text{-}\sum_{n=0,m=0}^{\infty^{\#}}\left(\|\mathbf{\Phi}_S^{\#}(f)^n\mathbf{\Phi}_S^{\#}(g)^m\psi\|_{\#}/n!m!\right)$ #-converges for all $t \in {}^{*}\mathbb{R}_c^{\#}$. Since ψ is an #-analytic vector for $\mathbf{\Phi}_S^{\#}(g)$, $Ext\text{-}\sum_{m=0}^{\infty^{\#}}((i\mathbf{\Phi}_S^{\#}(g)^m)/m!)\psi = (Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(g)])\psi$. Further, for each $n \in \mathbb{N}^{\#}$, $(Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(g)])\psi$ is in the domain of $\left(\overline{\mathbf{\Phi}_S^{\#}(f)}\right)^n$ since any finite and hyperfinite sum

$$Ext\text{-}\exp\sum_{m=0}^{M}\frac{(i\mathbf{\Phi}_S^{\#}(g)^m)}{m!}\psi$$

with $M \in \mathbb{N}^{\#}$ is in it and $\mathbf{\Phi}_S^{\#}(f)^n\left(Ext\text{-}\sum_{m=0}^{M}((i\mathbf{\Phi}_S^{\#}(g)^m)/m!)\psi\right)$ #-converges as $M \to \infty^{\#}$. Thus the estimate $Ext\text{-}\sum_{n=0,m=0}^{\infty^{\#},\infty^{\#}}\left(\|\mathbf{\Phi}_S^{\#}(f)^n\mathbf{\Phi}_S^{\#}(g)^m\psi\|_{\#}/n!m!\right)t^n t^m \leq \infty^{\#}$ shows that $(Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(g)])\psi$ is an #-analytic vector for $\mathbf{\Phi}_S^{\#}(f)$ and therefore can be computed by the external hyper infinite power series. Thus

$$(Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(f)])(Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(g)])\psi = Ext\text{-}\sum_{n=0,m=0}^{\infty^{\#},\infty^{\#}}\frac{(i\mathbf{\Phi}_S^{\#}(f))^n(i\mathbf{\Phi}_S^{\#}(g))^m}{n!m!}\psi. \tag{20}$$

Similarly one obtains

$$\begin{gathered}\left(Ext\text{-}\exp\left[-\frac{it^2}{2}\operatorname{Im}(f,g)_{\mathbf{H}^{\#}}\right]\right)(Ext\text{-}\exp[it\mathbf{\Phi}_S^{\#}(f+g)])\psi = \\ Ext\text{-}\sum_{n=0,m=0}^{\infty^{\#},\infty^{\#}}\frac{1}{n!m!}\left[\left(-\frac{it^2}{2}\operatorname{Im}(f,g)_{\mathbf{H}^{\#}}\right)^m(it\mathbf{\Phi}_S^{\#}(f+g))^n\right]\psi\end{gathered} \tag{21}$$

where the hyper infinite series in RHS of Eq.(21) #-converges absolutely. Direct computations using Eq.(11) now show that Eq.(12) holds by a term-by-term comparison of the #-convergent external hyper infinite power series.

To prove (d) let $\psi \in \mathbf{H}_s^{\#(k)}$ and suppose that $\#\text{-}\lim_{n\to\infty^{\#}} f_n = f$ in $\mathbf{H}^{\#}$. Then

$$\|\mathbf{\Phi}_S^{\#}(f_n)\psi - \mathbf{\Phi}_S^{\#}(f)\psi\| \leq \sqrt{2(k+1)}\,\|f_n - f\|\,\|\psi\| \tag{22}$$

so $\#\text{-}\lim_{n\to\infty^{\#}}\mathbf{\Phi}_S^{\#}(f_n) = \mathbf{\Phi}_S^{\#}(f)$. Thus, Φ5(/„) #-converges strongly to $\mathbf{\Phi}_S^{\#}(f)$ on F_0. Since F_0 is a core for all $\mathbf{\Phi}_S^{\#}(f_n)$ and $\mathbf{\Phi}_S^{\#}(f)$, Theorems VIII.21 and VIII.25 imply that $\#\text{-}\lim_{n\to\infty^{\#}}(Ext\text{-}\exp[it\mathbf{\Phi}_S^{\#}(f_n)]\psi) = Ext\text{-}\exp[it\mathbf{\Phi}_S^{\#}(f)]\psi$ for all $\psi \in \mathcal{F}_s(\mathbf{H}^{\#})$.

To prove (e), let $\eta \in \mathbf{H}^{\#(n)}$ be of the form $\eta = \psi_1 \otimes \cdots \otimes \psi_n$. Then $\Gamma^{\#}(U)b^{-}(f)\Gamma^{\#}(U)^{-1}\eta = \Gamma^{\#}(U)b^{-}(f)(U^{-1}\psi_2 \otimes \cdots \otimes U^{-1}\psi_n) =$ $\Gamma^{\#}(U)(f,U^{-1}\psi_1)(U^{-1}\psi_2 \otimes \cdots \otimes U^{-1}\psi_n) = (Uf,\psi_1)(\psi_2 \otimes \cdots \otimes \psi_n) = b^{-}(Uf)\eta$.

Since finite linear combinations of such η are dense in $\mathbf{H}^{\#(n)}$ and $b^{-}(g)$ has norm $\|g\|$, we conclude that $\Gamma^{\#}(U)b^{-}(f)\Gamma^{\#}(U)^{-1} = b^{-}(Uf)$. But $\mathbf{N}$ and $\mathbf{S}$ commute with $\Gamma^{\#}(U)$ so this immediately implies that $\Gamma^{\#}(U)a^{-}(f)\Gamma^{\#}(U)^{-1} = a^{-}(Uf)$ on F_0. Taking adjoints and restricting to F_0 we also have $\Gamma^{\#}(U)(a^{-}(f))^{*}\Gamma^{\#}(U)^{-1} = (a^{-}(Uf))^{*}$. Thus for $\psi \in F_0$, $\Gamma^{\#}(U)\mathbf{\Phi}_S^{\#}(f)\Gamma^{\#}(U)^{-1}\psi = \mathbf{\Phi}_S^{\#}(Uf)\psi$. Since the operators on both the right- and left-hand sides of this equality are essentially self-#-adjoint on F_0, we conclude that $\Gamma^{\#}(U)\overline{\mathbf{\Phi}_S^{\#}(f)}\Gamma^{\#}(U)^{-1} = \overline{\mathbf{\Phi}_S^{\#}(Uf)}$.

Remark 1.1. Henceforth we use $\mathbf{\Phi}_S^{\#}(f)$ to denote the #-closure of $\mathbf{\Phi}_S^{\#}(f)$.

Definition 1.1. For each $m > 0, m \in {}^{*}\mathbb{R}_{c,\mathbf{fin}}^{\#}$ let

$$H_m^{\#} = \{p \in {}^*\mathbb{R}_c^{\#4} p \cdot \tilde{p} = m^2, p_0 > 0\}, \tag{23}$$

where $\tilde{p} = (p^0, -p^1, -p^2, -p^3)$. The sets $H_m^{\#}$, which are called mass hyperboloids, are invariant under ${}^\sigma\mathcal{L}_+^{\uparrow}$. Let j_m be the #-homeomorphism of $H_m^{\#}$ onto ${}^*\mathbb{R}_c^{\#3}$ (or in the case $m = 0$ onto ${}^*\mathbb{R}_c^{\#3}\backslash\{0\}$) given by $j_m : \langle p_0, p_1, p_2, p_3\rangle \mapsto \langle p_1, p_2, p_3\rangle = \mathbf{p}$. Define a #-measure $\Omega_m^{\#}$ on $H_m^{\#}$ by

$$\Omega_m^{\#}(E) = Ext\text{-}\int\limits_{j_m(E)} \frac{d^{\#3}\mathbf{p}}{\sqrt{|\mathbf{p}|^2 + m^2}} \tag{24}$$

for any measurable set $E \subset H_m^{\#}$. The measure $\Omega_m^{\#}(E)$ can easily be seen to be ${}^\sigma\mathcal{L}_+^{\uparrow}$-invariant. In fact, up to a constant multiple, $\Omega_m^{\#}$ is the only ${}^\sigma\mathcal{L}_+^{\uparrow}$-invariant measure on $H_m^{\#}$. Furthermore, every polynomially bounded ${}^\sigma\mathcal{L}_+^{\uparrow}$-invariant measure on $\overline{V}_+$ is the sum of a multiple of δ and an integral of the measures $\Omega_m^{\#}$. We state this fact as a theorem.

Theorem 1.2. Let $\mu^{\#}$ be a polynomially bounded #-measure with support in $\overline{V}_+$. If $\mu^{\#}$ is ${}^\sigma\mathcal{L}_+^{\uparrow}$-invariant, there exists a polynomially bounded #-measure ρ on $[0, \infty^{\#})$ and a constant c so that for any $f \in S^{\#}({}^*\mathbb{R}_c^{\#4})$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} f\, d^{\#}\mu^{\#} = cf(0) + Ext\text{-}\int_0^{\infty^{\#}} d^{\#}\rho^{\#}(m)\left(Ext\text{-}\int_{H_m^{\#}} f d^{\#}\Omega_m^{\#}\right). \tag{25}$$

Theorem 1.3.

We can now use the Segal quantization to define the free Hermitian scalar field of mass m. We take $\mathbf{H}^{\#} = \mathcal{L}_2^{\#}(H_m^{\#}, d^{\#}\Omega_{m,x}^{\#})$, where $H_m^{\#}, m > 0$, is the mass hyperboloid in ${}^*\mathbb{R}_c^{\#4}$ consisting of those $p \in {}^*\mathbb{R}_c^{\#4}$ satisfying $p \cdot \tilde{p} - m^2 = 0$ and $p_0 > 0$, and $d^{\#}\Omega_m^{\#}$ is the Lorentz invariant #-measure.

For each $f \in S^{\#}({}^*\mathbb{R}_c^{\#4})$ we define $Ef \in \mathbf{H}^{\#}$ by $Ef = 2\pi_{\#}\widehat{f} \restriction H_m^{\#}$ where the Fourier transform

$$(2\pi_{\#})^{-2}\left(Ext\text{-}\int\left(Exp\text{-}\exp\left[i\left(p \cdot \tilde{x}\right)\right]\right)f(x)d^{\#4}x\right) \tag{26}$$

is defined in terms of the Lorentz invariant inner product $p \cdot \tilde{x}$. The reason for the extra $\sqrt{2\pi_{\#}}$ in our definition of E and the plus sign in the definition of Fourier transform is that if f is the distribution $f(x) = g(\mathbf{x})\delta^{\#}(t)$, then $\sqrt{2\pi_{\#}}\,\widehat{f}$ is the ordinary three-dimensional Fourier transform of g. If $\mathbf{\Phi}_S^{\#}(\cdot)$ is the Segal quantization over $\mathcal{L}_2^{\#}(H_m^{\#}, d^{\#}\Omega_{m,x}^{\#})$, we define for each ${}^*\mathbb{R}_c^{\#}$-valued $f \in S^{\#}({}^*\mathbb{R}_c^{\#4})$

$$\Phi_{m,x}^{\#}(f) = \mathbf{\Phi}_S^{\#}(Ef). \tag{27}$$

For ${}^*\mathbb{C}_c^{\#}$-valued function $f \in S^{\#}({}^*\mathbb{R}_c^{\#4})$ we define

$$\Phi_{m,x}^{\#}(f) = \Phi_{m,x}^{\#}(\operatorname{Re} f) + i\Phi_{m,x}^{\#}(\operatorname{Im} f) \tag{28}$$

The mapping $f \mapsto \Phi_m^{\#}(f)$ is called the free Hermitian scalar field of mass m.

On $\mathcal{L}_2^{\#}(H_m^{\#}, d^{\#}\Omega_m)$ we define the following unitary representation of the restricted Poincare group:

$$(U_m(a,\Lambda)\psi)(p) = \left(Exp\text{-}\exp\left[i\left(p \cdot \tilde{a}\right)\right]\right)\psi(\Lambda^{-1}p) \tag{29}$$

where we are using Λ to denote both an element of the abstract restricted Lorentz group and the corresponding element in the standard representation on ${}^*\mathbb{R}^4_{\mathbf{st}} = \mathbb{R}^4$.

Remark 1.3. Recall that a #-conjugation on a Hilbert space $\mathbf{H}^\#$ is an antilinear #-isometry $\mathbf{C}^\#$ so that $\mathbf{C}^{\#2} = \mathbf{I}$.

Definition 1.2. Let $\mathbf{H}^\#$ be a ${}^*\mathbb{C}^\#_c$-complex Hilbert space, $\mathbf{\Phi}^\#_S(\cdot)$ the associated Segal quantization. Let $\mathbf{C}^\#$ be a #-conjugation on $\mathbf{H}^\#$ and define $\mathbf{H}^\#_{\mathbf{C}^\#} = \{|\mathbf{C}^\# f = f\}$. For each $f \in \mathbf{H}^\#_{\mathbf{C}^\#}$ we define $\varphi^\#(f) = \mathbf{\Phi}^\#_S(f)$ and $\pi^\#(f) = \mathbf{\Phi}^\#_S(if)$. The map $f \mapsto \varphi^\#(f)$ is called the canonical free field over the doublet $\langle \mathbf{H}^\#, \mathbf{C}^\# \rangle$ and the map $f \mapsto \pi^\#(f)$ is called the canonical conjugate momentum. We often drop the $\langle \mathbf{H}^\#, \mathbf{C}^\# \rangle$ and just write $\mathbf{H}^\#$ if the intended #-conjugation is clear.

Remark 1.4.Note that the set of elements of $\mathbf{H}^\#$ for which the maps $f \mapsto \varphi^\#(f)$ and $f \mapsto \pi^\#(f)$ are defined depends on the #-conjugation $\mathbf{C}^\#$.

Theorem 1.4. Let $\mathbf{H}^\#$ be a ${}^*\mathbb{C}^\#_c$-complex Hilbert space with #-conjugation $\mathbf{C}^\#$. Let $\varphi^\#(\cdot)$ and $\pi^\#(\cdot)$ be the corresponding canonical fields. Then:

(i) For each $f \in \mathbf{H}^\#_{\mathbf{C}^\#}, \varphi^\#(f)$ is essentially self-adjoint on F_0.

(ii) $\{\varphi^\#(f) | f \in \mathbf{H}^\#_{\mathbf{C}^\#}\}$ is a commuting family of self-adjoint operators.

(iii) $\mathbf{\Omega}_0$ is a #-cyclic vector for the family $\{\varphi^\#(f) | f \in \mathbf{H}^\#_{\mathbf{C}^\#}\}$.

(iv) If $\#\text{-}\lim_{n\to\infty^\#} f_n = f$ in $\mathbf{H}^\#_{\mathbf{C}^\#}$, then

$$\#\text{-}\lim_{n\to\infty^\#} \varphi^\#(f_n)\psi = \varphi^\#(f)\psi \text{ for all } \psi \in F_0$$

and

$$\#\text{-}\lim_{n\to\infty^\#}(Exp\text{-}\exp[i\varphi^\#(f_n)]\psi) = Exp\text{-}\exp[i\varphi^\#(f)]\psi \text{ for all } \psi \in \mathcal{F}_s(\mathbf{H}^\#)$$

(v) Properties (i)-(iv) hold with $\varphi^\#(f)$ replaced by $\pi^\#(f)$.

(vi) If $f,g \in \mathbf{H}^\#_{\mathbf{C}^\#}$, then

$$\varphi^\#(f)\pi^\#(g)\psi - \pi^\#(g)\varphi^\#(f)\psi = i(f,g)\psi \tag{30}$$

for all $\psi \in F_0$ and

$$\begin{gathered}\left(Exp\text{-}\exp\left[\, i\varphi^\#(f)\,\right]\right)\left(Exp\text{-}\exp\left[\, i\pi^\#(g)\,\right]\right) = \\ (Exp\text{-}\exp[i(f,g)])\left(Exp\text{-}\exp\left[\, i\pi^\#(g)\,\right]\right)\left(Exp\text{-}\exp\left[\, i\varphi^\#(f)\,\right]\right).\end{gathered} \tag{31}$$

Proof. (i) and (iv) follow immediately from the corresponding properties of $\mathbf{\Phi}^\#_S(\cdot)$ proven in Theorem 1.1. To see that $\{\varphi^\#(f) | f \in \mathbf{H}^\#_{\mathbf{C}^\#}\}$ is a commuting family, notice that (12) implies

$$\begin{gathered}\left(Exp\text{-}\exp\left[\, it\varphi^\#(f)\,\right]\right)\left(Exp\text{-}\exp\left[\, is\varphi^\#(g)\,\right]\right) = \\ (Exp\text{-}\exp[-its\operatorname{Im}(f,g)])\left(Exp\text{-}\exp\left[\, is\varphi^\#(g)\,\right]\right)\left(Exp\text{-}\exp\left[\, it\varphi^\#(f)\,\right]\right)\end{gathered} \tag{32}$$

where we have used the fact that $\varphi^\#(\cdot)$ is real linear. If $f,g \in \mathbf{H}^\#_{\mathbf{C}^\#}$, then it follows from polarization that $(f,g) = (C^\# f, C^\# g) = (g,f)$, so $\operatorname{Im}(f,g) = 0$. Thus

$$\begin{gathered}\left(Exp\text{-}\exp\left[\, it\varphi^\#(f)\,\right]\right)\left(Exp\text{-}\exp\left[\, is\varphi^\#(g)\,\right]\right) = \\ \left(Exp\text{-}\exp\left[\, is\varphi^\#(g)\,\right]\right)\left(Exp\text{-}\exp\left[\, it\varphi^\#(f)\,\right]\right)\end{gathered} \tag{33}$$

for s and t. Therefore, by Theorem VIII. 13, $\varphi^\#(g)$ and $\varphi^\#(f)$ commute.

The proof of (b) is similar to the proof of (a). (X.70) and (X.71) follow immediately from (X.64), (X.65), and the fact that if $f,g \in \mathbf{H}^\#_{\mathbf{C}^\#}$, then $\operatorname{Im}(f,ig) = \operatorname{Re}(f,g) = (f,g)$.

Definition 1.3.We write $f \in \mathcal{L}^\#_2(H^\#_m, d^\#\Omega^\#_{m,x})$ as $f(p_0,\mathbf{p})$ and define now the #-conjugation by $(\mathbf{C}^\# f)(p_0,\mathbf{p}) = \overline{f(p_0,-\mathbf{p})}$.

Remark 1.4.Note that $\mathbf{C}^{\#}$ is well-defined on $\mathcal{L}_2^{\#}(H_m^{\#}, d^{\#}\Omega_{m,x}^{\#})$ since $\langle p_0, \mathbf{p}\rangle \in H_m^{\#}$ if and only if $\langle p_0, -\mathbf{p}\rangle \in H_m^{\#}$. $\mathbf{C}^{\#}$ is clearly a #-conjugation.

Definition 1.4.We denote the canonical fields corresponding to $\mathbf{C}^{\#}$ by $\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ and define $\varphi_m^{\#}(f) = \varphi^{\#}(Ef)$ and $\pi_m^{\#}(f) = \pi^{\#}(\mu Ef), \mu = \sqrt{\mathbf{p}^2 + m^2}$ for ${}^*\mathbb{R}_c^{\#}$-valued $f \in \mathcal{L}({}^*\mathbb{R}_c^{\#4})$, extending to all of $\mathcal{L}({}^*\mathbb{R}_c^{\#4})$ by linearity. In terms of $a^-(f)$,

$$\begin{aligned}\varphi_m^{\#}(f) &= \{(a^-(Ef))^* + a^-(\mathbf{C}^{\#}Ef)\}/\sqrt{2},\\ \pi_m^{\#}(f) &= i\{(a^-(Ef))^* + a^-(\mathbf{C}^{\#}\mu Ef)\}/\sqrt{2}.\end{aligned} \tag{34}$$

Remark 1.5.Note that the a's in these last formulas differ from those most often used in discussing the free field and that the correct space-time free field is $\mathbf{\Phi}_m^{\#}$ and not $\varphi_m^{\#}$ as we will discuss below, $\varphi_m^{\#}$ and $\pi_m^{\#}$ are useful for discussing the time-zero field. The maps $f \mapsto \varphi_m^{\#}(f)$ and $f \mapsto \pi_m^{\#}(f)$ are complex linear and $\varphi_m^{\#}(f), \pi_m^{\#}(f)$ are self-adjoint if and only if $Ef \in \mathbf{H}_{\mathbf{C}^{\#}}^{\#}$.

Because of the projection E we can extend the class of functions on which $\varphi_m^{\#}(\cdot)$ and $\pi_m^{\#}(\cdot)$ are defined to include distributions of the form $\delta(t - t_0)g(x_1, x_2, x_3)$ where $g \in {}^*\mathbb{R}_c^{\#3}$. In particular, if $t_0 = 0, g$ is ${}^*\mathbb{R}_c^{\#}$-lvalued, and $Ext\text{-}\widehat{g}$ is the usual Fourier transform on ${}^*\mathbb{R}_c^{\#3}$, then

$$\left(\mathbf{C}^{\#}E\widehat{\delta g}\right)(p_0, -\mathbf{p}) = (2\pi_{\#})^{-1/2}\overline{\widehat{g}(-\mathbf{p})} = (2\pi_{\#})^{-1/2}\widehat{g}(-\mathbf{p}) = E\widehat{\delta g}. \tag{35}$$

Thus $E(\delta g)$ and $\mu E(\delta g)$ are in $\mathbf{H}_{\mathbf{C}^{\#}}^{\#}$. Therefore $\varphi_m^{\#}(\delta g)$ and $\pi_m^{\#}(\delta g)$ are self-adjoint if $g \in \mathcal{L}({}^*\mathbb{R}_c^{\#3})$ is real. For obvious reasons, the maps $g \mapsto \varphi_m^{\#}(\delta g), g \mapsto \pi_m^{\#}(\delta g)$ are called the time-zero fields. From now on we will only use test functions of the form δg in $\varphi_m^{\#}(\cdot)$ and $\pi_m^{\#}(\cdot)$ and write $\varphi_m^{\#}(g)$ and $\pi_m^{\#}(g)$ if $g \in S^{\#}{}^*\mathbb{R}_c^{\#3}$ instead of $\varphi_m^{\#}(\delta g)$ and $\pi_m^{\#}(\delta g)$.

If f and g are ${}^*\mathbb{R}_c^{\#}$-valued functions in $\mathcal{L}({}^*\mathbb{R}_c^{\#3})$, then (X.70) implies that for $\psi \in F_0$,

$$[\varphi_m^{\#}(f), \pi_m^{\#}(g)]\psi = i\left(Ext\text{-}\int_{H_m} \overline{\widehat{f}(p)}\widehat{g}(p)\mu(p)\psi d\Omega_{m,\mathbf{x}}^{\#}\right). \tag{36}$$

For convenience and also so that our notation coincides with the standard terminology, we now transfer the fields we have constructed from the Fock space built up from $\mathcal{L}_2^{\#}(H_m^{\#}, d\Omega_{m,\mathbf{x}}^{\#})$ to the Fock space built up from $\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3})$. For notational simplicity, we define for $f \in \mathcal{L}_2^{\#}(H_m^{\#}, d\Omega_{m,\mathbf{x}}^{\#})$

$$a^{\dagger}(f) = (a^-(f))^*, a(f) = a^-(\mathbf{C}^{\#}f). \tag{37}$$

First notice that each function $f(p) \in \mathcal{L}_2^{\#}(H_m^{\#}, d\Omega_{m,\mathbf{x}}^{\#})$ is in a natural way a function $f(\mathbf{p}) = f(\mu(\mathbf{p}), \mathbf{p})$ on ${}^*\mathbb{R}_c^{\#3}$. For each $f \in \mathcal{L}_2^{\#}(H_m^{\#}, d\Omega_{m,\mathbf{x}}^{\#})$, we define

$$(Jf)(\mathbf{p}) = f(\mu(\mathbf{p}), \mathbf{p})/\sqrt{\mu(\mathbf{p})}. \tag{38}$$

J is a unitary map of $\mathcal{L}_2^{\#}(H_m^{\#}, d\Omega_{m,\mathbf{x}}^{\#})$ onto $\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3})$, so $\Gamma^{\#}(J)$ is a unitary map of $\mathcal{F}_s(\mathcal{L}_2^{\#}(H_m^{\#}, d\Omega_{m,\mathbf{x}}^{\#}))$ onto $\mathcal{F}_s(\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3}))$. The annihilation and creation operators on $\mathcal{F}_s(\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3}))$, $\widetilde{a}(\cdot)$, $\widetilde{a}^{\dagger}(\cdot)$, are related to $a(\cdot)$ and $a^{\dagger}(\cdot)$ by the formulas

$$\widetilde{a}\left(\frac{f(\mathbf{p})}{\sqrt{\mu(\mathbf{p})}}\right) = \Gamma^{\#}(J)a(f)\Gamma^{\#}(J)^{-1}$$
$$\widetilde{a}^{\dagger}\left(\frac{f(\mathbf{p})}{\sqrt{\mu(\mathbf{p})}}\right) = \Gamma^{\#}(J)a^{\dagger}(f)\Gamma^{\#}(J)^{-1} \tag{39}$$

We use the unitary map $\Gamma^{\#}(J)$ to carry the Wightman fields over to $\mathcal{F}_s(\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3}))$ by defining:(i) for ${}^*\mathbb{R}^{\#}_{c,\mathbf{fin}}$-valued $f \in \mathcal{L}^{\#}_{\mathbf{fin}}({}^*\mathbb{R}_c^{\#4})$

$$\widetilde{\Phi}_{m,x}(f) = \Gamma^{\#}(J)\Phi_{m,x}(f)\Gamma^{\#}(J)^{-1} =$$
$$\frac{1}{\sqrt{2}}\left\{\widetilde{a}\left(\tilde{\mathbf{C}}^{\#}\frac{Ef}{\sqrt{\mu}}\right) + \tilde{a}^{\dagger}\left(\frac{Ef}{\sqrt{\mu}}\right)\right\} \tag{40}$$

(ii) for ${}^*\mathbb{R}^{\#}_{c,\mathbf{fin}}$-valued $f \in \mathcal{L}^{\#}_{\mathbf{fin}}({}^*\mathbb{R}_c^{\#3})$

$$\widetilde{\varphi}_{m,x}(f) = \Gamma^{\#}(J)\varphi_{m,x}(f)\Gamma^{\#}(J)^{-1} =$$
$$\frac{1}{\sqrt{2}}\left\{\widetilde{a}\left(\tilde{\mathbf{C}}^{\#}\frac{E(f\delta)}{\sqrt{\mu}}\right) + \tilde{a}^{\dagger}\left(\frac{E(f\delta)}{\sqrt{\mu}}\right)\right\} \tag{41}$$

where $\tilde{\mathbf{C}}^{\#} = J\mathbf{C}^{\#}J^{-1}$ acts by $\left(\tilde{\mathbf{C}}^{\#}g\right)(\mathbf{p}) = \overline{g(-\mathbf{p})}$. Having established this correspondence,
we now drop the ~ and the bold face letters; from now on we will only deal with the fields
on $\mathcal{F}_s(\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3}))$ and three-dimensional momenta. Further, we recall that the restriction of
the four-dimensional Fourier transform that we have been using in this section to functions of the form $\delta(x_0)g(x_1,x_2,x_3)$ the usual three-dimensional Fourier transform. Notice that

$$\widetilde{f} = Ext\text{-}\check{h}, h = \left(\mathbf{C}^{\#}\hat{f}\right) \tag{42}$$

so $\mathbf{C}^{\#}\hat{f} = \hat{f}$ if and only if f is ${}^*\mathbb{R}_c^{\#}$-valued.
For f and g ${}^*\mathbb{R}_c^{\#}$-valued, (36) becomes

$$[\varphi_m^{\#}(f), \pi_m^{\#}(g)] \approx i\left(Ext\text{-}\int f(x)g(x)\right)d^{\#3}x. \tag{43}$$

(43) is the space form of the canonical commutation relations (CCR).
In the Appendix to this section we prove that for each $m > 0$, this representation of the CCR is irreducible and for different m, the representations are inequivalent. Thus, the time-zero fields in the free scalar field theories give rise to different representation of the CCR.
As a final topic before turning to interacting fields we will show how the structures developed above are related to the “fields” and “annihilation and creation operators” introduced in physics texts. We let now

$$D_{S^{\#}_{\mathbf{fin}}} = \left\{\psi|\psi \in F_0, \psi^{(n)} \in S^{\#}_{\mathbf{fin}}({}^*\mathbb{R}_c^{\#3n}), n \in \mathbb{N}\right\} \tag{44}$$

and for each $p \in {}^*\mathbb{R}_c^{\#3}$ we define an operator $a(p)$ on $\mathcal{F}_s(\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3}))$ with domain $D_{S^{\#}_{\mathbf{fin}}}$ by

$$(a(p)\psi)^{(n)}(k_1,\ldots,k_n) = \sqrt{n+1}\,\psi^{(n+1)}(p,k_1,\ldots,k_n). \tag{45}$$

The adjoint of the operator $a(p)$ is not a #-densely defined operator since it is given formally by

$$(a^{\dagger}(p)\psi)^{(n)}(k_1,\ldots,k_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta(p-k_i)\psi^{(n+1)}(p,k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n). \tag{46}$$

However, $a^{\dagger}(p)$ is a well-defined quadratic form on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \times D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$. For example, if $\psi_1 = \{0,\psi^{(1)},0,\ldots\}$, and $\psi_2 = \{0,0,\psi^{(2)},0,\ldots\}$, then

$$(\psi_2,a^{\dagger}(p)\psi_1) = \frac{1}{\sqrt{2}}\left\{Ext\text{-}\int\left[\overline{\psi^{(2)}(k_1,p)}\psi^{(1)}(k_1)+\overline{\psi^{(2)}(p,k_1)}\psi^{(1)}(k_1)\right]d^{\#}k_1\right\}. \tag{47}$$

Remark 1.1.Note that the formulas

$$a(g) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_{c}} a(p)g(-p)d^{\#}p \tag{48}$$

and

$$a^{\dagger}(g) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_{c}} a^{\dagger}(p)g(p)d^{\#3}p \tag{49}$$

hold for all $g \in S^{\#}_{\mathbf{fin}}({}^{*}\mathbb{R}^{\#3}_{c})$ if the equalities are understood in the sense of quadratic forms. That is, (48) means that for $\psi_1,\psi_2 \in D_{S^{\#}_{\mathbf{fin}}}$ we have

$$(\psi_1,a(g)\psi_2) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_{c}}(\psi_1,a(p)\psi_2)g(-p)d^{\#3}p \tag{50}$$

and similarly for (X.76b).
Since $a(p) : D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \to D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ the powers of $a(p)$ are well-defined operators on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$.
As before we can write down a formal expression for $(a^{\dagger}(p))^n$, but it does not make sense as operator, only as ${}^{*}\mathbb{C}^{\#}_{c}$-valued quadratic form on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \times D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$.
Notice that

$$(\psi_1,(a^{\dagger}(p))^n\psi_2) = ((a(p))^n\psi_1,\psi_2) \tag{51}$$

so for each n, $(a^{\dagger}(p))^n$ and $(a(p))^n$ are formally adjoints in the sense of ${}^{*}\mathbb{C}^{\#}_{c}$-valued quadratic forms. We could of course have defined the quadratic form $(a^{\dagger}(p))^n$ by (50) and then calculated that it arises by taking the n-th power of the formal object given by (45). Since $a(p_1) : D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \to D_{\mathcal{L}^{\#}_{\mathbf{fin}}}, (\psi_1,a^{\dagger}(p_2)a(p_1)\psi_2)$ is a well-defined ${}^{*}\mathbb{C}^{\#}_{c}$-valued quadratic form for all $\langle p_1,p_2\rangle \in {}^{*}\mathbb{R}^{\#3}_{c} \times {}^{*}\mathbb{R}^{\#3}_{c}$. Notice, however, that $(\psi_1,a(p_1)a^{\dagger}(p_2)\psi_2)$ does not make sense since $a^{\dagger}(p_2)$ is only a quadratic form. In general any product $\prod_{i=1}^{N_1}a(f_i)$ is a
well-defined operator from $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ to $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ and $\prod_{i=1}^{N_1}a^{\dagger}(f_i)$ is a well-defined quadratic form on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \times D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$. Thus

$$\left(\psi_1,\left(\prod_{i=N_1+1}^{N_2}a^{\dagger}(p_i)\right)\left(\prod_{i=1}^{N_1}a^{\dagger}(-p_i)\right)\psi_2\right) \tag{52}$$

is also well-defined ${}^{*}\mathbb{C}^{\#}_{c}$-valued quadratic form on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \times D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$. One can check directly that if $f \in \mathcal{L}^{\#}_{\mathbf{fin}}({}^{*}\mathbb{R}^{\#3}_{c})$ then as ${}^{*}\mathbb{C}^{\#}_{c}$-valued quadratic forms

$$\left(\prod_{i=N_1+1}^{N_2} a^\dagger(f_i)\right)\left(\prod_{i=1}^{N_1} a^\dagger(f_i)\right) =$$
$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3N_2}} \left(\prod_{i=N_1+1}^{N_2} a^\dagger(p_i)\right)\left(\prod_{i=1}^{N_1} a^\dagger(-p_i)\right)\left(\prod_{i=1}^{N_2} f_i(p_i)\right) d^\# p_1 \ldots d^\# p_{N_2} \tag{53}$$

and

$$N = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} a^\dagger(p)a(p)d^\# p \tag{54}$$

The generator of time translations in the free scalar field theory of mass m is given by

$$\mathbf{H}_0 = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \mu(p)a^\dagger(p)a(p)d^\# p \tag{54}$$

$\mathbf{H}_0$ is called the free Hamiltonian of mass m. (52), (53), and (54) involve no formal manipulations, but are mathematical statements about quadratic forms.

Theorem X.44 Let n_1 and n_2 be nonnegative integers and suppose that $W \in \mathcal{L}_2^\#\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$. Then there is a unique operator T_W on $\mathcal{F}_s(\mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#3}))$ so that $D_{\mathcal{L}^\#_{\mathbf{fin}}} \subset D(T_W)$ is a core for T_W and

(a) as ${}^*\mathbb{C}_c^\#$-valued quadratic forms on $D_{\mathcal{L}^\#_{\mathbf{fin}}} \times D_{\mathcal{L}^\#_{\mathbf{fin}}}$

$$T_W = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3(n_1+n_2)}} W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2})\left(\prod_{i=1}^{n_1} a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} a(p_i)\right) d^{\#n_1}k d^{\#n_2}p \tag{55}$$

(b) If m_1 and m_2 are nonnegative integers so that $m_1 + m_2 = n_1 + n_2$, then $(1+N)^{-m_1/2}T_W(1+N)^{-m2/2}$ is a bounded operator with

$$\|(1+N)^{-m_1/2}T_W(1+N)^{-m_2/2}\| < C(m_1,m_2)\|W\|_{\mathcal{L}_2^\#}. \tag{56}$$

In particular, if $m_1 = n_1$ and $m_2 = n_2$, then

$$\|(1+N)^{-n_1/2}T_W(1+N)^{-n_2/2}\| < C(m_1,m_2)\|W\|_{\mathcal{L}_2^\#}. \tag{57}$$

(c) As ${}^*\mathbb{C}_c^\#$-valued quadratic forms on $D_{\mathcal{L}^\#_{\mathbf{fin}}} \times D_{\mathcal{L}^\#_{\mathbf{fin}}}$

$$T_W^* = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3(n_1+n_2)}} \overline{W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2})}\left(\prod_{i=1}^{n_2} a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_1} a(p_i)\right) d^{\#n_1}k d^{\#n_2}p \tag{58}$$

(d) If $W_n \to_\# W$ in $\mathcal{L}_2^\#\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$, then $T_{W_n} \to_\# T_W$ strongly on $D_{\mathcal{L}^\#_{\mathbf{fin}}}$.

(e) F_0 is contained in $D(T_W)$ and $D(T_W^*)$, and on vectors in F_0, T_W and T_W^* are given by the explicit formulas

$$(T_W\psi)^{(l-n_2+n_1)} = K(l,n_1,n_2)\mathbf{S}\times$$
$$\left[Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3n_2}} W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2})\psi^{(l)}(p_1,\ldots,p_{n_2},k_{n_1+1},\ldots,k_{n_1+l-n_2})d^{\#n_2}p\right] \tag{59}$$

$(T_W\psi)^n = 0$ if $n < n_1 - n_2$

$$(T_W^*\psi)^{(l-n_1+n_2)} = K(l,n_2,n_1)\mathbf{S}\times$$
$$\left[Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3(n_1)}} \overline{W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2})}\psi^{(l)}(k_1,\ldots,k_{n_1},p_{n_2+1},\ldots,p_{n_2+l-n_1})d^{\#n_1}k\right] \tag{60}$$

$(T_W^*\psi)^n = 0$ if $n < n_2 - n_1$ where $\mathbf{S}$ is the symmetrization operator and

$$K(l,n_1,n_2) = \left[\frac{l!(l+n_1-n_2)!}{((l-n_2)!)^2} \right]^{1/2}. \tag{61}$$

Proof. For vectors in $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$, we define $T_W\psi$ by the formula (X.82a). By the Schwarz inequality and the fact that $\mathbf{S}$ is a projection,

$$\left\| (T_W\psi)^{(l-n_2+n_1)} \right\|^2 \leq K(l,n_1,n_2)\|\psi^{(l)}\|^2\|W\|^2. \tag{62}$$

If we now define an operator $T^*_W\psi$, on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ by using the formula in (62), then for all φ and ψ in $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ one easily verifies that $(\varphi, T_W\psi) = (T^*_W\varphi,\psi)$.
Thus, T_W is #-closable and T^*_W is the restriction of the adjoint of T_W to $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$.
From now on we will use T_W to denote $\overline{T}_W$ and T^*_W to denote the adjoint of T_W.
By the definition of $T_W, D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ is a #-core and further, since T_W is bounded on the l-particle vectors in $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$, we have $F_0 \subset D(T_W)$. Since the right-hand side of (59) is also bounded on the l-particle vectors, (X.82a) represents T_W on all l-particle vectors.
The proof of the statements in (e) about T^*_W are the same.
To prove (b), let $\psi \in D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$. Then by the above computation

$$\begin{gathered}\left\| ((1+N)^{-m_1/2}T_W(1+N)^{-m_2/2}\psi)^{(l-n_2+n_1)} \right\|^2 \leq \\ \left[\frac{K(l,n_1,n_2)}{(1+l-n_2+n_1)^{m_1/2}(1+l)^{m_2/2}} \right]^2 \|\psi^{(l)}\|^2\|W\|^2\end{gathered} \tag{63}$$

so that

$$\begin{gathered}\left\| ((1+N)^{-m_1/2}T_W(1+N)^{-m_2/2}\psi)^{(l-n_2+n_1)} \right\| \leq \\ \left[\sup_{l\in\mathbb{N}} \frac{K(l,n_1,n_2)}{(1+l-n_2+n_1)^{m_1/2}(1+l)^{m_2/2}} \right] \|\psi^{(l)}\|\|W\| \leq C(m_1,m_2)\|\psi^{(l)}\|\|W\|\end{gathered} \tag{64}$$

where

$$C(m_1,m_2) = \sup_{l\in\mathbb{N}} \frac{K(l,n_1,n_2)}{(1+l-n_2+n_1)^{m_1/2}(1+l)^{m_2/2}} < \infty^{\#} \tag{65}$$

since $m_1+m_2 = n_1+n_2$. In all the sup's only l so that $l-n_2+n_1 > 0$ occur since the other terms are annihilated by the action of T_W. Thus, $(1+N)^{-m_1/2}T_W(1+N)^{-m_2/2}$ extends to a hyper bounded operator on $\mathcal{F}_s(\mathbf{H}^{\#})$ with norm less than or equal to $C(m_1,m_2)$. If $m_1 = n_1$ and $m_2 = n_2$, then $C(m_1,m_2) = 1$.
To prove (d) we need only note that if $\psi = (0,\ldots,\psi^{(l)},0,\ldots) \in D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ and $W_n \to_{\#} W$ in $\mathcal{L}^{\#}_2$, then

$$\|T_{W_n}\psi - T_W\psi\| = \|(T_{W_n-W})\psi\| \leq K(l,n_1,n_2)\|W_n-W\|\|\psi\|, \tag{66}$$

where $\#\text{-}\lim_{n\to\infty^{\#}} K(l,n_1,n_2)\|W_n-W\|\|\psi\| = 0$.
Since $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ consists of finite linear combinations of such vectors, we have shown that T_{W_n} #-converges strongly on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ to T_W if $W_n \to_{\#} W$ in $\mathcal{L}^{\#}_2$.
To prove (a) let $\psi_1,\psi_2 \in D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ with $\psi_1 = (0,\ldots,\psi^{(l-n_2+n_1)},0,\ldots)$ and $\psi_1 = (0,\ldots,\psi^{(l)},0,\ldots)$.
Then, if $W = \left(\prod_{i=1}^{n_1} f_i(k_i)\right)\left(\prod_{i=1}^{n_2} g_i(k_i)\right)$ the definition of the form

$\left(\prod_{i=1}^{n_1} a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} a_i(k_i)\right)$ shows that

$$(\psi_1, T_W\psi_2) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3n_2}} W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2}) \times \left(\psi_1, \left(\prod_{i=1}^{n_1} a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} a_i(k_i)\right)\psi_2\right) d^{\#n_1}k d^{\#n_2}p \tag{67}$$

Since both sides of (X.83) are linear in W, the relationship continues to hold for the all such W's that are hyperfinite linear combinations of such products. Since

$$\left(\psi_1, \left(\prod_{i=1}^{n_1} a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} a_i(k_i)\right)\psi_2\right) \in \mathcal{L}_2^\#\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right) \tag{68}$$

and since (d) holds, both the right- and left-hand sides of (X.83) are continuous linear functionals on ${}^*\mathbb{R}_c^{\#3(n_1+n_2)}$. Since they agree on a #-dense set, they agree everywhere. Finally, (68) extends by linearity to all of $D_{\mathcal{L}^\#_{\mathbf{fin}}} \times D_{\mathcal{L}^\#_{\mathbf{fin}}}$.

This proves (a); the proof of (c) is similar. |

Finally, we note that as quadratic forms on $D_{\mathcal{L}^\#_{\mathbf{fin}}}$ we can express the free scalar field and the time zero fields in terms of $a^\dagger(k)$ and $a(k)$:

$$\Phi_{m,x}(x,t) = \frac{1}{(2\pi_\#)^{3/2}} \int_{|p|\leq x} \{[Ext\text{-}\exp(\mu(p)t - ipx)]a^\dagger(p) + [Ext\text{-}\exp(-\mu(p)t + ipx)]a(p)\} \frac{d^{\#3}p}{\sqrt{2\mu(p)}} \tag{69}$$

$$\varphi^\#_{m,x}(x) = \frac{1}{(2\pi_\#)^{3/2}} \int_{|p|\leq x} \{[Ext\text{-}\exp(-ipx)]a^\dagger(p) + [Ext\text{-}\exp(ipx)]a(p)\} \frac{d^{\#3}p}{\sqrt{2\mu(p)}} \tag{70}$$

$$\pi^\#_{m,x}(x) = \frac{1}{(2\pi_\#)^{3/2}} \int_{|p|\leq x} \{[Ext\text{-}\exp(-ipx)]a^\dagger(p) - [Ext\text{-}\exp(ipx)]a(p)\} \sqrt{\frac{\mu(p)}{2}}\, d^{\#3}p. \tag{71}$$

2.$Q^\#$-space representation of the Fock space structures

In this section the construction of $Q^\#$-space and $L_2^\#(Q^\#, d^\#\mu)$, another representation of the Fock space structures are presented. In analogy with the one degree of freedom case where $\mathcal{F}^\#({}^*\mathbb{R}_c^\#)$ is isomorphic to $L_2^\#({}^*\mathbb{R}_c^\#, d^\#x)$ in such a way that $\Phi_{\mathbf{S}}(1)$ becomes multiplication by x, we will construct a #-measure space $\langle Q^\#, \mu^\#\rangle$, with $\mu(Q^\#) = 1$, and a unitary map $S : \mathcal{F}_s^\#({}^*\mathbb{R}_c^\#) \to L_2^\#(Q^\#, d^\#\mu)$ so that for each $f \in \mathbf{H}^\#_{\mathbf{C}^\#}$, $S\varphi^\#(f)S^{-1}$ acts on $L_2^\#(Q, d^\#\mu^\#)$ by multiplication by a #-measurable function. We can then

show that in the case of the free scalar field of mass m in 4-dimensional space-time, $V = SH_{\mathbf{I}}(g)S^{-1}$ is just multiplication by a function $V(q)$ which is in $L_p^\#(Q, d^\#\mu)$ for each $p \in \mathbb{N}^\#$. Let $\{f_n\}_{n=1}^{\infty^\#}$ be an orthonormal basis for $\mathbf{H}^\#$ so that each $f_n \in \mathbf{H}^\#_{\mathbb{C}^\#}$ and let $\{g_k\}_{k=1}^N, N \in \mathbb{N}^\#$ be a finite or hyperfinite subcollection of the $\{f_n\}_{n=1}^{\infty^\#}$. Let $\mathbf{P}_N$ be a set of the all external hyperfinite polynomials $Ext\text{-}P[u_1,\ldots,u_N]$ and $\mathcal{F}_N^\#$ be the #-closure of the set

$$\{Ext\text{-}P[\varphi^\#(g_1),\ldots,\varphi^\#(g_N)] | P \in \mathbf{P}_N\} \tag{1}$$

in $\mathcal{F}_s^\#(\mathbf{H}^\#)$ and define $F_0^N = \mathcal{F}_N^\# \cap F_0$ From Theorem X.43 (and its proof) it follows that $\varphi^\#(g_k)$ and $\pi^\#(g_l)$, for all $1 \leq k,l \leq N$ are essentially self-adjoint on F_0^N and that

$$\begin{gathered}(Ext\text{-}\exp[it\varphi^{\#}(g_k)])(Ext\text{-}\exp[is\pi^{\#}(g_l)]) = \\ (Ext\text{-}\exp[-ist\delta_{kl}])(Ext\text{-}\exp[is\pi^{\#}(g_l)])(Ext\text{-}\exp[it\varphi^{\#}(g_k)]).\end{gathered} \tag{2}$$

Thus we have a representation of the generalized Weyl relations in which the vector $\mathbf{\Omega}_0$ satisfies $\left([\varphi^{\#}(g_k)]^2 + [\pi^{\#}(g_k)]^2 - 1\right)\mathbf{\Omega}_0 = 0$ and is #-cyclic for the operators $\{\varphi^{\#}(g_k)\}_{k=1}^{N}, N \in \mathbb{N}^{\#}$. Therefore there is a unitary map $\widetilde{\mathbf{S}}^{(N)} : \mathcal{F}_N^{\#} \to L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ so that

$$\begin{gathered}\widetilde{\mathbf{S}}^{(N)}\varphi^{\#}(g_k)\left(\widetilde{\mathbf{S}}^{(N)}\right)^{-1} = x_k \\ \widetilde{\mathbf{S}}^{(N)}\pi^{\#}(g_k)\left(\widetilde{\mathbf{S}}^{(N)}\right)^{-1} = \frac{1}{i}\frac{d^{\#}}{dx_k^{\#}}\end{gathered} \tag{3}$$

and

$$\widetilde{\mathbf{S}}^{(N)}\mathbf{Q}_0 = \pi_{\#}^{-N/4}\left\{Ext\text{-}\exp\left[-\left(Ext\text{-}\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right]\right\}. \tag{4}$$

It is convenient to use the Hilbert space

$$L_2^{\#}\left({}^*\mathbb{R}_c^{\#N}, \pi_{\#}^{-N/2}d^{\#N}x\left\{Ext\text{-}\exp\left[-\left(Ext\text{-}\textstyle\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right]\right\}\right)$$

instead of $L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ so let $d^{\#}\mu_k = \pi_{\#}^{-1/2}\exp(-x_k^2/2)\ d^{\#}x_k$ and define

$$(Tf)(x) = \pi_{\#}^{N/4}\left[Ext\text{-}\exp\left(Ext\text{-}\textstyle\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right]f(x). \tag{5}$$

Then T is a unitary map of $L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ onto $L_2^{\#}\left({}^*\mathbb{R}_c^{\#N}, Ext\text{-}\prod_{k=1}^{N}d^{\#}\mu_k^{\#}\right)$ and if we let $\mathbf{S}^{(N)} = T\widetilde{\mathbf{S}}^{(N)}$ we get

$$\begin{gathered}\mathbf{S}^{(N)} : \mathcal{F}_N^{\#} \to L_2^{\#}\left({}^*\mathbb{R}_c^{\#N}, Ext\text{-}\textstyle\prod_{k=1}^{N}d^{\#}\mu_k^{\#}\right), \\ \mathbf{S}^{(N)}\varphi^{\#}(g_k)(\mathbf{S}^{(N)})^{-1} = x_k, \\ \mathbf{S}^{(N)}\pi^{\#}(g_k)(\mathbf{S}^{(N)})^{-1} = -\frac{x_k}{i} + \frac{1}{i}\frac{d^{\#}}{d^{\#}x_k}, \\ \mathbf{S}^{(N)}\mathbf{Q}_0 = 1,\end{gathered} \tag{6}$$

where 1 is the function identically one.Note that each $\mu_k^{\#}$ has mass one, which implies that

$$\begin{gathered}\left\langle \mathbf{Q}_0, \left(Ext\text{-}\textstyle\prod_{k=1}^{N}P_k[\varphi^{\#}(g_k)]\right)\mathbf{Q}_0\right\rangle = \\ \int\limits_{{}^*\mathbb{R}_c^{\#N}}\left(Ext\text{-}\textstyle\prod_{k=1}^{N}P_k[x_k]\right)\left(Ext\text{-}\textstyle\prod_{k=1}^{N}d^{\#}\mu_k^{\#}\right) = \\ Ext\text{-}\textstyle\prod_{k=1}^{N}\int\limits_{{}^*\mathbb{R}_c^{\#}}P[x_k]d^{\#}\mu_k^{\#} = Ext\text{-}\textstyle\prod_{k=1}^{N}\int\limits_{{}^*\mathbb{R}_c^{\#}}\langle\mathbf{Q}_0, P_k[\varphi^{\#}(g_k)]\mathbf{Q}_0\rangle,\end{gathered} \tag{7}$$

where $P_1,\dots,P_N$ are external hyperfinite polynomials. This formula (7) can also be proven by direct computations on $\mathcal{F}_s^{\#}(\mathbf{H}^{\#})$.

Now it is easy to see how to construct $\langle Q^{\#}, d^{\#}\mu^{\#}\rangle$. We define $Q^{\#} = \times_{k=1}^{\infty^{\#}}\ {}^*\mathbb{R}_c^{\#}$. Take the $\sigma^{\#}$-algebra generated by hyper infinite products of #-measurable sets in ${}^*\mathbb{R}_c^{\#}$ and set $\mu^{\#} = \otimes_{k=1}^{\infty^{\#}}\mu_k^{\#}$. We denote the points of $Q^{\#}$ by $q = \langle q_1, q_2, \dots\rangle$. Then $\langle Q^{\#}, d^{\#}\mu^{\#}\rangle$ is a #-measure space and the set of functions of the form $P(q_1, q_2, \dots)$, where P is a polynomial and $n \in \mathbb{N}^{\#}$ is arbitrary, is #-dense in $\mathcal{L}_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$. Let P be a polynomial in $N \in \mathbb{N}^{\#}$ variables

$$P(x_{k_1}, \ldots, x_{k_N}) = Ext\text{-}\sum_{l_1,\ldots,l_N} c_{l_1,\ldots,l_N} x_{k_1}^{l_1}, \ldots, x_{k_N}^{l_N} \tag{8}$$

and define

$$\mathbf{S} : P(\varphi^{\#}(f_{k_1}), \ldots, \varphi^{\#}(f_{k_N}))\mathbf{Q}_0 \to P(q_{k_1}, \ldots, q_{k_N}). \tag{9}$$

Then

$$\begin{gathered} P(\varphi^{\#}(f_{k_1}), \ldots, \varphi^{\#}(f_{k_N}))\mathbf{Q}_0 = Ext\text{-}\sum_{l,\mathbf{m}} c_l \bar{c}_{\mathbf{m}} (\mathbf{Q}_0, \varphi^{\#}(f_{k_1})^{l_1+m_1}, \ldots, \varphi^{\#}(f_{k_N})^{l_N+m_N}\mathbf{Q}_0) = \\ Ext\text{-}\sum_{l,\mathbf{m}} c_l \bar{c}_{\mathbf{m}} \int_{{}^*\mathbb{R}_c^{\#N}} q_{k_1}^{l_1+m_1} \cdots q_{k_N}^{l_N+m_N} \left(Ext\text{-}\prod_{i=1}^{N} d^{\#}\mu_{k_i}^{\#} \right) = \int_{Q^{\#}} |P(x_{k_1}, \ldots, x_{k_N})|^2 d^{\#}\mu^{\#} \end{gathered} \tag{10}$$

by (**X**.**92**) and the fact that each $\mu_k^{\#}$ has mass one. Since $\mathbf{Q}_0$ is cyclic for polynomials in the fields (Theorem **X**.**42**), S extends to a unitary map of $\mathcal{F}_s^{\#}(\mathbf{H}^{\#})$ onto $\mathcal{L}_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$. Clearly

$$\mathbf{S}\varphi^{\#}(f_k)\mathbf{S}^{-1} = q_k \text{ and } \mathbf{S}\mathbf{Q}_0 = 1. \tag{11}$$

Theorem 1. Let $\varphi_{m,x}^{\#}(f), x \in {}^*\mathbb{R}_c^{\#} \backslash {}^*\mathbb{R}_{c,\mathbf{fin}}^{\#}$ be the free scalar field of mass m (in 4-dimensional space-time) at time zero. Let $g \in \mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#3}) \cap \mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3})$ and define

$$H_{I,x,\lambda}(g) = \lambda(x) \int g(x) : \varphi_{m,x}^{\#}(x)^4 : d^{\#3}x, \tag{12}$$

where $\lambda(x) \in {}^*\mathbb{R}_c^{\#}, \lambda(x) \approx 0$. Let $\mathbf{S}$ denote the unitary map of $\mathcal{F}_s^{\#}(\mathbf{H}^{\#})$ onto $\mathcal{L}_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ constructed above. Then $V = \mathbf{S}H_{I,x,\lambda}(g)\mathbf{S}^{-1}$ is multiplication by a function $V_{x,\lambda}(q)$ which satisfies:

(a) $V_{x,\lambda}(q) \in \mathcal{L}_p^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ for all $p \in \mathbb{N}^{\#}$.

(b) $Ext\text{-}\exp(-tV_{x,\lambda}(q)) \in \mathcal{L}_1^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ for all $t \in [0, \infty)$.

Proof. We will prove (a). By Eq.() we get

$$\varphi_{m,x}^{\#}(x) = \frac{1}{(2\pi_{\#})^{3/2}} \int_{|p| \leq x} \{[Ext\text{-}\exp(-ipx)]a^{\dagger}(p) + [Ext\text{-}\exp(ipx)]a(p)\} \frac{d^3p}{\sqrt{2\mu(p)}}. \tag{13}$$

Then $\varphi_{m,x}^{\#}(x)$ is a well-defined operator-valued function of $x \in {}^*\mathbb{R}_c^{\#3}$. We define $: \varphi_{m,x}^{\#}(x)^4 :$ by moving all the $a^{\dagger}$'s to the left in the formal expression for $\varphi_{m,x}^{\#}(x)^4$. By Theorem **X**.**44** $: \varphi_{m,x}^{\#}(x)^4 :$ is also a well-defined operator for each $x \in {}^*\mathbb{R}_c^{\#3}$ and $: \varphi_{m,x}^{\#}(x)^4 :$ takes F_0 into itself. Thus for each $x \in {}^*\mathbb{R}_c^{\#3}$,

$$: \varphi_{m,x}^{\#}(x)^4 : = \varphi_{m,x}^{\#}(x)^4 + d_2(x)\varphi_{m,x}^{\#}(x)^2 + d_0(x) \tag{14}$$

where the coefficients $d_2(x)$ and $d_0(x)$ are independent of x. For each $x \in {}^*\mathbb{R}_c^{\#3}$, $\mathbf{S}\varphi_{m,x}^{\#}(x)\mathbf{S}^{-1}$ is just the operator on #-measurable space $\mathcal{L}_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ which operates by multiplying by the function

$$Ext\text{-}\sum_{k=1}^{\infty^{\#}} c_k(x, x) q_k \tag{15}$$

where

$$c_k(x,\varkappa) = (2\pi_{\#})^{-3/2}\left(f_k, Ext\text{-}\exp(ipx)(\mu(p))^{-1/2}\right). \tag{16}$$

Furthermore,

$$Ext\text{-}\sum_{k=1}^{\infty^{\#}}|c_k(x,\varkappa)|^2 = (2\pi_{\#})^{-3/2}\left\|(\mu(p))^{-1/2}\right\|_2^2, \tag{17}$$

so $\mathbf{S}\varphi^{\#}_{m,\varkappa}(x)^4\mathbf{S}^{-1}$and $\mathbf{S}\varphi^{\#}_{m,\varkappa}(x)^2\mathbf{S}^{-1}$ are in $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ and the $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ norms are uniformly bounded in $\varkappa$. Therefore, since $g \in \mathcal{L}^{\#}_1({}^{*}\mathbb{R}^{\#3}_c)$, $\mathbf{S}H_{I,\varkappa,\lambda}(g)\mathbf{S}^{-1}$ operates on $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ by multiplication by an $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ function which we denote by $V_{\varkappa,\lambda}(q)$. Consider now the expression for $H_{I,\varkappa}(g)\mathbf{Q}_0$. This is a vector $(0,0,0,0,\psi^{(4)},0,\ldots)$

$$\begin{gathered}\psi^{(4)}(p_1,p_2,p_3,p_4) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_c}\frac{\lambda g(x)\left[Ext\text{-}\exp\left(-ix\sum_{i=1}^4 p_i\right)\right]d^{\#3}x}{(2\pi_{\#})^{3/2}\prod_{i=1}^4(2\mu(p_i))^{1/2}} = \\ = \frac{\lambda\widehat{g}\left(\sum_{i=1}^4 k_i\right)}{(2\pi_{\#})^{9/2}\prod_{i=1}^4(2\mu(p_i))^{1/2}}\end{gathered} \tag{18}$$

where $|p_i| \leq \varkappa, 1 \leq i \leq 4$. We choose now the parameter $\lambda = \lambda(\varkappa) \approx 0$ such that $\|\psi^{(4)}\|_2 \in \mathbb{R}$, thus

$$\|H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{Q}_0\|_2 \in \mathbb{R}, \tag{19}$$

since $\|H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{Q}_0\|_2 = \|\psi^{(4)}\|_2$. But, since $\mathbf{S}\mathbf{Q}_0 = 1$, we get

$$\|H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{Q}_0\|_2 = \|\mathbf{S}H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{-1}\|_{\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})} = \|V_{\varkappa,\lambda(\varkappa)}(q)\|_{\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})} \tag{20}$$

From (19) and Eq.(20) we get that $\|V_{\varkappa,\lambda(\varkappa)}(q)\|_{\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})}$ is finite. It is easily verify that each $P(q_1,q_2,\ldots,q_n), n \in \mathbb{N}^{\#}$ is in the domain of $V_{\varkappa,\lambda(\varkappa)}(q)$ and $\mathbf{S}H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{-1} = V_{\varkappa,\lambda(\varkappa)}(q)$ on that domain. Since $\mathbf{Q}_0$ is in the domain of $[H_{I,\varkappa,\lambda(\varkappa)}(g)]^p$ for all $n \in \mathbb{N}^{\#}$, 1 is in the domain of $[V_{\varkappa,\lambda(\varkappa)}(q)]^n$ for all $n \in \mathbb{N}^{\#}$. Thus, for all $n \in \mathbb{N}^{\#}, V_{\varkappa,\lambda(\varkappa)} \in \mathcal{L}^{\#}_{2n}(Q^{\#},d^{\#}\mu^{\#})$. Since $\mu^{\#}(Q^{\#}) < \infty^{\#}$, $V_{\varkappa,\lambda(\varkappa)} \in \mathcal{L}^{\#}_p(Q^{\#},d^{\#}\mu^{\#})$ for all $p < \infty^{\#}$.

Chapter X. A non-Archemedean Banach algebras and $\mathbf{C}^{\star}_{\#}$-Algebras.

§1. A non-Archemedean Banach algebra $B(H^{\#})$

§1.1. Basic Properties

Definition 1.1.An linear operator T on a non-Archemedean Hilbert space $H^{\#}$ is a linear map $H^{\#} \to H^{\#}$. We can define a #-norm by

$$\|T\|_{\#} \triangleq \sup_{v\in H^{\#}\setminus\{0\}}\frac{\|Tv\|_{\#}}{\|v\|_{\#}} \tag{1.1}$$

if supremum in RHS of (1.1) exists.

This is a #-norm since

1. By definition of the #-norm on $H^{\#}$, it is always positive.
2. We have that $T = 0 \Leftrightarrow \forall v \in H^{\#}, Tv = 0 \Leftrightarrow \forall v \in H^{\#} \setminus \{0\}$,
$\dfrac{\|Tv\|_{\#}}{\|v\|_{\#}} = 0 \Leftrightarrow \|T\|_{\#} = 0$.

3. $\|\lambda T\|_{\#} = \sup_{v \in H^{\#} \setminus \{0\}} \dfrac{\|\lambda T v\|_{\#}}{\|v\|_{\#}} = |\lambda| \sup_{v \in H^{\#} \setminus \{0\}} \dfrac{\|T v\|_{\#}}{\|v\|_{\#}} = |\lambda| \|T\|$

4. $\|T_1 + T_2\|_{\#} = \sup_{v \in H^{\#} \setminus \{0\}} \dfrac{\|T_1 v + T_2 v\|_{\#}}{\|v\|_{\#}} \leq \sup_{v \in H^{\#} \setminus \{0\}} \dfrac{\|T_1 v\|_{\#} + \|T_2 v\|_{\#}}{\|v\|_{\#}} \leq$

$\leq \sup_{v \in H^{\#} \setminus \{0\}} \dfrac{\|T_1 v\|_{\#}}{\|v\|_{\#}} + \sup_{v \in H^{\#} \setminus \{0\}} \dfrac{\|T_2 v\|_{\#}}{\|v\|_{\#}} = \|T_1\|_{\#} + \|T_2\|_{\#}$.

Definition 1.2. Let $H^{\#}$ be a non-Archemedean Hilbert space over ${}^*\mathbb{C}_c^{\#}$. A linear map A: $H^{\#} \to H^{\#}$ is called bounded in ${}^*\mathbb{R}_c^{\#}$ operator iff $\|A\|_{\#} < {}^*\infty$.

Definition 1.3. Let $H^{\#}$ be a non-Archemedean Hilbert space over ${}^*\mathbb{C}_c^{\#}$. We denote by $B(H^{\#})$ the set of all bounded in ${}^*\mathbb{R}_c^{\#}$ operators A: $H^{\#} \to H^{\#}$.

Definition 1.4. Algebra A is called an algebra over ${}^*\mathbb{C}_c^{\#}$ if it is a vector space over ${}^*\mathbb{C}_c^{\#}$ and a binary map $\bullet : A \times A \to A$ Satisfying:

1. Left distrubitivity: $\forall v, w, u \in A[(v + w) \bullet u = v \bullet u + w \bullet w]$
2. Right distrubitivity: $\forall v, w, u \in A[v \bullet (w + u) = v \bullet w + v \bullet u]$
3. $\forall v, w \in A, \forall \alpha, \beta \in {}^*\mathbb{C}_c^{\#}[\alpha\beta v \bullet w = (\alpha v) \bullet (\beta w)]$

We note that $B(H^{\#})$ is an algebra over ${}^*\mathbb{C}_c^{\#}$ where for $A, B \in B(H^{\#}), \lambda \in {}^*\mathbb{C}_c^{\#}$ we define:

λA: $H^{\#} \to H^{\#}, v \mapsto \lambda A v$

$A + B$: $H^{\#} \to H^{\#}, v \mapsto Av + Bv$

$A \bullet B$: $H^{\#} \to H^{\#}, v \mapsto A(B((v))$

In $B(H^{\#})$ we have the #-adjoint operator. This maps each A to the unique A^* such that for all $v, w \in H^{\#}$ we have $\langle Av, w\rangle_{\#} = \langle v, A^* w\rangle_{\#}$. We denote the adjoint of an operator A by A^* and define the adjoint of a subset $M \subset B(H^{\#})$ by $M^* \triangleq \{A^* \in B(H^{\#}) \mid A \in M\}$. The adjoint has the following key properties:

Lemma 1.4. Adjoint Properties (Algebraic)

$\forall B, A \in B(H^{\#})$ we have

1. A^* always exists is unique.
2. If A is bounded in ${}^*\mathbb{R}_c^{\#}$,then A^* is also bounded in ${}^*\mathbb{R}_c^{\#}$.
3. $A^{**} = A$ (Involutivity)
4. $\|A\|_{\#} = \|A^*\|_{\#}$
5. If A is invertible, A^* also is, with $(A^*)^{-1} = (A^{-1})^*$
6. $(A + B)^* = A^* + B^*, (\lambda A)^* = \bar{\lambda} A^*$
7. $(AB)^* = B^* A^*$
8. $\|A^* A\|_{\#} = \|A\|_{\#}$

Proof. 1. Let $x \in H^{\#}$ and consider the bounded in ${}^*\mathbb{R}_c^{\#}$ linear functional f: $H^{\#} \to {}^*\mathbb{C}_c^{\#}$, $f(v) \mapsto \langle Av, x\rangle_{\#}$ we have $\|f\| \leq \|A\|_{\#} \|x\|_{\#}$. By generalized Riesz representation theorem there exists a unique $y \in H^{\#}$ with $f(v) = \langle v, y\rangle_{\#} \forall v \in H^{\#}$. So we set $A^* x = y$. Then for any $y, z \in H^{\#}$ and $\forall \alpha \in {}^*\mathbb{C}_c^{\#}$ we have:

$\langle v, A^*(\alpha y + z)\rangle_{\#} = \langle Av, \alpha y + z\rangle_{\#} = \bar{\alpha}\langle Av, y\rangle_{\#} + \langle Av, z\rangle_{\#} = \bar{\alpha}\langle (v, A^* y\rangle_{\#} + \langle v, A^* z\rangle_{\#} =$

$= \langle v, \alpha A^* y + A^* z\rangle_{\#}\ \forall v \in H^{\#}$. In particular, if we choose $v = A^*(\alpha y + z) - \alpha A^* y + A^* z$, we see that $\|v\|_{\#} = 0 \Rightarrow v = 0 \Rightarrow A^*$ is linear.

2. Following from 1. we have

$\|A^* x\|_{\#} = \|y\|_{\#} = \|f\|_{\#} \leq \|A\|_{\#} \|y\|_{\#}$.

3. We can see this as

$\langle A^{**} v, w\rangle_{\#} = \langle v, A^* w\rangle_{\#} = \langle Av, w\rangle_{\#} \forall v, w \in H^{\#}$.

4. Combining the estimate from above and involutivity, we have
$\|A^{**}\|_{\#} \leq \|A^{*}\|_{\#} \leq \|A\|_{\#} = \|A^{**}\|_{\#}$.
So we must have equality everywhere.
5. We have $\langle v,(A^{-1})^{*}A^{*}w\rangle_{\#} = \langle A^{-1}v,A^{*}w\rangle_{\#} = \langle AA^{-1}v,w\rangle_{\#} = \langle v,w\rangle_{\#}\ \forall v,w \in H^{\#}$.
Hence, $(A^{-1})^{*}A^{*} = 1$. The argument for $A^{*}(A^{-1})^{*} = 1$ is the same.
6. This follows clearly from conjugate linearity in the second argument of an inner product.
7. This is clear since, $\langle ABv,w\rangle_{\#} = \langle Bv,A^{*}w\rangle_{\#} = \langle v,B^{*}A^{*}w\rangle_{\#}\ \forall v,w \in H^{\#}$.
8. For this we have $\|T\|_{\#}^{2} = \sup_{\|x\|_{\#}=1}\|Tx\|_{\#}^{2} = \sup_{\|x\|_{\#}=1}|\langle Tx,Tx\rangle_{\#}| =$
$\sup_{\|x\|_{\#}=1}|\langle T^{*}Tx,x\rangle|_{\#} \leq \sup_{\|x\|_{\#}=1}\|T^{*}Tx\|_{\#}\|x\|_{\#} = \|T^{*}T\|_{\#}$. But also,
$\|T^{*}T\|_{\#} \leq \|T^{*}\|_{\#}\|T\|_{\#} = \|T\|_{\#}^{2}$, and so there is equality everywhere.

§1.2 Types of Operators

Definition 1.2.1. A is called *normal* if $A^{*}A = AA^{*}$.
Definition 1.2.2. A is called positive if $A = B^{*}B$ for some $B \in B(H^{\#})$
Definition 1.2.3. A is called *self #-adjoint* if $A^{*} = A$.
Lemma 1.2.1. Let $A \in B(H^{\#})$. Then $A = A_1 + iA_2$ where A_1 and A_2 are both self #-adjoint.
Proof. Let $A_1 = \dfrac{A+A^{*}}{2}, A_2 = \dfrac{iA^{*}-iA}{2}$.
It is then clear from basic algebra.
Definition 1.2.4. U is called *unitary* if $U^{*}U = UU^{*} = 1$
Example 1.2.1. If U is unitary, we have $\forall h,k \in H^{\#}, \langle h,k\rangle_{\#} = \langle Uh,Uk\rangle_{\#}$. This is because $\langle Uh,Uk\rangle_{\#} = \langle h,U^{*}Uk\rangle_{\#} = \langle h,\mathbf{1}k\rangle_{\#} = \langle h,k\rangle_{\#}$.
Definition 1.2.5. A is called *isometric* if $A^{*}A = 1$.
We also have a relaxed definition, a partial isometry.
Definition 1.2.6. A is called a partial isometry if it is an isometry on the orthogonal complement of it's kernel,i.e. $A^{*}Av = v, \forall v \in \mathbf{Ker}(A)^{\perp} =$
$= \{v \in H^{\#} | \langle v,w\rangle_{\#} = 0, \forall w \in \mathbf{Ker}(A)\}$.
Definition 1.2.7. $p \in B(H^{\#})$ is called a *projection* if $p = p^{*} = p^{2}$.
Example 1.2.2. Consider $H^{\#} = l_2^{\#}({}^{*}\mathbb{N})$ the set of all square summable ${}^{*}\mathbb{C}_c^{\#}$-valued series. An example of a projection would be:
$p_n : H^{\#} \to H^{\#}, (a_1,a_2,\ldots\ldots,a_n,a_{n+1},an_{+2}\ldots) \mapsto (a_1,a_2,\ldots,a_n,0,0,\ldots.)$.
We see this is self #-adjoint as $\langle p_n a,b\rangle_{\#} = Ext\text{-}\sum_{k=1}^{n} a_k\overline{b_k} = \langle a,p_n b\rangle_{\#}$ and idempotent as $p_n^2 = p_n$.
Lemma 1.2.2. Multiplication and #-norm property
$\forall A,B \in B(H^{\#}), \|A \cdot B\|_{\#} \leq \|A\|_{\#}\|B\|_{\#}$
Proof. For all $h \in H^{\#}$, we always have the estimate $\|Ah\|_{\#} \leq \|A\|_{\#}\|h\|_{\#}$.
Using this we have
$\|AB\|_{\#} = \sup_{h \in H\setminus\{0\}}\|(AB)h\|_{\#}/\|h\|_{\#} = \sup_{h \in H\setminus\{0\}}\|A(Bh)\|_{\#}/\|h\|_{\#}$
$\leq \sup_{h \in H\setminus\{0\}}\|A\|_{\#}\|Bh\|_{\#}/\|h\|_{\#} = \|A\|_{\#}\|B\|_{\#}$
Lemma 1.2.3. $(B(H^{\#}),\|\cdot\|_{\#})$ is complete, i.e. if $(A_n)_{n \in {}^{*}\mathbb{N}} \subset B(H^{\#})$ is cauchy with respect to the operator #-norm $\|\cdot\|_{\#}$, it #-converges in #-norm to some element $A \in B(H^{\#})$.
Proof. Let $(A_n)_{n \in {}^{*}\mathbb{N}}$ be cauchy with respect to the operator #-norm. This means that $\forall \varepsilon(\varepsilon \approx 0, \varepsilon > 0)\ \exists N \in {}^{*}\mathbb{N}_{\infty}\ [n,m > N \Rightarrow \|A_n - A_m\|_{\#} < \varepsilon]$.

In particular then $\|A_n\|$ is bounded above, say by $K \in {}^*\mathbb{R}^{\#}_{c,+}$. Now fix $v \in H^{\#}$ and let $N,m,n \in {}^*\mathbb{N}_{\infty}$ be as before. We have that
$\|A_n v - A_m v\|_{\#} \leq \|A_n - A_m\|_{\#}\|v\|_{\#} \leq \|v\|_{\#}$.
Hence,$(A_n v)_{n \in {}^*\mathbb{N}}$ is cauchy in $H^{\#}$. By completeness of $H^{\#}$, we have a #-limit and can define $A{:}H^{\#} \to H^{\#}, v \mapsto \#\text{-}\lim_{n\to{}^*\infty} A_n v$, this is our candidate for our #-limit.
A is linear since (by algebra of #-limits)
$A(\alpha v + w) = \#\text{-}\lim_{n\to{}^*\infty} A_n(\alpha v + w) = \alpha\,(\#\text{-}\lim_{n\to{}^*\infty} A_n v) + \#\text{-}\lim_{n\to{}^*\infty} A_n w = \alpha A v + A w$
and bounded in ${}^*\mathbb{R}^{\#}_{c,+}$ because
$\|Av\|_{\#} = \#\text{-}\lim_{n\to{}^*\infty}\|A_n v\|_{\#} = \#\text{-}\lim_{n\to{}^*\infty}\|A_n v\|_{\#} \leq \#\text{-}\lim_{n\to{}^*\infty}\|A_n\|_{\#}\|v\|_{\#} \leq K\|v\|_{\#}$.
Hence, $A \in B(H^{\#})$. Finally we show convergence in #-norm. Fix $\varepsilon \approx 0, \varepsilon > 0$ and let $N \in {}^*\mathbb{N}_{\infty}$ be as in the definition of cauchy. If $n > N$ we have:
$\|A - A_n\|_{\#} = \sup_{\|v\|_{\#}=1} \|\#\text{-}\lim_{m\to{}^*\infty}(A_m - A_n)v\|_{\#} \leq$
$\sup_{\|v\|_{\#}=1} \#\text{-}\lim_{m\to{}^*\infty}\|(A_m - A_n)v\|_{\#}\|v\|_{\#} \leq \#\text{-}\lim_{m\to{}^*\infty} \quad = \varepsilon.$
Definition 1.2.8. $A \in B(H^{\#})$ is #-compact if for all bounded subsets β of $H^{\#}$ the image of A restricted to β has #-compact #-closure:
$A \in B(H^{\#})$ #-compact $\Leftrightarrow$ $\forall\beta$ bounded , #-$\overline{A\beta}$ is #-compact.
We denote by $K^{\#}(B(H^{\#}))$ the set of all #-compact operators in $B(H^{\#})$.
Lemma 1.2.4. A is #-compact iff
$\forall(v_{\alpha})_{\alpha\in X}$ bounded $\Rightarrow$ $(Av_{\alpha})_{\alpha\in X}$ has a #-convergent subsequence.
Definition 1.2.9. For $A \in B(H^{\#})$ the rank $\Re(A)$ of A is the dimension of the range of A
Lemma 1.2.5. If A has rank $N \in {}^*\mathbb{N}$, then we can write
$A(\bullet) \ = Ext\text{-}\sum_{n\leq N}\alpha_n\langle\bullet, v_n\rangle_{\#} w_n$ where $\{v_n\}_{n\leq N}, \{w_n\}_{n\leq N} \subset H^{\#}, \{\alpha_n\}_{n\leq N} \subset {}^*\mathbb{C}^{\#}_c$
Proof. This follows immediately from the generalized Riesz representation theorem noting that if $\{w_n\}_{n\leq N}$ forms a basis of $A(H^{\#})$, then $\langle A\,\bullet, w_n\rangle_{\#}$ is a linear functional $H^{\#} \to {}^*\mathbb{C}^{\#}_c$. So for some $v_n, \langle A\,\bullet, w_n\rangle_{\#} = \langle\bullet, v_n\rangle_{\#}$ so setting $\alpha_n = 1/\|w_n\|_{\#}$ we have
$A\,\bullet = Ext\text{-}\sum_{n}\alpha_n\langle A\,\bullet, w_n\rangle_{\#} w_n = Ext\text{-}\sum_{n\leq N}\alpha_n\langle\bullet, v_n\rangle_{\#} w_n.$
We denote by $\mathcal{F}^{\#}\Re(B(H^{\#}))$ the set of hyperfinite rank operators in $B(H^{\#})$.
Lemma 1.2.6. Any operator with hyperfinite rank is #-compact
Proof. Say A has hyperfinite rank, then if $(v_{\alpha})_{\alpha\in X}$ is bounded, then $(Av_{\alpha})_{\alpha\in X}$ is bounded, and lies in a hyperfinite dimensional Hilbert space. By generalized Bolzano Weirstrass theorem [1], we have that it omits a #-convergence subsequence so by lemma 1.2.5, A is #-compact.
Theorem 1.2.1.. Let $H_1 = \{h \in H^{\#} | \|h\|_{\#} < 1\}$ The following are equivalent:
1. $A \in \overline{\{C \mid \Re(C) < {}^*\infty\}} = \overline{\mathcal{F}^{\#}\Re(B(H^{\#}))}$ where the #-closure is with respect to the #-norm topology
2. $A \in K^{\#}(B(H^{\#}))$
3. $A(H_1)$ has #-compact #-closure
Proof. 1 $\Rightarrow$ 2.
If $A \in \overline{\{C \mid \Re(C) < {}^*\infty\}}$ then the #-compactness A is clear, since hyperfinite rank operators are in $K^{\#}(B(H^{\#}))$ and $K^{\#}(B(H^{\#}))$ must be #-closed with respect to the #-norm topology.
2 $\Rightarrow$ 3.
This is clear by definition since H_1 is bounded subset of $H^{\#}$.
3 $\Rightarrow$ 1.
Say this did not hold, i.e. we had some A that has property 3 but not 1. Then, let

$(P_\alpha)_{\alpha\in A}$ be a net of hyperfinite rank projections tending towards the identity map. We have that $P_\alpha A$ also must have hyperfinite rank, and so $P_\alpha A \nrightarrow_\# A$ in #-norm sense. Well, then there exists some $\varepsilon \approx 0, \varepsilon > 0$ and some $v_\alpha \in H_1$ such that $\|(A - P_\alpha A)v_\alpha\|_\# > \varepsilon$. Since these v_α are in H_1, we can apply 3 to get some subnet such that $Av_\alpha \rightarrow_\# v$ in #-norm. Then we have:

$0 < \varepsilon < \|(A - P_\alpha A)v_\alpha\|_\# = \|v - P_\alpha v + (\mathbf{1} - P_\alpha)(v_\alpha - v)\|_\# \leq \ldots$

$\ldots \leq \|v - P_\alpha v\|_\# + \|(\mathbf{1} - P_\alpha)(Av_\alpha - v)\|_\# \leq \|\mathbf{1} - P_\alpha\|_\# \left(\|v\|_\# + \|Av_\alpha - v\|_\#\right) \rightarrow_\# 0.$

A contradiction. Hence, 3 ⇒ 1.

Corollary 1.2.1. A #-compact ⇔ $A(\bullet = Ext\text{-}\sum_{n\in{}^*\mathbb{N}} \alpha_n\langle\bullet, v_n\rangle_\# w_n$ where $\{v_n\}_{n\in{}^*\mathbb{N}}$, $\{w_n\}_{n\in{}^*\mathbb{N}} \subset H^\#$, and $\{\alpha_n\}_{n\in{}^*\mathbb{N}} \subset {}^*\mathbb{C}_c^\#$ and s.t. #-$\lim_{n\to{}^*\infty} \alpha_n = 0$.

The #-convergence on the RHS is with respect to the operator #-norm.

§1.3 Basic Spectral Theory

Spectrum is a generalisation of eigenvalues which is crucial for understanding operator algebras.Much of it is built upon whether operators or aren't invertible.

Definition 1.3.1. $A \in B(H^\#)$ is said to be invertible if there exists a $B \in B(H^\#)$ such that $AB = BA = 1$. If X is an algebra, we define $\mathbf{Inv}(X) = \{x \in X \mid x \text{ is invertible }\}$.

Lemma 1.3.1. Neumann Series is #-convergent

Let $\|A\|_\# < 1$. Then, $\mathbf{1} - A$ is invertible with inverse

$$(\mathbf{1} - A)^{-1} = Ext\text{-}\sum_{n\in{}^*\mathbb{N}} A^n. \tag{1.3.1}$$

Where

$$Ext\text{-}\sum_{n\in{}^*\mathbb{N}} A^n = \#\text{-}\lim_{N\to{}^*\infty}\left(Ext\text{-}\sum_{n=0}^{N} A^n\right) \tag{1.3.2}$$

where $N \in {}^*\mathbb{N}_\infty$.

Proof. The first question to ask is whether the series on the right hand side even #-converges. It does as by lemma 1.2.2 one obtains

$$\|Ext\text{-}\textstyle\sum_{n\in{}^*\mathbb{N}} A^n\|_\# \leq Ext\text{-}\sum_{n\in{}^*\mathbb{N}} \|A^n\|_\# \leq Ext\text{-}\sum_{n\in{}^*\mathbb{N}} \|A\|_\#^n = (1 - \|A\|_\#)^{-1}$$

Say it #-converges to B. Then, we see that because we have a telescoping sum

$$Ext\text{-}\sum_{n=0}^{N} (A^n)(1 - A) = 1 - A^{N+1} = (1 - A)\left(Ext\text{-}\sum_{n=0}^{N} A^n\right)$$

Hence, it is sufficient to check that $\mathbf{1} - A^n \rightarrow_\# \mathbf{1}$ as $n \to {}^*\infty$. Fix $\varepsilon \approx 0, \varepsilon > 0$, and choose $N \in {}^*\mathbb{N}_\infty, N > Ext\text{-}\log_{\|A\|_\#}\varepsilon$ Then we have for $n > N$

$$\|\mathbf{1} - A^n - \mathbf{1}\|_\# = \|A^n\|_\# \leq \|A\|_\#^n \leq \|A\|_\#^N < \varepsilon$$

Lemma 1.3.2. $\mathbf{Inv}(B(H^\#))$ is #-open in $B(H^\#)$. Furthermore, the map

${}^{-1}\colon \mathbf{Inv}(B(H^\#)) \rightarrow \mathbf{Inv}(B(H^\#)), A \mapsto A^{-1}$

is #-continuous with respect to the operator #-norm.

Proof. Say $A \in \mathbf{Inv}(B(H^\#))$. Then, if $\|B - A\|_\# < \|A^{-1}\|_\#^{-1}$, we have

$\|BA^{-1} - \mathbf{1}\|_\# = \|(B - A)A^{-1}\|_\# \leq \|B - A\|_\# \|A^{-1}\| < 1$

Which by lemma **1.3.1** gives

$\mathbf{1} - (BA^{-1} - \mathbf{1}) = BA^{-1} \in \mathbf{Inv}(B(H^\#)) \Rightarrow BA^{-1}A = B \in \mathbf{Inv}(B(H^\#))$.

Then if we consider BA^{-1}, we can note that $1/(1 - \|\ \mathbf{1} - BA^{-1}\ \|) > 1$ and hence

$\|BA^{-1}\|_\# \leq Ext\text{-}\sum_{n\in{}^*\mathbb{N}} \|1 - BA^{-1}\|_\#^n = Ext\text{-}\sum_{n\in{}^*\mathbb{N}} \|(A - B)A^{-1}\|_\#^n \leq$

$\leq Ext\text{-}\sum_{n\in{}^*\mathbb{N}} \|(A - B)\|_\#^n \|A^{-1}\|_\#^n = 1/(1 - \|(A - B)\|_\# \|A^{-1}\|_\#)$.

Therefore $\| \|A^{-1} - B^{-1}\|_{\#} = \|A^{-1}(AB^{-1})^{-1}(B-A)A^{-1}\|_{\#} \leq$
$\|A^{-1}\|_{\#}^{2}\|B-A\|_{\#}\|(AB^{-1})^{-1}\|_{\#} \leq \|A^{-1}\|_{\#}^{2}\|B-A\|_{\#}[1/(1-\|(A-B)\|_{\#}\|A^{-1}\|_{\#})]$
We can see then that as $\|A-B\|_{\#} \to_{\#} 0,\ \|A^{-1}-B^{-1}\|_{\#} \to_{\#} 0$ as required.
Definition 1.3.2. (Spectrum)
Let $A \in B(H^{\#})$. We define the spectrum of A, denoted $\sigma(A)$ by
$\sigma(A) := \{\lambda \in {}^{*}\mathbb{C}_{c}^{\#} \mid (A - \lambda \cdot \mathbf{1}) \notin \mathbf{Inv}(B(H^{\#}))\}$, i.e. the set of all complex numbers such that $A - \lambda\mathbf{1}$ is not invertible. We denote the complement of $\sigma(A)$ by $\varphi(A)$.
Lemma 1.3.3. Spectrum is a generalisation of an eigenvalue ($\mathbf{eigen}(A) \subset \sigma(A)$), i.e. if λ is an eigenvalue of $A, \lambda \in \sigma(A)$
Proof. Say λ is an eigenvalue of A. Then $v \in H^{\#} \setminus \{0\}$ s.t. $(A-\lambda)v = 0$ However, by linearity, $(A-\lambda)0 = 0$. As $(A-\lambda)$ is not injective it cannot be invertible, hence $\lambda \in \sigma(A)$
Corollary 1.3.1. Let $H^{\#}$ be hyperfinitefinite dimensional. Then, if $A \in B(H^{\#})$ we have that spectrum agrees with the eigenvalues, i.e. $\mathbf{eigen}(A) = \sigma(A)$.
Proof. By lemma 1.3.3, we only need to check the other direction. Up to choosing bases, we can assume $H^{\#} = {}^{*}\mathbb{C}_{c}^{\#\dim(H^{\#})}$. In this case, $B(H^{\#})$ is just the $\dim(H^{\#}) \times \dim(H^{\#})$ square matrices. By standard results in linear algebra, we have that $(A-\lambda) \notin \mathbf{Inv}(B(H^{\#}))$ iff $Ext\text{-}\det(A-\lambda) = 0$ iff λ is an eigenvalue.
Lemma 1.3.4. If $A \in B(H^{\#})$ then $\sigma(A)$ is #-closed as a subset of the complex plane ${}^{*}\mathbb{C}_{c}^{\#}$.
Moreover, it is a subset of the disc of radius $\|A\|_{\#}$ centred at the origin.
Proof. Say $\lambda > \|A\|_{\#}$. Then, $-\lambda^{-1}\|A\|_{\#} < 1$ so $1 - \lambda^{-1}A$ is invertible by lemma 1.3.1.
Then, $\lambda \notin \sigma(A)$. Now examine $\sigma(A)^{c} = \varphi(A) = \{\lambda \in {}^{*}\mathbb{C}_{c}^{\#} | A - \lambda \in \mathbf{Inv}(B(H^{\#}))\}$.
Say $\lambda \in \varphi(A)$. By lemma 1.3.2 we have that there exists some $\varepsilon \approx 0, \varepsilon > 0$ such that $\|A - \lambda - B\|_{\#} < \varepsilon \Rightarrow B \in \mathbf{Inv}(B(H^{\#}))$.
Now, we see that if $\left|\lambda - \widehat{\lambda}\right| < \varepsilon$ we have:
$\left\|A - \lambda - (A - \widehat{\lambda})\right\|_{\#} = \left|\lambda - \widehat{\lambda}\right| < \varepsilon$
Hence, $A - \widehat{\lambda} \in \mathbf{Inv}(B(H^{\#}))$. Then $\varphi(A)$ is #-open, and so $\sigma(A)$ is #-closed.
We need the following lemmas to show that $\sigma(A) \neq \varnothing$ ever.
Lemma 1.3.5. Let $A \in B(H^{\#})$. Then let $\gamma : B(H^{\#}) \to {}^{*}\mathbb{C}_{c}^{\#}$ be an arbitrary linear functional ($\gamma \in B(H^{\#})^{*}$). We have that the map
$f_{A_{\gamma}} : \varphi(A) \to {}^{*}\mathbb{C}_{c}^{\#}, \lambda \mapsto \gamma(1/(A-\lambda))$ is #-analytic on $\varphi(A)$, and has $\#\text{-}\lim_{\lambda \to {}^{*}\infty} f_{A_{\gamma}}(\lambda) = 0$
Proof. For $\lambda, \lambda_0 \in \varphi(A)$ we have that
$$\frac{1}{A-\lambda} - \frac{1}{A-\lambda_0} = \frac{A - \lambda_0 - A + \lambda}{(A-\lambda)(A-\lambda_0)} = \frac{\lambda - \lambda_0}{(A-\lambda)(A-\lambda_0)}.$$
Then,
$$\#\text{-}\lim_{\lambda \to_{\#} \lambda_0} \frac{f_{A_{\gamma}}(\lambda) - f_{A_{\gamma}}(\lambda_0)}{\lambda - \lambda_0} = \gamma\left(\frac{1}{(A-\lambda)(A-\lambda_0)}\right) = \gamma\left(\#\text{-}\lim_{\lambda \to_{\#} \lambda_0} \frac{\lambda - \lambda_0}{(A-\lambda)(A-\lambda_0)}\right) = \ldots$$
Where we use linearity of γ in the first equality, and #-continiuty of γ in the second.
Then, by lemma 1.3.2 we have that
$$\ldots = \gamma\left(\frac{\lambda - \lambda_0}{\#\text{-}\lim_{\lambda \to_{\#} \lambda_0}(A-\lambda)(A-\lambda_0)}\right) = \gamma\left(\frac{1}{(A-\lambda_0)^2}\right).$$
Hence, $f_{A_{\gamma}}$ is #-analytic on $\varphi(A)$. By the estimate
$$\left\|\frac{1}{A-\lambda}\right\|_{\#} = \frac{1}{|\lambda|}\left\|\frac{1}{1-\lambda^{-1}A}\right\|_{\#} = \frac{1}{|\lambda|}\left(\left\|Ext\text{-}\sum\nolimits_{n \in {}^{*}\mathbb{N}}(\lambda^{-1}A)^{n}\right\|_{\#}\right) \leq$$

$\leq \frac{1}{|\lambda|}\left(Ext\text{-}\sum_{n\in{}^*\mathbb{N}}\|(\lambda^{-1}A)^n\|_{\#}^{n}\right)=\frac{1}{|\lambda|-\|A\|_{\#}}..$

It is clear that $\frac{1}{A-\lambda}$ $\to_{\#}$ 0 as $\lambda \to {}^*\infty$ and hence by #-continuity of f_{A_γ} we are done.

Theorem 1.3.1. If $A \in B(H^{\#})$ then $\sigma(A) \neq \varnothing$.

Proof. Say $\exists A \in B(H^{\#})$ suth that $\sigma(A) = \varnothing$. For this A, we have that f_{A_γ} is:

(i) $|f_{A_\gamma}(z)|$ bounded by positive constant $K \in {}^*\mathbb{R}^{\#}_{c,+}$

(ii) $f_{A_\gamma}(z)$ is #-entire function, that is $f_{A_\gamma}(z)$ is a ${}^*\mathbb{C}^{\#}_c$-valued function #-holomorphic on the whole ${}^*\mathbb{C}^{\#}_c$

(iii) $f_{A_\gamma}(z)$ has $\#\text{-}\lim_{\lambda\to{}^*\infty} f_{A_\gamma}(\lambda) = 0$.

The only map satisfying these three properties is the zero map. But since γ was arbitrary, this implies that an arbitrary functional is the zero functional, which is clearly a contradiction. Hence, $\sigma(A) \neq \varnothing$

In particular this means that $\sigma(A) \neq \mathbf{eigen}(A)$ if $\mathbf{eigen}(A)$ is empty.

Theorem 1.3.2. (Generalized Gelfand Mazur theorem)

If $\mathbf{Inv}(B(H^{\#})) = B(H^{\#}) \setminus \{0\}$, Then $B(H^{\#}) \cong {}^*\mathbb{C}^{\#}_c$.

Proof. Let $A \in B(H^{\#})$ then let $\lambda_A \in \sigma(A)$ we have $A - \lambda_A = 0$. So λ_A is unique. Our isomorphism is then $\psi : B(H^{\#}) \to {}^*\mathbb{C}^{\#}_c A \mapsto \lambda_A$.

Theorem 1.3.3. (Generalized Spectral Mapping Theorem)

Let $A \in B(H^{\#}), f \in {}^*\mathbb{C}^{\#}_c[z]$. Then we have: $\sigma(f(A)) = f(\sigma(A))$.

Proof. Let $\lambda \in \sigma(A) f(z) = Ext\text{-}\sum_{n=0}^{N} a_n z^n$. Then

$$f(A) - f(\lambda) = Ext\text{-}\sum_{n=0}^{N} a_n(A^n - \lambda^n) = (A-\lambda)\left(Ext\text{-}\sum_{n=0}^{N} a_n\left(Ext\text{-}\sum_{j<n}(A^j\lambda^{n-j-1})\right)\right).$$

So $f(\lambda) \in \sigma(f(A))$. Say $\mu \notin f(\sigma(A))$ Then, we can write $f(z) - \mu = a_N\left(Ext\text{-}\prod_{n=0}^{N}(z-\lambda_n)\right)$.

Then as $\mu - f(\lambda) \neq 0\ \forall\lambda \in \sigma(A)$ (the zero operator isn't invertible) we have that $\lambda_n \notin \sigma(A)(n \leq N)$. Therefore, $f(A) - \mu = a_N\left(Ext - \prod_{n=0}^{N}(A-\lambda_n)\right)$, must be invertible, and $\mu \notin \sigma(f(A))$.

This theorem has many forms and generalises much more than for f being a polynomial.

Definition 1.3.3. (Spectral Radius)

Given A in $B(H^{\#})$ the spectral radius, denoted $r(A)$, of A is defined by $r(A) = \sup_{\lambda\in\sigma(A)}|\lambda|$.

We note by lemma 1.3.4 the supremum exists and is attained. In fact, the following lemma tells us what the spectral radius of a given operator is in terms of a #-limit.

Lemma 1.3.6. $Let A \in B(H^{\#})$. Then the #-limit: $\#\text{-}\lim_{n\to{}^*\infty} \|A_n\|_{\#}^{1/n}$ exists, and is equal to $r(A)$, the spectral radius of A.

Proof. By theorem 1.3.4 and lemma 1.3.4 we have that

$[r(A)]^n = r(A^n) \leq \|A_n\|_{\#} \Rightarrow r(A) \leq \|A_n\|_{\#}^{1/n},\ n \in {}^*\mathbb{N} \Rightarrow r(A) \leq \#\text{-}\lim\inf_{n\to{}^*\infty}\|A_n\|_{\#}^{1/n}$.

For the other direction, examine again the function from lemma 1.3.5, but this time restricted to $\Omega = \{z \in {}^*\mathbb{C}^{\#}_c \| z| > r(A)\}$. We know that fAγ is analytic in $\Omega \subset \varphi(A)$. So it has laurent expansion $Ext\text{-}\sum_{n\in{}^*\mathbb{Z}_-} a_n z^n$and also that $\#\text{-}\lim_{z\to{}^*\infty} f_{A_\gamma}(z) = 0$. So in fact, we have laurent expansion $Ext\text{-}\sum_{n\in{}^*\mathbb{N}} \frac{a_n}{z^n}$. To determine the coefficients we know that for $z \in \Omega$, $\left\|\frac{A}{z}\right\|_{\#} < 1$ and hence, by lemma 1.3.1 one obtains

$$\frac{1}{z-A} = \frac{1}{z(1-z^{-1}A)} = \frac{1}{z}\left(Ext\text{-}\sum_{n=0}^{{}^*\infty}\frac{A^n}{z^n}\right) = Ext\text{-}\sum_{n=0}^{{}^*\infty}\frac{A^n}{z^{n+1}} = Ext\text{-}\sum_{n=1}^{{}^*\infty}\frac{A^{n-1}}{z^n}.$$

Hence, $f_{A_\gamma}(z) = \gamma\left(\frac{1}{z-A}\right) = \gamma\left(Ext\text{-}\sum_{n=1}^{^*\infty} \frac{A^{n-1}}{z^n}\right) = Ext\text{-}\sum_{n=1}^{^*\infty} \frac{\gamma(A^{n-1})}{z^n}$.
So we have limn→∞$\frac{\gamma(A^{n-1})}{z^n} = 0$, for all functionals $\gamma \in H^{\#*}$. It follows that
#-$\lim_{n\to^*\infty} \frac{\|A^{n-1}\|_\#}{|z|^n} = 0$ and so $\forall z \in \Omega\ |z| >$ #-$\limsup_{n\to^*\infty} \|A_n\|_\#^{1/n}$
then, $\forall z \in$ #-$\overline{\Omega}\ |z| \geq$ #-$\limsup_{n\to^*\infty} \|A_n\|_\#^{1/n}$. In particular then,
$r(A) \geq$ #-$\limsup_{n\to^*\infty} \|A_n\|_\#^{1/n}$ and so we are done.
Remark 1.3.1. If A is self adjoint, $\|A^2\|_\# = \|A\|_\#^2$ so by induction $\|A^{2^n}\|_\# = \|A\|_\#^{2^n}$
and therefore $r(A) =$ #-$\lim_{n\to^*\infty} \|A^{2^n}\|_\#^{1/2^n} = \|A\|$.

§1.4. $l_2^\#(G)$ and $B(l_2^\#(G))$.

Definition 1.4.1. Let G be a discrete, $*$-countable group. Then define

$$l_2^\#(G) = \left\{f{:}G \to {}^*\mathbb{C}_c^\# \ \middle|\ Ext\text{-}\textstyle\sum_{g\in G}\lceil f(g)\rceil^2 < {}^*\infty\right\} \tag{1.4.1}$$

This is a non-Archimedean Hilbert space with respect to the inner product

$$\langle f \mid h\rangle_\# = Ext\text{-}\textstyle\sum_{g\in G} f(g)\bar{h}(g). \tag{1.4.2}$$

Lemma 1.4.1. Let $g \in G, f \in l_2^\#(G)$ then define $g * f \in l_2^\#(G)$ by $g * f(h) = f(g^{-1}h)$. This defines a group action on $l_2^\#(G)$.
Proof. Fix $f \in l_2^\#(G)$. We verify directly, $\forall h, g_1, g_2 \in G$:
$(g_1 \cdot g_2) * f(h) = f((g_1 \cdot g_2)^{-1}h) = f(g_2^{-1}g_1^{-1}h) = g_2 * f(g_1^{-1}h) = g_1 * (g_2 * f(h))$.
Definition 1.4.1. Let $g \in G$, we define $T_g \in B(l_2^\#(G))$ as $T_g{:}l_2^\#(G) \to l_2^\#(G),\ f \mapsto g * f$.
Where $g * f$ is the group action as in lemma 1.4.1.
Lemma1.4.2. T_g has the following properties:
(i) $T_{g_1} \cdot T_{g_2} = T_{g_1 \cdot g_2}$ (ii) $T_g^* = T_{g^{-1}}$.
Proof. (i) This follows clearly from lemma 1.4.1.
(ii) Let $f, h \in l_2^\#(G)$. Then,
$\langle T_g f \mid h\rangle_\# = Ext\text{-}\sum_{a\in G} T_g f(a)\overline{h(a)} = Ext\text{-}\sum_{a\in G} g * f(a)\overline{h(a)} =$
$= \left[Ext\text{-}\sum_{a\in G} f(g^{-1}(ga))\overline{h(a)}\right]\left[Ext\text{-}\sum_{a\in G} f(g^{-1}a)\overline{h((ga))}\right] =$
$= Ext\text{-}\sum_{a\in G} f(a)\overline{g^{-1} * h(a)} = Ext\text{-}\sum_{a\in G} f(a)\overline{T_{g^{-1}}h(a)} = \langle f \mid T_{g^{-1}}h\rangle_\#$.

§1.5.Topologies on $B(H^\#)$.

In order to study a non-Archimedean von Neumann algebras, one needs to look into useful topologies on $B(H^\#)$. Since all operators are bounded in ${}^*\mathbb{R}_c^\#$ we have the operator #-norm and therefore the induced topology.
Definition 1.5.1. #-Norm Topology.
Using this norm, we can define a metric topology, using the induced ${}^*\mathbb{R}_c^\#$-valued metric $d{:}B(H^\#) \times B(H^\#) \to R, d(T_1, T_2) = \|T_1 - T_2\|_\#$.
This topology is useful for many reasons, but for the purposes of looking at non-Archimedean von Neumann Algebras is somehow too “fine”. We need coarser topologies to enable us to have nice examples.
Definition 1.5.2. Strong Operator Topology (s.o.t.)
We define the strong operator topology as the coarsest topology such that $\forall v \in H^\#$ the map $\psi_v{:}B(H^\#) \to {}^*\mathbb{R}_c^\#, T \mapsto \|Tv\|_\#$ is #-continuous.

Example 1.5.1. For H$^{\#}$ = ‘2(N), let Tn : H$^{\#}$ → H$^{\#}$ v ↦ (v, en)en We have that $T_n \to 0$ in the s.o.t. but not in the #-norm topology.
Tn →/ 0 in #-norm sense, since ‖Tn‖ ≥ ‖Tn(en)‖H$^{\#}$ = ‖en‖ = 1 ∀n.
However in the strong operator topology, we have that
$T_n \to_{\#} 0 \Leftrightarrow \psi_v(T_n) \to_{\#} 0 \forall v \in H^{\#}$
In this case, v ∈ ‘2, so in particular the entries of v always tend to zero, i.e.
$\psi_v(T_n) \to_{\#} 0 \forall v \in H^{\#}$.
This distinguishes the strong and the #-norm topologies. Making use of the adjoint, we define a finer topology
Definition 1.5.3. Strong-∗Operator Topology (s*.o.t.)
We define the strong-∗ operator topology as the coarsest topology such that $\forall v \in H^{\#}$ the two maps

$$\psi_v : B(H^{\#}) \to {}^*\mathbb{R}_c^{\#}, T \mapsto \|Tv\|_{\#} \tag{1.5.1}$$

and

$$\psi_v^* : B(H^{\#}) \to {}^*\mathbb{R}_c^{\#}, T \mapsto \|T^*v\|_{\#} \tag{1.5.2}$$

are both #-continuous.
And finally, making use of the inner product we define the weak operator topology:
Definition 1.5.4. Weak Operator Topology (w.o.t.)
We define the weak operator topology (w.o.t.) as the coarsest topology such that $\forall v, w \in H^{\#}$ the map $\psi_{vw} : B(H^{\#}) \to {}^*\mathbb{R}_c^{\#}, T \mapsto |\langle v, Tw \rangle_{\#}|$ is #-continuous.
Lemma 1.5.1. For the topologies as in Definitions 1.5.1, 1.5.2, 1.5.3 and 1.5.4 we have that w.o.t. ≺ s.o.t. ≺ s-∗.o.t ≺ #-norm topology.
Lemma 1.5.2.A basis for the strong operator topology is given by
$\beta = \{N(A, \{v_i\}_{i=1}^N,)| A \in B(H^{\#}) v_i \in H^{\#} > 0\}$
where $N(A, \{v_i\}_{i=1}^N,) := \{B \in B(H^{\#}) | \|(A - B)v_i\|_{\#} < i = 1, 2, \ldots, N\}$
Proof. First we need to check it is a basis for a topology.
(i) It covers $B(H^{\#})$ since for example
$N(A, \{0\}, 1) = \{B \in B(H^{\#}) | \|(A - B)\mathbf{0}\|_{\#} < 1\} = \{B \in B(H^{\#}) | \|\mathbf{0}\|_{\#} < 1\} = B(H^{\#})$.
(ii) It is closed under intersection since for $C \in N(A, \{v_i\}_{i=1}^N, \varepsilon) \cap N(B, \{\hat{v}_i\}_{i=1}^{\hat{N}}, \hat{\varepsilon})$.
We have that $C \in N(C, \{v_i\}_{i=1}^N \cup \{\hat{v}_i\}_{i=1}^{\hat{N}}, \min\{\varepsilon - \|(C - A)v_i\|_{\#}, \hat{\varepsilon}\} - \|(C - B)\hat{v}_i\|_{\#}\})$.
The only thing we need to verify for this is that (w.l.o.g.)
$\forall D \in N(C, \{v_i\}_{i=1}^N \cup \{\hat{v}_i\}_{i=1}^{\hat{N}}, \ \min\{\varepsilon - \|(C - A)v_i\|_{\#}, \hat{\varepsilon}\} - \|(C - B)\hat{v}_i\|_{\#}\})$,
$D \in N(A, \{v_i\}_{i=1}^N, \varepsilon)$. This is clear since for all i
$\|(D - A)v_i\|_{\#} \leq \|(D - C)v_i\|_{\#} + \|(C - A)v_i\|_{\#} \leq \varepsilon - \|(C - A)v_i\|_{\#} + \|(C - A)v_i\|_{\#} = \varepsilon$.
Now we need to show that for all topologies such that $\forall v \in H^{\#}$ the map
$\psi_v : B(H^{\#}) \to {}^*\mathbb{R}_c^{\#}, T \mapsto \|Tv\|_{\#}$ is #-continuous, subsets of this form are #-open.
Noting that $N(A, \{v_i\}_{i=1}^N,) \ = \bigcap_{i=1}^N \psi_{v_i}^{-1}(\|Av_i\|_{\#} - \varepsilon, \|Av_i\|_{\#} + \varepsilon)$.This is clear.
Lemma 1.5.3. A basis for the weak operator topology is given by:
$\beta = \{N(A, \{v_n\}_{n \leq N}, \{w_n\}_{n \leq N}, \varepsilon) \ | \ A \in B(H^{\#}) \{v_n\}_{n \leq N}, \{w_n\}_{n \leq N} \subset H^{\#}, \varepsilon > 0\}$.
Where $N(A, \{v_n\}_{n \leq N}, \{w_n\}_{n \leq N}, \varepsilon) = \{B \in B(H^{\#}) | |\langle (B - A)v_n, w_n \rangle_{\#}| < \varepsilon \forall n \leq N\}$.
We omit the proof of this result. It is similar to the proof of the basis of the SOT.
Lemma 1.5.4. Let $f : B(H^{\#}) \to {}^*\mathbb{C}_c^{\#}$ be a linear functional. The following are equivalent:
(i) $\exists \{v_n\}_{n \leq N}, \{w_n\}_{n \leq N} \subset H^{\#}$, such that $f(A) = Ext\text{-}\sum_{n \leq N} \langle Av_n, w_n \rangle_{\#} \ \forall A \in B(H^{\#})$.

(ii) f is #-continious in the weak sense
(iii) f is #-continious in the strong sense
Proof. It is clear that the first implies the second, and by lemma 1.5.1 that the second implies the third. Hence all we must show is that for all f is #-continious in the strong sense then we can find $\{v_n\}_{n \leq N}$, $\{w_n\}_{n \leq N} \subset H^{\#}$, such that $\forall A \in B(H^{\#})$: $f(A) = Ext\text{-}\sum_{n \leq N} \langle Av_n, w_n \rangle_{\#}\ \forall A \in B(H^{\#})$. Suppose f is #-continious in the strong sense, then the inverse image of the #-open ball in ${}^*\mathbb{C}_c^{\#}$ is #-open in the strong operator topology. Considering our basis elements, then there is some constant $\kappa > 0$ and $\{v_n\}_{n \leq N}$ such that $|f(A)|^2 \leq \kappa\left(Ext\text{-}\sum_n \|Tv_n\|_{\#}^2\right)$. Now consider the subspace of $Ext\text{-}H^{\#} \oplus H^{\#} \ldots \oplus H^{\#}$ given by $\{\oplus_{n \leq N} Av_n | A \in B(H^{\#})\}$ we can define a linear functional on this set by $\oplus_{n \leq N} Av_n \mapsto f(A)$. Then by the generalized Riesz representation theorem, $\exists \{w_n\}_{n \leq N}$ such that $f(A) = Ext\text{-}\sum_{n \leq N} (Av_n, w_n) \forall A \in B(H^{\#})$ as required.

§2.Non-Archemedean Banach algebras and $\mathbf{C}_{\#}^{\star}$-Algebras.

§2.1. Initial Definitions and #-Continious Functional Calculus.

von Neumann Algebras are a specific type of $\mathbf{C}_{\#}^{*}$ algebra, and so it is important to understand well the theory of $\mathbf{C}_{\#}^{*}$ algebras before non-Archemedean von Neumann Algebras.

Definition 2.1.1.A non-Archemedean Banach algebra $A_{\#}$ is a complex algebra over field
${}^*\mathbb{C}_c^{\#}$ which is a non-Archemedean Banach space under a ${}^*\mathbb{R}_c^{\#}$-valued #-norm which is submultiplicative:

$$\|xy\|_{\#} \leq \|x\|_{\#} \|y\|_{\#} \tag{2.1.1}$$

for all $x, y \in A_{\#}$.

Definition2.1.2.An involution on a non-Archemedean Banach algebra $A_{\#}$ is a conjugate-linear #-isometric antiautomorphism of order two, usually denoted $x \mapsto x^*$. In other words,

1.$(x^*)^* = x, \|x^*\| = \|x\|$
2.$(x+y)^* = x^* + y^*$,
3.$(xy)^* = y^* x^*$,
4.$(\lambda x)^* = \bar{\lambda} x^*$, for all $x, y \in A, \lambda \in {}^*\mathbb{C}_c^{\#}$.

Definition2.1.3.Spectrum (of an element of some a non-Archemedean algebra)
Let $A_{\#}$ be some a non-Archemedean algebra and $a \in A_{\#}$ we define $\sigma(a) = \{\lambda \in {}^*\mathbb{C}_c^{\#} \mid a - \lambda\mathbf{1}$ is not invertible $\}$.

Definition2.1.4.A Banach #-algebra is a non-Archemedean Banach algebra $A_{\#}$ with an
involution. An $\mathbf{C}_{\#}$-algebra is a Banach #-algebra $A_{\#}$ satisfying the $\mathbf{C}_{\#}^{*}$-axiom: for all $x \in A_{\#}$

$$\|x^* x\|_{\#} = \|x\|_{\#}^2. \tag{2.1.2}$$

Example 2.1.1. $B(H^{\#})$ is a $\mathbf{C}_{\#}^{*}$ Algebra)
We see this is an immediate consequence of **lemma 1.4**
Lemma 2.1.1. $K \subset B(H^{\#})$ is a $\mathbf{C}_{\#}^{*}$ Algebra iff

(i) K is an algebra over ${}^*\mathbb{C}_c^\#$
(ii) $K = K^*$
(iii) K is #-closed with respect to the #-norm topology.
Proof. It is clear that if K is a $\mathbf{C}_\#^*$ algebra it must be closed with respect to the #-norm topology and an algebra. To see the other direction, we note that the only conditions we must check are conditions of #-closure by **lemma 1.4** all of the operations work algebraically as they should. We have
1. K is #-closed under taking sums, scalar multiples and products as it is an algebra.
2. K is #-closed under taking adjoints by the second bullet point
3. K is #-closed with respect to the #-norm topology by the third bullet point.
Therefore, K is a $\mathbf{C}_\#^*$ algebra.
Example 2.2. $K(B(H^\#))$ is a $\mathbf{C}_\#^*$ algebra. This follows clearly from lemma 2.1.1 and theorem
1.22, as $K(B(H^\#)) = \{A \in B(H^\#) \mid \Re(A) < {}^*\infty\} = \mathcal{F}\Re(B(H^\#))$ and
$\mathcal{F}\Re(B(H^\#)) = \{A \in B(H^\#) \mid \Re(A) < {}^*\infty\}$ is a $*$-algebra.
Example 2.1.3. The set $\mathcal{F}\Re(B(H^\#))$ is in general a $*$-subalgebra of $B(H^\#)$ but is not a $\mathbf{C}_\#^*$ algebra if $H^\#$ is hyper infinite. This can be seen by considering an orthonormal basis $\{e_i\}_{i\in X}$ and
considering p_i to be the orthonormal projection into the line spanned by $e_i(p_i(e_j) = \delta_{ij})$ then the hyper infinite sequence $(q_N)N \in {}^*\mathbb{N}$ where $q_N = Ext\text{-}\sum_{i=1}^N p_i$ #-converges in #-norm to the identity, which would not be hyperfinite rank.
As promised, we return to spectral theory, with a more general version of theorem
1.34.
Theorem 2.1.1. #-Continious functional calculus
Let K_1, K_2 be $\mathbf{C}_\#^*$ algebras and $A \in K_1$ normal, then we have:
(i) The map $\psi: C^\#(\sigma(A)) \to K_1 f \mapsto f(A)$ is a homomorphism.
(ii) For all $f \in C^\#(\sigma(A))$ we have $\sigma(f(A)) = f(\sigma(A))$
If $\Psi: K_1 \to K_2$ is a $\mathbf{C}_\#^*$-homomorphism, then $\Psi(f(A)) = f(\Psi(A))$
This of course raises a few questions, how for example, would one take the square root of
an operator? For the purposes of these notes we don’t look too deeply into this, but one
way to define this we can take any sequence $f_n \in {}^*\mathbb{C}_c^\#[z]$ which approximates f locally uniformly well,and take $f(A) := \#\text{-}\lim_{n\to{}^*\infty} f_n(A)$.
Most of these definitions we get are intuitive, for example for $f(z) = |z|^2$, we take $f(A) = A^*A$

§ 2.2 ${}^*\mathbb{C}_c^\#$-valued States

Definition 2.2.1. If K is a $*$ algebra, a state is a linear ${}^*\mathbb{C}_c^\#$-valued functional that is positive and normalised. That is: $\omega: K \to {}^*\mathbb{C}_c^\#$ such that:
(i) $\omega(A^*A) \geq 0 \forall A \in K$
(ii) $\omega(1) = 1$.
Notation 2.2.1. We denote the space of all states on A by $\mathbf{S}^\#(A)$.
Throughout the rest of this subsection, K will refer to a $\mathbf{C}_\#^*$ algebra and we will consider states on K.
Example 2.2.2. Let $K = M_n({}^*\mathbb{C}_c^\#), n \in {}^*\mathbb{N}$ the $n \times n$ matrices with complex coefficients.

Then for all A positive, $\omega(A):K \to {}^*\mathbb{C}_c^{\#}, B \mapsto \dfrac{Ext\text{-}\mathbf{Tr}(AB)}{Ext\text{-}\mathbf{Tr}(A)}$.

Where $Ext\text{-}\mathbf{Tr}(C)$ is the external sum of the diagonal entries of C (or equivalently the external sum of the eigenvalues of C) . Indeed, since $Ext\text{-}\mathbf{Tr}(AB) = Ext\text{-}\mathbf{Tr}(BA)$ and $Ext\text{-}\mathbf{Tr}(A) \geq 0$ if A is positive,letting $A = C^*C$ we see also

$Ext\text{-}\mathbf{Tr}(AB^*B) = Ext\text{-}\mathbf{Tr}(BAB^*) = Ext\text{-}\mathbf{Tr}(BC^*CB^*) = Ext\text{-}\mathbf{Tr}((CB^*)^*(CB^*)) \ \geq 0.$

So $\omega(A)$ is positive, it is also normalised clearly and therefore a state.

Definition 2.2.2. We say that a linear${}^*\mathbb{C}_c^{\#}$-valued functional ψ is hermitian if $\forall A \in K\left[\psi(A^*) = \overline{\psi(A)}\right]$.

We for some state ω are interested in the bilinear form $f_\omega:K \times K \to {}^*\mathbb{C}_c^{\#}$, $(A,B) \mapsto \omega(B^*A)$.

This is because it has many properties similar to an inner product. The first we show is that states are hermitian, which implies something similar to conjugate symmetry for f_ω.

Lemma 2.2.1.Let $\omega \in \mathbf{S}^{\#}(A)$ then ω is hermitian.

Proof. First suppose $A = A^*$i.e. A is self #-adjoint. Then let

$A_+ = Ext\text{-}\sum_{\lambda\in\sigma(A),\lambda>0}\lambda p_\lambda, A_- = Ext\text{-}\sum_{\lambda\in\sigma(A),\lambda<0}(-\lambda)p_\lambda$

Noting that both of these are positive, we have that

$\omega(A) = \omega(A_+ - A_-) \in {}^*\mathbb{R}_c^{\#} \Rightarrow \omega(A^*) = \omega(A) = \overline{\omega(A)}$.

Then for any $A \in K$ we can write $A = A_1 + iA_2$ where A_1, A_2 are both self #-adjoint. Then we have

$\omega(A^*) = \omega(A_1) - i\omega(A_2) = \overline{\omega(A_1) + i\omega(A_2)} = \overline{\omega(A_1) + i\omega(A_2)} = \overline{\omega(A)}$.

Corollary 2.2.1. Let f_ω be the bilinear form as defined before. Then it is conjugate symmetric i.e. $f_\omega(A,B) = f_\omega(B,A)$.

Proof. Using that states are hermitian we see clearly

$f_\omega(BA) = \omega(A^*B) = \omega((A^*B)^*) = \omega(B^*A) = f_\omega(A,B)$.

Next, we show the cauchy schwarz for states.

Lemma 2.2.2. (Cauchy Schwarz)

Let $\omega \in \mathbf{S}^{\#}(K)$, then we have,$|\omega(AB^*)|^2 \leq \omega(A^*A)\omega(B^*B)$.

Proof. If $B = 0$ this is clear. Otherwise, by positivity we have for

$C = \omega(BB^*)A - \omega(AB^*)B$:

$0 \leq \omega(CC^*) = \omega((\omega(BB^*)A - \omega(AB^*)B)(\omega(BB^*)A^* - \omega(AB^*)B^*)) \ = \ldots\ldots =$

$= \omega(BB^*)(\omega(BB^*)\omega(AA^*) - \omega(AB^*)\omega(BA^*) - \omega(AB^*)\omega(AB^*) + \omega(AB^*)\omega(AB^*))$.

Then using that states are hermitian (lemma 2.13) we can simplify

$\ldots = \omega(BB^*)(\omega(BB^*)\omega(AA^*) - |\omega(AB^*)|^2)$.

Then by positivity, $\omega(BB^*) \geq 0$ and so $\omega(BB^*)\omega(AA^*) - |\omega(AB^*)|^2 \geq 0$ as required.

Corollary 2.2.2.$|f\omega(A,B)| \leq \sqrt{f_\omega(A,A)f_\omega(B,B)}$.

We see now that f_ω is very similar to an inner product, but fails on positive definiteness, as seen in the following example.

Example 2.2.3. In $M_2({}^*\mathbb{C}_c^{\#})$, we can set $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.Then as A is positive we can define the state ω_A as before: $\omega_A:M_2({}^*\mathbb{C}_c^{\#}) \to {}^*\mathbb{C}_c^{\#}, B \mapsto Ext\text{-}\mathbf{Tr}(AB)$.

Then for $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$ we have $\omega_A(B^*B) = \omega_A(B) = Ext\text{-}\mathbf{Tr}(AB) = Ext\text{-}\mathbf{Tr}(0) = 0$.

This motivates the next definition.

Definition 2.2.3. We define for each $\omega \in S^{\#}(K)$

$J_\omega = \{A \in K \mid \omega(A^*A) = f_\omega(A,A) = 0\}$.

The fact that our candidate is not positive definite is not an issue, so long as we can use some devices from abstract algebra (namely quotient objects) to "forget" about the problem areas. For this, we need to find an appropriate ideal of K.

Lemma 2.2.3. Let J_ω be as before. Then J_ω is a left ideal.

Proof. Say $A, B \in J_\omega$. Then

$(A+B)^*(A+B) \leq (A+B)^*(A+B) + (A-B)^*(A-B) = 2A^*A + 2B^*B$

so $0 \leq f_\omega(A+B, A+B) \leq 2f_\omega(A) + 2f_\omega(B) = 0$. So J_ω is a #-closed linear subspace. We also see that for all $A \in J_\omega$ and $B \in K (BA)^*(BA) \leq \|B\|_{\#}^2 A^*A$, and so $BA \in J_\omega$

Lemma 2.2.4. If ω is a positive linear functional on K, then the operator #-norm of ω, $\|\omega\|_{\#} = \sup_{A \in K \setminus \{0\}} \dfrac{\omega(A)}{\|\omega\|_{\#}}$ satisfies $\|\omega\|_{\#} = \omega(1)$.

Proof. We know that $\|\omega\|_{\#} \geq \omega(1)$ since $\|\mathbf{1}\|_{\#} = 1$. Now let $A \in K \setminus \{0\}$. We have that $\|A + A^*\|_{\#}\,\mathbf{1} - (A+A^*) \geq 0$ and so $\omega(A+A^*) \leq \|A+A^*\|_{\#}\omega(1)$. But also, we have $\left|\omega(\dfrac{A+A^*}{2})\right| \leq \dfrac{|\omega(A)| + |\omega(A^*)|}{2} = |\omega(A)| = \dfrac{\omega(A+A^*)}{2} + \dfrac{\omega(A-A^*)}{2} \leq \dfrac{\omega(A+A^*)}{2} =$
$= \left|\dfrac{\omega(A+A^*)}{2}\right|$. And so we have equality everywhere and that $|\omega(A)| = \left|\dfrac{\omega(A+A^*)}{2}\right|$.

Putting this together we have

$|\omega(A)| = \left|\dfrac{\omega(A+A^*)}{2}\right| \leq \dfrac{\|A+A^*\|_{\#}\omega(\mathbf{1})}{2} \leq \dfrac{\|A\|_{\#} + \|A^*\|_{\#}\omega(\mathbf{1})}{2} = \|A\|_{\#}\omega(\mathbf{1})$.

And so $\|\omega\|_{\#} \leq \omega(\mathbf{1})$. In fact this relationship is equivalent.

Lemma 2.2.5. Let ω be a linear functional on K. The following statements are equivalent

1. ω is positive
2. $\|\omega\|_{\#} = \omega(1)$.

Proof. 1.$\Rightarrow$ 2. is **lemma 2.18•** 2. $\Rightarrow$ 1.

Let A be positive, and say $\omega(A) = a + ib, a, b \in {}^*\mathbb{R}_c^{\#}$. Then for all $t \in {}^*\mathbb{R}_c^{\#}$ we have:

$a^2 + (b + t\|\omega\|_{\#})^2 = \|A + it\|_{\#}^2 \leq \|A + it\|_{\#}^2 \|\omega\|_{\#}^2 \leq (\|A\|_{\#}^2 + t^2)\ \|\omega\|_{\#}^2$.

Substracting t2 ‖ω‖2from both sides we have $2bt \leq \|A\|_{\#}^2$ and hence $b = 0$. Then,

$\|A\|_{\#}\|\omega\|_{\#} - a = \omega(\|A\|_{\#} - A) = \|\omega\|_{\#}\|\,\|A\|_{\#} - A\|_{\#} \leq \|\omega\|_{\#}\ \|A\|_{\#}$ So $a \geq 0$.

From the theory so far, we can relate the spectrum of some element $A \in K$ to some states on K.

Lemma 2.2.6. Let $A \in K$ then for each $\lambda \in \sigma(A)$ there exists a state $\omega_{A\lambda} : K \to {}^*\mathbb{C}_c^{\#}$ such that $\omega_{A\lambda}(A) = \lambda$.

Proof. We define the linear functional on the subspace ${}^*\mathbb{C}_c^{\#} \cdot A + {}^*\mathbb{C}_c^{\#} \cdot \mathbf{1}$ by $\omega_0(aA + b_1) = a\lambda + b$. It is clear then that $\omega_0(aA + b_1) = a\lambda + b \in \sigma(aA + b\mathbf{1})$ and hence by lemma **1.29** $1 = \omega_0(1) \leq \|\omega_0\|_{\#} = \sup_{a,b \in {}^*\mathbb{C}_c^{\#}}\left[\dfrac{|a\lambda + b|}{\|aA + b\mathbf{1}\|_{\#}} \leq 1\right]$.

Then by generalized Hahn-Banach theorem, there exists an extension of ω_0 to $K, \omega_{A\lambda}$ with $\|\omega_{A\lambda}\|_{\#} = 1 = \omega_{A\lambda}(1)$ by **lemma 2.19**, $\omega_{A\lambda}$ is a state.

The next lemma shows us how even though we don't have positive definiteness, we can conclude an equivalence between $A = 0$ and $\omega(A) = 0 \forall \omega$.

Lemma 2.21. Let K be a $\mathbf{C}_{\#}^*$ algebra, and $A \in K$. Then we have

$A = 0 \Leftrightarrow \omega(A) = 0 \forall\omega \in S^{\#}(K)$.
Proof. $\Rightarrow$ is clear by linearity of ω.
$\Leftarrow$ can be seen by the string of implications
$\omega(A) = 0 \forall\omega \in S^{\#}(K) \Rightarrow \sigma(A) = \{0\} \Rightarrow A = 0$.
We see in fact there are a huge number of results in analogy to those discussed in subsection **2.1** using that $\sigma(A) \subset \{\omega(A) \mid \omega \in S^{\#}(K)\}$. For example
Lemma 2.22. Let K be a $\mathbf{C}^*_{\#}$ algebra and let $A \in K$.
(i) $A = A^* \Leftrightarrow \omega(A) \in {}^*\mathbb{R}^{\#}_c \forall\omega \in S^{\#}(K)$
(ii) $A \geq 0 \Leftrightarrow \omega(A) \geq 0 \forall\omega \in S^{\#}(K)$.
Proof. (i)-(ii) $\Leftarrow$ follows since $\sigma(A) \subset \{\omega(A) \mid \omega \in S^{\#}(K)\}$ and $\Rightarrow$ since ω is hermitian.
(i)-(ii) $\Leftarrow$ follows since $\sigma(A) \subset \{\omega(A) \mid \omega \in S^{\#}(K)\}$ and $\Rightarrow$ since ω is positive.

§2.3.Representations and the generalized Gelfand--Naimark-Segal Construction.

Definition 2.3.1. Let K be a $\mathbf{C}^*_{\#}$ algebra. A representation is a $*$-homomorphism
$\pi{:}K \to B(H^{\#}) A \mapsto \pi[A]$
Definition 2.3.2. Let K be a $\mathbf{C}^*_{\#}$ algebra, represented in $B(H^{\#})$ by π. Suppose further that $H^{\#}_0 \subset H^{\#}$ is a subspace such that $\{\pi[A]H^{\#}_0\} A \in K \subset H^{\#}_0$ (i.e. π is stable in $H^{\#}_0$). Then the restriction of π to this subspace, $\pi_0{:}K \to B(H^{\#}_0) A \mapsto \pi[A]$ is called a subrepresentation.
Example 2.3.1. For all representations π we always have the trivial subrepresentations where we restrict the domain of $\pi[A]$ to $\{0\}$ or $H^{\#}$
Definition 2.3.3. A representation $\pi{:}K \to B(H^{\#}) A \mapsto \pi[A]$ is called irreducible if the only subrepresentations are the restrictions to $\{0\}$ or $H^{\#}$ there are no nontrivial subrepresentations.
Definition 2.3.4. (Equivalent Representations)
Say $\pi_1{:}K \to B(H^{\#}_1) A \mapsto \pi_1[A] \pi_2{:}K \to B(H^{\#}_2) A \mapsto \pi_2[A]$
Are two representations of the same $\mathbf{C}^*_{\#}$ algebra such that there exists a unitary linear map $U{:}H^{\#}_1 \to H^{\#}_2$ such that $\forall A \in K{:} U\pi_1(A) = \pi_2 U(A)$
Then they are called equivalent.
Example 2.3.2. (Direct Sum over representations)
Say $\pi_i{:}K \to B(H^{\#}_i), i \leq N \in {}^*\mathbb{N}$ are a finite or hyperfinite family of representations. Then we can define a representation $\pi{:}K \to B\ (Ext\text{-}\bigoplus_i\ H^{\#}_i)\ A \mapsto \pi[A]$ where if v is uniquely decomposed into $Ext\text{-}\sum_i v_i$ (where each $v_i \in H^{\#}_i$) we have
$\pi[A](v) = \sum i\pi_i[A](v_i)$. Then for each $i$we have a subrepresentation equivalent to the representation in $H^{\#}_i$, given by the restriction of π to the subspace
$0 \oplus \ldots \oplus 0 \oplus H^{\#}_i \oplus 0 \oplus \ldots \oplus 0$.We can imagine representations like this in terms of hyperfinite matrices:

$$\pi[A](v) = \begin{bmatrix} \pi_1[A] & 0 & \ldots & 0 \\ 0 & \pi_2[A] & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \pi_N[A] \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ . \\ v_N \end{bmatrix}$$

We explore this concept later in greater detail.
Lemma 2.3.1. Let $\pi{:}K \to B(H^{\#})$ be a representation and say $v \in H^{\#}$ has #-norm 1.

Then the map $\omega_v{:}K \to {}^*\mathbb{C}_c^\# A \mapsto (\pi[A]v,v)H^\#$ defines a state on K

Proof. It is clear that ωv is linear and by cauchy schwarz we have

$|\omega_v(A)| \leq \sqrt{\|\pi[A]v\|_\# \|v\|_\#} \leq \sqrt{\|\pi[A]v\|_\# \|v\|_\#^2} = \sqrt{\|\pi[A]v\|_\#}\,\|v\|_\#.$

So ω_v is bounded in ${}^*\mathbb{R}_c^\#$. It is positive since $\langle \pi[A^*A]v,v\rangle = \langle \pi[A]v, \pi[A]v\rangle = \|\pi[A]v\|_\#^2$.

And so by **lemma 2.19** we have

$\|\omega_v\|_\# = \langle \pi[1]v,v\rangle_\# = \langle v,v\rangle_\# = \|v\|_\#^2 = 1$ as required.

In fact, every state on K arises in this fashion, as shown in the GNS construction.

We break down the proof of the GNS construction into a few lemmas.

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Lemma 2.3.2. The non-Archimedean Hilbert space completion of the space K/J_ω with respect to the ${}^*\mathbb{C}_c^\#$-valued inner product

$\langle \cdot \mid \cdot \rangle_\# : K/J_\omega \times K/J_\omega \to {}^*\mathbb{C}_c^\#([A],[B]) \mapsto \omega(B^*A) = f_\omega(A,B)$

is a non-Archimedean Hilbert space.

Proof. We have seen in lemma **2.17** that J_ω is a left ideal. Therefore this quotient object makes sense, and furthermore the inner product is well defined. It is clearly linear in the first argument, as well as positive definite by lemma **2.14** and conjugate symmetric by cororallary **2.13.1**. Therefore it is an inner product on the quotient space, and the Hilbert space completion defines a Hilbert space clearly.

Definition 2.3.5. Given a $\mathbf{C}_\#^*$ algebra K and a state ω, we define the Hilbert space completion of K/J_ω with respect to the inner product to be $L_2^\#(K,\omega)$.

Lemma 2.3.3. Given K,ω as before, we can define a representation $\pi{:}K \to B(L_2^\#(K,\omega))$ Such that $\omega(A) = (\pi[A]1_\omega \mid 1_\omega)$. Where $1_\omega \in L_2^\#(K,\omega)$ is the unit cyclic vector

Proof. For $A \in K$ we consider the map $\pi_0(A){:}K/J_\omega \to K/J_\omega[B] \mapsto [AB]$.

It is clear to see since J_ω is a left ideal that this is well defined and since $\|\pi_0[A](B)\|_\#^2 = \omega((AB)^*(AB)) \leq \|A\|_\#^2\omega(B^*B)$ this extends to a bounded in ${}^*\mathbb{R}_c^\#$operator $\pi(A) \in B(L_2^\#(K,\omega))$ Then we have that the map $\pi{:}K \to B(L_2^\#(K,\omega)), A \mapsto \pi(A)$.

Is a homomorphism clearly but moreover for all $C,D \in K, ([C]\omega|\pi(A^*)[D]\omega) =$
$= \omega(D^*A^*C) = (\pi(A)[C]\omega|[D]\omega)$. And so π is a $*$-homomorphism. Also, since $1_\omega = [1]_\omega \in K/J_\omega$, we have $(\pi(A)1_\omega \mid 1_\omega) = \omega(1_\omega^* A 1_\omega) = \omega(A)$.

Theorem 2.33. (The non-Archimedean GNS construction)

Let K be a $\mathbf{C}_\#^*$ algebra. For every state $\omega \in \mathbf{S}^\#(K)$ then there is a non-Archimedean Hilbert space $L_2^\#(K,\omega)$ and a unique (up to equivalence) representation $\pi{:}K \to B(L_2^\#(K,\omega))$ such that $\omega(A) = (\pi[A]1_\omega \mid 1_\omega), A \in K$.

Where $1_\omega \in L_2^\#(K,\omega)$ is the unit cyclic vector.

Proof. By **lemma 2.32** it remains to show uniqueness.

Say $\rho{:}A \to B(H^\#)$ is representation with $\imath \in H^\#$ cyclic and $\omega(A) = \langle \rho(A)\imath, \imath\rangle_\#$ then we can consider the map $U_0{:}K/J_\omega \to H^\#, [A] \mapsto \rho(A)\imath$. We then would have

$(U_0(A), U_0(B)) = (\rho(A)\imath, \rho(B)\imath) = (\rho(B^*A)\imath, \imath) = \omega(B^*A) = ([A]|[B])$.

So U_0 is well defined, and an isometry. Furthermore for any $A,B \in K$ we have

$U_0(\pi(A)[B]\omega) = U_0([AB]\omega) = \rho(AB)\imath = \rho(A)U_0([B])$.

So U_0 extends to an isometry

$U{:}L_2^\#(K,\omega) \to H^\#$ suth that $\forall A \in K : U\pi(A) = \rho(A)U$

Since $\imath$ is cyclic , and we have $\rho(K)\imath \subset U(L_2^\#(K,\omega))$ it follows that U must be unitary.

The following cororallary tells us that we can think of any $\mathbf{C}_\#^*$ algebra as a subset of $B(H^\#)$ for some $H^\#$.

Corollary 2.33.1. Let K be a $\mathbf{C}_{\#}^{*}$ algebra. Then there exists a faithful representation of K.

Proof. Let π be the direct sum over all GNS representations corresponding to states. Then by **lemma 2.21** this representation is faithful.

This result is very deep, and shows that there is a one to one correspondence.

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