Basic non-Archimedean functional analysis over non-Archimedean field ${}^{*}\mathbb{R}_{c}^{\#}$. Application to constructive quantum field theory

Jaykov Foukzon

Center for Mathematical Sciences, Israel Institution of Technology, Haifa, Israel.
E-mail: jaykovfoukzon@list.ru
E-mail: jaikovfoukzon@gmail.com

https://orcid.org/0000-0002-6837-4004

Abstract. Definitions and theorems related to non-Archimedean functional analysis on non-Archemedean field ${}^{*}\mathbb{R}_{c}^{\#}$ and on complex field ${}^{*}\mathbb{C}_{c}^{\#} = {}^{*}\mathbb{R}_{c}^{\#} + i{}^{*}\mathbb{R}_{c}^{\#}$are considered.Definitions and theorems appropriate to analysis on non-Archemedean field ${}^{*}\mathbb{R}_{c}^{\#}$ and on complex field ${}^{*}\mathbb{C}_{c}^{\#} = {}^{*}\mathbb{R}_{c}^{\#} + i{}^{*}\mathbb{R}_{c}^{\#}$are given in [1]-[2]

Content.

Chapter I. ${}^{*}\mathbb{R}_{c}^{\#}$-Valued abstract measures

1.σ$^{\#}$-algebras

Definition 1.1 ($\sigma^{\#}$-algebra). Let X be any set. We denote by $2^{X} = P(X) = \{A : A \subset X\}$ the set of all subsets of X. A family $\mathcal{F} \subset 2^{X}$ is called a $\sigma^{\#}$-algebra (on X) if:
(i) $\varnothing \in \mathcal{F}$;
(ii) $\mathcal{F}$ is closed under complements, i.e. $A \in \mathcal{F}$ implies $X \backslash A \in \mathcal{F}$;
(iii) $\mathcal{F}$ is closed under hypercountable unions, i.e. if $(A_n)_{n\in\mathbb{N}^{\#}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcup_{n\in\mathbb{N}^{\#}} A_n \in \mathcal{F}$.

Proposition 1.1.If $\mathcal{F}$ is a $\sigma^{\#}$-algebra on X then:
1. $\mathcal{F}$ is closed under hypercountable intersections, i.e. if $(A_n)_{n\in\mathbb{N}^{\#}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcap\limits_{n\in\mathbb{N}^{\#}} A_n \in \mathcal{F}$.
2. $X \in \mathcal{F}$.
3. $\mathcal{F}$ is closed under hyperfinite unions and hyperfinite intersections.
4. $\mathcal{F}$ is closed under set differences.
5. $\mathcal{F}$ is closed under symmetric differences.

Proposition 1.2.Suppose $\mathcal{F} \subset 2^{X}$ is a family of subsets satisfying the following:
1. $\varnothing \in \mathcal{F}$;
2. $\mathcal{F}$ is closed under complements;
3. $\mathcal{F}$ is closed under hyperinfinite intersections.
Then $\mathcal{F}$ is a $\sigma^{\#}$-algebra.

Proposition 1.3.If $(\mathcal{F}_{\alpha})_{\alpha\in I}$ is a collection of $\sigma^{\#}$-algebras on X, then $\bigcap\limits_{\alpha}\mathcal{F}_{\alpha}$ is also a $\sigma^{\#}$-algebra on X.

Proposition 1.4.($\sigma^{\#}$-algebra generated by subsets). Let K be a collection of subsets of X. There exists a $\sigma^{\#}$-algebra, denoted $\sigma^{\#}(K)$ such that $K \subset \sigma^{\#}(K)$ and for every other $\sigma^{\#}$ algebra $\mathcal{F}$ such that $K \subset \mathcal{F}$ we have that $\sigma^{\#}(K) \subset \mathcal{F}$
We call $\sigma^{\#}(K)$ the $\sigma^{\#}$-algebra generated by K.
Proof. Define $\sigma^{\#}(K) \triangleq \bigcap\{\mathcal{F}|\mathcal{F}$ is a $\sigma^{\#}$-algebra on $X, K \subset \mathcal{F}\}$.
This is a $\sigma^{\#}$-algebra with the required properties.
Proposition 1.5.If $K \subset \mathcal{L}$ then $\sigma^{\#}(K) \subset \sigma^{\#}(\mathcal{L})$. Also, if $K \subset \mathcal{F}$ and $\mathcal{F}$ is a $\sigma^{\#}$-algebra, then $\sigma^{\#}(K) \subset \mathcal{F}$.
Definition 1.2. (Borel $\sigma^{\#}$-algebra). Given a topological space X, the Borel $\sigma^{\#}$-algebra is the $\sigma^{\#}$-algebra generated by the open sets. It is denoted $B^{\#}(X)$.
Specifically in the case $X = {}^{*}\mathbb{R}_c^{\#d}, d \in \mathbb{N}^{\#}$we have that
$B_d^{\#} \triangleq B^{\#}({}^{*}\mathbb{R}_c^{\#d}) = \sigma^{\#}(U|U$ is an #-open set).
A Borel-#-measurable set, i.e. a set in $B^{\#}(X)$, is called a #-Borel set.
Measurable functions. Let f be a ${}^{*}\mathbb{R}_c^{\#}$-valued function defined on a set X. We suppose that some $\sigma^{\#}$-algebra $\Omega \subseteq P(X)$ is fixed.
Definition 1.3. We say that f is #-measurable, if $f^{-1}([a,b]) \in \Omega$ for any hyperreals $a,b \in {}^{*}\mathbb{R}_c^{\#}$ such that $a < b$.
The following three propositions are obvious.
Proposition 1.7. Let $f : X \to {}^{*}\mathbb{R}_c^{\#}$ be a function. Then the following conditions are equivalent:
(a) f is #-measurable;
(b) $f^{-1}([0,b)) \in \Omega$ for any hyperreal $b \in {}^{*}\mathbb{R}_c^{\#}$;
(c) $f^{-1}((b,\infty)) \in \Omega$ for any hyperreal $b \in {}^{*}\mathbb{R}_c^{\#}$;
(d) $f^{-1}(B) \in \Omega$ for any $B \in B(R)$.
Proposition 1.8 Let f and g be #-measurable functions, then
(a) $\alpha \times f + \beta \times g$ is #-measurable for any $\alpha,\beta \in {}^{*}\mathbb{R}_c^{\#}$;
(b) functions $\max\{f,g\}$ and $f \times g$ are #-measurable.
In particular, functions $f^{+} := \max\{f,0\}, f^{-} := (-f)^{+}$, and $|f| := f^{+} + f^{-}$ are #-measurable. .

§2. #-Measures and measure #-space

Definition 2.1. A pair $(X,\mathcal{F})$ where $\mathcal{F}$ is a $\sigma^{\#}$-algebra on X is call a #-measurable space. Elements of $\mathcal{F}$ are called #-measurable sets.
Given a #-measurable space $(X,\mathcal{F})$, a function $\mu^{\#} : \mathcal{F} \to [0,\infty^{\#}]$ is called a #-measure on $(X,\mathcal{F})$ if
1. $\mu^{\#}(\varnothing) = 0$;
2. (Hyper infinite additivity) For all hyper infinite sequences $(A_n)_{n\in\mathbb{N}^{\#}} \subset \mathcal{F}$ of pairwise disjoint sets in $\mathcal{F}$, we have that $\mu^{\#}\left(\bigcup_{n\in\mathbb{N}^{\#}} A_n\right) = Ext\text{-}\sum_{n\in\mathbb{N}^{\#}} \mu^{\#}(A_n)$.
$(X,\mathcal{F},\mu^{\#})$ is called a #-measure space.
Definition 2.2. A measure space $(X,\mathcal{F},\mu^{\#})$ is called: (a) hyperfinite if $\mu^{\#}(X) < \infty^{\#}$.
(b)It is called $\sigma^{\#}$-hyperfinite if $X = \bigcup_{n\in\mathbb{N}^{\#}} A_n$ where $A_n \in \mathcal{F}$ and $\mu^{\#}(A_n) < \infty^{\#}$ for all $n \in \mathbb{N}^{\#}$.
Definition 2.3. Let Σ be a $\sigma^{\#}$-algebra of subsets of a set X, and let $E = (E, \|\cdot\|_{\#})$ be a non-Archimedean Banach space.A function $\mu^{\#} : \Sigma \to E \cup \{{}^{*}\infty\}$ is called a

vector-valued #-measure (or E-valued measure) if

(a) $\mu^{\#}(\varnothing) = 0$;

(b) $\mu^{\#}\left(\bigcup_{n\in\mathbb{N}^{\#}} A_n\right) = Ext\text{-}\sum_{n\in\mathbb{N}^{\#}} \mu^{\#}(A_n)$ for any pairwise disjoint sequence $A_n, n \in \mathbb{N}^{\#}$, $A_n \subseteq \Sigma$;

(c) for any $S \in \Sigma,\ \mu^{\#}(S) = \infty,$ there exists $B \in \Sigma$ such that $B \subseteq S$ and $0 < \|\mu^{\#}(B)\|_{\#} < {}^{*}\infty$.

Definition 2.4.(a) A function $\mu^{\#} : \mathcal{F} \to {}^{*}\mathbb{C}_c^{\#} \cup \{{}^{*}\infty\}$ is called a complex #-measure if

1.$\mu^{\#}(\emptyset) = 0$,

2.$\mu^{\#}\left(\bigcup_{n\in\mathbb{N}^{\#}} A_n\right) = Ext\text{-}\sum_{n\in\mathbb{N}^{\#}} \mu^{\#}(A_n)$ for any sequence $A_n, n \in \mathbb{N}^{\#}$ of pairwise disjoint sets from $\mathcal{F}$, and, for any $A \in \mathcal{F}, \mu^{\#}(A) = {}^{*}\infty$, there exists $B \in \mathcal{F}$ such that $B \subseteq A$ and $0 < |\mu^{\#}(B)|_{\#} < {}^{*}\infty$.

(b) A function $\mu^{\#} : \mathcal{F} \to {}^{*}\mathbb{R}_c^{\#} \cup \{{}^{*}\infty\}$ is called a signed #-measure if

$\mu^{\#}(\emptyset) = 0$

$\mu^{\#}\left(\bigcup_{n\in\mathbb{N}^{\#}} A_n\right) = Ext\text{-}\sum_{n\in\mathbb{N}^{\#}} \mu^{\#}(A_n)$ for any sequence $A_n, n \in \mathbb{N}^{\#}$ of pairwise disjoint sets from $\mathcal{F}$, , and, for any $A \in \mathcal{F}, \mu^{\#}(A) = {}^{*}\infty,$ there exists $B \in \mathcal{F}$ such that $B \subseteq A$ and $0 < |\mu^{\#}(B)| < {}^{*}\infty$.

Definition 2.5. If a certain property involving the points of #-measure space is true, except a subset having #-measure zero, then we say that this property is true #-almost everywhere (abbreviated as #-a.e.).

Proposition 2.5. Let $\mu^{\#}$ be a #-measure on a $\sigma^{\#}$-algebra $\mathcal{F}, A_n \in \mathcal{F}$, and $A_n \to A$. Then $A \in \mathcal{F}$ and $\mu^{\#}(A) = \#\text{-}\lim_{n\to{}^{*}\infty} \mu^{\#}(A_n)$. In particular, if $(B_n)_{n=1}^{{}^{*}\infty}$ is a decreasing hyper infinite sequence of elements of $\mathcal{F}$ such that $\bigcap_{n=1}^{{}^{*}\infty} B_n = \emptyset$, then $\mu^{\#}(B_n) \to_{\#} 0$.

Definition 2.6. If $\mathcal{F}$ is a $\sigma^{\#}$-algebra of subsets of X and $\mu^{\#}$ is a #-measure on $\mathcal{F}$, then the triple $(X, \mathcal{F}, \mu)$ is called a #-measure space. The sets belonging to $\mathcal{F}$ are called #-measurable sets because the #-measure is defined for them.

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§2.1.#-Convergence of functions and the generalized Egoroff theorem.

Definition 2.1.1. Let $f_n, n \in \mathbb{N}^{\#}$ be a hyper infinite sequence of ${}^{*}\mathbb{R}_c^{\#}$-valued functions defined on X. We say that:

1. $f_n \to_{\#} f$ pointwise, if $f_n(x) \to_{\#} f(x)$ for all $x \in X$;
2. $f_n \to_{\#} f$ almost #-everywhere (#-a.e.), if $f_n(x) \to_{\#} f(x)$ for all $x \in X$ except a set of #-measure 0;
3. $f_n \to_{\#} f$ uniformly, if for any $\varepsilon > 0, \varepsilon \approx 0$ there is $n(\varepsilon)$ such that $\sup\{|f_n(x) - f(x)| : x \in X\} \leq \varepsilon$ for all $n \geq n(\varepsilon)$.

Theorem 2.1.1. (generalized Egoroff 's theorem) Suppose that $\mu^{\#}(X) < {}^{*}\infty$, $\{f_n\}_{n=1}^{{}^{*}\infty}$ and f are #-measurable functions on X such that $f_n \to_{\#} f$ #-a.e. Then, for every $\varepsilon \approx 0, \varepsilon > 0,$ there exists $E \subseteq X$ such that $\mu^{\#}(E) < \varepsilon$ and $f_n \to_{\#} f$ uniformly on $E^c = X \backslash E$.

Proof: Without loss of generality, we may assume that $f_n \to_\# f$ everywhere on X and (by replacing f_n with $f_n - f$) that $f \equiv 0$. For $k,\ n \in {}^*\mathbb{N}$, let $E_n(k) := \bigcup_{m=n}^{^*\infty}\{x : |f_m(x)| \geq k-1\}$. Then, for a fixed $k, E_n(k)$ decreases as n increases, and $\bigcap_{n=1}^{^*\infty} E_n(k) = \emptyset$. Since $\mu^\#(X) < {}^*\infty$, we conclude that $\mu^\#(E_n(k)) \ \to_\# 0$ as $n \to {}^*\infty$. Given $\varepsilon \approx 0, \varepsilon > 0$ and ${}^*\mathbb{N}$, choose n_k such that $\mu^\#(En_k(k)) \ < \varepsilon \times 2^{-k}$, and set $E = \bigcup_{n=1}^{^*\infty} En_k(k)$. Then $\mu^\#(E) < \varepsilon$, and we have $|f_n(x)| < k^{-1}(\forall n > n_k, x \notin E)$. Thus $f_n \to_\# 0$ uniformly on $X \backslash E$.

Generalized exhaustion argument.

Let $(X,\Sigma,\mu^\#)$ be a $\sigma^\#$-finite #-measure space. Given a hyper infinite sequence $(U_n)_{n=1}^{^*\infty} \subseteq \Sigma$, a set $A \in \Sigma$ is called $(U_n)_n$-bounded if there exists $n \in {}^*\mathbb{N}$ such that $A \subseteq U_n$ $\mu^\#$-almost everywhere.

Theorem 2.1.2. (Generalized Exhaustion theorem) Let $(Y_n)_{n=1}^{^*\infty} \subseteq \Sigma$ be a hyper infinite sequence satisfying $Y_n \uparrow X$ and $\mu^\#(Y_n) < {}^*\infty$ for all $n \in {}^*\mathbb{N}$.
Let P be some property of $(Y_n)_n$-bounded #-measurable sets, such that $A \in P$ iff $B \in P$ for all $B, \mu^\#(A\Delta B) = 0$. Suppose that any $(Y_n)_n$-bounded set $A,\ \mu^\#(A) > 0$, has a subset $B \in \Sigma, \mu^\#(B) > 0$ with the property P. Moreover, assume that either
(a) $A_1 \cup A_2 \in P$ for every $A_1, A_2 \in P$, or
(b) $\cup_{n\in {}^*\mathbb{N}} B_n \in P$ for every at most hyper infinite family $(B_n)_n$ of pairwise disjoint sets possessing the property P.
Then there exists hyper infinite sequence $(X_n)_{n=1}^{^*\infty} \subseteq \Sigma$ such that $X_n \uparrow X$, and $P \ni X_n \subseteq Y_n$
for all $n \in {}^*\mathbb{N}$. Moreover, there exists a pairwise disjoint sequence $(A_n)_{n=1}^{^*\infty} \subseteq \Sigma$ such that $\cup_{n\in {}^*\mathbb{N}} A_n = X$ and $A_n \in P$ for all $n \in {}^*\mathbb{N}$.

Proof: Let A be a $(Y_n)_n$-bounded set with $\mu^\#(A) > 0$. Denote $P_A := \{B \in P : B \subseteq A\} \bigwedge m(A) := \sup\{\mu^\#(B) : B \in P_A\}$.

I(a) Suppose P satisfies (a). Then there exists a sequence $(F_n)_{n=1}^{^*\infty} \subseteq P_A$ such that $m(A) = \#\text{-}\lim_{n\to {}^*\infty} \mu^\#(F_n)$, We may assume, that $F_n \uparrow$. By Proposition 2.5 the set $F = \cup_{n=1}^{^*\infty} F_n$ satisfies $\mu^\#(F) = m(A)$. We show that $\mu^\#(A) = m(A)$. If not then $\mu^\#(A \backslash F) > 0$. The set $A \backslash F$ has a subset of positive #-measure $F_0 \in P$. Then $F_n \cup F_0 \in P_A$ and $\mu^\#(F_n \cup F_0) > m(A)$ for a sufficiently large $n \in {}^*\mathbb{N}$, which contradicts to the definition of $m(A)$. Therefore, $\mu^\#(A) = m(A)$.
Now we apply this for $A = Y_n$. Thus, there exists hyper infinite sequence $(X'_n)_n \subseteq \Sigma$ such that $X'_n \subseteq Y_n,\ X'_n, n \in P$, and $\mu^\#(Y_n \backslash X'_n) < n^{-1}$for all $n \in {}^*\mathbb{N}$. By (a), we may assume that $X'_n \uparrow$. The set $X'_0 = \cup_{n=1}^{^*\infty} X'_n$ satisfies $Y_n \backslash X'_0 \subseteq Y_n \backslash X'_n$, so $\mu^\#(Y_n \backslash X'_0) < n^{-1}$ for all $n \in {}^*\mathbb{N}$. Then $\mu^\#(Y_n \backslash X'_0) = 0$, and $\mu^\#((\cup_{n=1}^{^*\infty} Y_n) \backslash X'_0) = 0$, or $\mu^\#(X \backslash X'_0) = 0$.
Let $X_n = (X'_n \cup (X \backslash X'_0) \cap Y_n$, then the hyper infinite sequence $(X_n)_n$ has the required properties. The desired pairwise disjoint sequence $(A_n)_{n=1}^{^*\infty}$ is given recurrently by $A_1 = X_1$and $A_{k+1} = X_{k+1} \backslash \cup_{i=1}^k A_i$.

I(b) Suppose P satisfies (b). Let F_A be the family of all pairwise disjoint families of elements of P_A of nonzero measure. Then F_A is ordered by inclusion

and, obviously, satisfies the conditions of the Zorn lemma. Therefore, we have a maximal element in F_A, say Δ. Then Δ is at most hyper infinite family, say $\Delta = \{D_n\}_n$. By (b), its union $D = \cup_n D_n$ is an element of P_A as well. If D is a proper subset of A, then $\mu^{\#}(A \backslash D) > 0$. The set $A \backslash D$ has a subset $F \in P$ of the positive measure. Then $\Delta_1 := \Delta \cup \{F\}$ is an element of F_A which is strictly greater then Δ. The obtained contradiction, shows that $A \in P$ for every $(Y_n)_n$-bounded set A. So, we may take $X_n = Y_n$ for each $n \in {}^*\mathbb{N}$.
Now we apply this for $A = Z_m = Y_m \backslash \cup_{k=1}^{m-1} Y_k$ be a pairwise disjoint union, where $D_n^m \in P$ for all $n,m \in {}^*\mathbb{N}$. The family $\{D_n^m\}_{n,m}$ is an at most hyper infinite disjoint decomposition of X, say $\{D_n^m\}_{n,m} = (A_n)_{n=1}^{*\infty}$. The sequence $(A_n)_{n=1}^{*\infty}$ satisfies the required properties.
Theorem 2.1.3.(The generalized Borel-Cantelli lemma) Let $(X,\Sigma,\mu^{\#})$ be a #-measure space. Assume that $\{A_n\}_n \subseteq \Sigma$ and $Ext\text{-}\sum_{n=1}^{*\infty} \mu(A_n) < {}^*\infty$ then $\limsup_{n\to{}^*\infty} \mu^{\#}(A_n) = 0$.   .

§2.2.Vector-valued #-measures

In this section, we extend the notion of a measure. Then we study the basic operations with signed measures and present the Jordan decomposition theorem.

2.2.1. Vector-valued, signed and complex #-measures.

Let $\Sigma^{\#}$ be a $\sigma^{\#}$-algebra of subsets of a set X, and let $E^{\#} = (E^{\#}, \|\cdot\|_{\#})$ be a non-Archimedean Banach space.
Definition 2.2.1 A function $\mu^{\#} : \Sigma^{\#} \to E^{\#} \cup \{{}^*\infty\}$ is called a vector-valued #-measure (or $E^{\#}$-valued measure) if
(a) $\mu^{\#}(\emptyset) = 0$;
(b) $\mu^{\#}(\cup_{k=1}^{*\infty} A_k) = Ext\text{-}\sum_{k=1}^{*\infty} \mu^{\#}(A_k)$ for any pairwise disjoint sequence $(A_k)_k \subseteq \Sigma^{\#}$;
(c) for any $A \in \Sigma^{\#}, \mu^{\#}(A) = {}^*\infty$, there exists $B \in \Sigma^{\#}$ such that $B \subseteq A$ and $0 < \|\mu^{\#}(B)\|_{\#} < {}^*\infty$.
Example 2.2.1 Take $\Sigma^{\#} = P({}^*\mathbb{N})$, and $c_0^{\#}$ is the non-Archimedean Banach space of all #-convergent $\mathbb{C}_c^{\#}$-valuedhyper infinite sequences with a fixed element $(\alpha_n)_n \in c_0^{\#}$. Define now for any $A \subseteq \mathbb{N}\psi(A) := (\beta_n)_n$, where $\beta_n = \alpha_n$ if $n \in A$ and $\beta_n = 0$ if $n \notin A$. Then ψ is a $c_0^{\#}$-valued #-measure on $P({}^*\mathbb{N})$.
Example 2.2.2 Let X be a set and let Ω be a $\sigma^{\#}$-algebra in $P(X)$. Then for any family $\{\mu_k\}_{k=1}^m$of finite #-measures on Ω and for any family $\{w_k\}_{k=1}^m$ of vectors of $\mathbb{R}_c^{\#n}$, the $\mathbb{R}_c^{\#n}$-valued #-measure Ψ on Ω is defined by the formula $\Psi(E) = Ext\text{-}\sum_{k=1}^m \mu_k(E) \times w_k, (E \in \Omega)$.
Example2.2.3 Let X be a set and let Ω be a $\sigma^{\#}$-algebra in $P(X)$. Then for any family $\{\mu_k\}_{k=1}^m$of finite #-measures on Ω, for any family $\{A_k\}_{k=1}^m$ of pairwise disjoint sets in Ω, and for any family $\{w_k\}_{k=1}^m$ of $\mathbb{R}_c^{\#n}, n \in {}^*\mathbb{N}$, the $\mathbb{R}_c^{\#n}$-valued #-measure Φ on Ω is defined by the formula $\Phi(E) = Ext\text{-}\sum_{k=1}^m \mu_k(E \cap A_k) \times w_k$, $(E \in \Omega)$.

§3. The Lebesgue #-Integral

In the following consideration, we fix a $\sigma^{\#}$-finite #-measure space $(X,\mathcal{F},\mu^{\#})$.
Definition 3.1.Let $A_i \in \mathcal{F}, i = 1,\ldots,n \in {}^*\mathbb{N}$, be such that $\mu^{\#}(A_i) < {}^*\infty$ for all i, and

$A_i \cap A_j = \emptyset$ for all $i \neq j$. The external function

$$f(x) = Ext\text{-}\sum_{i=1}^{n} \lambda_i \chi_{A_i}(x), \tag{3.1}$$

$\lambda_i \in {}^*\mathbb{R}_c^\#$, is called a simple external function. The Lebesgue external integral (Lebesgue #-integral) of a simple external function $f(x)$ is defined as

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = Ext\text{-}\sum_{i=1}^{n} \lambda_i \mu^\#(A_i). \tag{3.2}$$

The Lebesgue external integral of a simple function is well defined.

Notation 3.1.Let $A_i \in \mathcal{F}, i = 1,\ldots,n \in {}^*\mathbb{N}$, be such that $\mu^\#(A_i) < {}^*\infty$ for all i, and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Let $f_1(x), f_2(x)$ be a simple external function such that (i) $0 \leq f_1(x) \leq f_2(x)$ and (ii) $f_1(x) = Ext\text{-}\sum_{i=1}^{n} \lambda_{1,i}\chi_{A_i}(x), f_2(x) = Ext\text{-}\sum_{i=1}^{n} \lambda_{2,i}\chi_{A_i}(x)$.

$$Ext\text{-}\sum_{i=1}^{n} \lambda_{1,i} \leq Ext\text{-}\sum_{i=1}^{n} \lambda_{2,i}, \tag{3.3}$$

then we will write $f_1(x) \leq_s f_2(x)$.

Definition 3.2. Suppose that $\mu^\#$ is hyperfinite. Let $f : X \to {}^*\mathbb{R}_c^\#$ be an arbitrary nonnegative bounded in ${}^*\mathbb{R}_c^\#$ #-measurable external function and let $(f_n)_{n\in{}^*\mathbb{N}}$, be a hyper infinite sequence of simple external functions which #-converges uniformly to f. Then the Lebesgue #-integral of f is

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = \#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_X f_n(x)d^\#\mu^\#\right). \tag{3.4}$$

Remark 3.1.It can be easily shown that the #-limit in Definition 3.2 exists and does not depend on the choice of a hyper infinite sequence $(f_n)_{n\in{}^*\mathbb{N}}$, and moreover, the hyper infinite sequence $(f_n)_{n\in{}^*\mathbb{N}}$can be chosen such that $0 \leq f_n \leq f$ for all $n \in {}^*\mathbb{N}$.

Notation 3.2.Let $f_1 : X \to {}^*\mathbb{R}_c^\#$ and $f_2 : X \to {}^*\mathbb{R}_c^\#$ be an arbitrary nonnegative bounded in ${}^*\mathbb{R}_c^\#$ #-measurable external functions and let $(f_{1,n})_{n\in{}^*\mathbb{N}}$ and $(f_{2,n})_{n\in{}^*\mathbb{N}}$be a hyper infinite sequences of simple external functions which #-converges uniformly to f_1 and to f_2 correspondingly. We assume that for all $n \in {}^*\mathbb{N}$ the inequality (3.3) is satisfied, then we will write $f_1(x) \leq_s f_2(x)$.

Definition 3.3. Let $f : X \to {}^*\mathbb{R}_c^\#$ be a #-measurable function. Then the Lebesgue #-integral of f is defined by

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = Ext\text{-}\int_X f^+(x)d^\#\mu^\# - Ext\text{-}\int_X f^-(x)d^\#\mu^\#. \tag{3.5}$$

If both of these terms are finite or hyperfinite then the function f is called #-integrable. In this case we write $f \in L_1^\# = L_1^\#(X,\mathcal{F},\mu^\#)$.

Notation 3.3.We will use the following notation. For any $A \in \mathcal{F}$:

$$Ext\text{-}\int_A f(x)d^\#\mu^\# = Ext\text{-}\int_X f(x)\chi_A(x)d^\#\mu^\#. \tag{3.6}$$

Lemma 3.1.(1) Let $f : X \to {}^*\mathbb{R}_c^\#$ be an arbitrary nonnegative #-measurable function then

$$\begin{gathered} Ext\text{-}\int_X f(x)d^\#\mu^\# = \\ \sup\left\{Ext\text{-}\int_X \varphi(x)d^\#\mu^\# \,\middle|\, \varphi \text{ is a simple function such that } 0 \leq \varphi(x) \leq_s f(x)\right\}. \end{gathered} \tag{3.7}$$

(2) If $f,g : X \to {}^*\mathbb{R}_c^\#$ are #-measurable, g is #-integrable, and $|f(x)| \leq_s g(x)$, then f

is #-integrable and

$$\left|Ext\text{-}\int_X f(x)d^{\#}\mu^{\#}\right| \leq Ext\text{-}\int_X g(x)d^{\#}\mu^{\#}. \tag{3.8}$$

(3) $Ext\text{-}\int_X |f(x)|d^{\#}\mu^{\#} = 0$ if and only if $f(x) = 0$ #-a.e.

(4) If $f_1, f_2, \ldots f_n : X \to {}^*\mathbb{R}_c^{\#}, n \in {}^*\mathbb{N}$ are integrable then, for $\lambda_1, \lambda_2, \ldots, \lambda_n \in {}^*\mathbb{R}_c^{\#}$, the linear combination $Ext\text{-}\sum_{i=1}^n \lambda_i f_i$ is #-integrable and

$$Ext\text{-}\int_X \left(Ext\text{-}\sum_{i=1}^n \lambda_i f_i\right) d^{\#}\mu^{\#} = Ext\text{-}\sum_{i=1}^n \left(Ext\text{-}\int_X \lambda_i f_i d^{\#}\mu^{\#}\right). \tag{3.9}$$

(5) Let $f \in L_1^{\#}(X, \mathcal{F}, \mu^{\#})$, then the formula

$$\nu^{\#}(A) = Ext\text{-}\int_A f(x)d^{\#}\mu^{\#} = Ext\text{-}\int_X f(x)\chi_A(x)d^{\#}\mu^{\#} \tag{3.10}$$

defines a signed #-measure on the $\sigma^{\#}$-algebra $\mathcal{F}$.

Remark 3.2. Assume that $f, g : X \to {}^*\mathbb{R}_c^{\#}$ are #-integrable functions and such that $0 \leq f \leq_s g$ #-a.e., then

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} \leq Ext\text{-}\int_X g(x)d^{\#}\mu^{\#}.$$

#-Convergence theorem

Definition 3.4. We say that a hyper infinite sequence $\{f_n\}_{n=1}^{*\infty}$ of #-integrable functions $L_1^{\#}$-#-converges to f (or #-converges in $L_1^{\#}(X, \mathcal{F}, \mu^{\#})$) if

$$Ext\text{-}\int_X |f_n - f| d^{\#}\mu^{\#} \to_{\#} 0 \text{ as } n \to {}^*\infty. \tag{3.11}$$

Theorem 3.1 (The monotone #-convergence theorem) If $\{f_n\}_{n=1}^{*\infty}$ is a hyper infinite sequence in $L_1^{\#+}(X, \mathcal{F}, \mu^{\#})$ such that $f_j \leq_s f_{j+1}$ for all j and $f(x) = \sup_{n \in {}^*\mathbb{N}} f_n(x)$ then

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} = \#\text{-}\lim_{n \to {}^*\infty} Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}. \tag{3.12}$$

Proof: The #-limit of the increasing sequence

$$\left(Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}\right)_{n=1}^{*\infty}$$

($*$-finite or $*$-infinite) exists. Moreover by (3.2),

$$Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#} \leq Ext\text{-}\int_X f(x)d^{\#}\mu^{\#}$$

for all $n \in {}^*\mathbb{N}$, so

$$\#\text{-}\lim_{n \to {}^*\infty}\left(Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}\right) \leq Ext\text{-}\int_X f(x)d^{\#}\mu^{\#}.$$

To establish the reverse inequality, fix $\alpha \in (0,1)$, let φ be a simple function with $0 \leq \varphi \leq f$, and let $E_n = \{x : f_n(x) \geq \alpha\varphi(x)\}$. Then $(E_n)_{n=1}^{*\infty}$ is an increasing hyper infinite sequence of #-measurable sets whose union is X, and we have

$$Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#} \geq Ext\text{-}\int_{E_n} f_n(x)d^{\#}\mu^{\#} \geq \alpha\left(Ext\text{-}\int_{E_n} \varphi(x)d^{\#}\mu^{\#}\right) \tag{3.13}$$

By (3.10) and by Proposition 2.5,

$$\#\text{-}\lim_{n \to {}^*\infty}\left(Ext\text{-}\int_{E_n} \varphi(x)d^{\#}\mu^{\#}\right) = Ext\text{-}\int_X \varphi(x)d^{\#}\mu^{\#}, \tag{3.14}$$

and hence

$$\#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_{E_n} f_n(x)d^\#\mu^\#\right) \geq \alpha\left(Ext\text{-}\int_X \varphi(x)d^\#\mu^\#\right). \tag{3.15}$$

Since this is true for all $\alpha, 0 < \alpha < 1$, it remains true for $\alpha = 1$:

$$\#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_{E_n} f_n(x)d^\#\mu^\#\right) \geq Ext\text{-}\int_X \varphi(x)d^\#\mu^\#. \tag{3.16}$$

Using Lemma 3.1.(1), we may take the supremum over all simple functions φ, $0 \leq \varphi \leq_s f$. Thus

$$\#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\int_{E_n} f_n(x)d^\#\mu^\#\right) \geq Ext\text{-}\int_X f(x)d^\#\mu^\#. \tag{3.17}$$

Proofs of the following two corollaries of Theorem 3.1 are straightforward.

Corollary 3.1 If $(f_n)_{n=1}^{*\infty}$ is a hyper infinite sequence in $L_+^1(X)$ and $f = Ext\text{-}\sum_{n=1}^{*\infty} f_n$ pointwise then

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = Ext\text{-}\sum_{n=1}^{*\infty}\left(Ext\text{-}\int_X f_n(x)d^\#\mu^\#\right). \tag{3.18}$$

Corollary 3.2 If $(f_n)_{n=1}^{*\infty}$ is a hyper infinite sequence in $L_+^1(X)$, f $\in L_+^1(X)$, and $f_n \to_\# f$ $\mu^\#$-a.e., then

$$Ext\text{-}\int_X f_n(x)d^\#\mu^\# \to_\# Ext\text{-}\int_X f(x)d^\#\mu^\#. \tag{3.19}$$

Theorem 3.2 (Generalized Fatou's lemma) If $(f_n)_{n=1}^{*\infty}$is any hyper infinite sequence in $L_+^1(X)$ then

$$Ext\text{-}\int_X \#\text{-}\liminf_{n\to{}^*\infty}(f_n(x))d^\#\mu^\# \leq \#\text{-}\liminf_{n\to{}^*\infty}\left(Ext\text{-}\int_X f_n(x)d^\#\mu^\#\right). \tag{3.20}$$

Theorem 3.3 (The dominated #-convergence theorem) Let f and g be #-measurable, let f_n be #-measurable for any $n \in {}^*\mathbb{N}$ such that $|f_n(x)| \leq_s g(x)$ #-a.e., and $f_n \to_\# f$ #-a.e. If g is #-integrable then f and f_n are also #-integrable and

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = \#\text{-}\lim_{n\to{}^*\infty} Ext\text{-}\int_X f_n(x)d^\#\mu^\#. \tag{3.21}$$

Proof: f is #-measurable and, since $|f| \leq_s g$ $\mu^\#$-a.e., we have $f \in L_+^1(X)$. We have that $g + f_n \geq 0$ $\mu^\#$-a.e. and $g - f_n \geq 0$ so, by Fatou's lemma,

$$\begin{gathered}
Ext\text{-}\int_X gd^\#\mu^\# + Ext\text{-}\int_X fd^\#\mu^\# \leq \#\text{-}\liminf_{n\to{}^*\infty}\left(Ext\text{-}\int_X [g+f_n]d^\#\mu^\#\right) = \\
Ext\text{-}\int_X gd^\#\mu^\# + \#\text{-}\liminf_{n\to{}^*\infty}\left(Ext\text{-}\int_X f_nd^\#\mu^\#\right), \\
Ext\text{-}\int_X gd^\#\mu^\# - Ext\text{-}\int_X fd^\#\mu^\# \leq \#\text{-}\liminf_{n\to{}^*\infty}\left(Ext\text{-}\int_X [g-f_n]d^\#\mu^\#\right) = \\
= Ext\text{-}\int_X gd^\#\mu^\# - \#\text{-}\limsup_{n\to{}^*\infty}\left(Ext\text{-}\int_X f_nd^\#\mu^\#\right)
\end{gathered} \tag{3.22}$$

Therefore

$$\#\text{-}\liminf_{n\to{}^*\infty}\left(Ext\text{-}\int_X f_nd^\#\mu^\#\right) \geq Ext\text{-}\int_X fd^\#\mu^\# \geq \#\text{-}\limsup_{n\to{}^*\infty}\left(Ext\text{-}\int_X f_nd^\#\mu^\#\right) \tag{3.23}$$

and the required result follows from (3.23).

§ 4. #-Convergence in #-measure.

Definition 4.1. We say that a hyper infinite sequence $(f_n)_{n=1}^{*\infty}$ of #-measurable functions on $(X, M, \mu^\#)$ is Cauchy in #-measure if, for every $\varepsilon \approx 0, \varepsilon > 0$,

$$\mu^{\#}(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) \to_{\#} 0 \text{ as } m,n \to {}^*\infty, \quad (4.1)$$

and that $(f_n)_{n=1}^{*\infty}$ #-converges in #-measure to f if, for every $\varepsilon \approx 0, \varepsilon > 0$,

$$\mu^{\#}(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to_{\#} 0 \text{ as } n \to {}^*\infty. \quad (4.2)$$

Proposition 4.1. If $f_n \to_{\#} f$ in L^1 then $f_n \to_{\#} f$ in #-measure.

Proof. Let $E_{n,\varepsilon} = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Then

$$Ext\text{-}\int_X \left| f_n - f \right| d\mu^{\#} \geq Ext\text{-}\int_{E_{n,\varepsilon}} \left| f_n - f \right| d\mu^{\#} \geq \varepsilon\mu^{\#}(E_{n,\varepsilon}),$$

so $\mu(E_{n,\varepsilon}) \leq \varepsilon^{-1} Ext\text{-}\int_X \left| f_n - f \right| d\mu^{\#} \to_{\#} 0.$

Theorem 3.1. Suppose that $(f_n)_{n=1}^{*\infty}$ is Cauchy in #-measure. Then there is a #-measurable function f such that $f_n \to_{\#} f$ in #-measure, and there is a hyper infinite subsequence $(f_{n_j})_{j \in {}^*\mathbb{N}}$ that #-converges to f #-a.e. Moreover, if $f_n \to_{\#} g$ in #-measure then $g = f$ #-a.e.

Proof. We can choose a hyper infinite subsequence $(g_j)_j = (f_{n_j})_j$ of $(f_n)_{n=1}^{*\infty}$ such that if $E_j = \{x : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}$ then $\mu^{\#}(E_j) \leq 2^{-j}$. If $F_k = \bigcup_{j=k}^{*\infty} E_j$ then $\mu^{\#}(F_k) \leq Ext\text{-}\sum_{j=k}^{*\infty} 2^{-j} = 2^{1-k}$, and if $x \notin F_k$ we have for $i \geq j \geq k$

$$|g_j(x) - g_i(x)| \leq Ext\text{-}\sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq Ext\text{-}\sum_{l=j}^{i-1} \leq 2^{1-j}. \quad (4.3)$$

Thus $(g_j)_j$ is pointwise Cauchy on F_k^c. Let $F = \bigcap_{k=1}^{*\infty} F_k = \limsup_j E_j$. Then $\mu^{\#}(F) = 0$, and if we set $f(x) = \lim_{j \to {}^*\infty} g_j(x)$ for $x \notin F$, and $f(x) = 0$ for $x \in F$, then f is #-measurable and $g_j \to_{\#} f$ a.e. By (4.3), we have that $|g_j(x) - f(x)| \leq 2^{1-j}$for $x \notin F_k$ and $j \geq k$. Since $\mu^{\#}(F_k) \to_{\#} 0$ as $k \to {}^*\infty$, it follows that $g_j \to_{\#} f$ in #-measure, because

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq \{x : |f_n(x) - g_j(x)| \geq (1/2)\varepsilon\} \cup \{x : |g_j(x) - f(x)| \geq (1/2)\varepsilon\},$$

and the sets on the right both have infinte small #-measure when n and j are infinte large. Likewise, if $f_n \to_{\#} g$ in #-measure

$$\{x : |f(x) - g(x)| \geq \varepsilon\} \subseteq \{x : |f(x) - f_n(x)| \geq (1/2)\varepsilon\} \cup \{x : |f_n(x) - g(x)| \geq (1/2)\varepsilon\}$$

for all $n \in {}^*\mathbb{N}$, hence $\mu^{\#}(\{x : |f(x) - g(x)| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$, and $f = g$ #-a.e.

Theorem 3.2 Let $f_n \to_{\#} f$ in $L_1^{\#}$ then there is a hyper infinite subsequence $(f_{n_k})_k$ such that $f^{n_k} \to_{\#} f$ #-a.e.

Proof. Let $E_{n,\varepsilon} = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Then

$$Ext\text{-}\int_X |f_n - f| d^{\#}\mu \geq Ext\text{-}\int_{E_{n,\varepsilon}} |f_n - f| d^{\#}\mu \geq \varepsilon\mu^{\#}(E_{n,\varepsilon}),$$

so $\mu^{\#}(E_{n,\varepsilon}) \to_{\#} 0.$ Then, by Theorem 3.1, there is a hyper infinite subsequence $(f_{n_k})_k$ such that $f_{n_k} \to f$ #-a.e.

§ 5.The Extension of #-Measure

§ 5.1.Outer #-measures.

Definition 5.1.1. Let X be a nonempty set. An outer #-measure

(or #-submeasure) on X is a function $\xi^{\#} : \widetilde{P}(X) \to [0, {}^*\infty], \widetilde{P}(X) \subset P(X)$ that satisfies:
(a) $\xi^{\#}(\emptyset) = 0$;
(b) $\xi^{\#}(A) \leq \xi(B)$ if $A \subseteq B$;
(c) $\xi^{\#}\left(\bigcup_{j=1}^{{}^*\infty} A_j\right) \leq Ext\text{-}\sum_{j=1}^{{}^*\infty} \xi^{\#}(A_j)$ for all hyper infinite sequences $(A_j)_{j=1}^{{}^*\infty}$ in $\widetilde{P}(X)$.

The common way to obtain an outer #-measure is to start with a family G of "elementary sets" on which a notion of measure is defined (such as rectangles or cubes in ${}^*\mathbb{R}_c^{\#n}$ and then approximate arbitrary sets from the outside by hyper infinite unions of members of G.

Proposition 5.1.1 Let $G \subseteq \widetilde{P}(X)$ be a set such that $\emptyset \in G, X \in G$ and let $\rho : G \to [0, {}^*\infty]$ be a function such that $\rho(\emptyset) = 0$. For any $A \subseteq X$, define

$$\xi^{\#}(A) = \rho^*(A) = \inf\left\{ Ext\text{-}\sum_{j=1}^{{}^*\infty} \rho(G_j) : G_j \in G \text{ and } A \subseteq \bigcup_{j=1}^{{}^*\infty} G_j \right\}. \tag{5.1.1}$$

if $\rho^*(A)$ exists. Then $\xi^{\#}$ is an outer #-measure.

Definition 5.1.2.We will say that $A \subseteq X$ is admissible if $\rho^*(A)$ exists.

Proof. For any admissible $A \subseteq X$, $\xi^{\#}(A)$ is well defined. Obviously $\xi^{\#}(\emptyset) = 0$. To prove $*$-countable subadditivity, suppose $\{A_j\}_{j=1}^{{}^*\infty} \subseteq \widetilde{P}(X)$ and $\varepsilon \approx, \varepsilon > 0$.

For each $j \in {}^*\mathbb{N}$, there exists $\{G_k^j\}_{k=1}^{{}^*\infty} \subseteq G$ such that $A_j \subseteq \bigcup_{k=1}^{{}^*\infty} G_k^j$ and

$Ext\text{-}\sum_{k=1}^{{}^*\infty} \rho(G_k^j) \leq \xi^{\#}(A_j) + \varepsilon 2^{-j}$. Then if $A = \bigcup_{j=1}^{{}^*\infty} A_j$, we have $A \subseteq \bigcup_{j,k=1}^{{}^*\infty} G_k^j$ and

$Ext\text{-}\sum_{j,k=1}^{{}^*\infty} \rho(G_k^j) \leq \sum_{j=1}^{{}^*\infty} \xi(A_j) + \varepsilon$, whence $\xi^{\#}(A) \leq Ext\text{-}\sum_{j=1}^{{}^*\infty} \xi^{\#}(A_j) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have done.

Definition 5.1.3. A set $A \subseteq X$ is called $\xi^{\#}$-measurable if $\rho^*(A)$ exists and $\forall B \subseteq X$ such that $\rho^*(B)$ exists the equality (5.1.2) holds

$$\xi^{\#}(B) = \xi^{\#}(B \cap A) + \xi^{\#}(B \cap (X \backslash A)). \tag{5.1.2}$$

Of course, the inequality $\xi^{\#}(B) \leq \xi^{\#}(B \cap A) + \xi^{\#}(B \cap (X \backslash A))$ holds for any (admissible) set A and B.

So, to prove that A is $\xi^{\#}$-measurable, it suffices to prove the reverse inequality, which is trivial if $\xi^{\#}(B) = {}^*\infty$. Thus, we see that A is $\xi^{\#}$-measurable iff for any admissible $B \subseteq X, \xi^{\#}(B) < {}^*\infty$

$$\xi^{\#}(B) \geq \xi^{\#}(B \cap A) + \xi^{\#}(B \cap (X \backslash A)). \tag{5.1.3}$$

Theorem 5.1.1 (Generalized Caratheodory's theorem) Let $\xi^{\#}$ be an outer #-measure on X. Then the family Σ of all $\xi^{\#}$-measurable sets is a $\sigma^{\#}$-algebra, and the restriction of $\xi^{\#}$ to Σ is a complete #-measure.

Proof: First, we observe that Σ is closed under complements, since the definition of $\xi^{\#}$-measurability of A is symmetric in A and $A^c \triangleq X \backslash A$. Next, if $A, B \in \Sigma$ and $E \subseteq X$,

$\xi^{\#}(E) = \xi^{\#}(E \cap A) + \xi^{\#}(E \cap A^c) = \xi^{\#}(E \cap A \cap B) + \xi^{\#}(E \cap A \cap B^c) + \xi^{\#}(E \cap Ac \cap B) +$
$+\xi^{\#}(E \cap A^c \cap B^c)$.

But $(A \cup B) = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$ so, by subadditivity,

$\xi^{\#}(E\cap A\cap B)+\xi^{\#}(E\cap A\cap B^{c})+\xi^{\#}(E\cap A^{c}\cap B)\geq\xi^{\#}(E\cap(A\cup B))$,
and hence $\xi^{\#}(E)\geq\xi^{\#}(E\cap(A\cup B))\ +\xi^{\#}(E\cap(A\cup B)^{c})$.
It follows that $A\cup B\in\Sigma$, so Σ is an algebra. Moreover, if $A,B\in\Sigma$ and $A\cap B=\emptyset,\xi^{\#}(A\cup B)=\xi^{\#}((A\cup B)\cap A)+\xi^{\#}((A\cup B)\cap A^{c})=\xi^{\#}(A)+\xi^{\#}(B)$, so $\xi^{\#}$ is hyperfinitely additive on Σ.
To show that Σ is a $\sigma^{\#}$-algebra, it suffices to show that Σ is closed under $*$-countable disjoint unions. If $(A_j)_{j=1}^{*\infty}$ is a sequence of disjoint sets in Σ, set $B_n=\bigcup_{j=1}^{n}A_j\wedge B=\bigcup_{j=1}^{*\infty}A_j$. Then, for any admissible $E\subseteq X$,
$\xi^{\#}(E\cap B_n)=\xi^{\#}(E\cap B_n\cap A_n)+\xi^{\#}(E\cap B_n\cap A_n^{c})=\xi^{\#}(E\cap A_n)+\xi^{\#}(E\cap B_{n-1})$,
so a hyperfinite induction shows that $\xi^{\#}(E\cap B_n)=Ext\text{-}\sum_{j=1}^{n}\xi^{\#}(E\cap A_j)$. Therefore

$$\xi^{\#}(E)=\xi^{\#}(E\cap B_n)+\xi^{\#}(E\cap B_n^{c})\geq Ext\text{-}\sum_{j=1}^{n}\xi^{\#}(E\cap A_j)+\xi(E\cap B^{c})$$

and, letting $n\to{}^{*}\infty$, we obtain
$\xi^{\#}(E)\geq Ext\text{-}\sum_{j=1}^{*\infty}\xi^{\#}(E\cap A_j)+\xi^{\#}(E\cap B^{c})\geq\xi^{\#}\left(\bigcup_{j=1}^{*\infty}E\cap A_j\right)+\xi^{\#}(E\cap B^{c})=\xi^{\#}(E\cap B)+$
$+\xi^{\#}(E\cap B^{c})\geq\xi^{\#}(E)$.
Thus the inequalities in this last calculation become equalities. It follows $B\in\Sigma$.
Taking $E=B$ we have $\xi^{\#}(B)=Ext\text{-}\sum_{j=1}^{*\infty}\xi^{\#}(A_j)$, so $\xi^{\#}$ is $\sigma^{\#}$-additive on Σ. Finally, if $\xi^{\#}(A)=0$ then we have for any admissible set $E\subseteq X$
$\xi^{\#}(E)\leq\xi^{\#}(E\cap A)+\xi^{\#}(E\cap A^{c})=\xi^{\#}(E\cap A^{c})\leq\xi^{\#}(E)$, so $A\in\Sigma$.
Therefore $\xi^{\#}(E\cap A)=0$ and $\xi^{\#}|_{\Sigma}$ is a complete #-measure.
Combination of Proposition 5.1.1 and Theorem 5.1.1 gives the following corollary which is also called generalized Caratheodory's theorem.
Corollary 5.1.1 Let $G\subseteq P(X)$ be a set such that $\emptyset\in G,X\in G$, and let $\rho:G\to[0,{}^{*}\infty]$ satisfy $\rho(\emptyset)=0$. Then the family Σ of all ρ^{*} #-measurable sets (where $\rho^{*}$is given by (5.1.1)) is a $\sigma^{\#}$-algebra, and the restriction $\rho^{*}|_{\Sigma}$ of ρ^{*} to Σ is a complete #-measure.
Definition 5.1.4 Let $\overline{A}$ be an algebra of subsets of X, i.e. $\overline{A}$ contains $\emptyset$ and X, and $\overline{A}$ is closed under hyperfinite intersections and complements. A function $\zeta:\overline{A}\to[0,{}^{*}\infty]$ is called a #-premeasure if $\zeta(\emptyset)=0$ and $\zeta\left(\bigcup_{j=1}^{*\infty}A_j\right)=Ext\text{-}\sum_{j=1}^{*\infty}\zeta(A_j)$ for any disjoint sequence $(A_j)_{j\in{}^{*}\mathbb{N}}$ of elements of $\overline{A}$ such that $\bigcup_{j=1}^{*\infty}A_j\in\overline{A}$.
Theorem 5.1.2 If ζ is a #-premeasure on an algebra $\overline{A}\subseteq P(X)$ and $\zeta^{*}:P(X)\to[0,{}^{*}\infty]$ is given by (5.1.1) then $\zeta^{*}|_{A}=\zeta$ and every $A\in\overline{A}$ is ζ^{*} #-measurable.

§ 5.2.The Lebesgue and Lebesgue – Stieltjes #-measure on ${}^{*}\mathbb{R}_c^{\#}$.

The most important application of generalized Caratheodory's theorem is the construction of the Lebesgue #-measure on ${}^{*}\mathbb{R}_c^{\#}$. Take G as the set of all intervals $[a,b]$, where $a,b\in{}^{*}\mathbb{R}_c^{\#}\cup\{-{}^{*}\infty,+{}^{*}\infty\}$ and $[a,b]=\emptyset$ if $a>b$. Define the function $\rho : G \to {}^*\mathbb{R}_c^\# \cup \{{}^*\infty\}$ by

$$\forall a \forall b (a \leq b)\ [\rho([a,b]) \ = b - a] \text{ and } \forall a \forall b (a > b)[\rho([a,b]) \ = 0]. \qquad (5.2.1)$$

The function ρ has the obvious extension (which we denote also by ρ) to the algebra A generated by all intervals, and this extension is a #-premeasure on A. The $\sigma^\#$-algebra Σ given by Corollary **2.1.1** is called the the Lebesgue $\sigma^\#$-algebra in R, and the restriction of ρ^* to $\Sigma = \Sigma({}^*\mathbb{R}_c^\#)$ is called the Lebesgue #-measure on ${}^*\mathbb{R}_c^\#$ and is denoted by $\mu^\#$. By Theorem 5.1.2, $\mu^\#$ is the unique extension of ρ. By the construction, $B^\#({}^*\mathbb{R}_c^\#) \subseteq \Sigma({}^*\mathbb{R}_c^\#)$. Hence the Lebesgue #-measure is a Borel #-measure. It can be shown that $B^\#({}^*\mathbb{R}_c^\#) \neq \Sigma({}^*\mathbb{R}_c^\#)$ and that the Lebesgue #-measure can be obtained also as the completion of any Borel #-measure $\omega^\#$ such that $\omega^\#([a,b]) \ = b - a (\forall a \leq b)$.

The notion of the Lebesgue measure on ${}^*\mathbb{R}_c^\#$ has the following generalization. Suppose that $\mu^\#$ is a $\sigma^\#$-finite Borel measure on ${}^*\mathbb{R}_c^\#$, and let $\forall x \in^* \mathbb{R}_c^\#$

$$F(x) = \mu^\#((-{}^*\infty, x]) \qquad (5.2.2)$$

Then F is increasing and right #-continuous . Moreover, if $b > a$, $(-{}^*\infty, b] = (-{}^*\infty, a] \cup (a,b]$, so $\mu^\#((a,b]) \ = F(b) - F(a)$.

Our procedure used above can be to turn this process around and construct a measure μ starting from an increasing, right-continuous function F. The special case $F(x) = x$ will yield the usual Lebesgue #-measure. As building blocks we can use the left-#-open, right-#-closed intervals in ${}^*\mathbb{R}_c^\#$ i.e. sets of the form $(a,b]$ or $(a, {}^*\infty)$ or $\emptyset$, where $-{}^*\infty \leq a < b < {}^*\infty$. We call such sets h-intervals. The family A of all finite disjoint unions of h-intervals is an algebra, moreover, the $\sigma^\#$-algebra generated by A is the #-Borel algebra $B^\#({}^*\mathbb{R}_c^\#)$.

Lemma 5.2.1 Given an increasing and right #-continuous function $F :^* \mathbb{R}_c^\# \to {}^*\mathbb{R}_c^\#$, if $(a_j, b_j](j = 1,\ldots,n), n \in {}^*\mathbb{N}$ are disjoint h-intervals, let

$$\mu_0^\#\left(\bigcup_{j=1}^{n}(a_j, b_j]\right) = Ext\text{-}\sum_{j=1}^{n}[F(b_j) - F(a_j)], \qquad (5.2.3)$$

and let $\mu_0^\#(\emptyset) = 0$. Then $\mu_0^\#$ is a #-premeasure.

Theorem 5.2.1 If $F :^* \mathbb{R}_c^\# \to {}^*\mathbb{R}_c^\#$ is any increasing, right #-continuous function, there is a unique Borel #-measure $\mu_F^\#$ on ${}^*\mathbb{R}_c^\#$ such that $\forall a \forall b (a, b \in {}^*\mathbb{R}_c^\#)$

$$\mu_F^\#((a,b]) \ = F(b) - F(a).$$

If G is another such function then $\mu_F^\# = \mu_G^\#$ iff $F - G$ is constant.

Conversely, if $\mu^\#$ is a Borel #-measure on ${}^*\mathbb{R}_c^\#$ that is gyperfinite on all #-bounded #-Borel sets, and we define $F(x) = \mu^\#((0,x])$ if $x > 0, F(x) = 0$ if $x = 0$, $F(x) = -\mu^\#((x,0])$ if $x < 0$,

then F is increasing and right #-continuous function, and $\mu^\# = \mu_F^\#$.

Proof: Each F induces a #-premeasure on $B^\#({}^*\mathbb{R}_c^\#)$ by Lemma 5.1.1. It is clear that F and G induce the same #-premeasure iff $F - G$ is constant, and that these #-premeasures are $\sigma^\#$-finite (since ${}^*\mathbb{R}_c^\# = \bigcup_{-{}^*\infty}^{{}^*\infty}(j, j+1])$. The first two assertions follow now from **Exercise 2.1.11**. As for the last one, the monotonicity of $\mu^\#$ implies the monotonicity of F, and the #-continuity of $\mu^\#$ from above and from below implies the right #-continuity of F for $x \geq 0$ and $x < 0$. It is evident that $\mu^\# = \mu_F^\#$ on algebra A, and hence $\mu^\# = \mu_F^\#$ on $B^\#({}^*\mathbb{R}_c^\#)$ (accordingly to Lemma 5.2.3).

Lebesgue – Stieltjes #-measures possess some important and useful regularity properties.
Let us fix a complete Lebesgue – Stieltjes #-measure $\mu^{\#}$ on ${}^*\mathbb{R}^{\#}_c$ associated to an increasing, right #-continuous function F. We denote by $\Sigma_{\mu^{\#}}$ the Lebesgue algebra correspondent to $\mu^{\#}$. Thus, for any $E \in \Sigma_{\mu^{\#}}$,

$$\begin{aligned}\mu^{\#}(E) &= \inf\left\{ Ext\text{-}\sum_{j=1}^{{}^*\infty}[F(b_j) - F(a_j)] \,\middle|\, E \subseteq \bigcup_{j=1}^{{}^*\infty}(a_j, b_j] \right\} = \\ &= \inf\left\{ Ext\text{-}\sum_{j=1}^{{}^*\infty}\mu^{\#}_F((a_j, b_j]) \,\middle|\, E \subseteq \bigcup_{j=1}^{{}^*\infty}(a_j, b_j] \right\}\end{aligned} \quad (5.2.4)$$

if infinum in RHS of (5.2.4) exists. Since $B^{\#}({}^*\mathbb{R}^{\#}_c) \subseteq \Sigma_{\mu^{\#}}$, we may replace in the second formula for $\mu^{\#}(E)$ h-intervals by #-open intervals, namely
Lemma 5.2.2 For any $E \in \Sigma_{\mu^{\#}}$,

$$\mu^{\#}(E) = \inf\left\{ Ext\text{-}\sum_{j=1}^{{}^*\infty}\mu^{\#}_F((a_j, b_j)) \,\middle|\, E \subseteq \bigcup_{j=1}^{{}^*\infty}(a_j, b_j) \right\}. \quad (5.2.5)$$

Theorem 5.2.2 If $E \in \Sigma_{\mu^{\#}}$ then

$$\begin{aligned}E \in \Sigma_{\mu^{\#}} &= \inf\{\mu^{\#}(U) : U \supseteq E \text{ and } U \text{ is \#-open}\} = \\ &= \sup\{\mu^{\#}(K) : K \subseteq E \text{ and } K \text{ is \#-compact}\}.\end{aligned} \quad (5.2.6)$$

Proof. By Lemma 5.2.2, for any $\varepsilon \approx, \varepsilon > 0$, there exist intervals (a_j, b_j) such that $E \subseteq \bigcup_{j=1}^{{}^*\infty}(a_j, b_j)$ and $\mu^{\#}(E) \leq Ext\text{-}\sum_{j=1}^{{}^*\infty}\mu^{\#}((a_j, b_j)) + \varepsilon$. If $U = \bigcup_{j=1}^{{}^*\infty}(a_j, b_j)$ then U is #-open, $E \subseteq U$, and $\mu^{\#}(U) \leq \mu^{\#}(E) + \varepsilon$. On the other hand, $\mu^{\#}(U) \geq \mu^{\#}(E)$ whenever $E \subseteq U$ so the first equality is valid.
For the second one, suppose first that E is bounded in ${}^*\mathbb{R}^{\#}_c$. If E is #-closed then E is #-compact and the equality is obvious. Otherwise, given $\varepsilon \approx, \varepsilon > 0$, we can choose an #-open U, $(\#\text{-}\overline{E})\backslash E \subseteq U$, such that $\mu^{\#}(U) \leq \mu^{\#}((\#\text{-}\overline{E})\backslash E) + \varepsilon$.
Let $K = (\#\text{-}\overline{E})\backslash U$. Then K is #-compact, $K \subseteq E$, and
$\mu^{\#}(K) = \mu^{\#}(E) - \mu^{\#}(E \cap U) = \mu^{\#}(E) - [\mu^{\#}(U) - \mu^{\#}(U \backslash E)] \geq$
$\geq \mu^{\#}(E) - \mu^{\#}(U) + \mu^{\#}((\#\text{-}\overline{E})\backslash E) \geq \mu^{\#}(E) - \varepsilon$.
If E is unbounded in ${}^*\mathbb{R}^{\#}_c$, let $E_j = E \cap (j, j+1]$. By the preceding argument, for any $\varepsilon \approx, \varepsilon > 0$, there exist a #-compact $K_j \subseteq E_j$ with $\mu^{\#}(K_j) \geq \mu^{\#}(E_j) - \varepsilon 2^{-j}$. Let $H_n = \bigcup_{j=-n}^{j=n} K_j$. Then H_n is #-compact, $H_n \subseteq E$, and $\mu^{\#}(H_n) \geq \mu^{\#}\bigcup_{j=-n}^{j=n}(E_j) - \varepsilon$.
Since $\mu^{\#}(E) = \#\text{-}\lim_{n\to{}^*\infty}\mu^{\#}\left(\bigcup_{j=-n}^{j=n} E_j\right)$, the result follows.
Theorem 5.2.3. If $E \subseteq {}^*\mathbb{R}^{\#}_c$, the following are equivalent:
(a) $E \in \Sigma_{\mu^{\#}}$;
(b) $E = V \backslash N_1$, where V is a $G_{\delta^{\#}}$–set and $\mu^{\#}(N_1) = 0$;
(c) $E = H \cup N_2$, where H is an $F_{\sigma^{\#}}$–set and $\mu^{\#}(N_2) = 0$.
Theorem 5.2.4. If $E \in \Sigma_{\mu^{\#}}$ and $\mu^{\#}(E) < {}^*\infty$ then, for every $\varepsilon \approx, \varepsilon > 0$, there is

a set A that is a hyperfinite union of #-open intervals such that $\mu^{\#}(E\Delta A) < \varepsilon$.

Lemma 5.2.3 Let $A \subseteq P(X)$ be an algebra, let $\mu_0^{\#}$ be a $\sigma^{\#}$-finite #-premeasure on A, and let Ω be the $\sigma^{\#}$-algebra generated by A. Then there exists a unique extension of $\mu_0^{\#}$ to a #-measure $\mu^{\#}$ on Ω.

§ 5.3. Product #-measures.

Definition 5.3.1.Let $\{(X_\alpha, \mathcal{F}_\alpha, \mu_\alpha^{\#})\}_{\alpha\in\Delta}$ be a nonempty family of #-measure spaces. We define the family Ω of blocks:

$$A(A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}) := \\ = A_{\alpha_1} \times A_{\alpha_2} \times \cdots \times A_{\alpha_n} \times Ext\text{-}\prod_{\alpha\in\Delta, \alpha\neq\alpha_k, 1\leq k\leq n} X_\alpha, \tag{5.3.1}$$

where $A_{\alpha_k} \in \mathcal{F}_{\alpha_k}$ and define a function

$$\mu_\Omega^{\#} : \Omega \to {}^*\mathbb{R}_c^{\#} \cup \{{}^*\infty\} := \\ \mu^{\#}\left(A_{\alpha_1}\right) \times \mu^{\#}(A_{\alpha_2}) \times \cdots \times \mu^{\#}(A_{\alpha_n}) \times \left[Ext\text{-}\prod_{\alpha\in\Delta, \alpha\neq\alpha_k, 1\leq k\leq n} \mu^{\#}(X_\alpha) \right]. \tag{5.3.2}$$

This function possesses an extension (by #-additivity) on the #-algebra A generated by Ω. It is easily to show that $\mu_\Omega^{\#}$ is a #-premeasure on A.

Definition 5.3.2 The #-measure $\widehat{\mu}^{\#}$ on the $\sigma^{\#}$-algebra Σ generated by A accordingly to **Theorem 2.1.3** is called the product #-measure of $\{\mu_\alpha^{\#}\}_{\alpha\in\Delta}$, and the triple $\left(\prod_{\alpha\in\Delta} X_\alpha, \Sigma, \widehat{\mu}^{\#}\right)$ is called the product of #-measure spaces $(X_\alpha, \Sigma_\alpha, \mu_\alpha^{\#})$.

We denote the $\sigma^{\#}$-algebra Σ by $\bigotimes_{\alpha\in\Delta} \Sigma_\alpha$, and the #-measure $\widehat{\mu}^{\#}$ by $\bigotimes_{\alpha\in\Delta} \mu_\alpha^{\#}$.

Definition 5.3.3.If $E \subseteq X_1 \times X_2$ and $x_1 \in X_1, x_2 \in X_2$, we define

$E_{x_1} = \{x \in X_2 : (x_1, x) \in E\}$ and $E^{x_2} = \{x \in X_1 : (x, x_2) \in E\}$.

If $f : X_1 \times X_2 \to {}^*\mathbb{R}_c^{\#}$is a function, we define $f_{x_1} : X_2 \to {}^*\mathbb{R}_c^{\#}$ and $f^{x_2} : X_1 \to {}^*\mathbb{R}_c^{\#}$ by $f_{x_1}(x) = f(x_1, x)$ and $f^{x_2}(x) = f(x, x_2)$.

Theorem 5.3.1 (The generalized Fubini's theorem) Let $\mu_1^{\#}, \mu_2^{\#}$ be $\sigma^{\#}$-hyperfinite #-measures on $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$,

$$(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1^{\#} \otimes \mu_2^{\#}) = (X_1, \mathcal{F}_1, \mu_1^{\#}) \times (X_2, \mathcal{F}_2, \mu_2^{\#}), \tag{5.3.3}$$

and let $f \in L_1^{\#}(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1^{\#} \otimes \mu_2^{\#})$.Then $f_{x_1} \in L_1^{\#}(X_2, \mathcal{F}_2, \mu_2^{\#})$ $\mu_1^{\#}$-#-a.e., and $f^{x_2} \in L_1^{\#}(X_1, \mathcal{F}_1, \mu_1^{\#})$ $\mu_2^{\#}$-#-a.e., and

$$Ext\text{-}\int_{X_1\times X_2} f d^{\#}(\mu_1^{\#} \otimes \mu_2^{\#}) = Ext\text{-}\int_{X_2}\left[Ext\text{-}\int_{X_1} f^{x_2} d^{\#}\mu_1^{\#}\right] d^{\#}\mu_2^{\#} = \\ = Ext\text{-}\int_{X_1}\left[Ext\text{-}\int_{X_2} f_{x_1} d^{\#}\mu_2^{\#}\right] d^{\#}\mu_1^{\#} \tag{5.3.4}$$

Lemma 5.3.1. Let $(X_1, \Sigma_1, \mu_1^{\#})$ and $(X_2, \Sigma_2, \mu_2^{\#})$ be #-measure spaces, $E \in \Sigma_1 \otimes \Sigma_2$, and let f be a $\Sigma_1 \otimes \Sigma_2$-measurable function on $X_1 \times X_2$, then:

(a) $E_{x_1} \in \Sigma_2$ for all $x_1 \in X_1$ and $E_{x_2} \in \Sigma_1$ for all $x_2 \in X_2$;

(b) $f_{x_1}$is Σ_2-measurable and $f_{x_2}$is Σ_1-measurable for all $x_1 \in X_1$ and $x_2 \in X_2$.

Proof. Denote by A the collection of all $A \subseteq X_1 \times X_2$ such that $A_{x_1} \in \Sigma_2$and $A^{x_2} \in \Sigma_1 (\forall x_1 \in X_1, x_2 \in X_2)$.

The family A contains all rectangles. Thus, since

$$\left[\bigcup_{n=1}^{*\infty} A_n\right]_{x_1} = \bigcup_{n=1}^{*\infty}[A_n]_{x_1}, [B_n]^{x_2} = [B_n]^{x_2} \quad (5.3.5)$$

and

$$[X_1 \times X_2 \backslash A]x_1 = X_2 \backslash A_{x_1}, [X_1 \times X_2 \backslash A]^{x_2} = X_1 \backslash A^{x_2}, \quad (5.3.6)$$

A is a $\sigma^{\#}$-algebra. So $\Sigma_1 \otimes \Sigma_2 \subseteq A$, and (a) is proved.Now the part (b) follows from (a) due to $f_{x_1}^{-1}(A) = [f^{-1}(A)]_{x_1}$ and $[f^{x_2}]^{-1}(A) = [f^{-1}(A)]^{x_2}(\forall A \subseteq {}^*\mathbb{R}_c^{\#})$.

Definition 5.3.4 A family $M \subseteq P(X)$ is called a monotone class if M is closed under $*$-countable increasing unions and $*$-countable decreasing intersections.

Lemma 5.3.2. If $A \subseteq P(X)$ is an algebra then the monotone class generated by A coincides with the $\sigma^{\#}$-algebra generated by A.

Lemma 5.3.3. Let $(X_1, \Sigma_1, \mu_1^{\#})$ and $(X_2, \Sigma_2, \mu_2^{\#})$ be #-measure spaces, $E \in \Sigma_1 \otimes \Sigma_2$. Then the functions $x_1 \to \mu_2^{\#}(E_{x_1})$ and $x_2 \to \mu_1^{\#}(E^{x_2})$ are #-measurable on (X_1, Σ_1) and (X_2, Σ_2), and

$$\mu_1^{\#} \otimes \mu_2^{\#}(E) = Ext\text{-}\int_{X_2} \mu_1^{\#}(E^{x_2}) d^{\#}\mu_2^{\#} = Ext\text{-}\int_{X_1} \mu_2^{\#}(E_{x_1}) d^{\#}\mu_1^{\#}. \quad (5.3.7)$$

Proof. First we consider the case when $\mu_1^{\#}$ and $\mu_2^{\#}$ are finite. Let A be the family of all $E \in \Sigma_1 \otimes \Sigma_2$ for which (5.3.7) is true. If $E = A \times B$, then $\mu_1^{\#}(E^{x_2}) = \mu_1^{\#}(A)\chi_B(x_2)$ and $\mu_2^{\#}(E_{x_1}) = \mu_2^{\#}(B)\chi_A(x_1)$, so $E \in A$. By additivity, it follows that gyperfinite disjoint unions of rectangles are in A so, by Lemma 5.3.2,bit will suffice to show that A is a monotone class. If $(E_n)_{n=1}^{*\infty}$ is an increasing hyper infinite sequence in A and $E = \bigcup_{n=1}^{*\infty} E_n$, then the function $f_n(x_2) = \mu_1^{\#}((E_n)^{x_2})$ are #-measurable and increase pointwise to $f(y) = \mu_1^{\#}(E^{x_2})$. Hence f is #-measurable and, by the monotone convergence theorem,

$$\begin{gathered} Ext\text{-}\int_{X_2} \mu_1^{\#}(E^{x_2}) d\mu_2^{\#} = \#\text{-}\lim_{n \to {}^*\infty} ExtX_1 \int_{X_2} \mu_1^{\#}((E_n)^{x_2}) d\mu_2^{\#} = \\ \#\text{-}\lim_{n \to {}^*\infty} \mu_1^{\#} \times \mu_2^{\#}(E_n) = \mu_1^{\#} \times \mu_2^{\#}(E). \end{gathered} \quad (5.3.8)$$

Likewise $\mu_1^{\#} \times \mu_2^{\#}(E) = Ext\text{-}\int_{X_1} \mu_2^{\#}(E_x) d\mu_1^{\#}$, so $E \in A$. Similarly, if $(E_n)_{n=1}^{*\infty}$ is a decreasing hyper infinite sequence in A and $E = \bigcap_{n=1}^{*\infty} E_n$, the function $x_2 \to \mu_1^{\#}((E_1)^{x_2})$ is in $L_1^{\#}(\mu_2^{\#})$ because $\mu_1^{\#}((E_1)x_2) \le \mu_1^{\#}(X_1) < {}^*\infty$ and $\mu_2^{\#}(X_2) < {}^*\infty$, so the dominated convergence theorem can be applied to show that $E \in A$. Thus, A is a monotone class, and the proof is complete for the case of finite #-measure spaces.

Finally, if $\mu_1^{\#}$ and $\mu_2^{\#}$ are $\sigma^{\#}$-finite, we can write $X_1 \times X_2$ as the union of an increasing hyper infinite sequence $(X_1^j \times X_2^j)_{j=1}^{*\infty}$ of rectangles of finite or hyperfinite #-measure. If $E \in \Sigma_1 \otimes \Sigma_2$, the preceding argument applies to $E \cap (X_1^j \times X_2^j)$ for each j gives us

$$\mu_1^{\#} \times \mu_2^{\#}(E \cap (X_1^j \times X_2^j)) = Ext\text{-}\int_{X_2} \mu_1^{\#}(E^{x_2} \cap X_1^j)\mu_2^{\#} = Ext\text{-}\int_{X_1} \mu_2^{\#}(E_{x_1} \cap X_2^j)\mu_1^{\#}. \quad (5.3.9)$$

The application of the monotone convergence theorem then yields the desired result.

Chapter II. ${}^*\mathbb{R}_c^\#$-valued distributions.

§1. ${}^*\mathbb{R}_c^\#$-valued test functions and distributions

Definitions and theorems appropriate to analysis on non-Archemedean field ${}^*\mathbb{R}_c^\#$ and on complex field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$are given in [1]-[2].

Definition 1.1.[3].(i) Let U be a free ultrafilters on $\mathbb{N}$ and introduce an equivalence relation on sequences in $\mathbb{R}^{\mathbb{N}}$ as $f_1 \sim_U f_2$ iff $\{i \in \mathbb{N} | f_1(i) = f_2(i)\} \in U$.

(ii) $\mathbb{R}^{\mathbb{N}}$ divided out by the equivalence relation $\sim_U$ gives us the nonstandard extension ${}^*\mathbb{R}$, the hyperreals; in symbols, ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\sim_U$ and similarly $\mathbb{N}^{\mathbb{N}}$ divided out by the equivalence relation $\sim_U$ gives us the nonstandard extension ${}^*\mathbb{N}$, the hyperintegers; in symbols, ${}^*\mathbb{N} = \mathbb{N}^{\mathbb{N}}/\sim_U$.

Abbreviation 1.1.If $f \in \mathbb{R}^{\mathbb{N}}$, we denote its image in ${}^*\mathbb{R}$ by $[f]$, i.e., $[f] = \{g \in \mathbb{R}^{\mathbb{N}} | g \sim_U f\}$.

Remark 1.1.For any real number $r \in \mathbb{R}$ let $\mathbf{r}$ denote the constant function $\mathbf{r} : \mathbb{N} \to \mathbb{R}$ with value r, i.e., $\mathbf{r}(n) = r$, for all $n \in \mathbb{N}$. We then have a natural embedding

$*(\cdot) : \mathbb{R} \hookrightarrow {}^*\mathbb{R}$

by setting ${}^*r = [\mathbf{r}(n)]$ for all $r \in \mathbb{R}$. We denote it image $*(\mathbb{R})$ in ${}^*\mathbb{R}$ by ${}^*\mathbb{R}_{\mathbf{st}}$.

Definition 1.2.[3]. An element $x \in {}^*\mathbb{R}$ is called finite if $|x| < r$ for some $r \in \mathbb{Q}, r > 0$.

Abbreviation 1.2.For $x \in {}^*\mathbb{R}$ we abbreviate $x \in {}^*\mathbb{R}_{\mathbf{fin}}$ if x is finite.

Remark 1.2.[3]. Let $x \in {}^*\mathbb{R}_{\mathbf{fin}}$ be finite. Let D_1, be the set of $r \in \mathbb{Q}$ such that $r < x$ and D_2 the set of $r' \in \mathbb{Q}$ such that $x < r'$. The pair (D_1, D_2) forms a Dedekind cut in $\mathbb{R}$, hence determines a unique $r_0 \in \mathbb{R}$. A simple argument shows that $|x - r_0|$ is infinitesimal, i.e., $|x - r_0| \approx 0$.

Definition 1.3.[1].This unique r_0 is called the standard part of x and is denoted by ${}^\circ x$ or $st(x)$.

The following notation will be used throughout this paper.

$n \in \mathbb{N}^\#$ is a fixed positive integer and $U \subset {}^*\mathbb{R}_c^{\#n}$ is a fixed non-empty #-open subset of lnear space ${}^*\mathbb{R}_c^{\#n}$ over non Archemedan field ${}^*\mathbb{R}_c^\#$.

$\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the standard natural numbers.

k will denote a non-negative integer or $\infty^\#$.

If f is a function then $\mathbf{Dom}(f)$ will denote its domain and the support of f, denoted by $supp(f)$, is defined to be the closure of the set $\{x \in \mathbf{Dom}(f) : f(x) \neq 0\}$ in $\mathbf{Dom}(f)$.

For two functions $f, g : U \to {}^*\mathbb{C}_c^\#$, the following notation defines external canonical pairing:

$$\langle f, g \rangle = Ext\text{-}\int_U f(x) g(x) d^\# x. \tag{1.1}$$

A multi-index of size $n \in \mathbb{N}^\#$ is an element in $\mathbb{N}^{\#n}$, if the size of multi-indices is omitted then the size should be assumed to be n. The length of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{\#n}$ is defined as $Ext\text{-}\sum_{i=1}^n \alpha_i$ and denoted by $|\alpha|$. Multi-indices are particularly useful when

dealing with functions of several variables, in particular we introduce the following canonical notations for a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{\#n}$,

$$x^{\alpha} = x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$
$$\partial^{\#\alpha} = \frac{\partial^{\#|\alpha|}}{\partial^{\#}x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \tag{1.2}$$

We also introduce a partial order of all multi-indices by $\beta \geq \alpha$ if and only if $\beta_i \geq \alpha_i$ for all $1 \leq i \leq n$. When $\beta \geq \alpha$ we define their multi-index binomial coefficient as:
$\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1}\cdots\binom{\beta_n}{\alpha_n}$.

1.Let $k \in \mathbb{N}^{\#} \cup \infty^{\#}$.

2.Let $C^{\#k}(U)$ denote the vector space of all k-times #-continuously #-differentiable ${}^*\mathbb{R}_c^{\#}$-valued or ${}^*\mathbb{C}_c^{\#}$-valued functions on U.

For any #-compact subset $K \subseteq U$, let $C^{\#k}(K)$ and $C^{\#k}(K;U)$ both denote the vector space of all those functions $f \in C^{\#k}(U)$ such that $\mathbf{supp}(f) \subseteq K$.

Note that $C^{\#k}(K)$ depends on both K and U but we will only indicate K, where in particular, if $f \in C^{\#k}(K)$ then the domain of f is U rather than K. We will use the notation

$C^{\#k}(K;U)$ only when the notation $C^{\#k}(K)$ risks being ambiguous.

Every $C^{\#k}(K)$ contains the constant 0 map, even if $K = \varnothing$.

Let $C_c^{\#k}(U)$ denote the set of all $f \in C^{\#k}(U)$ such that $f \in C^{\#k}(K)$ for some #-compact subset K of U.

Equivalently, $C_c^{\#k}(U)$ is the set of all $f \in C^{\#k}(U)$ such that f has #-compact support. $C_c^{\#k}(U)$ is equal to the union of all $C^{\#k}(K)$ as $K \subseteq U$ ranges over all #-compact subsets of U. If f is a ${}^*\mathbb{R}_c^{\#}$-valued function on U, then f is an element of $C_c^{\#k}(U)$ if and only if f is a $C^{\#k}$ bump function. Every ${}^*\mathbb{R}_c^{\#}$-valued test function on U is always also a ${}^*\mathbb{C}_c^{\#}$-valued test function on U.

For all $j,k \in \mathbb{N}$ and any #-compact subsets K and L of U, we have:

$C^{\#k}(K) \subseteq C_c^{\#k}(U) \subseteq C^{\#k}(U)$;

$C^{\#k}(K) \subseteq C^{\#k}(L)$ if $K \subseteq L C^{\#k}(K) \subseteq C^{\#j}(K)$ if $j \leq k$;

$C_c^{\#k}(U) \subseteq C_c^{\#j}(U)$ if $j \leq k$;

$C^{\#k}(U) \subseteq C^{\#j}(U)$ if $j \leq k$.

Definition1.1. Elements of $C_c^{\#\infty^{\#}}(U)$ are called ${}^*\mathbb{R}_c^{\#}$-valued test functions on U and $C_c^{\#\infty^{\#}}(U)$ is
called the space of ${}^*\mathbb{R}_c^{\#}$-valued test functions on U. We will use both $D(U)$ and $C_c^{\#\infty^{\#}}(U)$ to denote this space.

Definition1.2. Distributions on U are #-continuous ${}^*\mathbb{R}_c^{\#}$-valued linear functionals on $C_c^{\#\infty^{\#}}(U)$ when this vector space is endowed with a particular topology called the canonical
LF-topology.

The following proposition states two necessary and sufficient conditions for the #-continuity of a linear functional on $C_c^{\#\infty^{\#}}(U)$ that are often straightforward to verify.

Proposition1.1. A linear functional T on $C_c^{\#\infty^{\#}}(U)$ is #-continuous, and therefore a distribution, if and only if either of the following equivalent conditions are satisfied:

1.For every #-compact subset $K \subseteq U$ there exist constants $C > 0$ and $N \in \mathbb{N}$ dependent on K such that for all $f \in C_c^{\#\infty^{\#}}(U)$ with support contained in K
$|T(f)| \leq C\sup\{|\partial^{\#\alpha}f(x)| : x \in U, |\alpha| \leq N\}$.

2.For every #-compact subset $K \subseteq U$ and every sequence $\{f_i\}_{i=1}^{\infty^{\#}}$ in $C_c^{\#\infty^{\#}}(U)$ whose supports are contained in K, if $\{\partial^{\#\alpha} f_i\}_{i=1}^{\infty}$ #-converges uniformly to zero on U for every multi-index α, then $\#\text{-}\lim_{i\to\infty^\#} T(f_i) = 0$.

. .

§ 2.The non-Archimedian external ${}^*\mathbb{R}_c^\#$-Valued Schwartz distributions.

Defined below are the tempered distributions, which form a subspace of $\mathcal{D}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$, the space of distributions on ${}^*\mathbb{R}_c^{\#n}$. This is a proper subspace: while every tempered distribution is a distribution and an element of $\mathcal{D}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ the converse is not true. Tempered distributions are useful if one studies the Fourier transform since all tempered distributions have a Fourier transform, which is not true for an arbitrary distribution in $\mathcal{D}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$.

§ 2.1.Schwartz space $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$.

Definition 2.1. A function $f : X \to {}^*\mathbb{R}_c^\#$ defined on some set X is called finitely bounded (or bounded) if the set of its values is finitely bounded, i.e., $f(X) \subset [a,b]$ where $a,b \in {}^*\mathbb{R}^\#_{c,\mathbf{fin}}$. In other words, there exists a finite hyperreal number $M \in {}^*\mathbb{R}^\#_{c,\mathbf{fin}}$ such that

$$|f(X)| \leq M. \tag{2.1}$$

Definition 2.2.A function $f : X \to {}^*\mathbb{R}_c^\#$ defined on some set X is called hyper finitely bounded (or hyper bounded) if the set of its values is hyper finitely bounded, i.e., $f(X) \subset [a,b]$ where $a,b \in {}^*\mathbb{R}_c^\# \backslash {}^*\mathbb{R}^\#_{c,\mathbf{fin}}$. In other words, there exists a hyperfinite hyperreal number $M \in {}^*\mathbb{R}_c^\# \backslash {}^*\mathbb{R}^\#_{c,\mathbf{fin}}$ such that $|f(X)| \leq M$.

Definition 2.3.For $n \in \mathbb{N}^\#$, an #-integrable function $\phi : {}^*\mathbb{R}_c^{\#n} \to {}^*\mathbb{R}_c^\#$is called #-rapidly decreasing if for all $\alpha \in \mathbb{N}^{\#n}$ the product function $x \mapsto x^\alpha \phi(x)$ is a finitely bounded or hyper finitely bounded function.

Remark 2.1.If ϕ is a #-rapidly decreasing function, then its integral exists

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} \phi(x) d^{\#n}x < \infty^\# \tag{2.2}$$

In fact for all $\alpha \in \mathbb{N}^{\#n}$ the integral of $x \mapsto x^\alpha \phi(x)$ exists

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} x^\alpha \phi(x) d^{\#n}x < \infty^\#. \tag{2.3}$$

Definition 2.4.The Schwartz space, $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$, is the space of all #-smooth functions in $C^{\#\infty^\#}({}^*\mathbb{R}_c^{\#n})$ that are rapidly decreasing at #-infinity along with all partial #-derivatives. Thus
$\phi : {}^*\mathbb{R}_c^{\#n} \to {}^*\mathbb{R}_c^\#$ is in the Schwartz space provided that any #-derivative of ϕ, multiplied with any power of $|x|$, #-converges to 0 as $|x| \to \infty^\#$. These functions form a #-complete TVS with a suitably defined family of seminorms. More precisely, for any multi-indices α and β define:

$$p_{\alpha,\beta}(\phi) = \sup_{x \in {}^*\mathbb{R}_c^{\#n}} |x^\alpha \partial^{\#\beta} \phi(x)|. \quad (2.1)$$

Then ϕ is in the Schwartz space $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$ if all the values satisfy: $p_{\alpha,\beta}(\phi) < \infty^\#$.
Thus

$$\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{R}_c^\#) \triangleq \left\{\phi \in C^{\infty^\#}({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{R}_c^\#) | \forall \alpha, \beta \in \mathbb{N}^{\#n}(p_{\alpha,\beta}(\phi) < \infty^\#)\right\}.$$

Similarly

$$\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) \triangleq \left\{\phi \in C^{\infty^\#}({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) | \forall \alpha, \beta \in \mathbb{N}^{\#n}(p_{\alpha,\beta}(\phi) < \infty^\#)\right\}$$

The family of seminorms $p_{\alpha,\beta}(\cdot)$ defines a locally convex topology on the Schwartz space $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$.
For $n = 1$, the seminorms are norms on the Schwartz space $\mathcal{S}^\#({}^*\mathbb{R}_c^\#)$. One can also use the following family of seminorms to define the topology:

$$|f|_{m,k} = \sup_{|p| \leq m}\left(\sup_{x \in {}^*\mathbb{R}_c^{\#n}} \{(1+|x|)^k |(\partial^{\#\alpha} f)(x)|\}\right), k, m \in \mathbb{N}^\#. \quad (2.2)$$

Otherwise, one can define a norm on $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$ by

$$\|\phi\|_k = \max_{|\alpha|+|\beta| \leq k} \sup_{x \in {}^*\mathbb{R}_c^{\#n}} |x^\alpha \partial^\beta \phi(x)|, k \geq 1. \quad (2.3)$$

The Schwartz space $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$ is a Fréchet space (that is, a #-complete metrizable locally convex space). Because the Fourier transform changes $\partial^{\#\alpha}$ into multiplication by x^α and vice versa, this symmetry implies that the Fourier transform of a Schwartz function is also a Schwartz function.
Definition 2.5. A sequence $\{f_i\}_{i=1}^{\infty^\#}$ #-converges to 0 in $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$ if and only if the functions $(1+|x|)^k(\partial^{\#p} f_i)(x)$ #-converge to 0 uniformly in the whole of ${}^*\mathbb{R}_c^{\#n}$, which implies that such a sequence must converge to zero in $C^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$.
The subset of all #-analytic Schwartz functions is #-dense in $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#n})$
The Schwartz space is nuclear and the tensor product of two maps induces a canonical surjective TVS-isomorphisms $\mathcal{S}^\#({}^*\mathbb{R}_c^{\#m}) \widehat{\otimes} \mathcal{S}^\#({}^*\mathbb{R}_c^{\#n}) \to \mathcal{S}^\#({}^*\mathbb{R}_c^{\#m+n})$, where $\widehat{\otimes}$ represents the #-completion of the injective tensor product

§ 2.2.Schwartz space $\mathcal{S}^\#_{\mathbf{fin}}({}^*\mathbb{R}^{\#n}_{c,\mathbf{fin}})$

Definition 2.6.For $n \in \mathbb{N}$, an ${}^*\mathbb{R}^{\#n}_{c,\mathbf{fin}}$-valued and #-integrable function $\phi : {}^*\mathbb{R}_c^{\#n} \to {}^*\mathbb{R}^\#_{c,\mathbf{fin}}$is called #-rapidly decreasing if for all $\alpha \in \mathbb{N}^n$ the product function $x \mapsto x^\alpha \phi(x)$ is a finitely bounded function.
Remark 2.2.If ϕ is a #-rapidly decreasing ${}^*\mathbb{R}^{\#n}_{c,\mathbf{fin}}$-valued function, then its integral exists and finite,i.e.,

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} \phi(x) d^{\#n}x \in {}^*\mathbb{R}^\#_{c,\mathbf{fin}}. \quad (2.2)$$

In fact for all $\alpha \in \mathbb{N}^n$ the integral of $x \mapsto x^\alpha \phi(x)$ exists and finite,i.e.,

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} x^\alpha \phi(x) d^{\#n}x \in {}^*\mathbb{R}^\#_{c,\mathbf{fin}}. \quad (2.3)$$

It follows from () that for all $\alpha \in \mathbb{N}^n$ and for any $R \in {}^*\mathbb{R}_c^\# \backslash {}^*\mathbb{R}^\#_{c,\mathbf{fin}}$

$$Ext\text{-}\int\limits_{{}^*\mathbb{R}_c^{\#n}\setminus B(R)} x^\alpha \phi(x) d^{\#n}x \approx 0 \tag{2.3}$$

where $B(R) \triangleq \{x \in {}^*\mathbb{R}_c^{\#} \| |x| \leq R\}$

Definition 2.7.The Schwartz space, $\mathcal{S}^{\#}_{\mathbf{fin}}\left({}^*\mathbb{R}^{\#n}_{c,\mathbf{fin}}\right)$, is the space of all ${}^*\mathbb{R}^{\#n}_{c,\mathbf{fin}}$- valued #-smooth functions that are rapidly decreasing at #-infinity along with all partial #-derivatives any finite order $1 \leq m < \infty$.

Thus

$\phi : {}^*\mathbb{R}_c^{\#n} \to \mathbb{R}_c^{\#}$ is in the Schwartz space provided that any #-derivative of ϕ, multiplied with any power of $|x|$, #-converges to 0 as $|x| \to \infty^{\#}$. These functions form a #-complete TVS with a suitably defined family of seminorms. More precisely, for any multi-indices α and β define:

$$p_{\alpha,\beta}(\phi) = \sup_{x \in {}^*\mathbb{R}_c^{\#n}} |x^\alpha \partial^{\#\beta} \phi(x)|. \tag{2.1}$$

§ 2.3.Non-Archimedian tempered distributions $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$.

A non-Archimedian tempered distribution is a distribution $u \in \mathcal{D}'({}^*\mathbb{R}_c^{\#n})$ that does not "grow too fast" – at most polynomial (or tempered) growth – at #-infinity in all directions; in particular it is only defined on ${}^*\mathbb{R}_c^{\#n}$, not on any #-open subset. Formally, a tempered distribution is a #-continuous linear functional on the Schwartz space $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ of smooth functions with #-rapidly decreasing #-derivatives. The space of tempered distributions (with its natural topology) is denoted $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$. Every #-compactly supported distribution is a tempered distribution , yielding an inclusion $\mathcal{E}^{\#\prime}({}^*\mathbb{R}_c^{\#n}) \hookrightarrow \mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$.

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§ 3. The Fourier transform on $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n}), \mathcal{S}^{\#}_{\mathbf{fin}}({}^*\mathbb{R}_c^{\#n})$

We begin by defining the Fourier transform, and the inverse transform, on $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$,

$n \in \mathbb{N}^{\#}$, the Schwartz space of $C^{\infty^{\#}}$ functions of rapid decrease.

Definition 3.1. Suppose $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. The Fourier transform of $f(x)$ is the function $\widehat{f}(\lambda)$
given by

$$\widehat{f}(\lambda) = \frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int\limits_{{}^*\mathbb{R}_c^{\#n}} f(x)[Ext\text{-}\exp(-ix\cdot\lambda)]d^{\#n}x\right), \tag{3.1}$$

where $\mathbf{x}\cdot\boldsymbol{\lambda} = Ext\text{-}\sum_{i=1}^{n} x_i\lambda_i$. The inverse Fourier transform of f, denoted by $\check{f}$, is the function

$$\check{f}(\lambda) = \frac{1}{(2\pi_{\#})^{n/2}}\left(Ext\text{-}\int\limits_{{}^*\mathbb{R}_c^{\#n}} f(x)[Ext\text{-}\exp(ix\cdot\lambda)]d^{\#n}x\right). \tag{3.2}$$

We will usualy write $\hat{f} = \mathcal{F}[f]$ and $\check{f} = \mathcal{F}^{-1}[f]$.

Since every function in Schwartz space is in $\mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#n})$, the above integrals (1.1) and (1.2) make sense.

We will use the standard multi-index notation. A multi-index $\alpha = \langle\alpha_1,\ldots,\alpha_n\rangle, n \in \mathbb{N}^{\#}$ is an n-tuple of nonnegative integers. The collection of all multi-indices will be denoted by I_+^n. The symbols $|\alpha|, x^\alpha, D^{\#\alpha}$, and x^2 are defined as follows:

$$|\alpha| = Ext\text{-}\sum_{i=1}^{n}\alpha_i$$

$$x^\alpha = Ext\text{-}\prod_{i=1}^{n} x_i^{\alpha_i} \text{ or } Ext\text{-}(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}) \text{ or simbolically } x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$$

$$D^{\#\alpha}f(x) = Ext\text{-}\prod_{i=1}^{n}\frac{\partial^{\#\alpha_i}}{\partial^{\#}x^{\alpha_i}}f(x) \text{ or simbolically } D^{\#\alpha}f(x) = \frac{\partial^{\#|\alpha|}f(x)}{\partial^{\#}x^{\alpha_1}\partial^{\#}x^{\alpha_2}\cdots\partial^{\#}x^{\alpha_n}} \quad (3.3)$$

$$x^2 = Ext\text{-}\sum_{i=1}^{n}x_i^2.$$

Lemma 1.1.The maps $f\mapsto\hat{f}$ and $f\mapsto\check{f}$ are #-continuous linear transformations of $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ into $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$.Furthermore, if α and β are multi-indices, then

$$\left((i\lambda)^\alpha D^{\#\beta}\hat{f}\right)(\lambda) = \widehat{D^{\#\alpha}\left((-ix)^\beta f(x)\right)}(\lambda). \quad (3.4)$$

Proof The map $f\mapsto\hat{f}$ is clearly linear. Since

$$\left((i\lambda)^\alpha D^{\#\beta}\hat{f}\right)(\lambda) =$$

$$\frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}(\lambda^\alpha)(-ix)^\beta f(x)[Ext\text{-}\exp(-ix\cdot\lambda)]f(x)d^{\#n}x\right) =$$

$$\frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}\frac{1}{(-i)^\alpha}(D_x^{\#\alpha}[Ext\text{-}\exp(-ix\cdot\lambda)])(-ix)^\beta f(x)d^{\#n}x\right) = \quad (3.5)$$

$$\frac{(-i)^\alpha}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}x}^{\#\beta}[Ext\text{-}\exp(-ix\cdot\lambda)]D_x^{\#\alpha}\left((-ix)^\beta f(x)\right)d^{\#n}x\right).$$

We conclude that

$$\left\|\hat{f}\right\|_{\alpha,\beta} = \sup_{\lambda\in{}^*\mathbb{R}_c^{\#n}}\left|\lambda^\alpha\left(D^{\#\beta}\hat{f}\right)(\lambda)\right| \leq \frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}|D_x^{\#\alpha}(x^\beta f(x))|d^{\#n}x\right) < \infty^\# \quad (3.6)$$

so $f\mapsto\hat{f}$ takes $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ into $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, and we have also proven (1.4).Furthermore, if k is large enough, $\int(1+x^2)^{-k}d^{\#n}x < \infty^\#$ so that

$$\left\|\hat{f}\right\|_{\alpha,\beta} \leq \frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}\frac{(1+x^2)^{-k}}{(1+x^2)^{-k}}\left|D_x^{\#\alpha}\left((-ix)^\beta f(x)\right)\right|d^{\#n}x\right) \leq$$

$$\frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}(1+x^2)^{-k}d^{\#n}x\right)\sup_{x\in{}^*\mathbb{R}_c^{\#n}}\left\{(1+x^2)^{+k}\left|D_x^{\#\alpha}\left((-ix)^\beta f(x)\right)\right|\right\}. \quad (3.7)$$

Using generalized Leibnitz's rule we easily conclude that there exist multi-indices α_j,β_j and constants c_j so that

$$\left\|\hat{f}\right\|_{\alpha,\beta} \le \sum_{j=1}^{M} c_j \|f\|_{\alpha_j,\beta_j}. \quad (1.8)$$

Thus the map $f \mapsto \hat{f}$ is bounded and therefore #-continuous. The proof for $f \mapsto \check{f}$ is the same.

Theorem 1.1. (Generalized Fourier inversion theorem) The Fourier transform (3.1) is a linear bicontinuous bijection from $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ onto $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. Its inverse map is the inverse Fourier transform, i.e.,$\mathcal{F}^{-1}(\mathcal{F}[f]) = f$ and $\mathcal{F}(\mathcal{F}^{-1}[f]) = f$.

Proof. We will prove that $\mathcal{F}^{-1}(\mathcal{F}[f]) = f$. The proof that $\mathcal{F}(\mathcal{F}^{-1}[f]) = f$ is similar. $\mathcal{F}(\mathcal{F}^{-1}[f]) = f$ implies that $\mathcal{F}[f]$ is surjective and $\mathcal{F}^{-1}(\mathcal{F}[f]) = f$ implies that $\mathcal{F}[f]$ is injective. Since $\mathcal{F}[f]$ and $\mathcal{F}^{-1}[f]$ are #-continuous maps of $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ onto $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, it is sufficient to prove that $\mathcal{F}^{-1}(\mathcal{F}[f]) = f$ for f contained in the dense set $C_0^{\infty^{\#}}({}^*\mathbb{R}_c^{\#n})$.
Let $C_\varepsilon, \varepsilon \approx 0$ be the cube of volume $(2/\varepsilon)^n$ centered at the origin in ${}^*\mathbb{R}_c^{\#n}$.Choose $\varepsilon \approx 0$ infinite small enough so that the support of f is contained in C_ε. Let
$\mathbf{K}_\varepsilon = \{\mathbf{k} \in {}^*\mathbb{R}_c^{\#n} |$ each $k_i/\varepsilon\pi_{\#} \in \mathbf{k}$ is an integer $\}$

$$f(x) = Ext\text{-}\sum_{\mathbf{k}\in\mathbf{K}_\varepsilon} \left(\left(\frac{1}{2}\varepsilon\right)^{n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)], f\right)\left(\frac{1}{2}\varepsilon\right)^{n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)] \quad (3.9)$$

is just the hyper infinite Fourier series of f which #-converges uniformly in C_ε to f since f is #-continuously #-differentiable. Thus

$$f(x) = Ext\text{-}\sum_{\mathbf{k}\in\mathbf{K}_\varepsilon} \frac{\widehat{f}(k)[Ext\text{-}\exp(i\mathbf{k}\cdot x)]}{(2\pi_{\#})^{n/2}}(\varepsilon\pi_{\#})^n. \quad (3.10)$$

Since ${}^*\mathbb{R}_c^{\#n}$ is the disjoint union of the cubes of volume $(\varepsilon\pi_{\#})^n$ centered about the points in $\mathbf{K}_\varepsilon$, the right-hand side of (1.10) is just a hyper finite Riemann sum for the integral of the function $\widehat{f}(k)[Ext\text{-}\exp(i\mathbf{k}\cdot x)]$. By the lemma 3.1, $\widehat{f}(k)[Ext\text{-}\exp(i\mathbf{k}\cdot x)] \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, so the hyperfinite Riemann sums (1.10) #-converge to the integral. Thus $\mathcal{F}^{-1}(\mathcal{F}[f]) = f$.

Corollary 3.1.Suppose $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. Then

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |f(x)|^2 d^{\#n}x = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |f(k)|^2 d^{\#n}k. \quad (3.11)$$

Proof. This is really a corollary of the proof rather than the statement of Theorem 1.1. If f has #-compact support, then for $\varepsilon \approx 0$ small enough,

$$f(x) = Ext\text{-}\sum_{\mathbf{k}\in\mathbf{K}_\varepsilon} \left(\left(\frac{1}{2}\varepsilon\right)^{n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)], f\right)\left(\frac{1}{2}\varepsilon\right)^{n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)] \quad (3.12)$$

Since $\left\{\left(\frac{1}{2}\varepsilon\right)^{n/2n/2}[Ext\text{-}\exp(i\mathbf{k}\cdot x)]\right\}_{\mathbf{k}\in\mathbf{K}_\varepsilon}$ is an orthonormal basis for $\mathcal{L}_2^{\#}(C_\varepsilon)$,

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |f(x)|^2 d^{\#n}x = Ext\text{-}\int_{C_\varepsilon} |f(x)|^2 d^{\#n}x = \sum_{\mathbf{k}\in\mathbf{K}_\varepsilon}\left|\left(\tfrac{1}{2}\varepsilon\right)^{n/2}([Ext\text{-}\exp(i\mathbf{k}\cdot x)], f(x))\right|^2 =$$
$$\sum_{\mathbf{k}\in\mathbf{K}_\varepsilon}\left|\widehat{f}(k)\right|^2(\varepsilon\pi_{\#})^n \underset{\varepsilon\to_{\#}0}{\to_{\#}} Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |f(k)|^2 d^{\#n}k. \quad (3.13)$$

This proves the corollary for $f \in C_0^{\infty^{\#}}({}^*\mathbb{R}_c^{\#n})$. Since $f \mapsto \widehat{f}$ and $\|\cdot\|_2$ are #-continuous on $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ and $C_0^{\infty^{\#}}({}^*\mathbb{R}_c^{\#n})$ is #-dense, the result holds for all of $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$.

Definition 3.2. Let $T \in \mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$the Fourier transform of $T,$denoted by $\hat{T}$ or $\mathcal{F}[T],$ is the tempered distribution defined by $\hat{T}(\varphi) = T\left(\widehat{\varphi}\right)$.

Suppose that $h,\varphi \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n}),$then by the polarization identity and the corollary to Theorem 1.1 we have $(h,\varphi) = \left(\widehat{h},\widehat{\varphi}\right)$. Substituting $\overline{\mathcal{F}[g]} = \mathcal{F}^{-1}[\bar{g}]$ for $h,$ we obtain

$$T_{\widehat{g}}(\varphi) = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#n}} \widehat{g}(x)\varphi(x)d^{\#n}x = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#n}} g(x)\widehat{\varphi}(x)d^{\#n}x = T_g\left(\widehat{\varphi}\right) = \hat{T}_g(\varphi).$$

where $T_{\widehat{g}}$ and T_g are the distributions corresponding to the functions $\widehat{g}$ and g respectively. This shows that the Fourier transform on $\mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$ extends the transform we previously defined on $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$.

Theorem 3.2. The Fourier transform is a one-to-one linear bijection from $\mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$ to $\mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$ which is the unique weakly #-continuous extension of the Fourier transform on $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$.

Proof. If hyper infinite sequence $\{\varphi_n\}_{n\in\mathbb{N}^{\#}}$ #-convergence to $\varphi \in \mathcal{S}^{\#},$ then by Theorem 1.1, hyper infinite sequence $\left\{\widehat{\varphi}_n\right\}_{n\in\mathbb{N}^{\#}}$ #-convergence to $\widehat{\varphi} \in \mathcal{S}^{\#},$so $T\left(\widehat{\varphi}_n\right) \to_{\#} T\left(\widehat{\varphi}\right)$ for each $T \in \mathcal{S}^{\#\prime}$. Thus #-$\lim_{n\to\infty^{\#}} T\left(\widehat{\varphi}_n\right) = T\left(\widehat{\varphi}\right)$, which shows that T is a #-continuous linear functional on $\mathcal{S}^{\#}$. Furthermore, if $T_n \overset{w}{\to}_w T,$ then $\widehat{T}_n \overset{w}{\to}_w \widehat{T}$ because $T\left(\widehat{\varphi}_n\right) \to_{\#} T\left(\widehat{\varphi}\right)$ implies $\widehat{T}(\varphi_n) \to_{\#} \widehat{T}(\varphi)$. Thus $T \mapsto \widehat{T}$ is weakly #-continuous.

Definition 3.3. Suppose that $f,g \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$. Then the convolution of f and $g,$ denoted by $f * g,$ is the function

$$(f * g)(y) = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#n}} f(y-x)g(x)d^{\#n}x. \tag{3.14}$$

Convolutions frequently occur when one uses the Fourier transform because the Fourier transform takes products into convolutions.

Theorem 3.3.(a) For each $f \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$, $g \mapsto f * g$ is a #-continuous map of $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$ into $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$.

(b) $\widehat{fg} = (2\pi_{\#})^{-n/2}\widehat{f} * \widehat{g}$ and $\widehat{f * g} = (2\pi_{\#})^{n/2}\widehat{f}\widehat{g}$.

(c) For $f,g,h \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$,$f * g = g * f$ and $f * (g * h) = (f * g) * h$.

Definition 3.4. Suppose that $f \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n}), T \in \mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$ and let $\widetilde{f}(x)$ denote the function, $f(-x)$. Then, the convolution of T and f denoted $T * f$ is the distribution in , $\mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$ given by $(T * f)(\varphi) = T\left(\widetilde{f} * \varphi\right)$ for all $\varphi \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$.

The fact that g→ $\widetilde{f} * g$ is a #-continuous transformation guarantees that $T * f \in \mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$.

Abbreviation 3.1.Let f_y denote the function $f_y(x) = f(x-y)$ and $\widetilde{f}_y$ the function $f(y-x)$.When f is given by a longe expression $(\bullet\bullet\bullet),$ we will sometimes write $(\bullet\bullet\bullet)^{\sim}$ rather than $(\bullet\tilde{\bullet}\bullet)$.

Theorem 3.4. For each $f \in \mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$ the map $T \to T * f$ is a weakly #-continuous map of $\mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$ into $\mathcal{S}^{\#\prime}({}^{*}\mathbb{R}_c^{\#n})$ which extends the convolution on $\mathcal{S}^{\#}({}^{*}\mathbb{R}_c^{\#n})$. Furthermore,

(a) $T * f$ is a polynomially bounded $C^{\infty^{\#}}$ function. In fact, $(T * f)(y) = T\left(\widetilde{f}_y\right)$ and $D^{\#\beta}(T * f) = (D^{\#\beta}T) * f = T * D^{\#\beta}f$;

(b) $(T * f) * g = T * (f * g)$;

(c) $\widehat{T * f} = (2\pi_{\#})^{n/2}\widehat{f}\widehat{T}$.

Theorem 3.5. Let $T \in \mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ and $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. Then $\widehat{fT} \in O_M^n$ and $\widehat{fT}\,(k) = (2\pi_\#)^{n/2}T(f[Ext\text{-}\exp(-ik\cdot x)])$. In particular, if T has #-compact support and $\psi \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ is identically one on a #-neighborhood of the support of T, then

$$\widehat{T}(k) = (2\pi_\#)^{n/2}T(\psi[Ext\text{-}\exp(-ik\cdot x)]). \qquad (3.15)$$

Proof By Theorem 3.4.c and the Fourier inversion formula we have $\widehat{fT} = (2\pi_\#)^{n/2}\widehat{f} * \widehat{T}$. Thus $\widehat{fT} \in O_M^n$ and $\widehat{fT}\,(k) = (2\pi_\#)^{n/2}\widehat{T}\left(\widetilde{\widehat{f}}_k\right) =$ $(2\pi_\#)^{n/2}T(f[Ext\text{-}\exp(-ik\cdot x)])$.

Remark 3.1.We remark that one can also define the convolution of a distribution $T \in \mathcal{D}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ with an $f \in \mathcal{D}^{\#}({}^*\mathbb{R}_c^{\#n})$ by $(T * f)(y) = T\left(\widetilde{f}_y\right)$.

Definition 3.5. Let $j(x)$ be a positive $C^{\infty^\#}$ function whose support lies in the sphere of radius one about the origin in ${}^*\mathbb{R}_c^{\#n}$ and which satisfies $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} j(x)d^{\#n}x = 1$. The function $j_\varepsilon(x) = \varepsilon^{-n}j(x/\varepsilon), \varepsilon \approx 0$ is called an approximate identity.

Proposition 3.1. Suppose $T \in \mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$ and let $j_\varepsilon(x)$ be an approximate identity. Then $T * j_\varepsilon(x) \to_\# T$ weakly as $\varepsilon \to_\# 0$.

Proof. If $\varphi \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, then $(T * j_\varepsilon)(\varphi) = T\left(\widetilde{j}_\varepsilon * \varphi\right)$, so it is sufficient to show that $\widetilde{j}_\varepsilon * \varphi \to_\# \varphi$ in $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. To do this it is sufficient to show that $(2\pi_\#)^{n/2}\widehat{j}_\varepsilon\widehat{\varphi} \to_\# \widehat{\varphi}$ in $\mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$. Since $\widehat{j}_\varepsilon(\lambda) = j(\varepsilon\lambda)$ and $j(0) = (2\pi_\#)^{n/2}$, it follows that $(2\pi_\#)^{n/2}\widehat{j}_\varepsilon(x)$ #-converges to 1 uniformly on #-compact sets and is uniformly bounded. Similarly, $D^{\#\alpha}\widehat{j}_\varepsilon$ #-converges uniformly to zero. We conclude that $(2\pi_\#)^{n/2}\widehat{j}_\varepsilon\widehat{\varphi} \to_\# \widehat{\varphi}$.

Theorem 3.6 (The generalized Plancherel theorem) The Fourier transform extends uniquely to a unitary map of $\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#n})$ onto $\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#n})$. The inverse transform extends uniquely to its adjoint.

Proof The corollary to Theorem 3.1 states that if $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, then $\|f\|_2 = \left\|\widehat{f}\right\|_2$. Since $\mathcal{F}[\mathcal{S}^\#] = \mathcal{S}^\#$ is a surjective isometry on $\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#n})$.

Theorem 3.7 (The generalized Riemann-Lebesgue lemma) The Fourier transform extends uniquely to a bounded map from $\mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#n})$ into $C^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$, the #-continuous functions vanishing at $\infty^\#$.

Proof For $f \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, we know that $\widehat{f} \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ and thus $\widehat{f} \in C^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$. The estimate is trivial. The Fourier transform is thus a bounded linear map from a #-dense set of $\mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#n})$ into $C^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$. By the generalized B.L.T. theorem, extends uniquely to a bounded linear transformation of $C^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$ into $C^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$.

Remark 3.2.We remark that the Fourier transform takes $\mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#n})$ into, but not onto $C^{\infty^\#}({}^*\mathbb{R}_c^{\#n})$.

A simple argument with test functions shows that the extended transform on $\mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#n})$ and $\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#n})$ is the restriction of the transform on $\mathcal{S}^{\#\prime}({}^*\mathbb{R}_c^{\#n})$, but it is useful to have an explicit integral representation. For $f \in \mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#n})$, this is easy since we can find $f_m \in \mathcal{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ so that $\#\text{-}\lim_{m\to\infty^\#}\|f - f_m\|_1 = 0$. Then, for each λ,

$$f(\lambda) = \#\text{-}\lim_{m\to\infty^\#}\left(\widehat{f}_m(\lambda)\right) =$$

$$\#\text{-}\lim_{m\to\infty^\#}\left\{\frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}[Ext\text{-}\exp(-ik\cdot x)]f_m(x)d^\#x\right)\right\} = \qquad (3.16)$$

$$\frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}}[Ext\text{-}\exp(-ik\cdot x)]f(x)d^\#x\right).$$

So, the Fourier transform of a function in $\mathcal{L}_1^\#({}^*\mathbb{R}_c^{\#n})$ is given by the usual formula.
Next, suppose $f \in \mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#n})$ and let

$$\chi_R(x) = \begin{cases} 1 & \text{if } |x| \leq R \\ 0 & \text{if } |x| > R \end{cases} \qquad (3.17)$$

Then $\chi_R f \in \mathcal{L}_1^\#({}^*\mathbb{R}_c^{\#n})$ and #-$\lim_{R\to\infty^\#}\chi_R f = f$ in $\mathcal{L}_2^\#$, so by the generalized Plancherel theorem #-$\lim_{R\to\infty^\#}\widehat{\chi_R f} = \widehat{f}$ in $\mathcal{L}_2^\#$. Thus

$$f(\lambda) = \#\text{-}\lim_{R\to\infty^\#}\frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int_{|x|\leq R}[Ext\text{-}\exp(-ik\cdot x)]f(x)d^\#x\right) \qquad (3.18)$$

where by #-$\lim_{R\to\infty^\#}$ we mean the #-limit in the $\mathcal{L}_2^\#$-norm. Sometimes we will dispense with $|x| \leq R$ and just write

$$f(\lambda) = \#\text{-}\lim_{R\to\infty^\#}\frac{1}{(2\pi_\#)^{n/2}}\left(Ext\text{-}\int[Ext\text{-}\exp(-ik\cdot x)]f(x)d^\#x\right) \qquad (3.19)$$

for functions $f \in \mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#n})$.
We have proven above that $\mathcal{F} : \mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#n}) \to \mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#n})$ and $\mathcal{F} : \mathcal{L}_1^\#({}^*\mathbb{R}_c^{\#n}) \to \mathcal{L}_{\infty^\#}^\#({}^*\mathbb{R}_c^{\#n})$ and in both cases is a bounded operator.
Theorem 3.8 (Generalized Hausdorff-Young inequality) Suppose $1 \leq q \leq 2$, and $p^{-1} + q^{-1} = 1$. Then the Fourier transform is a bounded map of $\mathcal{L}_p^\#({}^*\mathbb{R}_c^{\#n})$ to $\mathcal{L}_q^\#({}^*\mathbb{R}_c^{\#n})$ and its norm is less than or equal to $(2\pi_\#)^{n(1/2-1/q)}$.

Chapter III.Hilbert Spaces over field ${}^*\mathbb{C}_c^\#$.

§ 1. Hilbert Spaces over field ${}^*\mathbb{C}_c^\#$ Basics.

Definition 1. Let H be external hyper infinite dimensional vector space over field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c + i{}^*\mathbb{R}_c$.An inner product on H is a ${}^*\mathbb{C}_c^\#$-valued function,
$\langle\cdot,\cdot\rangle_\# : H \times H \to {}^*\mathbb{C}_c$, such that
(1) $\langle ax + by, z\rangle_\# = a\langle x,z\rangle + b\langle y,z\rangle_\#$, i.e. $x \to \langle x,z\rangle_\#$ is linear.
(2) $\overline{\langle x,y\rangle_\#} = \langle y,x\rangle_\#$.
(3) $\|x\|_\#^2 \equiv \langle x,x\rangle_\# \geq 0$ with equality $\|x\|_\#^2 = 0$ iff $x = 0$.
Notice that combining properties (1) and (2) that x $\to \langle z,x\rangle$ is anti-linear for fixed $z \in H$, i.e. $\langle z, ax + by\rangle_\# = \bar{a}\langle z,x\rangle_\# + \bar{b}\langle z,y\rangle_\#$.
The following formula useful:

$$\|x+y\|_\#^2 = \langle x+y, x+y\rangle_\# = \|x\|_\#^2 + \|y\|_\#^2 + \langle x,y\rangle_\# + \langle y,x\rangle_\# = \|x\|_\#^2 + \|y\|_\#^2 + 2\,\mathrm{Re}\langle x,y\rangle_\# \quad (1.1)$$

Theorem 1. (Generalized Schwarz Inequality). Let $(H,\langle\cdot,\cdot\rangle_\#)$ be an inner product space, then for all $x,y \in H$

$$|\langle x,y\rangle_{\#}| \leq \|x\|_{\#}\|y\|_{\#} \qquad (1.2)$$

and equality holds iff x and y are linearly dependent.

Proof. If $y = 0$, the result holds trivially. So assume that $y \neq 0$. First off notice that if $x = \alpha y$ for some $\alpha \in {}^*\mathbb{C}_c^{\#}$, then $\langle x,y\rangle = \alpha\|y\|_{\#}^2$ and hence $|\langle x,y\rangle_{\#}| = |\alpha|\|y\|_{\#}^2 = \|x\|_{\#}\|y\|_{\#}$ Note that in this case $\alpha = \langle x,y\rangle\|y\|^2$. Now suppose that $x \in H$ is arbitrary, let $z \equiv x - \|y\|_{\#}^{-2}\langle x,y\rangle_{\#}y$. So z is the orthogonal projection of x onto y. Then

$$\begin{aligned}0 \leq \|z\|_{\#}^2 = \left\|x - \frac{\langle x,y\rangle_{\#}}{\|y\|_{\#}^2}y\right\|_{\#}^2 = \|x\|_{\#}^2 + \frac{|\langle x,y\rangle_{\#}|^2}{\|y\|_{\#}^4}\|y\|_{\#}^2 - 2\operatorname{Re}\left\langle x, \frac{\langle x,y\rangle_{\#}}{\|y\|_{\#}^2}y\right\rangle = \\ = \|x\|_{\#}^2 - \frac{|\langle x,y\rangle_{\#}|^2}{\|y\|_{\#}^2}.\end{aligned} \qquad (1.3)$$

from (1.3) it follows that $0 \leq \|y\|_{\#}^2\|x\|_{\#}^2 - |\langle x,y\rangle_{\#}|^2$ with equality iff $z = 0$ or equivalently iff $x = \|y\|_{\#}^{-2}\langle x,y\rangle_{\#}y$.

Corollary 1. Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space and $\|x\|_{\#} := \sqrt{\langle x,x\rangle_{\#}}$. Then $\|\cdot\|_{\#}$ is a ${}^*\mathbb{R}_c$-valued #-norm on H. Moreover $\langle\cdot,\cdot\rangle_{\#}$ is #-continuous on $H \times H$, where H is viewed as the #-normed space $(H,\|\cdot\|_{\#})$.

Proof. The only non-trivial thing to verify that $\|\cdot\|_{\#}$ is a #-norm is the triangle inequality:
$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x,y\rangle_{\#} \leq \|x\|^2 + \|y\|^2 + 2\|x\|_{\#}\|y\|_{\#} = (\|x\|_{\#} + \|y\|_{\#})^2$
where we have made use of Schwarz's inequality. Taking the square root of this inequality shows $\|x+y\| \leq \|x\| + \|y\|$. For the #-continuity assertion:
$|\langle x,y\rangle_{\#} - \langle x',y'\rangle_{\#}| = |\langle x-x',y\rangle_{\#} + \langle x',y-y'\rangle_{\#}| \leq \|y\|_{\#}\|x-x'\|_{\#} + \|x'\|_{\#}\|y-y'\|_{\#}$
$\leq \|y\|_{\#}\|x-x'\|_{\#} + (\|x\|_{\#} + \|x-x'\|_{\#})\|y-y'\|_{\#} = \|y\|_{\#}\|x-x'\|_{\#} + \|x\|_{\#}\|y-y'\|_{\#}$
$+ \|x-x'\|_{\#}\|y-y'\|_{\#}$ from which it follows that $\langle\cdot,\cdot\rangle$ is #-continuous.

Definition 2. Let $(H,\langle\cdot,\cdot\rangle_{\#})$ be an inner product space, we say $x,y \in H$ are orthogonal and write $x \perp y$ iff $\langle x,y\rangle_{\#} = 0$. More generally if $A \subset H$ is a set, $x \in H$ is orthogonal to A and write $x \perp A$ iff $\langle x,y\rangle = 0$ for all $y \in A$. Let $A_{\perp} = \{x \in H : x \perp A\}$ be the set of vectors orthogonal to A. We also say that a set $S \subset H$ is orthogonal if $x \perp y$ for all $x,y \in S$ such that $x \neq y$. If S further satisfies, $\|x\|_{\#} = 1$ for all $x \in S$, then S is said to be orthonormal.

Proposition 1. Let $(H,\langle\cdot,\cdot\rangle_{\#})$ be an inner product space then
(1) (Parallelogram Law)

$$\|x+y\|_{\#}^2 + \|x-y\|_{\#}^2 = 2\|x\|_{\#}^2 + 2\|y\|_{\#}^2 \qquad (1.4)$$

for all $x,y \in H$.
(2) (Pythagorean Theorem) If $S \subset H$ is a finite orthonormal set, then

$$\left\|\sum_{x\in S} x\right\|_{\#}^2 = \sum_{x\in S}\|x\|_{\#}^2 \qquad (1.5)$$

(3) If $A \subset H$ is a set, then $A_{\perp}$ is a #-closed linear subspace of H.

Proof. I will assume that H is a complex Hilbert space with ${}^*\mathbb{C}_c$ -valued inner product, the real case being easier. Statements (1) and (2) are proved by the following elementary computations:

$\|x+y\|_{\#}^{2} = \|x\|_{\#}^{2} + \|y\|_{\#}^{2} + 2\,\mathrm{Re}\langle x,y\rangle_{\#} + \|x\|_{\#}^{2} + \|y\|_{\#}^{2} - 2\,\mathrm{Re}\langle x,y\rangle_{\#} = 2\|x\|_{\#}^{2} + 2\|y\|_{\#}^{2}$ and

§ 3.#-Analytic vectors.Generalized Nelson's #-analytic vector theorem.

Let $\mathbf{H}^{\#}$ be a #-complex Hilbert space over field ${}^{*}\mathbb{C}_{c}^{\#}$. The most natural way to construct

a #-continuous one-parameter unitary group $U(t) : \mathbf{H}^{\#} \to \mathbf{H}^{\#}$ is to try to make sense of the power series $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(itA)^{n}$ on a #-dense set of vectors. Notice that this can certainly be done if A is self-adjoint. For let E_{Ω} be the family of spectral projections for A. Then on each of the spaces $E_{[-M,M]}$, A is a bounded operator and $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(itA)^{n}/n!$ #-converges to $Ext\text{-}\exp(itA)$ in norm. In particular, for any $\varphi \in \bigcup_{M\geq 0} E_{[-M,M]}$,

$$\#\text{-}\lim_{N\to\infty^{\#}}\left(Ext\text{-}\sum_{n=0}^{N}\frac{(itA)^{n}}{n!}\right) = Ext\text{-}\exp(itA). \quad (3.1)$$

Since $\bigcup_{M\geq 0} E_{[-M,M]}$ is #-dense in $\mathbf{H}^{\#}$, we see that the group generated by a self-adjoint operator A is completely determined by the well-defined action of the hyper infinite series $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(itA)^{n}/n!$ on a #-dense set. We will prove the #-converse: namely, if A is symmetric and has a #-dense set of vectors to which $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(itA)^{n}/n!$ can be applied, then A is essentially self-#-adjoint. We need several definitions.

Definition1.1. Let A be an operator on a non-Archimedean Hilbert space $\mathbf{H}^{\#}$. The set $\mathbf{C}^{\infty^{\#}}(A) = \bigcap_{n=0}^{\infty^{\#}} D(A^{n})$ is called the $\mathbf{C}^{\infty^{\#}}$-vectors for A. A vector $\varphi \in \mathbf{C}^{\infty^{\#}}(A)$ is called an #-analytic vector for A if

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}}\frac{\|A^{n}\varphi\|t^{n}}{n!} < {}^{*}\infty \quad (3.2)$$

for some $t > 0$. If A is self-adjoint, then $\mathbf{C}^{\infty^{\#}}(A)$ will be #-dense in $D(A)$. However, in general, a symmetric operator may have no $\mathbf{C}^{\infty^{\#}}$-vectors at all even if A is essentially self-#-adjoint. We caution the reader to remember that #-analytic vectors and vectors of

uniqueness (defined below) must be $\mathbf{C}^{\infty^{\#}}$- vectors for A. A vector $\varphi \in D(A)$ can be an #-analytic vector for an extension of A but fail to be an #-analytic vector for A because it is not in $\mathbf{C}^{\infty^{\#}}(A)$.

Definition1.2.Suppose that A is symmetric. For each $\varphi \in \mathbf{C}^{\infty^{\#}}(A)$, define

$$D_{\varphi} = \left\{Ext\text{-}\sum_{n=0}^{N}\alpha_{n}A^{n}\varphi \,\middle|\, N \in {}^{*}\mathbb{N}, \alpha_{n} \in {}^{*}\mathbb{C}_{c}^{\#}\right\}. \quad (3.3)$$

Let $\mathbf{H}_{\varphi}^{\#} = \#\text{-}\overline{D_{\varphi}}$ and define $A_{\varphi} : D_{\varphi} \to D_{\varphi}$ by $A_{\varphi}\left(Ext\text{-}\sum_{n=0}^{N}\alpha_{n}A^{n}\varphi\right) = Ext\text{-}\sum_{n=0}^{N}\alpha_{n}A^{n+1}\varphi$. φ is called a vector of #-uniqueness if and only if A_{φ} is essentially self-#-adjoint on D_{φ} as an operator on $\mathbf{H}_{\varphi}^{\#}$.

Finally, a subset $S \subset \mathbf{H}^{\#}$ is called #-total if the set of hyperfinite linear combinations of elements of S is #-dense in $\mathbf{H}^{\#}$.

Lemma (Generalized Nussbaum's lemma) Let A be a symmetric operator and suppose that $D(A)$ contains a #-total set of vectors of #-uniqueness. Then A is essentially self-#-adjoint.

Proof We will show that $\mathbf{Ran}(A \pm i)$ are #-dense in $\mathbf{H}^{\#}$. By the fundamental criterion this will show that A is essentially self-#-adjoint. Suppose $\psi \in \mathbf{H}^{\#}$ and $\varepsilon > 0$ are given and let S denote the set of vectors of #-uniqueness. Since S is #-total we can find $(\alpha_n)_{n=1}^{N}$ and $(\psi_n)_{n=1}^{N}$ with $\psi_n \in S$ so that

$$\left\| \psi - Ext\text{-}\sum_{n=1}^{N} \alpha_n \psi_n \right\|_{\#} \leq \varepsilon/2. \tag{3.4}$$

Since ψ_n is a vector of #-uniqueness, there is a $\varphi_n \in D_{\psi_n}$ so that

$$\|\psi_n - (A+i)\varphi_n\|_{\#} \leq \frac{\varepsilon}{2}\left(Ext\text{-}\sum_{n=1}^{N}|\alpha_n|\right)^{-1}. \tag{3.5}$$

Setting $\varphi = Ext\text{-}\sum_{n=1}^{N} \alpha_n \varphi_n$ we have $\varphi \in D(A)$ and $\|\psi - (A+i)\varphi\|_{\#} < \varepsilon$.

Thus $\mathbf{Ran}(A+i)$ is #-dense. The proof for $(A-i)$ is the same.

Theorem 3.1. (Generalized Nelson's #-analytic vector theorem) Let A be a symmetric operator on a non-Archimedean Hilbert space $\mathbf{H}^{\#}$. If $D(A)$ contains a #-total set of #-analytic vectors, then A is essentially self-#-adjoint.

Proof By Generalized Nussbaum's lemma, it is enough to show that each #-analytic vector ψ is a vector of #-uniqueness. First notice that A_ψ always has self-#-adjoint extensions, since the operator

$$C : Ext\text{-}\sum_{n=0}^{N} \alpha_n A^n \psi \tag{3.6}$$

extends to a conjugation on $\mathbf{H}_{\psi}^{\#}$ which commutes with A_ψ. Suppose that B is a self-#-adjoint extension of A_ψ on $\mathbf{H}_{\psi}^{\#}$ and let $\mu^{\#}$ be the spectral #-measure for B associated to ψ. Since ψ is an #-analytic vector for A,

$$Ext\text{-}\sum_{n=0}^{N} \|A^n\psi\|_{\#}/n! < {}^*\infty \tag{3.7}$$

for some $t > 0$. Let $0 < s < t$. Then

$$\begin{gathered} Ext\text{-}\sum_{n=0}^{{}^*\infty} \frac{s^n}{n!}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} |x|^n d^{\#}\mu^{\#} \right) \leq \\ \leq Ext\text{-}\sum_{n=0}^{{}^*\infty} \frac{s^n}{n!}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} x^{2n} d^{\#}\mu^{\#} \right)^{1/2}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} d^{\#}\mu^{\#} \right)^{1/2} = \\ \|\psi\|_{\#} Ext\text{-}\sum_{n=0}^{{}^*\infty} \frac{s^n}{n!}\|A^n\psi\|_{\#} < {}^*\infty. \end{gathered} \tag{3.8}$$

Therefore by generalized Fibini's theorem

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \left(Ext\text{-}\sum_{n=0}^{{}^*\infty} \frac{s^n}{n!}|x|^n \right) d^{\#}\mu^{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} Ext\text{-}(s|x|) d^{\#}\mu^{\#} < {}^*\infty. \tag{3.9}$$

As a result, the function

$$\langle \psi, [Ext\text{-}\exp(itB)]\psi \rangle_{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} [Ext\text{-}\exp(itx)] d^{\#}\mu^{\#} \tag{3.10}$$

has an #-analytic continuation

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}[Ext\text{-}\exp(izx)]d^\#\mu^\# \qquad (3.11)$$

to the region $|\mathrm{Im}\,z|< t.$ Since

$$\left[\left(\frac{d^\#}{d^\# z}\right)^k\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}[Ext\text{-}\exp(izx)]d^\#\mu^\#\right)\right]_{z=0} = \qquad (3.12)$$
$$= Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}\left[Ext\text{-}\exp(ix)^k\right]d^\#\mu^\# = \left\langle\psi,(iA)^k\psi\right\rangle_\#,$$

we obtain

$$\langle\psi,[Ext\text{-}\exp(isB)]\psi\rangle_\# = Ext\text{-}\sum_{n=0}^{{}^*\infty}\frac{(is)^n}{n!} = \left\langle\psi,(iA)^k\psi\right\rangle_\# \qquad (3.13)$$

for $|s|< t.$ Thus, for $|s|< t$ (and therefore for all s), the function $\langle\psi_1,[Ext\text{-}\exp(isB)]\psi_2\rangle_\#$ is completely determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_\#, n\in{}^*\mathbb{N}$.
Similar proof shows that $\langle\psi_1,[Ext\text{-}\exp(isB)]\psi_2\rangle_\#$ is determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_\#, n\in{}^*\mathbb{N}$for any $\psi_1,\psi_2\in D_\psi.$ Since D_ψ is #-dense in $\mathbf{H}_\psi^\#$ and $Ext\text{-}\exp(isB)$ is unitary, $Ext\text{-}\exp(isB)$ is completely determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_\#, n\in{}^*\mathbb{N}$ for any $\psi_1,\psi_2\in D_\psi.$Thus, all self-#-adjoint extensions of A_ψ generate the same unitary group, so by generalized Stone's theorem A_ψ has at most one self-#-adjoint extension. As we have already remarked, A_ψ has at least one self-#-adjoint extension. Thus A_ψ is essentially self-#-adjoint and ψ is a vector of uniqueness.
Corollary 3.1 A #-closed symmetric operator A is self-#-adjoint if and only if $D(A)$ contains a #-dense set of #-analytic vectors.
The statement of Corollary 1 is not true if "self-#-adjoint" is replaced by "essentially self-#-adjoint." A self-#-adjoint operator A may be essentially self-#-adjoint on a domain $D\subset D(A)$ and D may not even contain any #-vectors.
Corollary 3.2 Suppose that A is a symmetric operator and let D be a #-dense linear set contained in $D(A).$ Then, if D contains a #-dense set of #-analytic vectors and if D is invariant under $A,$ then A is essentially self-#-adjoint on $D.$
Proof Since D is invariant under A, each #-analytic vector for A in D is also an #-analytic vector for $A\restriction D.$ Thus, by Theorem 3.1 $A\restriction D$ is essentially self-#-aadjoint.
The reason that one needs the invariance condition in Corollary 2 is that for a vector $\psi\in D$ to be #-analytic for $A\restriction D,$ it must first be $C^{*\infty}$ for $A\restriction D.$ This requires that $A^n\in D$ for all $n\in{}^*\mathbb{N}.$

§4.The generalized Spectral Theorem

§ 4.1.The #-continuous functional calculus

In this section, we will discuss the generalized spectral theorem in its many guises. This structure theorem is a concrete description of all self-#-adjoint operators. There are several apparently distinct formulations of the spectral theorem. In some sense they are all equivalent.

The form we prefer says that every bounded self-#-adjoint operator is a multiplication operator. (We emphasize the word bounded since we will deal extensively with unbounded self-#-adjoint operators in the next chapter; there is a spectral theorem for unbounded operators which we discuss in Section § 4.3)

This means that given a bounded self-#-adjoint operator A on a non-Archimedean Hilbert space $\mathbf{H}^{\#}$ over field ${}^{*}\mathbb{R}_{c}^{\#}$ or ${}^{*}\mathbb{C}_{c}^{\#}$, we can always find a #-measure $\mu^{\#}$ on a #-measure space M and a unitary operator $U : \mathbf{H}^{\#} \to L_{2}^{\#}(M, d^{\#}\mu^{\#})$ so that

$$(UAU^{-1}f)(x) = F(x)f(x) \qquad (4.1.1)$$

for some bounded real-valued #-measurable function F on M.

In practice, M will be a union of copies of ${}^{*}\mathbb{R}_{c}^{\#}$ and F will be x so the core of the proof of the theorem will be the construction of certain #-measures. This will be done in Section

§ 4.2 by using the generalized Riesz-Markov theorem. Our goal in this section will be to

make sense out of $f(A)$, for f a #-continuous function.

In the next section, we will consider the #-measures defined by the functionals

$$f \mapsto \langle \psi, f(A)\psi \rangle_{\#} \qquad (4.1.2)$$

for fixed $\psi \in \mathbf{H}^{\#}$.

Given a fixed operator A, for which f can we define $f(A)$? First, suppose that A is an arbitrary bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ operator. If $f(x) = Ext\text{-}\sum_{n=1}^{N} c_n x^n$, $N \in {}^{*}\mathbb{N}$ is a polynomial, we let $f(A) = Ext\text{-}\sum_{n=1}^{N} c_n A^n$. Suppose that $f(x) = Ext\text{-}\sum_{n=1}^{{}^{*}\infty} c_n x^n$ is a hyper infinite power series with radius of #-convergence R. If $\|A\|_{\#} < R$ then hyper infinite power series $Ext\text{-}\sum_{n=1}^{{}^{*}\infty} c_n A^n$ #-converges in $\mathcal{L}(H^{\#})$ so it is natural to set

$$f(A) = Ext\text{-}\sum_{n=1}^{{}^{*}\infty} c_n A^n \qquad (4.1.3)$$

In this last case, f was a function #-analytic in a domain including all of $\sigma(A)$.

The functional calculus we have talked about thus far works for any operator in any Banach space. The special property of self-adjoint operators or more generally normal operators is that $\|P(A)\|_{\#} = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$ for any polynomial P, so that one can use the B.L.T. theorem to extend the functional calculus to #-continuous functions. Our major goal in this section is the proof of:

Theorem 4.1.1. (#-continuous functional calculus) Let A be a self-#-adjoint operator on a Hilbert space $H^{\#}$. Then there is a unique map $\phi : C^{\#}(\sigma(A)) \to \mathcal{L}(H^{\#})$ with the following properties:

(a) ϕ is an algebraic $*$-homomorphism, that is,
$\phi(fg) = \phi(f)\phi(g), \phi(\lambda f) = \lambda\phi(f), \phi(1) = I, \phi(\bar{f}) = \phi(f)^{*}$.

(b) ϕ is #-continuous, that is, $\|\phi(f)\|_{\mathcal{L}(H^{\#})} \leq C\|f\|_{{}^{*}\infty}$.

(c) Let f be the function $f(x) = x$; then $\phi(f) = A$.

Moreover, ϕ has the additional properties:

(d) If $A\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$.

(e) $\sigma[\phi(f)] = \{f(\lambda) | \lambda \in \sigma(A)\}$ [spectral mapping theorem].

(f) If $f \geq 0$, then $\phi(f) \geq 0$.

(g) $\|\phi(f)\|_{\#} = \|f\|_{{}^{*}\infty}$. [this strengthens (b)].

The proof which we give below is quite simple, (a) and (c) uniquely determine $\phi(P)$ for any hyperfinite polynomial $P(x)$. By the generalized Weierstrass theorem, the set of polynomials is #-dense in $C^{\#}(\sigma(A))$ so the main part of the proof is showing that

$$\|P(A)\|_{\#op} = \|P(x)\|_{C^{\#}(\sigma(A))} = \sup_{\lambda\in\sigma(A)}|P(\lambda)|. \tag{4.1.4}$$

The existence and uniqueness of ϕ then follow from the generalized B.L.T. theorem. To prove the crucial equality, we first prove a special case of (e) (which holds for arbitrary bounded operators):

Lemma 4.1.1.Let $P(x) = Ext\text{-}\sum_{n=1}^{N} c_n x^n,\ N \in {}^*\mathbb{N}$. Let $P(A) = Ext\text{-}\sum_{n=1}^{N} c_n A^n$. Then

$$\sigma(P(A)) = \{P(\lambda)|\lambda \in \sigma(A)\}. \tag{4.1.5}$$

Proof Let $\lambda \in \sigma(A)$. Since $x = \lambda$ is a root of $P(x) - P(\lambda)$, we have $P(x) - P(\lambda) = (x-\lambda)Q(x)$, so $P(A) - P(\lambda) = (A-\lambda)Q(A)$. Since $(A-\lambda)$ has no inverse neither does $P(A) - P(\lambda)$ that is, $P(\lambda) \in \sigma(P(A))$.
Conversely, let $\mu \in \sigma(P(A))$ and let $\lambda_1,\ldots,\lambda_n$ be the roots of $P(x) - \mu$, that is, $P(x) - \mu = a\left(Ext\text{-}\prod_{i=1}^{n}(x-\lambda_i)\right)$. If $\lambda_1,\ldots,\lambda_n \notin \sigma(A)$, then

$$(P(A)-\mu)^{-1} = a^{-1}\left(Ext\text{-}\prod_{i=1}^{n}(A-\lambda_i)^{-1}\right) \tag{4.1.6}$$

so we conclude that some $\lambda_i \in \sigma(A)$ that is, $\mu = P(\lambda)$ for some $\lambda \in \sigma(A)$.

Definition Let $r(A) = \sup_{\lambda\in\sigma(A)}|\lambda|$. Then $r(A)$ is called the spectral radius of A.

Theorem 4.1.2. Let X be a Banach space, $A \in \mathcal{L}(X)$ Then $\lim_{n\to{}^*\infty}\sqrt[n]{\|A^n\|_{\#op}}$ exists and is equal to $r(A)$. If X is a Hilbert space and A is self-#-adjoint, then $r(A) = \|A\|_{\#op}$.

Lemma 4.1.2. Let A be a bounded self-#-adjoint operator. Then

$$\|P(A)\|_{\#} = \sup_{\lambda\in\sigma(A)}|P(\lambda)|. \tag{4.1.7}$$

Proof By Theorem 4.1.2 and by Lemma 4.1.1 we obtain

$$\|P(A)\|_{\#}^2 = \|P(A)^*P(A)\|_{\#} = \|(\overline{P}P)(A)\|_{\#} = \sup_{\lambda\in\sigma\left((\overline{P}P)(A)\right)}|\lambda| = \sup_{\lambda\in\sigma(A)}|\overline{P}P(\lambda)| = \left(\sup_{\lambda\in\sigma(A)}|P(\lambda)|\right)^2.$$

Proof of **Theorem 4.1.1**. Let $\phi(P) = P(A)$. Then $\|\phi(P)\|_{\mathcal{L}(H^{\#})} = \|P\|_{C^{\#}(\sigma(A))}$ so ϕ has a unique linear extension to the #-closure of the polynomials in $C^{\#}(\sigma(A))$. Since the polynomials are an algebra containing $\mathbf{I}$, containing complex conjugates, and separating points, this #-closure is all of $C^{\#}(\sigma(A))$. Properties (a), (b), (c), (g) are obvious and if $\widetilde{\phi}$ obeys (a), (b), (c) it agrees with ϕ on polynomials and thus by #-continuity on $C^{\#}(\sigma(A))$ To prove (d), note that $\phi(P)\psi = P(\lambda)\psi$ and apply #-continuity. To prove (f), notice that if $f \geq 0$, then $f = g^2$ with g ${}^*\mathbb{R}_c^{\#}$-valued and $g \in C^{\#}(\sigma(A))$. Thus $\phi(f) = \phi(g)^2$ with $\phi(g)$ self-#-adjoint, so $\phi(f) \geq 0$.

Remark 4.1.1. In addition:
(1) $\phi(f) \geq 0$ if and only if $f \geq 0$.
(2) Since $fg = gf$ for all f,g, $\{f(A)|f \in C^{\#}(\sigma(A))\}$ forms an abelian algebra closed under adjoints. Since $\|\phi(f)\|_{\#} = \|f\|_{{}^*\infty}$ and $C^{\#}(\sigma(A))$ is #-complete, $\{f(A)|f \in C^{\#}(\sigma(A))\}$ is #-norm-#-closed. It is thus an non-Archimedean abelian $\mathbf{C}^*$ algebra of operators.
(3) $\mathbf{Ran}(\phi)$ is actually the non-Archimedean $\mathbf{C}^*$ algebra generated by A that is, the smallest $\mathbf{C}^*$-algebra containing A.
(4) This result, that $C^{\#}(\sigma(A))$ and the non-Archimedean $\mathbf{C}^*$-algebra generated by A

are #-isometrically isomorphic
(5) (b) actually follows from (a) and Proposition 4.1.1. Thus (a) and (c) alone determine ϕ uniquely.
Proposition 4.1.1. Suppose that ϕ: $C^{\#}(X) \to \mathcal{L}(H^{\#})$ is an algebraic $*$-homomorphism, X a #-compact metric space. Then
(a) If $f \geq 0$, then $\phi(f) \geq 0$.
(b) $\|\phi(f)\|_{\#} \leq \|f\|_{*\infty}$.
Theorem 4.1.2. (Generalized Weierstrass Approximation Theorem). Let $f \in C^{\#}([a,b], {}^*\mathbb{R}_c^{\#})$. Then there is a hyper infinite sequence of polynomials $p_n(x), n \in {}^*\mathbb{N}$ that #-converges uniformly to $f(x)$ on $[a,b]$.
Definition 4.1.1 (Hyperfinite Bernstein Polynomials). For each $n \in {}^*\mathbb{N}$, the n-th Bernstein Polynomial $B_n^{\#}(x,f)$ of a function $f \in C^{\#}([a,b], {}^*\mathbb{R}_c^{\#})$ is defined as

$$B_n^{\#}(x,f) = Ext\text{-}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^k. \tag{4.1.3}$$

Theorem 4.1.3.(Generalized B.L.T.theorem) Suppose that Z is a normed space, Y is a non-Archimedean Banach space, and $S \subset Z$ is a #-dense linear subspace of Z. If $T : S \to Y$ is a bounded linear transformation (i.e. there exists $C < {}^*\infty$ such that $\|Tz\|_{\#} \leq C\ \|z\|_{\#}$ for all $z \in S$), then T has a unique extension to an element of $\mathcal{L}(Z,Y)$.

§ 4.2.The spectral #-measures

Theorem 4.2.1.(Generalized Riesz-Markov theorem) Let X be a locally #-compact non-Archimedean metric space endowed with ${}^*\mathbb{R}_c^{\#}$-valued metric.Let $C_c^{\#}(X)$ be the space of #-continuous #-compactly supported ${}^*\mathbb{C}_c^{\#}$-valued functions on X.
For any positive linear functional Φ on $C_c^{\#}(X)$, there is a unique #-measure $\mu^{\#}$ on X such that

$$\forall f \in C_c^{\#}(X) : \Phi(f) = Ext\text{-}\int_X f(x)d^{\#}\mu^{\#}(x).$$

Theorem 4.2.2.(Generalized Riesz lemma) Let Y be a #-closed proper vector subspace of a normed space $(X, \|\cdot\|_{\#})$ and let $\alpha \in {}^*\mathbb{R}_c^{\#}$ be any real number satisfying $0 < \alpha < 1$.Then there exists a vector $u \in X$ of unit #-norm $\|u\|_{\#} = 1$ such that $\|u-y\|_{\#} \geq \alpha$ for all $y \in Y$.
We are now introduce the #-measures corresponding to bounded in${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operators. Let A be an bounded in${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator. Let $\psi \in \mathbf{H}^{\#}$. Then

$$f \mapsto \langle\psi, f(A)\psi\rangle_{\#} \tag{4.2.1}$$

is a positive linear functional on $C^{\#}(\sigma(A))$. Thus, by the generalized Riesz-Markov theorem, there is a unique #-measure $\mu_{\psi}^{\#}(\cdot)$ on the #-compact set $\sigma(A)$ with the property

$$\langle\psi, f(A)\psi\rangle_{\#} = Ext\text{-}\int_{\sigma(A)} f(\lambda)d^{\#}\mu_{\psi}^{\#}. \tag{4.2.2}$$

Definition 4.2.1.The #-measure $\mu_{\psi}^{\#}(\cdot)$ is called the spectral #-measure associated with the vector $\psi \in \mathbf{H}^{\#}$.
The first and simplest application of the $\mu_{\psi}^{\#}(\cdot)$ is to allow us to extend the functional calculus to $B^{\#}({}^*\mathbb{R}_c^{\#})$, the bounded in${}^*\mathbb{R}_c^{\#}$ #-Borel functions on ${}^*\mathbb{R}_c^{\#}$. Let $g \in B^{\#}({}^*\mathbb{R}_c^{\#})$.

It is natural to define $g(A)$ so that $\langle\psi,g(A)\psi\rangle_{\#} = Ext\text{-}\int_{\sigma(A)} g(\lambda)d^{\#}\mu_{\psi}^{\#}$. The polarization identity lets us recover $\langle\psi,g(A)\varphi\rangle_{\#}$ from the proposed $\langle\psi,g(A)\psi\rangle_{\#}$ and then the Generalized Riesz lemma lets us construct $g(A)$.

Theorem 4.2.1.(spectral theorem-functional calculus form) Let A be a bounded in${}^{*}\mathbb{R}_c^{\#}$ self-#-adjoint operator on $\mathbf{H}^{\#}$. There is a unique map $\widehat{\phi} : B^{\#}({}^{*}\mathbb{R}_c^{\#}) \to \mathcal{L}(\mathbf{H}^{\#})$ so that

(a) $\widehat{\phi}$ is an algebraic $*$-homomorphism.

(b) $\widehat{\phi}$ is #-norm #-continuous: $\left\|\widehat{\phi}(f)\right\|_{\mathcal{L}(\mathbf{H}^{\#})} \leq \|f\|_{{}^{*}\infty}$.

(c) Let f be the function $f(x) = x$; then $\widehat{\phi}(f) = A$.

(d) Suppose $f_n(x) \to_{\#} f(x)$ for each x as $n \to {}^{*}\infty$and hyper infinite sequence $\|f_n\|_{{}^{*}\infty}, n \in {}^{*}\mathbb{N}$ is bounded in${}^{*}\mathbb{R}_c^{\#}$. Then $\widehat{\phi}(f_n) \to_{\#} \widehat{\phi}(f)$ as $n \to {}^{*}\infty$strongly.

Moreover $\widehat{\phi}(\cdot)$ has the properties:

(e) If $A\psi = \lambda\psi$, then $\widehat{\phi}(f) = f(\lambda)\psi$.

(f) If $f \geq 0$,then $\widehat{\phi}(f) \geq 0$.

(g) If $BA = AB$ then $\widehat{\phi}(f)B = B\widehat{\phi}(f)$.

Remark 4.2.1. Note that: (i) Theorem 4.2.1 can be proven directly by extending Theorem 4.1.1, part (d) requires the dominated #-convergence theorem. Or, Theorem 4.2.1 can be proven by an easy corollary of Theorem 4.2.3 below. The proof of Theorem 4.2.3 uses only the #-continuous functional calculus, $\widehat{\phi}$ extends ϕ and as before we write $\widehat{\phi}(f) = f(A)$. As in the #-continuous functional calculus, one has $f(A)g(A) = g(A)f(A)$.

(ii) Since $B^{\#}({}^{*}\mathbb{R}_c^{\#})$ is the smallest family closed under #-limits of form (d) containing all of $C^{\#}({}^{*}\mathbb{R}_c^{\#})$, we know that any $\widehat{\phi}(f)$ is in the Smallest non Archimedean C^{*}-algebra containing A which is also strongly #-closed; such an algebra is called a von Neumann #-algebra or non Archimedean W^{*}-algebra. When we study von Neumann #-algebras we will see that this follows from (g).

(iii) The #-norm equality of Theorem 4.2.1 carries over if we define $\|f\|'_{{}^{*}\infty}$ to be the $L^{\#}_{{}^{*}\infty}$ #-norm with respect to a suitable notion of "#-almost everywhere." Namely, pick an orthonormal basis $\{\psi_n\}_{n=1}^{{}^{*}\infty}$ and say that a property is true #-a.e. if it is true #-a.e. with respect to each $\mu_{\psi_n}^{\#}$ Then $\left\|\widehat{\phi}(f)\right\|_{\mathcal{L}(\mathbf{H}^{\#})} = \|f\|'_{{}^{*}\infty}$.

Definition 4.2.2. A vector $\psi \in \mathbf{H}^{\#}$ is called a cyclic vector for A if gyperfinite linear combinations of the elements $\{A^n\psi\}_{n=0}^{{}^{*}\infty}$ are #-dense in $\mathbf{H}^{\#}$.

Not all operators have cyclic vectors, but if they do.

Lemma 4.2.1. Let A be a bounded in${}^{*}\mathbb{R}_c^{\#}$ self-#-adjoint operator with cyclic vector ψ. Then, there is a unitary operator $U : \mathbf{H}^{\#} \to L_2^{\#}(\sigma(A), d^{\#}\mu_{\psi}^{\#})$, with $(UAU^{-1}f)(\lambda) = \lambda f(\lambda)$ where equality holds is in the sense of elements of $L_2^{\#}(\sigma(A), d^{\#}\mu_{\psi}^{\#})$.

Proof Define U by $U\phi(f) = f$ where f is #-continuous. U is essentially the inverse of the map ϕ of Theorem 4.1.1. To show that U is well defined operator we compute $\|\phi(f)\psi\|_{\#}^2 = \langle\psi,\phi^{*}(f)\phi(f)\psi\rangle_{\#} = \left\langle\psi,\phi\left(\overline{f}\times f\right)\psi\right\rangle_{\#} = Ext\text{-}\int|f(\lambda)|^2 d^{\#}\mu_{\psi}^{\#}$.

Therefore, if $f = g$ a.e. with respect to $\mu_{\psi}^{\#}$, then $\phi(f)\psi = \phi(g)\psi$. Thus U is well defined on $\{\phi(f)\psi | f \in C^{\#}(\sigma(A))\}$ and is #-norm preserving. Since ψ is cyclic it #-closure #-$\overline{\{\phi(f)\psi | f \in C^{\#}(\sigma(A))\}} = \mathbf{H}^{\#}$ so by the generalized B.L.T. theorem U

extends to an #-isometric map of $\mathbf{H}^{\#}$ into $L_2^{\#}(\sigma(A), d^{\#}\mu_{\psi}^{\#})$. Since $C^{\#}(\sigma(A))$ is #-dense in $L_2^{\#}$, $\mathbf{Ran}\ U = L_2^{\#}(\sigma(A), d^{\#}\mu_{\psi}^{\#})$. Finally, if $f \in C^{\#}(\sigma(A))$ one obtains
$(UAU^{-1}f)(\lambda) = [UA\phi(f)](\lambda) = [U\phi(xf)](\lambda) = \lambda f(\lambda)$.
By #-continuity, this extends from $C^{\#}(\sigma(A))$ to $L_2^{\#}$.
To extend this lemma to arbitrary Ay we need to know that A has a family of invariant subspaces spanning $\mathbf{H}^{\#}$ so that A is cyclic on each subspace:
Lemma 4.2.2. Let A be a self-adjoint operator on a $*$-separable Hilbert space $\mathbf{H}^{\#}$. Then there is a direct sum decomposition $\mathbf{H}^{\#} = \bigoplus_{n=1}^{N} \mathbf{H}_n^{\#}$ with $N \in {}^*\mathbb{N}$ or $\mathbf{H}^{\#} = \bigoplus_{n=1}^{^*\infty} \mathbf{H}_n^{\#}$
so that:
(a) A leaves each $\mathbf{H}_n^{\#}$ invariant, that is, $\psi \in \mathbf{H}_n^{\#}$ implies $A\psi \in \mathbf{H}_n^{\#}$
(b) For each $n \in {}^*\mathbb{N}$, there is a $\phi_n \in \mathbf{H}_n^{\#}$ which is cyclic for $A \upharpoonright \mathbf{H}_n^{\#}$, i.e.
$\mathbf{H}_n^{\#} = \#\text{-}\overline{\{f(A)\phi_n | f \in C^{\#}(\sigma(A))\}}$
Theorem 4.2.3 (spectral theorem-multiplication operator form) Let A be a bounded in ${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator on $\mathbf{H}^{\#}$, a $*$-separable Hilbert space. Then, there exist #-measures $\{\mu_n^{\#}\}_{n=1}^{N}$ with $N \in {}^*\mathbb{N}$ or $\{\mu_n^{\#}\}_{n=1}^{^*\infty}$ on $\sigma(A)$ and a unitary operator $U : \mathbf{H}^{\#} \to \bigoplus_{n=1}^{N} L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu_n^{\#})$ or $U : \mathbf{H}^{\#} \to \bigoplus_{n=1}^{^*\infty} L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu_n^{\#})$
so that $(UAU^{-1}\psi)_n(\lambda) = \lambda\psi_n(\lambda)$
where we write an element $\psi \in \bigoplus_{n=1}^{N} L_2^{\#}({}^*\mathbb{R}_c^{\#}, d^{\#}\mu_n^{\#})$ as an N-tuple $\langle \psi_1(\lambda), \ldots, \psi_N(\lambda) \rangle$ or $*$-tuple
This realization of A is called a spectral representation.
Proof. Use Lemma 4.2.2 to find the decomposition and then use Lemma 4.2.1 on each component.
This theorem tells us that every bounded self-#-adjoint operator is a multiplication operator on a suitable #-measure space; what changes as the operator changes are the underlying #-measures. Explicitly:
Corolarly 4.2.1. Let A be a bounded in ${}^*\mathbb{R}_c^{\#}$ self-adjoint operator on a $*$-separable Hilbert space $\mathbf{H}^{\#}$. Then there exists a finite in ${}^*\mathbb{R}_c^{\#}$ measure space $\langle M, \mu^{\#} \rangle$, a bounded in ${}^*\mathbb{R}_c^{\#}$ function F on M, and a unitary map, $U : \mathbf{H}^{\#} \to L_2^{\#}(M, d^{\#}\mu^{\#})$ so that $(UAU^{-1}f)(m) = F(m)f(m)$.
Proof Choose the cyclic vectors ϕ_n so that $\|\phi_n\|_{\#} = 2^{-n}$. Let $M = \cup_{n=1}^{N*} \mathbb{R}_c^{\#}$ i.e. the union of $N \in {}^*\mathbb{N}$ copies of ${}^*\mathbb{R}_c^{\#}$. Define μ by requiring that its restriction to the n-th copy of ${}^*\mathbb{R}_c^{\#}$ be μ_n. Since $\mu(M) = Ext\text{-}\sum_{n=1}^{N} \mu_n^{\#}({}^*\mathbb{R}_c^{\#}) < {}^*\infty$, μ_n is finite in ${}^*\mathbb{R}_c^{\#}$. We also notice that this last theorem is essentially a rigorous form of the formaal Dirac notation. If we write $\phi_n = \phi(x;n)$, we see that in the "new representation defined by U" one has
$\langle \psi, \phi \rangle_{\#} = Ext\text{-}\sum_n Ext\text{-}\int d^{\#}\mu_n^{\#}\overline{\psi(\lambda;n)}\phi(\lambda;n)$ and
$\langle \psi, A\phi \rangle_{\#} = Ext\text{-}\sum_n Ext\text{-}\int d^{\#}\mu_n^{\#}\overline{\psi(\lambda;n)}\lambda\phi(\lambda;n)$.
These are the Dirac type formulas familiar to physicists except that the formal sums of Dirac are replaced with integrals over spectral measures, where we define:
Definition 4.2.3. The #-measures $d^{\#}\mu_n$ are called spectral measures; they are just $d^{\#}\mu_{\psi}$ for suitable ψ.
Notice these #-measures are not uniquely determined.

§ 4.3. Spectral projections

In the last section, we constructed a functional calculus, $f \mapsto f(A)$ for any #-Borel function and any bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operator A. The most important functions gained in passing from the continuous functional calculus to the #-Borel functional calculus are the characteristic functions of sets.

Definition 4.3.1. Let A be a bounded self-#-adjoint operator and Ω a #-Borel set of ${}^*\mathbb{R}_c^\#$. $P_\Omega = \chi_\Omega(A)$ is called a spectral projection of A.

As the definition suggests, P_Ω is an orthogonal projection since $\chi_\Omega = \chi_\Omega^2 = 1$ pointwise. The properties of the family of projections $\{P_\Omega | \Omega \text{ an arbitrary #-Borel set}\}$ is given by the following elementary translation of the functional calculus.

Proposition 4.3.1. The family $\{P_\Omega\}$ of spectral projections of a bounded self-#-adjoint operator A, has the following properties:

(a) Each P_Ω is an orthogonal projection.

(b) $P_\varnothing = 0$; $P_{(-a,a)} = I$ for some $a \in {}^*\mathbb{R}_c^\#$.

(c) If $\Omega = Ext\text{-}\bigcup_{n=1}^{*\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \varnothing$ for all $n \neq m$ then

$$P_\Omega = s\text{-#-}\lim_{N \to {}^*\infty}\left(Ext\text{-}\sum_{n=1}^{N}\right). \quad (4.3.1)$$

(d) $P_{\Omega_1}P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Definition 4.3.2. A family of projections obeying (a)-(c) is called a projection-valued #-measure (p.v.#-m.).

We remark that (d) follows from (a) and (c) by abstract considerations.

As one might guess, one can integrate with respect to a p.v.m. If P_Ω is a p.v.m., then $\langle \phi, P_\Omega \phi \rangle_\#$ is an ordinary #-measure for any ϕ. We will use the symbol $d^\#\langle \phi, P_\lambda \phi \rangle_\#$ to mean integration with respect to this #-measure. By generalized Riesz lemma methods, there is a unique operator B with $\langle \phi, B\phi \rangle_\# = Ext\text{-}\int f(\lambda) d^\#\langle \phi, P_\lambda \phi \rangle_\#$.

Theorem 4.3.1. If P_Ω is a p.v.#-m. and f a bounded in ${}^*\mathbb{R}_c^\#$ #-Borel function on $\mathbf{supp}(P_\Omega)$, then there is a unique operator B which we denote $Ext\text{-}\int f(\lambda) d^\# P_\lambda$ so that

$$\langle \phi, B\phi \rangle_\# = Ext\text{-}\int f(\lambda) d^\#\langle \phi, P_\lambda \phi \rangle_\#. \quad (4.3.2)$$

Theorem 4.3.2.(spectral theorem-p.v.#-m. form) There is a one-one correspondence between (bounded) self-#-adjoint operators A and (bounded) projection valued #-measures $\{P_\Omega\}$ given by

$$A \mapsto \{P_\Omega\} = \{\chi_\Omega(A)\} \quad (4.3.3)$$

and

$$\{P_\Omega\} \mapsto A = Ext\text{-}\int \lambda d^\# P_\lambda. \quad (4.3.4)$$

Spectral projections can be used to investigate the spectrum of A.

Proposition 4.3.1. $\lambda \in \sigma(A)$ if and only if $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ for any $\varepsilon > 0$.

The essential element of the proof is that $\|(A-\lambda)^{-1}\|_\# = [dist(\lambda, \sigma(A))]^{-1}$.

This suggests that we distinguish between two types of spectrum.

Definition 4.3.3. We say that (i) $\lambda \in \sigma_{ess}(A)$, the essential spectrum of A if and only if $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ is hyper infinite dimensional for all $\varepsilon > 0$.

(ii) If $\lambda \in \sigma(A)$ but $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ is hyperfinite dimensional for some $\varepsilon > 0$, we say

$\lambda \in \sigma_{disc}(A)$, the discrete spectrum of $A.P$ is hyper infinite dimensional means $\mathbf{Ran}(P)$ is hyper infinite dimensional.
Thus, we have a second decomposition of $\sigma(A)$. Unlike the first, it is a decomposition into two necessarily disjoint subsets. We note that σ_{disc} is not necessarily #-closed, but notice that.

Theorem 4.3.3 $\sigma_{ess}(A)$ is always #-closed.

Proof Let $\lambda_n \to_{\#} \lambda$ with each $\lambda_n \in \sigma_{ess}(A)$. Since any #-open interval I about λ contains an interval about some $\lambda_n, P_I(A)$ is hyper infinite dimensional.
The following three theorems give alternative descriptions of σ_{disc} and σ_{ess};

Theorem 4.3.4 $\lambda \in \sigma_{disc}$ if and only if both the following hold:
(a) λ is an #-isolated point of $\sigma(A)$ that is, for some $\varepsilon \approx 0$, $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(A) = \{\lambda\}$.
(b) λ is an eigenvalue of hyperfinite multiplicity, i.e., $\{\psi | A\psi = \lambda\psi\}$ is hyperfinite dimensional.

Theorem 4.3.5 $\lambda \in \sigma_{ess}$ if and only if one or more of the following holds:
(a) $\lambda \in \sigma_{cont}(A) \leftrightarrow \sigma_{ac}(A) \cup \sigma_{sing}(A)$.
(b) λ is a #-limit point of $\sigma_{pp}(A)$.
(c) λ is an eigenvalue of hyper infinite multiplicity.

Theorem 4.3.6 (Generalized Weyl's criterion) Let A be a bounded in ${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator. Then (i) $\lambda \in \sigma(A)$ if and only if there exists $\{\psi_n\}_{n=1}^{*\infty}$ with $\|\psi_n\|_{\#} = 1$and #-$\lim_{n\to{}^*\infty} \|(A - \lambda)\psi_n\|_{\#} = 0$.
(ii)$\lambda \in \sigma_{ess}(A)$ if and only if the above $\{\psi_n\}$ can be chosen to be orthogonal.
As one might guess, the essential spectrum cannot be removed by essentially hyperfinite dimensional perturbations. In Section 4.4, we will prove a general theorem which implies that $\sigma_{ess}(A) = \sigma_{ess}(B)$ if $A \backslash B$ is #-compact.
Finally, we discuss one useful formula relating the resolvent and spectral projections.
It is a matter of computation to see that

$$f_\varepsilon(x) \to_{\#} \begin{cases} 0 & \text{if} \quad x \notin [a,b] \\ 1/2 & \text{if} \quad x = a \vee x = b \\ 1 & \text{if} \quad x \in (a,b) \end{cases}$$

if $\varepsilon \to_{\#} 0$, where

$$f_\varepsilon(x) = (2\pi_{\#} i)^{-1}\left(Ext\text{-}\int_a^b \left[(x - \lambda - i\varepsilon)^{-1} - (x - \lambda + i\varepsilon)^{-1} \right] d^{\#}\lambda \right). \tag{4.3.5}$$

Moreover, $|f_\varepsilon(x)|$ is bounded in $x \in {}^* \mathbb{R}_c^{\#}$ uniformly in $\varepsilon \approx 0$, so by the functional calculus, one obtains that.

Theorem 4.3.7 (Generalized Stone's formula) Let A be a bounded in ${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operator. Then

$$\begin{aligned} s\text{-}\lim_{\varepsilon\to_{\#} 0}(2\pi_{\#} i)^{-1}\left(Ext\text{-}\int_a^b \left[(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1} \right] d^{\#}\lambda \right) = \\ = \frac{1}{2}[P_{[a,b]} + P_{(a,b)}]. \end{aligned} \tag{4.3.6}$$

§ 4.4.The #-continuous functional calculus related to

unbounded in $^*\mathbb{R}_c^\#$ self-#-adjoint operators

In this section we will show how the spectral theorem for bounded in $^*\mathbb{R}_c^\#$ self-#-adjoint operators which we developed in § 4.3 can be extended to unbounded in $^*\mathbb{R}_c^\#$ self-#-adjoint operators. To indicate what we are aiming for, we first prove the following:

Proposition 4.4.1. Let $\langle M, \mu^\# \rangle$ be a #-measure space with $\mu^\#$ a hyperfinite #-measure. Suppose that f is a #-measurable, $^*\mathbb{R}_c^\#$-valued function on M which is finite or hyperfinite a.e.$\mu^\#$. Then the operator $T_f : \varphi \to f\varphi$ on $L_2^\#(M, d^\#\mu^\#)$ with domain

$$D(T_f) = \{\varphi | f\varphi \in L_2^\#(M, d^\#\mu^\#)\} \tag{4.4.1}$$

is self-#-adjoint and $\sigma(T_f)$ is the essential range of T_f.

Proof T_f is clearly symmetric. Suppose that $\psi \in D(T_f^*)$ and let

$$\chi_N = \begin{cases} 1 & \text{if} \quad |f(m)| \leq N \\ 0 & \text{otherwise} \end{cases}$$

Then, using the generalized monotone #-convergence theorem,

$$\|T_f^*\psi\|_\# = \#\text{-lim}_{N \to {}^*\infty} \|\chi_N T_f^*\psi\|_\# = \#\text{-lim}_{N \to {}^*\infty}\left(\sup_{\|\varphi\|_\#=1} \left|\left\langle \varphi, \chi_N T_f^*\psi \right\rangle_\#\right|\right) =$$

$$\#\text{-lim}_{N \to {}^*\infty}\left(\sup_{\|\varphi\|_\#=1} |\langle \chi_N T_f\varphi, \psi \rangle_\#|\right) = \#\text{-lim}_{N \to {}^*\infty}\left(\sup_{\|\varphi\|_\#=1} |\langle \varphi, \chi_N f\psi \rangle_\#|\right) =$$

$$\#\text{-lim}_{N \to {}^*\infty} \|\chi_N f\psi\|_\#$$

Thus, $f\psi \in L_2^\#(M, d^\#\mu^\#)$, so $\psi \in D(T_f)$ and therefore T_f is self-#-adjoint. That $\sigma(T_f)$ is the essential range of f follows as in the bounded case.

With more information about f, one can say something about the domains on which T_f is essentially self-#-adjoint:

Proposition 4.4.2. Let f and T_f obey the conditions in Proposition 4.4.1. Suppose in addition that $f \in L_p^\#(M, d^\#\mu^\#)$ for $2 < p < {}^*\infty$. Let D be any #-dense set in $L_q^\#(M, d^\#\mu^\#)$ where $q^{-1} + p^{-1} = 1/2$. Then D is a #-core for T_f.

Proof Let us first show that $L_q^\#$ is a #-core for T_f. By the generalized Holder's inequality $\|g\|_{\#2} \leq \|1\|_{\#p} \cdot \|g\|_{\#q}$, and $\|fg\|_{\#2} \leq \|f\|_{\#p} \cdot \|g\|_{\#q}$ so $L_p^\# \subset D(T_f)$. Moreover, if $g \in D(T_f)$ let $g_n, n \in {}^*\mathbb{N}$ be that function which is zero where $|g(m)| > n$ and equal to g otherwise. By the generalized dominated convergence theorem, $g_n \to_\# g$ and $fg_n \to_\# fg$ in $L_2^\#$. Since each g_n is in $L_q^\#$, we conclude that $L_q^\#$ is a #-core for T_f. Now let D be #-dense in $L_q^\#$ and let $g \in L_q^\#$. Find $g_n \in D$ with $g_n \to_\# g$ in $L_q^\#$. Since $\|g_n - g\|_{\#2} \leq \|1\|_{\#p} \cdot \|g_n - g\|_{\#q}$ and $\|T_f(g_n - g)\|_{\#2} \leq \|f\|_{\#p} \cdot \|g_n - g\|_{\#q}$, $g \in \#\text{-}\overline{D(T_f \upharpoonright D)}$.

Thus $L_q^\# \subset D(T_f \upharpoonright D)$ so D is a #-core. Unless f$\in L_{^*\infty}^\#(M, d^\#\mu^\#)$ the operator T_f described in Propositions 4.4.1 and 4.4.2 will be unbounded.

Thus, we have found a large class of unbounded self-#-adjoint operators. In fact, we have found them all.

Theorem 4.4.1. (spectral theorem-multiplication operator form) Let A be a self-adjoint operator on a $^*\infty$-dimensional a non-Archimedean Hilbert space $\mathbf{H}^\#$ with domain $D(A)$. Then there is a #-measure space $\langle M, \mu^\# \rangle$ with $\mu^\#$ a hyperfinite #-measure, a unitary operator $U : \mathbf{H}^\# \to L_2^\#(M, d^\#\mu^\#)$, and a $^*\mathbb{R}_c^\#$-valued function f

on M which is finite or hyperfinite $\mu^{\#}$-a.e. so that

(a) $\psi \in D(A)$ if and only if $f(\cdot)(U\psi)(\cdot) \in L_2^{\#}(M, d^{\#}\mu^{\#})$.

(b) If $\varphi \in U[D(A)]$, then $(UAU^{-1}\varphi)(m) = f(m)\varphi(m)$.

Proof It easily verify that $A + i$ and $A - i$ are one to one correspondence and $\mathbf{Ran}(A \pm i) = \mathbf{H}^{\#}$. Since $A \pm i$ are #-closed, $(A \pm i)^{-1}$ are #-closed and therefore bounded in ${}^*\mathbb{R}_c^{\#}$. Note that the operators $(A+i)^{-1}$ and $(A-i)^{-1}$ commute. The equality $\langle (A-i)\psi, (A+i)^{-1}(A+i)\varphi\rangle_{\#} = \langle (A+i)^{-1}(A-i)\psi, (A+i)\varphi\rangle_{\#}$and the fact that $\mathbf{Ran}(A \pm i) = \mathbf{H}^{\#}$ shows that $((A+i)^{-1})^* = (A-i)^{-1}$. Thus the operator $(A+i)^{-1}$ is normal.

We now use the easy extension of the spectral theorem for bounded in ${}^*\mathbb{R}_c^{\#}$ self-#-adjoint operators to bounded in ${}^*\mathbb{R}_c^{\#}$ normal operators. The proof of this extension is a straightforward. We conclude that there is a #-measure space $\langle M, \mu^{\#}\rangle$ with $\mu^{\#}$ a hyperfinite #-measure, a unitary operator $U : \mathbf{H}^{\#} \to L_2^{\#}(M, d^{\#}\mu^{\#})$, and a #-measurable, bounded, in ${}^*\mathbb{R}_c^{\#}$ ${}^*\mathbb{C}_c^{\#}$-valued function $g(m)$ so that $U(A+i)^{-1}U^{-1}\varphi(m) = g(m)\varphi(m)$ for all $\varphi \in L_2^{\#}(M, d^{\#}\mu^{\#})$.Since $\mathbf{Ker}((A+i)^{-1})$ is empty, $g(m) \neq 0$ a.e.$\mu^{\#}$, so the function $f(m) = g^{-1}(m) - i$ is hyperfinite a.e.$\mu^{\#}$. Now, suppose $\psi \in D(A)$. Then $\psi = (A+i)^{-1}\varphi$ for some $\varphi \in \mathbf{H}^{\#}$ and $U\psi = gU\varphi$. Since fg is bounded in ${}^*\mathbb{R}_c^{\#}$, we conclude that $f(U\psi) \in L_2^{\#}(M, d^{\#}\mu^{\#})$. Conversely, if $f(U\psi) \in L_2^{\#}(M, d^{\#}\mu^{\#})$, then there is a $\varphi \in \mathbf{H}^{\#}$ so that $U\varphi = (f+i)U\psi$. Thus, $gU\varphi = g(f+i)U\psi = U\psi$, so $\psi = (A+i)^{-1}\varphi$ which shows that $\psi \in D(A)$. This proves (a).

To prove (b) notice that if $\psi \in D(A)$ then $\psi = (A+i)^{-1}\varphi$ for some $\varphi \in \mathbf{H}^{\#}$ and $A\psi = \varphi - i\psi$. Therefore, $(UA\psi)(m) = (U\varphi)(m) - i(U\psi)(m) = (g^{-1}(m) - i)(U\psi)(m)$ $= f(m)(U\psi)(m)$. Finally, if $\operatorname{Im}(f) > 0$ on a set of nonzero Lebesgue #-measure, there is a bounded in ${}^*\mathbb{R}_c^{\#}$ set B in the upper half plane so that $S = \{x | f(x) \in B\}$ has nonzero Lebesgue #-measure. If $\chi(x)$ is the characteristic function of S then $f\chi \in L_2^{\#}(M, d^{\#}\mu^{\#})$ and $\operatorname{Im}\langle \chi, f\chi\rangle > 0$. This contradicts the fact that multiplication by /is self-adjoint (since it is unitarily equivalent to A). Thus f is ${}^*\mathbb{R}_c^{\#}$-valued function.

There is a natural way to define functions of a self-#-adjoint operator by using the above theorem. Given a bounded in ${}^*\mathbb{R}_c^{\#}$ #-Borel function h on ${}^*\mathbb{R}_c^{\#}$ we define

$$h(A) = UT_{h(f)}U^{-1} \tag{4.4.2}$$

where $T_{h(f)}$ is the operator on $L_2^{\#}(M, d^{\#}\mu^{\#})$ which acts by multiplication by the function $h(f(m))$. Using this definition the following theorem follows easily from Theorem 4.4.1.

Theorem 4.4.2. (spectral theorem-functional calculus form) Let A be a self-#-adjoint operator on $\mathbf{H}^{\#}$. Then there is a unique map $\widehat{\phi}$ from the bounded #-Borel functions on ${}^*\mathbb{R}_c^{\#}$ into $\mathcal{L}(\mathbf{H}^{\#})$ so that

(a) $\widehat{\phi}$ is an algebraic $*$-homomorphism.

(b) $\widehat{\phi}$ is #-norm #-continuous, that is, $\left\| \widehat{\phi}(h) \right\|_{\mathcal{L}(\mathbf{H}^{\#})} \leq \|h\|_{{}^*\infty}$

(c) Let $h_n(x), n \in {}^*\mathbb{N}$ be a hyper infinite sequence of bounded in ${}^*\mathbb{R}_c^{\#}$ #-Borel functions with $\#\text{-}\lim_{n \to {}^*\infty} h_n(x) = x$ for each x and $|h_n(x)| \leq |x|$ for all x and $n \in {}^*\mathbb{N}$. Then, for any $\psi \in D(A)$, $\#\text{-}\lim_{n \to {}^*\infty} \widehat{\phi}(h_n)\psi = A\psi$.

(d) If $h_n(x) \to_{\#} h(x)$ pointwise and if the hyper infinite sequence $\|h_n\|_{{}^*\infty}, n \in {}^*\mathbb{N}$ is bounded in ${}^*\mathbb{R}_c^{\#}$, then $\widehat{\phi}(h_n) \to_{\#} \widehat{\phi}(h)$ strongly.

In addition:

(e) If $A\psi = \lambda\psi$ then $\widehat{\phi}(h) = h(\lambda)\psi$.
(f) If $h \geq 0$, then $\widehat{\phi}(h) \geq 0$.
The functional calculus is very useful. For example, it allows us to define the exponential $Ext\text{-}\exp(itA)$ and prove easily many of its properties as a function of t (see the next section). In the case where A is bounded in ${}^*\mathbb{R}_c^\#$ we do not need the functional calculus to define the exponential since we can define $Ext\text{-}\exp(itA)$ by the power series which #-converges in #-norm.
The functional calculus is also used to construct spectral #-measures and can be used to develop a multiplicity theory similar to that for bounded self-#-adjoint operators.
A vector $\psi \in \mathbf{H}^\#$ is said to be cyclic for A if $\{g(A)\psi | g \in C^{*\infty}({}^*\mathbb{R}_c^\#)\}$ is #-dense in $\mathbf{H}^\#$. If ψ is a cyclic vector, then it is possible to represent $\mathbf{H}^\#$ as $L_2^\#({}^*\mathbb{R}_c^\#, d^\#\mu_\psi^\#)$ where $\mu_\psi^\#$ is the measure satisfying $Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(x)d^\#\mu_\psi^\#(x) = \langle\psi, g(A)\psi\rangle_\#$in such a way that A becomes multiplication by x. In general, $\mathbf{H}^\#$ decomposes into a direct sum of cyclic subspaces so the #-measure space, M in Theorem 4.4.1 can be realized as a union of copies of ${}^*\mathbb{R}_c^\#$. As in the case of bounded in ${}^*\mathbb{R}_c^\#$ operators we can define $\sigma_{\mathbf{ac}}(A), \sigma_{\mathbf{pp}}(A), \sigma_{\mathbf{sing}}(A)$ and decompose $\mathbf{H}^\#$ accordingly.
Finally, the spectral theorem in its projection-valued #-measure form follows easily from the functional calculus. Let P_Ω be the operator $\chi_\Omega(A)$ where χ_Ω is the characteristic function of the measurable set $\Omega \subset {}^*\mathbb{R}_c^\#$. The family of operators $\{P_\Omega\}$ has the following properties:
(a) Each P_Ω is an orthogonal projection.
(b) $P_\varnothing = 0$; $P_{(-{}^*\infty, {}^*\infty)} = I$.
(c) If $\Omega = Ext\text{-}\bigcup_{n=1}^{*\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \varnothing$ for all $n \neq m$ then

$$P_\Omega = s\text{-}\#\text{-}\lim_{N \to {}^*\infty}\left(Ext\text{-}\sum_{n=1}^{N} P_{\Omega_n}\right). \quad (4.4.3)$$

(d) $P_{\Omega_1}P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.
Definition 4.4.1.Such a family is called a projection-valued #-measure (p.v.#-m.).
Remark 4.4.1. This is a generalization of the notion of bounded in ${}^*\mathbb{R}_c^\#$ projection-valued #-measure introduced in § 4.3.In that we only require $P_{(-{}^*\infty, {}^*\infty)} = I$ rather than $P_{(-a,a)} = I$ for some $a \in {}^*\mathbb{R}_c^\#$. For $\varphi \in \mathbf{H}^\#, \langle\varphi, P_\Omega\varphi\rangle_\#$ is a well-defined Borel #-measure on ${}^*\mathbb{R}_c^\#$ which we denote by $d^\#\langle\varphi, P_\lambda\varphi\rangle_\#$ as in § 4.3.
The complex ${}^*\mathbb{C}_c^\#$-valued #-measure $d^\#\langle\varphi, P_\lambda\psi\rangle_\#$ is defined by polarization. Thus, given a bounded in ${}^*\mathbb{R}_c^\#$ #-Borel function g we can define $g(A)$ by

$$\langle\varphi, g(A)\varphi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi, P_\lambda\varphi\rangle_\# \quad (4.4.4)$$

It is not difficult to show that this map $g \mapsto g(A)$ has the properties (a)-(d) of Theorem 4.4.1, so $g(A)$ as defined by (4.4.4) coincides with the definition of $g(A)$ given by Theorem 4.4.1. Now, suppose g is an unbounded ${}^*\mathbb{C}_c^\#$-valued #-Borel function and let

$$D_g = \left\{\varphi | Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi, P_\lambda\varphi\rangle_\# < {}^*\infty\right\}. \quad (4.4.5)$$

Then, D_g is #-dense in $H^\#$ and an operator $g(A)$ is defined on D_g by

$$\langle\varphi, g(A)\varphi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi, P_\lambda\varphi\rangle_\#. \quad (4.4.6)$$

As in § 4.3, we write symbolically

$$g(A) = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda) d^\# P_\lambda. \qquad (4.4.7)$$

In particular, for $\varphi, \psi \in D(A)$,

$$\langle \varphi, A)\psi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda) d^\# \langle \varphi, P_\lambda \varphi\rangle_\#. \qquad (4.4.8)$$

if g is ${}^*\mathbb{R}_c^\#$-valued, then $g(A)$ is self-#-adjoint on D_g. We summarize:

Theorem 4.4.3. (spectral theorem-projection valued measure form) There is a one-to-one correspondence between self-#-adjoint operators A and projection-valued #-measures $\{P_\Omega\}$ on $\mathbf{H}^\#$ the correspondence being given by

$$A = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} \lambda d^\# P_\lambda. \qquad (4.4.9)$$

We use the functional calculus developed above to define $Ext\text{-}\exp(itA)$.

Theorem 4.4.4. Let A be a self-#-adjoint operator and define $U(t) = Ext\text{-}\exp(itA)$. Then

(a) For each $t \in {}^*\mathbb{R}_c^\#, U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in {}^*\mathbb{R}_c^\#$.

(b) If $\varphi \in \mathbf{H}^\#$ and $t \to_\# t_0$, then $U(t)\varphi \to_\# U(t_0)\varphi$.

(c) For any $\psi \in D(A)$: $\dfrac{U(t)\psi - \psi}{t} \to_\# iA\psi$ as $t \to_\# 0$.

(d) If $\#\text{-}\lim_{t \to_\# 0} \dfrac{U(t)\psi - \psi}{t}$ exists, then $\psi \in D(A)$.

Proof (a) follows immediately from the functional calculus and the corresponding statements for the complex-valued function $Ext\text{-}\exp(it\lambda)$. To prove (b) observe that

$$\|Ext\text{-}\exp(itA)\varphi - \varphi\|_\#^2 = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} |Ext\text{-}\exp(it\lambda) - 1|^2 d^\# \langle P_\lambda \varphi, \varphi\rangle_\#. \qquad (4.4.10)$$

Since $|Ext\text{-}\exp(it\lambda) - 1|^2$ is dominated by the #-integrable function $g(\lambda) = 2$ and since for each $\lambda \in {}^*\mathbb{R}_c^\#$: $|Ext\text{-}\exp(it\lambda) - 1|^2 \to_\# 0$ as $t \to_\# 0$ we conclude that $\|U(t)\varphi - \varphi\|_\#^2 \to_\# 0$ as $t \to_\# 0$, by the generalized Lebesgue dominated-#-convergence theorem. Thus $t \mapsto U(t)$ is strongly #-continuous at $t = 0$, which by the group property proves $t \mapsto U(t)$ is strongly #-continuous everywhere. The proof of (c), which again uses the dominated #-convergence theorem and the estimate $|Ext\text{-}\exp(ix) - 1|^2 \leq |x|$.

To prove (d), we define

$$D(B) = \left\{ \psi \,\middle|\, \#\text{-}\lim_{t \to_\# 0} \frac{U(t)\psi - \psi}{t} \text{ exists} \right\} \qquad (4.4.11)$$

and let

$$iB\psi = \#\text{-}\lim_{t \to_\# 0} \frac{U(t)\psi - \psi}{t}. \qquad (4.4.12)$$

A simple computation shows that B is symmetric.By (c), $B \supset A$, so $B = A$.

Definition 4.4.2. An operator-valued function $U(t)$ satisfying (a) and (b) is called a strongly #-continuous one-parameter unitary group.

Definition 4.4.3. If $U(t)$ is a strongly #-continuous one-parameter unitary group, then the self-#-adjoint operator A with $U(t) = Ext\text{-}\exp(itA)$ is called the infinitesimal generator of $U(t)$.

Suppose that $U(t)$ is a weakly #-continuous one-parameter unitary group. Then $\|U(t)\varphi - \varphi\|_\#^2 = \|U(t)\varphi\|_\#^2 - \langle U(t)\varphi, \varphi\rangle_\# - \langle \varphi, U(t)\varphi\rangle_\# + \|\varphi\|_\#^2 \to_\# 0$ as $t \to_\# 0$. Thus

$U(t)$ is actually strongly #-continuous. As a matter of fact, to conclude that $U(t)$ is strongly #-continuous one need only show that $U(t)$ is weakly #-measurable,that is, that $\langle U(t)\varphi,\psi\rangle_{\#}$ is #-measurable for each φ and ψ. This startling result sometimes useful since in applications one can often show that $\langle U(t)\varphi,\psi\rangle_{\#}$ is the #-limit of a hyper infinite sequence of #-continuous functions;$\langle U(t)\varphi,\psi\rangle_{\#}$ is therefore #-measurable and by generalized von Neumann's theorem $U(t)$ is then strongly #-continuous.

Theorem 4.4.5. Let $U(t)$ be a one-parameter group of unitary operators on a hy infinite dimensional Hilbert space $\mathbf{H}^{\#}$. Suppose that for all $\varphi,\psi \in \mathbf{H}^{\#},\langle U(t)\psi,\varphi\rangle_{\#}$ is #-measurable. Then $U(t)$ is strongly #-continuous.

Proof. Let $\psi \in \mathbf{H}^{\#}$.Then for all $\varphi \in \mathbf{H}^{\#}$, $\langle U(t)\psi,\varphi\rangle_{\#}$ is a bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ #-measurable function and $\varphi \mapsto \int_{0}^{a}\langle U(t)\psi,\varphi\rangle_{\#}d^{\#}t$ is a linear functional on $\mathbf{H}^{\#}$ of #-norm less than or equal to $a\|\varphi\|_{\#}$. Thus, by the generalized Riesz lemma there is a $\psi_{a} \in \mathbf{H}^{\#}$ so that

$$\langle\psi_{a},\varphi\rangle_{\#} = \int_{0}^{a}\langle U(t)\psi,\varphi\rangle_{\#}d^{\#}t. \qquad (4.4.13)$$

Note that

$$\begin{gathered}\langle U(b)\psi_{a},\varphi\rangle_{\#} = \langle\psi_{a},U(-b)\varphi\rangle_{\#} = \int_{0}^{a}\langle U(t)\psi,U(-b)\varphi\rangle_{\#}d^{\#}t = \\ \int_{0}^{a}\langle U(t+b)\psi,\varphi\rangle_{\#}d^{\#}t = \int_{b}^{a+b}\langle U(t)\psi,\varphi\rangle_{\#}d^{\#}t.\end{gathered} \qquad (4.4.14)$$

From (4.1.14) we obtain

$$\begin{gathered}|\langle U(b)\psi_{a},\varphi\rangle_{\#}-\langle\psi_{a},\varphi\rangle_{\#}| = \\ = \left|\int_{0}^{b}\langle U(t)\psi,\varphi\rangle_{\#}d^{\#}t\right| + \left|\int_{b}^{a+b}\langle U(t)\psi,\varphi\rangle_{\#}d^{\#}t\right| \leq 2a\|\varphi\|_{\#}\|\psi\|_{\#}\end{gathered} \qquad (4.4.15)$$

and therefore #-$\lim_{b\to_{\#}0}\langle U(b)\psi_{a},\varphi\rangle_{\#} = \langle\psi_{a},\varphi\rangle_{\#}$ so that $U(b)$ is weakly and therefore strongly #-continuous on the set of vectors of the form $\{\psi_{a}|\psi \in \mathbf{H}^{\#}\}$. It remains only to show that this set is #-dense, since by by an $\varepsilon \approx 0,\varepsilon/3$ argument we can then conclude that $t \mapsto U(t)$ is strongly #-continuous on $\mathbf{H}^{\#}$. Suppose that $\varphi \in \{\psi_{a}|\psi \in \mathbf{H}^{\#},a \in {}^{*}\mathbb{R}_{c}^{\#}\}^{\mathsf{T}}$ and let $\{\psi^{(n)}\}_{n\in{}^{*}\mathbb{N}}$ be an orthonormal basis for $\mathbf{H}^{\#}$. Then for each $n \in {}^{*}\mathbb{N}$

$$Ext\text{-}\int_{0}^{a}\langle U(t)\psi^{(n)},\varphi\rangle_{\#}d^{\#}t = \left\langle\psi_{a}^{(n)},\varphi\right\rangle_{\#} = 0 \qquad (4.4.16)$$

for all $a \in {}^{*}\mathbb{R}_{c}^{\#}$ which implies that $\langle U(t)\psi^{(n)},\varphi\rangle_{\#} = 0$ except for $t \in S_{n},$ a set of Lebesgue #-measure zero. Choose $t_{0} \notin \cup_{n\in{}^{*}\mathbb{N}} S_{n}$. Then $\langle U(t_{0})\psi^{(n)},\varphi\rangle_{\#} = 0$ for all $n \in {}^{*}\mathbb{N}$ which implies that $\varphi = 0,$ since $U(t_{0})$ is unitary.

Theorem 4.4.6.Suppose that $U(t)$ is a strongly continuous one-parameter unitary group. Let D be a #-dense domain which is invariant under $U(t)$ and on which $U(t)$ is strongly #-differentiable. Then i^{-1} times the strong #-derivative of $U(t)$ is essentially

self-#-adjoint on D and its #-closure is the #-infinitesimal generator of $U(t)$.
This theorem has a reformulation which is sufficiently important that we state it as a theorem.
Theorem 4.4.7. Let A be a self-adjoint operator on $\mathbf{H}^{\#}$ and D be a #-dense linear set contained in $D(A)$. If for all t, $Ext\text{-}\exp(itA) : D \to D$ then D is a #-core for A.
Theorem 4.4.8.Let $U(t)$ be a strongly #-continuous one-parameter unitary group on a Hilbert space $\mathbf{H}^{\#}$. Then, there is a self-#-adjoint operator A on $\mathbf{H}^{\#}$ so that $U(t) = Ext\text{-}\exp(itA)$.
Proof Part (d) of Theorem 4.4.4 suggests that we obtain A by differentiating $U(t)$ at $t = 0$. We will show that this can be done on a #-dense set of especially nice vectors and then show that the #-limiting operator is essentially self-#-adjoint by using the basic criterion. Finally, we show that the exponential of this #-limiting operator is just $U(t)$. Let $f \in C_0^{*\infty}({}^*\mathbb{R}_c^{\#})$ and for each $\varphi \in \mathbf{H}^{\#}$ define

$$\varphi_f = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} f(t)U(t)\varphi d^{\#}t. \tag{4.4.17}$$

Since $U(t)$ is strongly #-continuous the integral in (4.4.7) can be taken to be a Riemann integral. Let D be the set of hyperfinite linear combinations of all such φ_f with $\varphi \in \mathbf{H}^{\#}$ and $f \in C_0^{*\infty}({}^*\mathbb{R}_c^{\#})$. If $j_\varepsilon(t)$ is the approximate identity then

$$\begin{gathered}\|\varphi_{j_\varepsilon} - \varphi\|_{\#} = \left\| Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} j_\varepsilon(t)[U(t)\varphi - \varphi]d^{\#}t \right\|_{\#} \leq \\ \leq \left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} j_\varepsilon(t)d^{\#}t \right) \sup_{t\in[-\varepsilon,\varepsilon]} \|U(t)\varphi - \varphi\|_{\#}.\end{gathered} \tag{4.4.18}$$

Since $U(t)$ is strongly #-continuous, D is #-dense in $\mathbf{H}^{\#}$. We have used the inequality

$$\left\| Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} h(t)d^{\#}t \right\|_{\#} \leq Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \|h(t)\|_{\#} d^{\#}t \tag{4.4.19}$$

for non-Archimedean Banach space-valued #-continuous functions on the real line ${}^*\mathbb{R}_c^{\#}$ (which can be proven using the approximate partial sums as in the ${}^*\mathbb{R}_c^{\#}$-valued case). For $\varphi_f \in D$ we obtain that

$$\begin{gathered}\left(\frac{U(s) - I}{s} \right)\varphi_f = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} f(t)\left(\frac{U(s+t) - U(t)}{s} \right)\varphi d^{\#}t = \\ Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \frac{f(\tau - s) - f(\tau)}{s} U(\tau)\varphi d^{\#}\tau \to_{\#} -Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} f^{\#'}(\tau)U(\tau)\varphi d^{\#}\tau = \varphi_{-f^{\#'}}\end{gathered} \tag{4.4.20}$$

since $[f(t-s) - f(t)]/s$ #-converges to $-f^{\#'}(t)$ uniformly. For $\varphi_f \in D$ we define $A\varphi_f = i^{-1}\varphi_{-f^{\#'}}$. Note that $U(t) : D \to D, A : D \to D$ and $U(t)A\varphi_f = AU(t)\varphi_f$ for $\varphi_f \in D$. Futhermore if $\varphi_f, \varphi_g \in D$ we obtain that

$$\begin{gathered}\langle A\varphi_f, \varphi_g \rangle_{\#} = \#\text{-}\lim_{s\to_{\#}0} \left\langle \left(\frac{U(s) - I}{is} \right)\varphi_f, \varphi_g \right\rangle_{\#} = \\ = \#\text{-}\lim_{s\to_{\#}0} \left\langle \varphi_f, \left(\frac{I - U(-s)}{is} \right)\varphi_g \right\rangle_{\#} = \frac{1}{i}\left\langle \varphi_f, \varphi_{-g^{\#'}} \right\rangle_{\#} = \langle \varphi_f, A\varphi_g \rangle_{\#}\end{gathered} \tag{4.4.21}$$

so A is symmetric. Now we show that A is essentially self-#-adjoint. Suppose that there is a $u \in D(A^*)$ so that $A^*u = iu$. Then for each $\varphi \in D(A) = D$

$$\frac{d^\#}{d^\# t}\langle U(t)\varphi, u\rangle_\# = \langle iAU(t)\varphi, u\rangle_\# = -i\langle U(t)\varphi, A^*u\rangle_\# = -i\langle U(t)\varphi, iu\rangle_\# = \langle U(t)\varphi, u\rangle_\# \quad (4.4.22)$$

Thus, the ${}^*\mathbb{C}_c^\#$-valued function $f(t) = \langle U(t)\varphi, u\rangle_\#$ satisfies the ordinary differential equation $f^{\#'} = f$ so $f(t) = f(0)[Ext\text{-}\exp(t)]$. Since $U(t)$ has #-norm one, $|f(t)|$ is bounded, in ${}^*\mathbb{R}_c^\#$ which implies that $f(0) = \langle \varphi, u\rangle_\# = 0$. Since D is #-dense, $u = 0$. A similar proof shows that $A^*u = -iu$ can have no nonzero solutions. Therefore A is essentially self-#-adjoint on D.

Let $V(t) = Ext\text{-}\exp(it(\#\text{-}\overline{A}))$. It remains to show that $U(t) = V(t)$. Let $\varphi \in D(A)$. Since $\varphi \in D((\#\text{-}\overline{A}))$, $V(t)\varphi \in D((\#\text{-}\overline{A}))$ and $V^{\#'}(t)\varphi = iAV(t)\varphi$ by (c) of Theorem 4.4.4, We already know that $U(t)\varphi \in D \subset D(\#\text{-}\overline{A})$ for all$\in {}^*\mathbb{R}_c^\#$. Let $w(t) = U(t)\varphi - V(t)\varphi$. Then $w(t)$ is a strongly #-differentiable vector-valued function and

$$w^{\#'}(t) = iAU(t)\varphi - i(\#\text{-}\overline{A})V(t)\varphi = iAw(t). \quad (4.4.23)$$

Thus

$$\frac{d^\#}{d^\# t}\|w(t)\|_\#^2 = -i\langle(\#\text{-}\overline{A})w(t), w(t)\rangle_\# + i\langle w(t), (\#\text{-}\overline{A})w(t)\rangle_\#. \quad (4.4.24)$$

Therefore $w(t) = 0$ for all $t \in {}^*\mathbb{R}_c^\#$ since $w(t) = 0$. This implies that $U(t)\varphi = V(t)\varphi$ for all $t \in {}^*\mathbb{R}_c^\#, \varphi \in D$. Since D is #-dense in $\mathbf{H}^\#, U(t) = V(t)$.

Remark 4.4.2.Finally, we have the following generalization of Stone's theorem 4.4.8. If g is a ${}^*\mathbb{R}_c^\#$-valued #-Borel function on ${}^*\mathbb{R}_c^\#$, then

$$g(A) = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\# P_\lambda \quad (4.4.25)$$

defined on D_g (4.4.5) is self-#-adjoint. If g is bounded, $g(A)$ coincides with $\widehat{\phi}(g)$ in Theorem 4.4.2.

We conclude with several remarks. First, generalized Stone's formula, given in Theorem 4.3.7 relates the resolvent and the projection-valued measure associated with any self-#-adjoint operator. The proof is the same as in the bounded in ${}^*\mathbb{R}_c^\#$ case. The spectrum of an unbounded self-#-adjoint operator is an unbounded subset of the real axis ${}^*\mathbb{R}_c^\#$. One can define discrete and essential spectrum; Theorem 4.3.6 (Generalized Weyl's criterion) still holds if one adds the criterion that the vectors $\{\psi_n\}$ must be in the domain of A.

Finally, we note that the measure space of Theorem 4.4.1 can always be chosen so that

Proposition **4.4.2** is applicable.

The following theorem says that every strongly #-continuous unitary group arises as the exponential of a self-#-adjoint operator.

Theorem 4.4.9. Let $U(\mathbf{t}) = U(t_1,\ldots,t_n)$ be a strongly continuous map of ${}^*\mathbb{R}_c^{\#n}$ into the unitary operators on a hyper infinite dimensional Hilbert space $\mathbf{H}^\#$ satisfying $U(\mathbf{t}+\mathbf{s}) = U(\mathbf{t})U(\mathbf{s})$ Let D be the set of hyperfinite linear combinations of vectors of the form

$$\varphi_f = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} f(\mathbf{t})U(\mathbf{t})d^{\#n}t \quad (4.4.26)$$

where $\varphi \in \mathbf{H}^\#, f \in C_0^{\#^*\infty}({}^*\mathbb{R}_c^{\#n})$.Then D is a domain of essential self-#-adjointness for each of the generators A_j of the one-parameter subgroups $U(0,0,\ldots,t_j,\ldots,0)$, each

$A_j : D \to D$ and the A_j commute, $j = 1,\dots,n$. Furthermore, there is a projection-valued #-measure P_Ω on ${}^*\mathbb{R}_c^{\#n}$ so that

$$\langle\varphi, U(\mathbf{t})\psi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} Ext\text{-}\exp(i\langle\mathbf{t},\boldsymbol{\lambda}\rangle)d^\#\langle\varphi, P_{\boldsymbol{\lambda}}\psi\rangle_\# \qquad (4.4.27)$$

for all $\varphi,\psi \in \mathbf{H}^\#$.

Proof Let A_j be the infinitesimal generator of $U_j(t_j) = U(0,\dots,t_j,\dots,0)$. The procedure used in the proof of Theorem 4.4.8 shows that $D \subset D(A_j)$, $A_j : D \to D$, and $U_j(t_j) : D \to D$. Theorem 4.4.7 shows that A_j is essentially self-#-adjoint on D. Because of the relation $U(\mathbf{t}+\mathbf{s}) = U(\mathbf{t})U(\mathbf{s})$, $U_j(t_j)$ commutes with $U_i(t_i)$ for all $t_j, t_i \in {}^*\mathbb{R}_c^\#$.
Therefore, it follows from Theorem 4.5.1, that A_i and A_j commute in the sense that is, their spectral projections commute.Let P_Ω^j be the projection-valued #-measure on ${}^*\mathbb{R}_c^\#$ corresponding to A_j. Define a projection valued #-measure P_Ω on ${}^*\mathbb{R}_c^{\#n}$ by defining it first on rectangles $r_n = Ext\text{-}\prod_{i=1}^n (a_i,b_i)$ by $P_{r_n} = Ext\text{-}\prod_{i=1}^n P^i_{(a_i,b_i)}$ and then letting P_Ω be the unique extension to the smallest $\sigma^\#$-algebra containing the rectangles, namely the #-Borel sets. Notice that, by Theorem 4.5.1, the $P^j_{\Omega_j}$ commute since the groups U_j commute. For each $\varphi,\psi \in \mathbf{H}^\#$, $\langle\varphi, P_{\boldsymbol{\Omega}}\psi\rangle_\#$ is a ${}^*\mathbb{C}_c^\#$-valued #-measure of hyperfinite mass which we denote by $d^\#\langle\varphi, P_{\boldsymbol{\lambda}}\psi\rangle_\#$.
Applying generalized Fubini's theorem we conclude that

$$\langle\varphi, U(\mathbf{t})\psi\rangle_\# = \left\langle\varphi, Ext\text{-}\prod_{i=1}^n U(t_i)\psi\right\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} Ext\text{-}\exp(i\langle\mathbf{t},\boldsymbol{\lambda}\rangle)d^\#\langle\varphi, P_{\boldsymbol{\lambda}}\psi\rangle_\#. \qquad (4.4.28)$$

§ 4.5.

Suppose that A and B are two unbounded self-#-adjoint operators on a non-Archimedean Hilbert space $H^\#$. We would like to find a reasonable definition for the statement: "A and B commute."
This cannot be done in the straightforward way since $AB - BA$ may not make sense on any vector $\psi \in H^\#$ for example, one might have $(\mathbf{Ran}(A)) \cap D(B) = \varnothing$ in which case BA does not have a meaning. This suggests that we find an equivalent formulation of commutativity for bounded self-#-adjoint operators. The spectral theorem for bounded self-#-adjoint operators A and B shows that in that case $AB - BA = 0$ if and only if all their projections, $\{P_\Omega^A\}$ and $\{P_\Omega^B\}$, commute, We take this as our definition in the unbounded case.

Definition 4.5.1.Two possibly unbounded in${}^*\mathbb{R}_c^\#$ self-#-adjoint operators A and B are said to commute if and only if all the projections in their associated projection-valued #-measures commute.

Remark 4.5.1.The spectral theorem shows that if A and B commute, then all the bounded in${}^*\mathbb{R}_c^\#$ #-Borel functions of A and B also commute. In particular, the resolvents $R_\lambda(A)$ and $R_\mu(B)$ commute and the unitary groups $Ext\text{-}\exp(itA)$ and $Ext\text{-}\exp(isA)$ commute.
The converse statement is also true and this shows that the above definition of "commute" is reasonable:

Theorem 4.5.1. Let A and B be self-#-adjoint operators on a non-Archimedean Hilbert spaceHilbert space $H^\#$.

Then the following three statements are equivalent:

(a) Spectral projections $P^A_{(a,b)}$ and $P^B_{(c,d)}$, commute.

(b) If $\operatorname{Im}\lambda$ and $\operatorname{Im}\mu$ are nonzero, then $R_\lambda(A)R_\mu(B) - R_\mu(B)R_\lambda(A) = 0$.

(c) For all $s,t \in {}^*\mathbb{R}^\#_c$, $[Ext\text{-}\exp(itA)][Ext\text{-}\exp(isB)] = [Ext\text{-}\exp(isB)][Ext\text{-}\exp(itA)]$.

Proof The fact that (a) implies (b) and (c) follows from the functional calculus. The fact that (b) implies (a) easily follows from the formula which expresses the spectral projections of A and B as strong #-limits of the resolvents (generalized Stone's formula) together with the fact that

$$s\text{-}\#\text{-}\lim_{\varepsilon\to_\# 0}[i\varepsilon R_{a+i\varepsilon}(A)] = P^A_{\{a\}}. \tag{4.5.1}$$

To prove that (c) implies (a), we use some simple facts about the Fourier transform. Let $f \in S^\#({}^*\mathbb{R}^\#_c)$. Then, by generalized Fubini's theorem,

$$\begin{aligned} &Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} f(t)\langle[Ext\text{-}\exp(itA)]\varphi,\psi\rangle_\# d^\# t = \\ &= Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} f(t)\left(Ext\text{-}\int_{{}^*\mathbb{R}^\#_c}\left([Ext\text{-}\exp(-it\lambda)]d^\#_\lambda\langle P^A_\lambda\varphi,\psi\rangle_\#\right)\right)d^\# t = \\ &= \sqrt{2\pi_\#}\left(Ext\text{-}\int_{{}^*\mathbb{R}^\#_c}\widehat{f}(\lambda)d^\#_\lambda\langle P^A_\lambda\varphi,\psi\rangle_\#\right) = \sqrt{2\pi_\#}\left\langle\varphi,\widehat{f}(A)\psi\right\rangle_\#. \end{aligned} \tag{4.5.2}$$

Thus, using (c) and generalized Fubini's theorem again,

$$\begin{aligned} &\left\langle\varphi,\widehat{f}(A)\widehat{g}(B)\psi\right\rangle_\# = \\ &Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} f(t)g(s)\langle\varphi,[Ext\text{-}\exp(-itA)][Ext\text{-}\exp(-isB)]\psi\rangle_\# d^\# s d^\# t = \\ &= \left\langle\varphi,\widehat{g}(B)\widehat{f}(A)\psi\right\rangle_\# \end{aligned} \tag{4.5.3}$$

so, for all $f,g \in S^\#({}^*\mathbb{R}^\#_c)$, $\widehat{f}(A)\widehat{g}(B) - \widehat{g}(B)\widehat{f}(A) = 0$.

Since the Fourier transform maps $S^\#({}^*\mathbb{R}^\#_c)$ onto $S^\#({}^*\mathbb{R}^\#_c)$ we conclude that $f(A)g(B) = g(B)f(A)$ for all $f,g \in S^\#({}^*\mathbb{R}^\#_c)$. But, the characteristic function, $\chi_{(a,b)}$ can be expressed as the pointwise #-limit of a hyperinfinite sequence $f_n, n \in {}^*\mathbb{N}$ of uniformly bounded functions in $S^\#({}^*\mathbb{R}^\#_c)$. By the functional calculus,

$$s\text{-}\#\text{-}\lim_{n\to{}^*\infty} f_n(A) = P^A_{(a,b)}. \tag{4.5.4}$$

Similarly,we find uniformly bounded $g_n \in S^\#({}^*\mathbb{R}^\#_c)$ #-converging pointwise to $\chi_{(c,d)}$ and

$$s\text{-}\#\text{-}\lim_{n\to{}^*\infty} g_n(B) = P^B_{(c,d)}. \tag{4.5.5}$$

Since the f_n and g$_n$ are uniformly bounded in ${}^*\mathbb{R}^\#_c$ and

$$f_n(A)g_n(B) = g_n(B)f_n(A) \tag{4.5.6}$$

for each $n \in {}^*\mathbb{N}$, we conclude that $P^A_{(a,b)}$ and $P^B_{(c,d)}$, commute which proves (a).

.

Chapter IV.Non-Archimedean Banach spaces endroved

with ${}^*\mathbb{R}_c^\#$-valued norm.

1.Definitions and examples

A non-Archimedean normed space with ${}^*\mathbb{R}_c^\#$-valued norm (#-norm) is a pair $(X,\|\cdot\|_\#)$ consisting of a vector space X over a non-Archimedean scalar field ${}^*\mathbb{R}_c^\#$ or complex field ${}^*\mathbb{C}_c^\#$ together with a distinguished norm $\|\cdot\|_\# : X \to {}^*\mathbb{R}_c^\#$. Like any norms, this #-norm induces a translation invariant distance function, called the canonical or (norm) induced non-Archimedean ${}^*\mathbb{R}_c^\#$-valued metric for all vectors $x,y \in X$, defined by

$$d^\#(x,y) = \|x-y\|_\# = \|y-x\|_\#. \quad (1.1)$$

Thus (1.1) makes X into a metric space $(X,d^\#)$. A hyper infinite sequence $(x_n)_{n=1}^{\infty^\#}$ is called $d^\#$-Cauchy or Cauchy in $(X,d^\#)$ or $\|\cdot\|_\#$ -Cauchy if for every hyperreal $r \in {}^*\mathbb{R}_c^\#$, $r > 0$, there exists some $N \in \mathbb{N}^\#$ such that

$$d^\#(x_n,x_m) = \|x_n - x_m\|_\# < r, \quad (1.2)$$

where m and n are greater than N. The canonical metric $d^\#$ is called a #-complete metric if the pair $(X,d^\#)$ is a #-complete metric space, which by definition means for every $d^\#$-Cauchy sequence $(x_n)_{n=1}^{\infty^\#}$ in $(X,d^\#)$, there exists some $x \in X$ such that

$$\#\text{-}\lim_{n\to\infty^\#}\|x_n - x\|_\# = 0 \quad (1.3)$$

where because $\|x_n - x\|_\# = d^\#(x_n,x)$, this hyper infinite sequence's #-convergence to x can equivalently be expressed as: #-$\lim_{n\to\infty^\#} x_n = x$ in $(X,d^\#)$.

Definition 1.1. The normed space $(X,\|\cdot\|_\#)$ is a non-Archimedean Banach space endroved with ${}^*\mathbb{R}_c^\#$-valued norm if the #-norm induced metric $d^\#$ is a #-complete metric, or said differently, if $(X,d^\#)$ is a #-complete metric space. The #-norm $\|\cdot\|_\#$ of a #-normed space $(X,\|\cdot\|_\#)$ is called a #-complete #-norm if $(X,\|\cdot\|_\#)$ is a non-Archimedean Banach space endroved with ${}^*\mathbb{R}_c^\#$-valued #-norm.

Remark 1.1.For any #-normed space $(X,\|\cdot\|_\#)$, there exists an L-semi-inner product $\langle\cdot,\cdot\rangle_\# : X \times X \to {}^*\mathbb{R}_c^\#$ such that $\|x\|_\# = \sqrt{\langle x,x\rangle_\#}$ for all $x \in X$; in general, there may be infinitely many L-semi-inner products that satisfy this condition. L-semi-inner products are a generalization of inner products, which are what fundamentally distinguish non-Archimedean Hilbert spaces from all other non-Archimedean Banach spaces. Characterization in terms of hyper infinite series,see ref. [1].

The vector space structure allows one to relate the behavior of hyper infinite Cauchy sequences to that of #-converging hyper infinite series of vectors.

Remark 1.2.A #-normed space X is a non-Archimedean Banach space if and only if each absolutely #-convergent hyper infinite series $Ext\text{-}\sum_{n=1}^{\infty^\#} v_n$ in X #-converges in X,i.e., $Ext\text{-}\sum_{n=1}^{\infty^\#}\|v_n\| < \infty^\#$ implies that $Ext\text{-}\sum_{n=1}^{\infty^\#} v_n$ #-converges in X.

2.Linear operators,isomorphisms

If X and Y are #-normed spaces over the same ground field ${}^*\mathbb{R}_c^\#$, the set of all #-continuous ${}^*\mathbb{R}_c^\#$-linear maps $T : X \to Y$ is denoted by $B^\#(X,Y)$. In hyper infinite-dimensional spaces, not all linear maps are #-continuous. A linear mapping from a #-normed space X to another normed space is #-continuous if and only if it is bounded or hyper bounded on the #-closed unit ball of X. Thus, the vector space

$B^{\#}(X,Y)$ can be endroved with the operator norm

$$\|T\| = \sup\{\|Tx\|_{\#Y} \mid x \in X, \|x\|_{\#X} \leq 1\}. \quad (2.1)$$

For Y a non-Archimedean Banach space, the space $B^{\#}(X,Y)$ is a Banach space with respect to this #-norm.

If X is a non-Archimedean Banach space, the space $B^{\#}(X) = B^{\#}(X,X)$ forms a unital Banach algebra; the multiplication operation is given by the composition of linear maps.

Definition 2.1.If X and Y are #-normed spaces, they are #-isomorphic #-normed spaces if there exists a linear bijection $T : X \to Y$ such that T and its inverse T^{-1} are #-continuous. If one of the two spaces X or Y is #-complete then so is the other space. Two #-normed spaces X and Y are #-isometrically isomorphic if in addition, T is an #-isometry, that is, $\|T(x)\| = \|x\|$ for every $x \in X$.

Definition 2.2.Let $\{X, \|\cdot\|\}$ be standard Banach space.For $x \in {}^*X$ and $\varepsilon > 0, \varepsilon \approx 0$ we define the open $\approx$-ball about x of radius ε to be the set $B_\varepsilon(x) = \{y \in {}^*X | {}^*\|x - y\| < \varepsilon\}$.

Definition 2.3.Let $\{X, \|\cdot\|\}$ be standard Banach space, $Y \subset X$ thus ${}^*Y \subseteq {}^*X$ and let $x \in {}^*X$. Then x is an $*$-accumulation point of ${}^*X$ if for every $\varepsilon > 0, \varepsilon \approx 0, Y \cap (B_\varepsilon(x) \backslash \{x\}) \neq \varnothing$.

Definition 2.4.Let $\{X, \|\cdot\|\}$ be a standard Banach space, let $Y \subseteq {}^*X, Y$ is $*$-closed if every $*$-accumulation point of Y is an element of Y.

Definition 2.5.Let $\{X, \|\cdot\|\}$ be standard Banach space.We shall say that internal hyper infinite sequence $\{x_n\}_{n=1}^{n={}^*\infty}$ in ${}^*X$ $*$-converges to $x \in {}^*X$ as $n \to {}^*\infty$if for any $\varepsilon > 0, \varepsilon \approx 0$ there is $N \in {}^*\mathbb{N}$ such that for any $n > N$: ${}^*\|x_n - x\| < \varepsilon$.

Definition 2.6.Let $\{X, \|\cdot\|\}, \{Y, \|\cdot\|\}$ be a standard Banach spaces. A linear internal operator $A : D(A) \subseteq {}^*X \to {}^*Y$is $*$-closed if for every internal hyper infinite sequence $\{x_n\}_{n=1}^{n={}^*\infty}$ in $D(A)$ $*$-converging to $x \in {}^*X$ such that $Ax_n \to y \in {}^*Y$ as $n \to {}^*\infty$ one has $x \in D(A)$ and $Ax = y$. Equivalently, A is $*$-closed if its graph is $*$-closed in the direct sum ${}^*X \oplus {}^*Y$.

Given a linear operator $A : {}^*X \to {}^*Y$, not necessarily $*$-closed, if the $*$-closure of its graph in ${}^*X \oplus {}^*Y$ happens to be the graph of some operator, that operator is called the $*$-closure of A, and we say that A is $*$-closable. Denote the $*$-closure of A by $*$-$\bar{A}$. It follows that A is the restriction of $*$-$\bar{A}$ to $D(A)$.

A $*$-core (or $*$-essential domain) of a $*$-$\bar{A}$closable operator is a subset $C \subset D(A)$ such that the $*$-closure of the restriction of A to C is $*$-$\bar{A}$.

Definition 2.7. The graph of the linear transformation $T : H \to H$ is the set of pairs $\{\langle\varphi, T\varphi\rangle | (\varphi \in D(T))\}$.

The graph of T, denoted by $\Gamma(T)$, is thus a subset of $H \times H$ which is a non-Archimedean Hilbert space with inner product $(\langle\varphi_1, \psi_1\rangle, \langle\varphi_2, \psi_2\rangle)$.

T is called a #-closed operator if $\Gamma(T)$ is a #-closed subset of $H \times H$.

Definition 2.8. Let T_1 and T be operators on H. If $\Gamma(T_1) \supset \Gamma(T)$, then T_1 is said to be an extension of T and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$

and $T_1\varphi = T\varphi$ for all $\varphi \in D(T)$.

Definition 2.9. An operator T is #-closable if it has a #-closed extension. Every #-closable operator has a smallest #-closed extension, called its #-closure, which we denote by #-$\overline{T}$.

Theorem 2.1.If T is #-closable, then $\Gamma(\#\text{-}\overline{T}) = \#\text{-}\overline{\Gamma(T)}$.

Definition 2.10.Let T be a #-densely defined linear operator on a non-Archimedean Hilbert space H. Let $D(T^*)$ be the set of $\varphi \in H$ for which there is an $\xi \in H$ with $(T\psi,\varphi) = (\psi,\xi)$ for all $\psi \in D(T)$.

For each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$. T^* is called the #-adjoint of T. Note that $\varphi \in D(T^*)$ if and only if $|(T\psi,\varphi)| \leq C\|\psi\|$ for all $\psi \in D(T)$. We note that $S \subset T$ implies $T^* \subset S^*$.

Theorem 2.2. Let T be a #-densely defined operator on a non-Archimedean Hilbert space H.

Then:(i) T^* is #-closed.

(ii) T is #-closabie if and only if $D(T^*)$ is #-dense in which case $T = T^{**}$.

(iii) If T is #-closabie, then $(\# - \overline{T})^* = T^*$.

Definition 2.11. Let T be a #-closed operator on a Hilbert space H. A complex number $\lambda \in {}^*\mathbb{C}_c^{\#}$ is in the resolvent set,$\rho(T)$,if $\lambda I - T$ is a bijection of $D(T)$ onto H with a a finitely or hyper finitely bounded inverse. If $\lambda \in \rho(T)$, $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.

The definitions of spectrum, point spectrum, and residual spectrum are the same for unbounded operators as they are for bounded operators. We will sometimes refer to the spectrum of nonclosed, but closabie operators. In this case we always mean the spectrum of the closure.

3. Symmetric and self-#-adjoint operators: the basic criterion for self-#-adjointness.

Definition 3.1. A #-densely defined operator T on a non-Archimedean Hilbert space is called symmetric (or Hermitian) if $T \subset T^*$, that is, if $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$.

Equivalently, T is symmetric if and only if $(T\varphi,\psi) = (\varphi,T\psi)$ for all $\varphi,\psi \in D(T)$

Definition 3.2. T is called self-adjoint if $T = T^*$, that is, if and only if T is symmetric and $D(T) = D(T^*)$.

A symmetric operator is always #-closable, since $D(T^*) \supset D(T)$ is #-dense in H. If T is symmetric, T^* is a closed extension of T so the smallest #-closed extension T^{**} of T must be contained in T^*. Thus for symmetric operators, we have $T \subset T^{**} \subset T^*$.For #-closed symmetric operators,$T = T^{**} \subset T^*$ and, for self-adjoint operators,$T = T^{**} = T^*$

From this one can easily see that a #-closed symmetric operator T is self-adjoint if and only if T^* is symmetric.

The distinction between #-closed symmetric operators and self-adjoint operators is very important. It is only for self-adjoint operators that the spectral theorem holds and it is only self-adjoint operators that may be #-exponentiated to give the one-parameter unitary groups which give the dynamics in

QFT. Chapter X is mainly devoted to studying methods for proving that operators are self-adjoint. We content ourselves here with proving the basic criterion for selfadjointness.

First, we introduce the useful notion of essential self-adjointness.

Definition 3.3 A symmetric operator T is called essentially self- #-adjoint if its #-closure #-$\bar{T}$ is self- #-adjoint. If T is #-closed, a subset $D \subset D(T)$ is called a core for T if

$\overline{\# - T \restriction D} = T.$

If T is essentially self-#-adjoint, then it has one and only one self-#-adjoint extension. The importance of essential self-#-adjointness is that one is often given a nonclosed symmetric operator $T.$ If T can be shown to be essentially self-#-adjoint, then there is uniquely associated to Ta self-adjoint operator $T = T^{**}.$ Another way of saying this is that if A is a self-#-adjoint operator, then to specify A uniquely one need not give the exact domain of A (which is often difficult), but just some #-core for A

Chapter V. Semigroups of operators on a non-Archimedean Banach spaces.

§1.Semigroups on non-Archimedean Banach spaces and their generators.

A family of #-bounded operators $\{T(t)|0 < t < \infty^{\#}\}$ on external hyper infinite dimensional non-Archimedean Banach space X endoved with ${}^{*}\mathbb{R}^{\#}_{c,+}$ - valued norm $\|\cdot\|_{\#}$ is called a strongly #-continuous semigroup if:

(a) $T(0) = I$

(b) $T(s)T(t) = T(s+t)$ for all $s,t \in {}^{*}\mathbb{R}^{\#}_{c,+}$

(c) For each $\varphi \in X, t \mapsto T(t)$ is #-continuous mapping.

We will see that strongly continuous semigroups are the “exponentials,” $T(t) = Ext\text{-}\exp(-tA)$, of a certain class of operators. .

We begin by studying a special class of semigroups:

Definition 1.1. A family $\{T(t)|0 < t < \infty^{\#}\}$ of bounded or hyper bounded operators on external hyper infinite dimensional Banach space X is called a contraction semigroup if it is a strongly #-continuous semigroup and moreover $\|T(t)\|_{\#} < 1$ for all $t \in [0,\infty^{\#})$. Note that the all theorems about general strongly #-continuous semigroups are easy generalizations of the corresponding theorems for #-contraction semigroups. Thus, we study the special case first. We then briefly discuss the general theory and conclude the section by studying another special class, #-holomorphic semigroups.

Proposition 1.1. Let $T(t)$ be a strongly #-continuous semigroup on a non-Archimedean Banach space X and set $A\varphi = \#\text{-}\lim_{r \to_{\#} 0} A_r\varphi$ where $D(A) = \{\varphi|\ \#\text{-}\lim_{r \to_{\#} 0} A_r\varphi$ exists$\}.$ Then A is #-closed and #-densely defined. A is called the infinitesimal generator of $T(t).$ We will also say that A generates $T(t)$ and write $T(t) = Ext\text{-}\exp(-tA).$

Proof.Let $T(t)$ be a contraction semigroup on a Banach space $X.$ We obtain the generator of $T(t)$ by #-differentiation. Set $A_t = t^{-1}(I - T(t))$ and define

$$D(A) = \{\varphi|\ \#\text{-}\lim_{t \to_{\#} 0} A_t\varphi \text{ exists}\}.$$

For $\varphi \in D(A),$ we define $A\varphi = \#\text{-}\lim_{t \to_{\#} 0} A_t\varphi$. Our first goal is to show that $D(A)$ is

#-dense. For $\varphi \in X$, we set

$$\varphi_s = Ext\text{-}\int_0^s T(t)\varphi d^{\#}t. \tag{2.1}$$

For any $r > 0$, we get

$$T(r)\varphi_s = Ext\text{-}\int_0^s T(t+r)\varphi d^{\#}t \tag{2.2}$$

thus

$$\begin{gathered} A_r\varphi_s = -\frac{1}{r}\left(Ext\text{-}\int_0^s [T(t+r)\varphi - T(t)\varphi] d^{\#}t \right) = \\ -\frac{1}{r}\left(Ext\text{-}\int_s^{r+s} T(t)\varphi d^{\#}t \right) + \frac{1}{r}\left(Ext\text{-}\int_s^r T(t)\varphi d^{\#}t \right). \end{gathered} \tag{2.3}$$

From Eq.(2.3) one obtains $\#\text{-}\lim_{r \to_{\#} 0} A_r\varphi_s = -T(s)\varphi + \varphi$. Therefore, for each $\varphi \in X$

and $s > 0$, $\varphi_s \in D(A)$. Since $s^{-1}\varphi_s \to_{\#} \varphi$ as $\to_{\#} 0$, A is #-densely defined.
Furthermore, if $\varphi \in D(A)$, then $A_r T(t)\varphi = T(t)A_r\varphi$, so $T(t) : D(A) \to D(A)$ and

$$\frac{d^{\#}}{d^{\#}t}T(t)\varphi = -AT(t)\varphi = -T(t)A\varphi \tag{2.4}$$

A is also #-closed, for if $\varphi_n \in D(A)$, $\#\text{-}\lim_{n\to\infty^{\#}} \varphi_n = \varphi$, and $\#\text{-}\lim_{n\to\infty^{\#}} A\varphi_n = \psi$, then

$$\begin{gathered} \#\text{-}\lim_{r \to_{\#} 0} A_r\varphi = \#\text{-}\lim_{r \to_{\#} 0} \#\text{-}\lim_{n\to\infty^{\#}} \left[-\frac{1}{r}(T(r)\varphi_n - \varphi_n) \right] = \\ \#\text{-}\lim_{r \to_{\#} 0} \#\text{-}\lim_{n\to\infty^{\#}} \frac{1}{r}\left(Ext\text{-}\int_s^r T(t)A\varphi_n d^{\#}t \right) = \\ \#\text{-}\lim_{r \to_{\#} 0} \frac{1}{r}\left(Ext\text{-}\int_s^r T(t)\psi d^{\#}t \right) \end{gathered} \tag{2.5}$$

so $\varphi \in D(A)$ and $A\varphi = \psi$.
The formal Laplace transform

$$\frac{1}{\lambda + A} = -\left(Ext\text{-}\int_0^{\infty^{\#}} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))d^{\#}t \right) \tag{2.6}$$

suggests that all $\mu \in {}^*\mathbb{C}_c^{\#}$ with $\operatorname{Re}\mu < 0$ are in $\rho(A)$. This is in fact true and the formula (2.6) holds in the strong sense. For suppose that $\operatorname{Re}\lambda > 0$. Then, since $\|Ext\text{-}\exp(-tA)\| < 1$, the formula (2.7)

$$R\varphi = Ext\text{-}\int_0^{\infty^{\#}} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA)\varphi)d^{\#}t \tag{2.7}$$

defines a hyper bounded linear operator of #-norm less than or equal to $(\operatorname{Re}\lambda)^{-1}$.

Moreover, for $r > 0$,

$$A_r R\varphi = -\frac{1}{r}\left(Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-(t+r)A) - Ext\text{-}\exp(-tA))\varphi d^\# t \right) =$$

$$\frac{1 - Ext\text{-}\exp(\lambda r)}{r}\left(Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^\# t \right) + \tag{2.8}$$

$$\frac{Ext\text{-}\exp(\lambda r)}{r}\left(Ext\text{-}\int_0^{r} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^\# t \right)$$

so as $r \to_\# 0, A_r R\varphi \to_\# (\varphi - \lambda R\varphi)$. Thus $R\varphi \in D(A)$ and $AR\varphi = \varphi - \lambda R\varphi$ which implies $(\lambda + A)R\varphi = \varphi$. In addition, for $\varphi \in D(A)$ we have $AR\varphi = RA\varphi$ since

$$A\left(Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^\# t \right) =$$

$$Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))A(Ext\text{-}\exp(-tA))\varphi d^\# t = \tag{2.9}$$

$$Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))A\varphi d^\# t.$$

The first equality follows by approximation with external hyperfinite Riemann sums (see [1]) from the facts that $(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi$ and $A(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))$ are #-integrable, A is #-closed. Thus, for $\varphi \in D(A)$, $R(\lambda + A)\varphi = \varphi = (\lambda + A)R\varphi$ which implies that

$$R = (\lambda + A)^{-1}. \tag{2.10}$$

The properties of A which we have derived are also sufficient to guarantee that A generates a contraction semigroup. In fact, we only need information about real positive A.

Theorem 1.1. (Generalized Hille-Yosida theorem) A necessary and sufficient condition that a #-closed
linear operator A on a Banach space X generate a contraction semigroup is that
(i) $(-\infty^\#, 0) \subset \rho(A)$
(ii) $\|(\lambda + A)^{-1}\|_\#$ for all $\lambda > 0$.
Furthermore, if A satisfies (i) and (ii), then the entire #-open left half-plane is contained in $\rho(A)$ and

$$(\lambda + A)^{-1}\varphi = -Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))d^\# t \tag{2.11}$$

for all $\varphi \in X$ and λ with $\operatorname{Re}\lambda > 0$. Finally, if $T_1(t)$ and $T_2(t)$ are contraction semigroups generated by A_1 and A_2 respectively, then $T_2(t) \neq T_1(t)$ for some t implies that $A_1 \neq A_2$.

Proof. Since we showed above that conditions (i) and (ii) are necessary and that (2.11) holds, we need only show sufficiency. So, suppose that A is a #-closed operator on X satisfying (i) and (ii). For $\lambda > 0$, define $A^{(\lambda)} = \lambda - \lambda^2(\lambda + A)^{-1}$. We will show that as $\lambda \to \infty^{\#}$, $A^{(\lambda)} \to_{\#} A$ strongly on $D(A)$ and then construct $Ext\text{-}\exp(-tA)$ as the strong #-limit of the semigroups $Ext\text{-}\exp(-tA^{(\lambda)})$. For $\varphi \in D(A)$, $A^{(\lambda)}\varphi = \lambda(\lambda + A)^{-1}A\varphi$. Moreover, by (ii),

$$\#\text{-}\lim_{\lambda\to\infty^{\#}}[\lambda(\lambda + A)^{-1}\varphi - \varphi] = \#\text{-}\lim_{\lambda\to\infty^{\#}}[-(\lambda + A)^{-1}A\varphi] = 0. \quad (2.12)$$

By condition (ii) the family $\{\lambda(\lambda + A)^{-1}|\lambda > 0\}$ is #-uniformly hyperfinitely bounded in #-norm, so since $D(A)$ is #-dense, $\#\text{-}\lim_{\lambda\to\infty^{\#}}[\lambda(\lambda + A)^{-1}\psi] = \psi$ for all $\psi \in X$. Thus $\#\text{-}\lim_{\lambda\to\infty^{\#}} A^{(\lambda)}\varphi = A\varphi$ for all $\varphi \in D(A)$. Since A is hyperfinitely bounded, the semigroups $Ext\text{-}\exp(-tA^{(\lambda)})$ can be defined by hyper infinite power series. Since

$$\begin{gathered}\|Ext\text{-}\exp(-tA^{(\lambda)})\|_{\#} = \|(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(t\lambda^2(\lambda + A)^{-1}))\|_{\#} \leq \\ \leq (Ext\text{-}\exp(-\lambda t))\left(Ext\text{-}\sum_{n=0}^{\infty^{\#}}\frac{t^n\lambda^{2n}}{n!}\|(\lambda + A)^{-1}\|_{\#}^{n}\right) \leq 1\end{gathered} \quad (2.13)$$

they are contraction semigroups. For all $\mu, \lambda, t > 0$, and all $\varphi \in D(A)$, we have

$$\begin{gathered}[Ext\text{-}\exp(-tA^{(\lambda)})]\varphi - [Ext\text{-}\exp(-tA^{(\mu)})]\varphi = \\ Ext\text{-}\int_0^t \frac{d^{\#}}{d^{\#}s}(Ext\text{-}\exp(-sA^{(\lambda)}))((Ext\text{-}\exp(-(t-s)A^{(\lambda)}))\varphi)d^{\#}s\end{gathered} \quad (2.14)$$

so,

$$\begin{gathered}\|[Ext\text{-}\exp(-tA^{(\lambda)})]\varphi - [Ext\text{-}\exp(-tA^{(\mu)})]\varphi\|_{\#} \leq \\ Ext\text{-}\int_0^t \|(Ext\text{-}\exp(-sA^{(\lambda)}))((Ext\text{-}\exp(-(t-s)A^{(\lambda)})))\|_{\#}\|A^{(\mu)}\varphi - A^{(\lambda)}\varphi\|_{\#}d^{\#}s \leq \\ \leq t\|A^{(\mu)}\varphi - A^{(\lambda)}\varphi\|_{\#}.\end{gathered} \quad (2.15)$$

We have used the fact that $Ext\text{-}\exp(-tA^{(\lambda)})$ and $[Ext\text{-}\exp(-(t-s)A^{(\mu)})]$ commute since $\{A^{(\lambda)}|\lambda > 0\}$ is a commuting family. Since we have proven above that $\#\text{-}\lim_{\lambda\to\infty^{\#}} A^{(\lambda)}\varphi = A\varphi$, $\{Ext\text{-}\exp(-tA^{(\lambda)})\}$ is Cauchy as $\lambda \to \infty^{\#}$ for each $t > 0$ and $\varphi \in D(A)$. Since $D(A)$ is #-dense and the $Ext\text{-}\exp(-tA^{(\lambda)})$ are uniformly hyperfinitely bounded, the same statement holds for all $\varphi \in X$. Now, define

$$T(t)\varphi = \#\text{-}\lim_{\lambda\to\infty^{\#}}[Ext\text{-}\exp(-tA^{(\lambda)})\varphi]. \quad (2.16)$$

$T(t)$ is a semigroup of contraction operators since these properties are preserved under strong #-limits. The above inequality shows that the #-convergence in Eq.(2.16) is uniform for t restricted to a hyperfinite interval, so $T(t)$ is strongly #-continuous since $Ext\text{-}\exp(-tA^{(\lambda)})$ is. Thus, $T(t)$ is a contraction semigroup.It remains to show that the infinitesimal generator of $T(t)$, call it $\widetilde{A}$, is equal to A. For all t and $\varphi \in D(A)$,

$$[Ext\text{-}\exp(-tA^{(\lambda)})\varphi] - \varphi == -\left[Ext\text{-}\int_0^t\right] \quad (2.17)$$

so, since $\#\text{-}\lim_{\lambda\to\infty^{\#}} A^{(\lambda)}\varphi = A\varphi$, we have

$$T(t)\varphi - \varphi = -\left[Ext\text{-}\int_0^t T(s)A\varphi d^\# s \right]. \quad (2.18)$$

Thus, $\widetilde{A}_t\varphi \to_\# A\varphi$ as $t \to_\# 0$. Therefore $D\left(\widetilde{A}\right) \supset D(A)$ and $\widetilde{A} \restriction D(A) = A$. For $\lambda > 0$, $(\lambda + A)^{-1}$ exists by hypothesis and $\left(\lambda + \widetilde{A}\right)^{-1}$ exists by the necessity part of the theorem.

.

§2 Hypercontractive semigroups

In the previous section we discussed $\mathcal{L}_\#^p$-contractive semigroups. In this section we will prove a self-adjointness theorem for operators of the form $A + V$ where V is a multiplication operator and A generates an $\mathcal{L}_\#^p$-contractive semigroup that satisfies a strong additional property.

Definition 2.1. Let $\langle M, \mu^\# \rangle$ be a #-measure space with $\mu^\#(M) = 1$ and suppose that A is a positive self-adjoint operator on $\mathcal{L}_\#^2(M, d^\#\mu^\#)$. We say that $Ext\text{-}\exp(-tA)$ is a hypercontractive semigroup if:

(i) $Ext\text{-}\exp(-tA)$ is $\mathcal{L}_\#^p$-contractive;

(ii) for some $b > 2$ and some constant C_b, there is a $T > 0$ so that $\|Ext\text{-}\exp(-tA)\varphi\|_b \leq C_b\|\varphi\|_2$ for all $\varphi \in \mathcal{L}_\#^2(M, d^\#\mu^\#)$.

By Theorem X.55, condition (i) implies that $Ext\text{-}\exp(-tA)$ is a strongly #-continuous contraction semigroup for all $p < \infty^\#$. Holder's inequality shows that

$$\|\cdot\|_q \leq \|\cdot\|_p \quad (1)$$

if $p \geq q$. Thus the $\mathcal{L}_\#^p$-Spaces are a nested family of spaces which get smaller as p gets larger; this suggests that (ii) is a very strong condition. The following proposition shows that b plays no special role.

Proposition 2.1. Let $Ext\text{-}\exp(-tA)$ be a hypercontractive semigroup on $\mathcal{L}_\#^2(M, d^\#\mu^\#)$. Then for all $p, q \in (1, \infty^\#)$, there is a constant C_{pq} and a $t_{pq} > 0$ so that if $t > t_{pq}$ then $\|Ext\text{-}\exp(-tA)\varphi\|_p \leq C_{pq}\|\varphi\|_q$ for all $\varphi \in \mathcal{L}_\#^q$.

Proof. The case where $p < q$ follows immediately from (i) and (1). So suppose that $p > q$. Since $Ext\text{-}\exp(-tA) : \mathcal{L}_\#^2 \to \mathcal{L}_\#^b$ and $Ext\text{-}\exp(-tA) : \mathcal{L}_\#^{\infty^\#} \to \mathcal{L}_\#^{\infty^\#}$, the generalized Riesz-Thorin theorem implies that there is a constant C so that for all $r \geq 2$, $\|Ext\text{-}\exp(-tA)\varphi\|_r \leq C\|\varphi\|_{br/2}$. We now consider two cases. First, if $q \geq 2$ we choose n large enough so that $2(b/2)^n > p$. Then $\|Ext\text{-}\exp(-nTA)\varphi\|_{2(b/2)^n} \leq C\|\varphi\|_2$ so the conclusion follows if $2 < q, p > 2(b/2)^n$, by using (1), and hypothesis (i). If $1 < q < 2$, then we choose n large enough so that $2(b/2)^n > p$ and $q > c$ where $c^{-1} + (2(b/2)^n)^{-1} = 1$. Since A is self-adjoint and $Ext\text{-}\exp(-nTA)\varphi$ is a bounded or hyper bounded map from $\mathcal{L}_\#^2$ to $\mathcal{L}_\#^{2(b/2)^n}$, $(Ext\text{-}\exp(-nTA))^* = Ext\text{-}\exp(-nTA)$ is a bounded or hyper bounded map from $\mathcal{L}_\#^c$ to $\mathcal{L}_\#^2$. Thus $Ext\text{-}\exp(-2nTA)$ is a bounded or hyper bounded map from $\mathcal{L}_\#^c$ to $\mathcal{L}_\#^{2(b/2)^n}$. Since $c < q < p < 2(b/2)^n$, (1) implies the proposition.

Theorem 2.1. The operator $-\frac{1}{2}d^{\#2}/d^\# x^2 + xd^\#/d^\# x$ on $\mathcal{L}_\#^2\left({}^*\mathbb{R}_c^\#, \pi_\#^{-1/2} Ext\text{-}\exp(-x^2)d^\# x\right)$ is positive and essentially self-adjoint on the set of hyperfinite linear combinations of Hermite polynomials, and generates a hypercontractive semigroup.

As a preparation for our main theorem, we prove the following result.
Theorem 2.2 Let $\langle M,\mu\rangle$ be a #-measure space with $\mu(M) = 1$ and let H_0 be the generator of a hypercontractive semigroup on $\mathcal{L}^2_{\#}(M,d\mu)$. Let V be a real-valued measurable function on $\langle M,\mu^{\#}\rangle$ such that $V \in \mathcal{L}^p_{\#}(M,d^{\#}\mu^{\#})$ for all $p \in [1,\infty^{\#})$ and $Ext\text{-}e^{-tV} \in \mathcal{L}^1_{\#}(M,d^{\#}\mu^{\#})$ for all $t > 0$. Then $H_0 + V$ is essentially self-#-adjoint on $C^{\infty^{\#}}(H_0) \cap D(V)$ and is bounded below. $C^{\infty^{\#}}(H_0) = \bigcap_{p\in\mathbb{N}^{\#}} D(H_0^p)$
.

Chapter VI. Singular Perturbations of Selfadjoint Operators on a non-Archimedean Hilbert space.

§1. Introduction

We study the sum $A+B$ of two #-selfadjoint operators on a non-Archimedean Banach spaces, and we find sufficient conditions for $C = A+B$ to be #-selfadjoint. Our technique is to approximate B by a hyperinfinite sequence of bounded #-selfadjoint
operators $B_n, n \in {}^*\mathbb{N}$ and so to approximate C by #-selfadjoint operators $C_n = A+B_n$.
We answer three questions separately:
1.When do the operators C_n have a #-$\lim C$? 2.When is C a #-selfadjoint operator? 3.When is $C = A+B$?
In Theorem 8 we give a set of estimates on the relative size of A and B which ensure a positive answer to all three questions. Hence these estimates show that $A+B = C$ is #-selfadjoint. In another paper [5], we use Theorem 2.8 to prove the existence of a self-interacting, causal quantum field in 4-dimensional space-time. Formally this field theory is Lorentz covariant and has non-trivial scattering; this application was the motivation for the present work.
In order to investigate the meaning of #-$\lim_{n\to{}^*\infty} C_n$, we give a new definition for the strong #-convergence of a hyperinfinite sequence of operators. Consequences of this definition
are worked out in Section 2. In Section 3 we give estimates on operators C_n which are sufficient to ensure that the #-$\lim_{n\to{}^*\infty} C_n = C$ exists and that C is maximal symmetric or #-selfadjoint. This result is given in Theorem 5 and Corollary 6.
In Section 4 we investigate whether #-$\lim_{n\to{}^*\infty} C_n = C$ is equal to $A+B$.
We combine this work in Theorem 8, our second main theorem, where B is a singular, but nearly positive #-selfadjoint perturbation of a positive #-selfadjoint operator A. To illustrate this theorem, let $A \geq I$ and let B be essentially #-selfadjoint on

$$D^{\#} = \bigcap_{n\in{}^*\mathbb{N}} D(A^n). \quad (1.0)$$

Assume now that, for some $\beta > 0$ and some α,

$$A^{-(1-\beta)}BA^{-(1-\beta)} \text{ and } A^{\beta}BA^{\alpha} \quad (1.1)$$

are #-densely defined, bounded operators. Also, for some positive $a,\varepsilon \in {}^*\mathbb{R}^{\#}_{c+}$ satisfying $2a+\varepsilon < 1$, suppose that there is a constant $b \in {}^*\mathbb{R}^{\#}_c$ such that, as bilinear forms on $D \times D$,

$$0 \leq aA + B + b \quad (1.2)$$

and

$$0 \le \varepsilon A^2 + [A^{1/2},[A^{1/2},B]] + b. \tag{1.3}$$

Then $A+B$ is #-selfadjoint.
We see from this example that neither the operator B nor the bilinear form B need be bounded relative to A.
While it may not appear evident, the conditions (1.1)-(1.3) are closely related to a more easily understandable estimate on $D^\# \times D^\#$,

$$A^2 + B^2 c(A+B)^2 + c. \tag{1.4}$$

In fact, estimates (1.1)-(1.3) are chosen because they allow us not only to prove (1.4), but also the similar inequality where B is replaced by B_n.
Let us now see that if $A+B$ is #-selfadjoint, then (1.4) must hold for every vector in $D(A+B) = D(A) \cap D(B)$.
Proposition 1.1. Let A and B be #-closed operators. Then $A+B$ is #-closed if and only if there is a constant $c \in {}^*\mathbb{R}_c^\#$ such that for all $\psi \in D(A+B)$

$$\|A\psi\|_\# + \|B\psi\|_\# \le \|(A+B)\psi\|_\# + c\|\psi\|_\# \tag{1.5}$$

and (1.5) is equivalent to (1.4) on $D(A+B) \times D(A+B)$.
Proof: Certainly (1.5) implies that $A+B$ is #-closed. Conversely, assume that $A+B$ is #-closed and introduce the #-norms on $D(A+B) = D(A) \cap D(B)$,

$$\|\psi\|_{\#1} \triangleq \|\psi\|_\# + \|A\psi\|_\# + \|B\psi\|_\# \tag{1.6}$$

and

$$\|\psi\|_{\#2} \triangleq \|\psi\|_\# + \|(A+B)\psi\|_\# \tag{1.7}$$

Then $D(A+B), \|\bullet\|_{\#2}$ is a non-Archimedean Banach space because $A+B$ is #-closed.
The identity map from $D(A+B), \|\bullet\|_{\#2}$ to $D(A+B), \|\bullet\|_{\#1}$ has a #-closed graph because $A, B,$ and $A+B$ are c#-losed. By the #-closed graph theorem, the identity map is #-continuous; hence

$$\|\psi\|_{\#1} \le c\|\psi\|_{\#2}. \tag{1.7$'$}$$

Proposition 1.2.Let $A \ge I, B$ be #-selfadjoint operators with $D^\# \subset D$(B) and suppose (1.2) and (1.3) hold. Then (1.4) is valid on $D^\# \times D^\#$.
Proof The operators $A^2, B^2, AB, BA,$ and $A^{1/2}BA^{1/2}$ define bilinear forms on $D^\# \times D^\#$. Using (1.2) and (1.3), we have the inequality:
$A^2 + B^2 = (A+B)^2 - 2A^{1/2}BA^{1/2} - [A^{1/2},[A^{1/2},B]] \ \le (A+B)^2 + (2a+\varepsilon)A^2 + 2Ab + b$
which establishes (1.4).

§2. Strong #-Convergence of Operators

Let $\mathcal{L}(C)$ be the graph of the operator C. For any hyperinfinite sequence $\{C_n\}, n \in {}^*\mathbb{N}$ of #-densely defined operators we define

$$\mathcal{L}_{^*\infty}(C) = \{\phi, \chi | \phi = \#\text{-}\lim_{n \to {}^*\infty} \phi_n, \phi_n \in D(C_n), \chi = \#\text{-}\lim_{n \to {}^*\infty} C_n\phi_n\}. \tag{8}$$

In general, $\mathcal{L}_{^*\infty}$ will not be the graph of an operator. If the hyperinfinite sequence $\{C_n^*\}, \quad n \in {}^*\mathbb{N}$#-converges strongly on a #-dense domain D to an operator C^*, namely,

$$C^*\psi = \#\text{-}\lim_{n \to {}^*\infty} C_n^*\psi, \psi \in D,$$

then $\mathcal{L}_{^*\infty}$ is the graph of some operator C^*. In particular, if each C_n is self #-adjoint, and if the C_n #-converge on a #-dense set D to an operator C defined on D, then $\mathcal{L}_{^*\infty} = \mathcal{L}_{^*\infty}(C_{^*\infty})$ and $C_{^*\infty}$ is a symmetric extension of C.

Definition 2.1. G #-CONVERGENCE. The hyperinfinite sequence of operators $C_n, n \in {}^*\mathbb{N}$ #-converge strongly to $C_{^*\infty}$ in the sense of graphs, written

$$C_n \to_{\#G} C_{^*\infty} \qquad (8')$$

if $\mathcal{L}_{^*\infty}$ is the graph of a #-densely defined operator $C_{^*\infty}$.

Remark 2.1.Note that for a hyperinfinite sequence of uniformly bounded operators $\{C_n^*\}_{n \in {}^*\mathbb{N}}$ such that $C_n \to_{\#G} C_{^*\infty}$, $C_{^*\infty}$ is the usual strong #-limit of the operators $C_n, n \in {}^*\mathbb{N}$ and is everywhere defined.

Definition 2.2.R #-CONVERGENCE. Let the resolvents $R_n(z) = (C_n - z)^{-1}, n \in {}^*\mathbb{N}$ exist for some $z \in {}^*\mathbb{C}_c^\#$, and be uniformly bounded in n. The operators C_n #-converge strongly to $C_{^*\infty}$ in the sense of resolvents, written

$$C_n \to_{\#R} C_{^*\infty} \qquad (8'')$$

if the resolvents $R_n(z)$ #-converge strongly to an operator $R(z)$, which has a #-densely defined inverse.

Remark 2.2.Note thatIn that case, the operator $C_{^*\infty} = R^{-1}(z) + z$ exists for all $z \in {}^*\mathbb{C}_c^\#$ for which the strong #-limit of the $R_n(z)$ exists, and $R^{-1}(z) + z$ is independent of z.

Remark 2.3.Note that G #-convergence is weaker than R #-convergence, in the case $C_n = C_n^*$ at least, because, as we shall show, in this case $C_n \to_{\#R} C_{^*\infty}$ implies $C_n \to_{\#G} C_{^*\infty}$. It seems likely that G #-convergence is strictly weaker than R #-convergence; this could be established by giving an example for which $C_n^* = C_n \to_{\#G} C_{^*\infty}$ with $C_{^*\infty}$ not maximal symmetric. The importance of G #-convergence is that it is technically easier to verify-and gives less information about the #-limit-than R #-convergence, while automatically selecting the correct domain in the case that R #-convergence also holds. The most familiar examples of G #-convergence occur where there is C_n strong #-convergence on a #-dense domain. A less trivial example occurs where there is $D(C_n)$ is independent of n,but apparently

$$D(C) \cap D(C_n) = \{0\}.$$

We have the following connection between G and R #-convergence for a hyperinfinite sequence of #-selfadjoint operators.

Proposition 3.Let $C_n, n \in {}^*\mathbb{N}$ be #-selfadjoint.

(a) The domain $D_{^*\infty} = \left\{\phi | \{\phi, \chi\} \in \mathcal{L}_{^*\infty} \text{ for some } \chi\right\}$ is #-dense in H and oniy if $C_n \to_{\#G} C_{^*\infty}$,and in this case $C_{^*\infty}$ is necessarily symmetric.

(b) If $R_n(z) = (C_n - z)^{-1}, n \in {}^*\mathbb{N}$ #-converges to a bounded operator $R(z)$ for an unbounded set of z's with $\|zR,(z)\|_\#$ bounded uniformly in $z \in {}^*\mathbb{C}_c^\#$ and $n \in {}^*\mathbb{N}$ and if $C_n \to_{\#G} C_{^*\infty}$,then each $R(z)$ is invertible.

(c) If $R_n(z)$ #-converges to an invertible $R(z)$, then $C_n \to_{\#R} C$.

(d) If $C_n \to_{\#R} C$,then $C_n \to_{\#G} C_{^*\infty}, \mathcal{L}_{^*\infty} = \mathcal{L}(C)$,and C is maximal symmetric.

(e) Conversely, if $C_n \to_{\#G} C$, where C is maximal symmetric, then $C_n \to_{\#R} C$.

In case the #-limit of the $C_n, n \in {}^*\mathbb{N}$ is actually selfadjoint, there are further connections between G and R #-convergence.

Theorem 4.

(a) $C_n \to_{\#G} C$, and $C = C^*$.

(b) $C_n \to_{\#R} C$, and $C = C^*$.
(c) The hyper infinite sequences $\{R_n(z)\}$ and $\{[R_n(z)]^*\}, n \in {}^*\mathbb{N}$ #-converge strongly and $\#\text{-}\lim_{n \to {}^*\infty} R_n(z)$ is invertible for some z.
(d) Statement (c) holds for all non-real $z \in {}^*\mathbb{C}_c^\#$

§3.Estimates on a G #-convergent hyper infinite sequence

In this section we give estimates which are sufficient to assure that it G #-convergent sequence of operators is R #-convergent, and that the limit is maximal symmetric or selfadjoint. In order to measure the rate of #-convergence, we introduce a selfadjoint operator $N \geq I$ and the associated non-Archimedean Hilbert spaces H_λ with the scalar product

$$\langle \psi, \psi \rangle_{\#\lambda} = \langle N^{\lambda/2}\psi, N^{\lambda/2}\psi \rangle_{\#}. \tag{3.1}$$

By standard identifications we have for $\lambda \geq 0 : H_\lambda \subset H_0 \subset H_{-1}$ and $H_0 = H$.
If $D : H_\alpha \to H_\beta$ is a #-densely defined, bounded operator from H_α to H_β, we let $\|D\|_{\#\alpha,\beta}$ denote its #-norm. Setting $\|D\|_\# = \|D\|_{\#0,0}$ we obtain

$$\|D\|_{\#\alpha,\beta} = \|N^{\beta/2} D N^{-\alpha/2}\|. \tag{3.2}$$

Let $C_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of selfadjoint operators, and consider the following three conditions.
(i) Suppose that $C_n - C_m$ is a #-densely defined, bounded operator from H_λ to $H_{-\lambda}$, for some λ, and that as $n,m \to {}^*\infty$

$$\|C_n - C_m\|_{\#\lambda,-\lambda} \to_\# 0. \tag{3.3}$$

(ii) Suppose that, for some p and for an unbounded set of $z = x + iy \in {}^*\mathbb{C}_c^\#$ in the sector $|x| \leq const \times |y|$,

$$\|R_n(z)\|_{\#\mu,\lambda} \leq M(z), \tag{3.4}$$

where the bound $M(z)$ is uniform in $n \in {}^*\mathbb{N}$.
(iii) Suppose that, for the above z's,

$$\|R_n(\bar{z})\|_{\#\mu,\lambda} \leq M(z). \tag{3.5}$$

Theorem 5. Let $C_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of #-selfadjoint operators with a common domain, such that

$$C_n \to_{\#G} C.$$

If conditions (i) and (ii) hold, then

$$C_n \to_{\#R} C$$

and C is maximal symmetric.
Corollary 6. If in addition to the hypothesis of Theorem 5, condition (iii) also holds, then C is #-selfajoint.
Remark 3.1.(1) If $\mu = 0$ in (ii), then the resolvents #-converge uniformly.
(2) If the C_n are uniformly semibounded from below, then we may choose the z in condition (ii) to be infinite large negative numbers. In that case the conclusion of Theorem 5 is that $C_n \to_{\#R} C = C^*$.

§ 4.Estimates for singular perturbations

In this section we consider a singular perturbation B of a #-selfadjoint operator A. We give estimates on B which ensure that the sum $A+B$ is #-selfadjoint.

Abbreviation 4.1.We abbreviate $A^{\#-}$ instead #-$\overline{A}$.

Definition 4.1. A #-core of an operator C is a domain D contained in $D(C)$ such that $C = (C \upharpoonright D)^{\#-}$.

Lemma 7. Let $A, A_n, n \in {}^*\mathbb{N}, B, B_n, n \in {}^*\mathbb{N}$ and $C_n = A, +B_n, n \in {}^*\mathbb{N}$ be #-selfadjoint operators with a common #-core D. Assume the hypotheses of Theorem 5 and Corollary 6 for $C_n, n \in {}^*\mathbb{N}$ and suppose also that, for $\theta \in D$,

$$\|(A-A_n)\theta\|_\# + \|(B-B_n)\theta\|_\# \to_\# 0 \text{ as } n \to {}^*\infty \tag{4.9}$$

and

$$\|A_n\theta\|_\#^2 + \|B_n\theta\|_\#^2 \leq const. \times \|\theta\|_\#^2 + const. \times \|C_n\theta\|_\#^2, \tag{4.10}$$

with constants independent of n. Then $A+B$ is #-selfadjoint and $C_n \to_{\#R} A+B$.

Remark 4.1.As hypothesis for our next theorem, our second main result, we assume that $N \leq A$ and that N and A commute. Let

$$D^{*\infty}(A) = \bigcap_{n \in {}^*\mathbb{N}} A(A^n) \tag{4.11}$$

the elements of $D^{*\infty}(A)$ are called $C^{*\infty}$ vectors for A. Assume that $D^{*\infty}(A)$ is a #-core for the #-selfadjoint operator B. Also assume that, as bilinear forms on $D^{*\infty} \times D^{*\infty}$, and for some α and ε in the indicated ranges,

$$0 \leq \alpha N + B + const., 0 \leq \alpha < 1/2 \tag{4.12}$$

and

$$0 \leq \varepsilon A^2 + const \times B + [A^{1/2}, [A^{1/2}, B]] + const., 2\alpha + \varepsilon < 1. \tag{4.13}$$

Let B be a bounded operator from H_ν to $H_{-\nu}$ and from H_α to H_β for some α, β and $\nu, \beta > 0$ (H_α is defined following Theorem 4.) If $\nu \geq 2$, assume that for all $\varepsilon > 0$

$$0 \leq \varepsilon N^{\mu+2} + [N^{(\mu+1)/2}, [N^{(\mu+1)/2}, B]] + const. \tag{4.14}$$

as bilinear forms on $D^{*\infty} \times D^{*\infty}$, for some $\mu > \nu - 2$.

Theorem 8. Under the above hypothesis, $A+B$ is #-selfadjoint.

Chapter V.

§1.Free scalar field

Let $\mathbf{H}^\#$ be a #-complex Hilbert space over field $\mathbb{C}_c^\#$ and let $\mathcal{F}(\mathbf{H}^\#) = \bigoplus_{n=0}^{\infty^\#} \mathbf{H}_\#^{(n)}$

(where $\mathbf{H}_\#^{(n)} = \bigoplus_{k=1}^{n} \mathbf{H}^\#$) be the Fock space over $\mathbf{H}^\#$. Our goal is to define the abstract free field on $\mathcal{F}_\mathbf{s}(\mathbf{H}^\#)$, the Boson subspace of $\mathcal{F}(\mathbf{H}^\#)$; to do this we need to introduce several other families of operators and some terminology. Let $f \in \mathbf{H}^\#$ be fixed. For vectors in $\mathbf{H}_{\#}^{(n)}$ of the form $\eta = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n$ we define a map $b^{-}(f)$: $\mathbf{H}_{\#}^{(n)} \to \mathbf{H}_{\#}^{(n-1)}$ by

$$b^{-}(f)\eta = (f, \psi_1)(\psi_2 \otimes \cdots \otimes \psi_n) \tag{1}$$

$b^{-}(f)$ extends by linearity to finite linear combinations of such η, the extension is well defined, and $\|b^{-}(f)\eta\| \leq \|f\| \times \|\eta\|$. Thus $b^{-}(f)$ extends to a bounded map (of norm $\|f\|$) of $\mathbf{H}_{\#}^{(n)}$ into $\mathbf{H}_{\#}^{(n-1)}$. Since this is true for each n (except for $n = 0$ in which case we define $b^{-}(f) : \mathbf{H}_{\#}^{(0)} \to 0$), $b^{-}(f)$ is in a natural way a bounded operator of norm $\|f\|$ from $\mathcal{F}(\mathbf{H}^{\#})$ to

$\mathcal{F}(\mathbf{H}^{\#})$. It is easy to check that $b^{+}(f) = (b^{-}(f))^{*}$ takes each $\mathbf{H}_{\#}^{(n)}$ into $\mathbf{H}_{\#}^{(n+1)}$ with the action

$$b^{+}(f)\eta = f \otimes \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \tag{2}$$

on product vectors. Notice that the map $f \mapsto b^{+}(f)$ is linear, but $f \mapsto b^{-}(f)$ is antilinear. Let $\mathbf{S}_n$ be the symmetrization operators introduced in Section II.4. Then $\mathbf{S} = \bigoplus_{n=0}^{\infty^{\#}} \mathbf{S}_n$ is

the projection onto the symmetric Fock space $\mathcal{F}_s(\mathbf{H}^{\#}) = \bigoplus_{n=0}^{\infty^{\#}} \mathbf{S}_n \mathbf{H}^{\#(n)}$ We will write

$\mathbf{S}_n \mathbf{H}^{\#(n)} = \mathbf{H}_s^{\#(n)}$ and call $\mathbf{H}_s^{\#(n)}$ the n-particle subspace of $\mathcal{F}_s(\mathbf{H}^{\#})$. Notice that $b^{-}(f)$ takes $\mathcal{F}_s(\mathbf{H}^{\#})$ into itself, but that $b^{+}(f)$ does not. A vector $\Psi = \{\psi^{(n)}\}_{n=1}^{\infty^{\#}}$ for which $\psi^{(n)} = 0$ for all except finitely many n is called a finite particle vector. We will denote the set of finite particle vectors by F_0. The vector $\mathbf{\Omega}_0 = \langle 1, 0, 0, \ldots \rangle$ plays a special role; it is called the vacuum.

Let A be any self-adjoint operator on $\mathbf{H}^{\#}$ with domain of essential selfadjointness D. Let $D_A = \left\{\Psi \in F_0 | \psi^{(n)} \in \otimes_{k=1}^{n} D \text{ for each } n \in \mathbb{N}^{\#}\right\}$ and define $d\Gamma^{\#}(A)$ on $D_A \cap \mathbf{H}_s^{\#(n)}$ as

$$d\Gamma^{\#}(A) = A \otimes I \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + \otimes I \cdots \otimes I \otimes A. \tag{3}$$

Note that $d\Gamma^{\#}(A)$ is essentially self-adjoint on D_A ; $d\Gamma^{\#}(A)$ is called the second quantization of A. For example, let $A = I$. Then its second quantization $N = d\Gamma^{\#}(I)$ is essentially self-adjoint on F_0 and for $\psi \in \mathbf{H}_s^{\#(n)}, N\psi = n\psi$. N is called the number operator. If U is a unitary operator on $\mathbf{H}^{\#}$, we define $d\Gamma^{\#}(U)$ to be the unitary operator on $\mathcal{F}_s(\mathbf{H}^{\#})$ which equals $Ext\text{-}\otimes_{k=1}^{n} U$ when restricted to $\mathbf{H}_s^{\#(n)}$ for $n > 0$, and which equals

the identity on $\mathbf{H}_s^{\#(0)}$. If $Ext\text{-}\exp(itA)$ is a #-continuous unitary group on $\mathbf{H}^{\#}$, then $\Gamma^{\#}(Ext\text{-}\exp(itA))$ is the group generated by $d\Gamma^{\#}(A)$, i.e., $\Gamma^{\#}(Ext\text{-}\exp(itA)) = Ext\text{-}\exp[itd\Gamma^{\#}(A)]$.

Deinition1.1. We define the annihilation operator $a^{-}(f)$ on $\mathcal{F}_s(\mathbf{H}^{\#})$ with domain F_0 by

$$a^{-}(f) = \sqrt{N+1}\, b^{-}(f) \tag{4}$$

$a^{-}(f)$ is called an annihilation operator because it takes each $(n+1)$-particle subspace into the n-particle subspace. For each ψ and η in F_0,

$$\left(\sqrt{N+1}\, b^{-}(f)\psi, \eta\right) = \left(\psi, Sb^{+}(f)\sqrt{N+1}\right). \tag{5}$$

Then Eq.(5) implies that

$$(a^{-}(f))^{*} \restriction F_0 = Sb^{+}(f)\sqrt{N+1} \tag{6}$$

The operator $(a^{-}(f))^{*}$ is called a creation operator. Both $a^{-}(f)$ and $a^{-}(f)^{*} \restriction F_0$ are

#-closable; we denote their #-closures by $a^{-}(f)$ and $a^{-}(f)^{*}$ also.

Example 1.1. If $\mathbf{H}^{\#} = L_2^{\#}(M, d^{\#}\mu)$, then $\bigotimes_{i=1}^{n} L_2^{\#}(M, d^{\#}\mu) = L_2^{\#}(\times_{i=1}^{n} M, \otimes_{i=1}^{n} d^{\#}\mu)$ and that $S\bigotimes_{i=1}^{n} L_2^{\#}(M, d^{\#}\mu) = L_{2,s}^{\#}(\times_{i=1}^{n} M, \otimes_{i=1}^{n} d^{\#}\mu)$, where $L_{2,s}^{\#}$ is the set of functions in $L_2^{\#}$ which are invariant under permutations of the coordinates. The operators $a^{-}(f)$ and $a^{-}(f)^{*}$ are given by

$$\begin{aligned} a^{-}(f)\psi^{(n)}(m_1,\ldots,m_n) &= \sqrt{n+1}\left(Ext\text{-}\int_M \tilde{f}(m)\psi^{(n+1)}(m,m_1,\ldots,m_n)d^{\#}\mu\right) \\ a^{-}(f)^{*}\psi^{(n)}(m_1,\ldots,m_n) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(m_i)\psi^{(n-1)}(m_1,\ldots,\tilde{m}_i,\ldots,m_n) \end{aligned} \tag{7}$$

where $\tilde{m}_i$ means that m_i is omitted. If A operates on $L_2^{\#}(M, d^{\#}\mu)$ by multiplication by the ${}^{*}\mathbb{R}_c^{\#}$-valued function $\omega(m)$, then

$$(d\Gamma^{\#}(A)\psi)^{(n)}(m_1,\ldots,m_n) = \left(\sum_{i=1}^{n}\omega(m_i)\right)\psi^{(n)}(m_1,\ldots,m_n) \tag{8}$$

Eq.(6) implies that the Segal field operator $\mathbf{\Phi}_S^{\#}(f)$ on F_0 defined by

$$\mathbf{\Phi}_S^{\#}(f) = \frac{1}{\sqrt{2}}[a^{-}(f) + a^{-}(f)^{*}] \tag{9}$$

is symmetric and essentially self-#-adjoint. The mapping from $\mathbf{H}^{\#}$ to the self-#-adjoint operators on $\mathcal{F}_s(\mathbf{H}^{\#})$ given by

$$f \mapsto \mathbf{\Phi}_S^{\#}(f) \tag{10}$$

is called the Segal quantization over $\mathbf{H}^{\#}$. Notice that the Segal quantization is a real (but not complex) linear map since $f \mapsto a^{-}(f)$ is antilinear and $f \mapsto a^{-}(f)^{*}$ is linear. The following theorem gives the properties of the Segal quantization.

Theorem 1.1. Let $\mathbf{H}^{\#}$ be hyper infinite dimensional Hilbert space over field ${}^{*}\mathbb{C}_c = {}^{*}\mathbb{R}_c^{\#} + i{}^{*}\mathbb{R}_c^{\#}$ and $\mathbf{\Phi}_S^{\#}(f)$ the corresponding Segal quantization. Then:

(a) (self-adjointness) For each $f \in \mathbf{H}^{\#}$ the operator $\mathbf{\Phi}_S^{\#}(f)$ is essentially self-adjoint on F_0,
the hyperfinite particle vectors.

(b) (cyclicity of the vacuum) $\mathbf{\Omega}_0$ is in the domain of all hyperfinite products $\prod_{i=1}^{n}\mathbf{\Phi}_S^{\#}(f_i), n \in \mathbb{N}^{\#}$
and the set $\left\{\prod_{i=1}^{n}\mathbf{\Phi}_S^{\#}(f_i)\mathbf{\Omega}_0 \middle| f_i \text{ and } n \text{ arbitrary}\right\}$ is #-total in $\mathcal{F}_s(\mathbf{H}^{\#})$.

(c) (commutation relations) For each $\psi \in F_0$ and $f,g \in \mathbf{H}^{\#}$

$$[\mathbf{\Phi}_S^{\#}(f)\mathbf{\Phi}_S^{\#}(g) - \mathbf{\Phi}_S^{\#}(g)\mathbf{\Phi}_S^{\#}(f)]\psi = i\,\mathrm{Im}(f,g)_{\mathbf{H}^{\#}}\psi. \tag{11}$$

Further, if $W(f)$ denotes the external unitary operator $Ext\text{-}\exp(i\mathbf{\Phi}_S^{\#}(f))$ then

$$W(f+g) = \left[Ext\text{-}\exp\left(\frac{-i\,\mathrm{Im}(f,g)_{\mathbf{H}^{\#}}}{2}\right)\right]W(f)W(g) \tag{12}$$

(d) (#-continuity) If $\{f_n\}_{n=1}^{\infty^{\#}}$ is hyper infinite sequence such as $\#\text{-}\lim_{n\to\infty^{\#}} f_n = f$ in $\mathbf{H}^{\#}$, then: (i) $\#\text{-}\lim_{n\to\infty^{\#}} W(f_n)\psi$ exists for all $\psi \in \mathcal{F}_s(\mathbf{H}^{\#})$ and

$$\#\text{-}\lim_{n\to\infty^{\#}} W(f_n)\psi = W(f)\psi \tag{13}$$

(ii) $\#\text{-}\lim_{n\to\infty^{\#}} \mathbf{\Phi}_S^{\#}(f_n)\psi$ exists for all $\psi \in F_0$ and

$$\#\text{-}\lim_{n\to\infty^{\#}} \mathbf{\Phi}_S^{\#}(f_n)\psi = \mathbf{\Phi}_S^{\#}(f)\psi. \tag{14}$$

(e) For every unitary operator U on $\mathbf{H}^{\#}, \Gamma^{\#}(U) : D\left(\overline{\mathbf{\Phi}_S^{\#}(f)}\right) \to D\left(\overline{\mathbf{\Phi}_S^{\#}(Uf)}\right)$ and for $\psi \in D\left(\overline{\mathbf{\Phi}_S^{\#}(Uf)}\right)$

$$\Gamma^{\#}(U)\overline{\mathbf{\Phi}_S^{\#}(f)}\Gamma^{\#}(U)^{-1}\psi = \overline{\mathbf{\Phi}_S^{\#}(Uf)}\psi \tag{15}$$

for all $f \in \mathbf{H}^{\#}$.

Proof. Let $\psi \in \mathbf{H}_s^{\#(n)}$. Since $\mathbf{\Phi}_S^{\#}(f) : F_0 \to F_0$, ψ is in $C^{\infty^{\#}}(\mathbf{\Phi}_S^{\#}(f))$. Further, it follows from Eq.(5)-Eq.(6), and the fact that $\|b^{-}(f)\| = \|f\|$, that

$$\left\| (a^{\bigstar}(f))^k \psi \right\|_{\#} \leq \left(Ext\text{-}\prod_{i=1}^{k} \sqrt{p+i} \right) \|f\|_{\#}^{k} \|\psi\|_{\#} \tag{16}$$

where $a^{\bigstar}(f)$ represents either $a^{-}(f)$ or $a^{-}(f)^{*}$. Therefore,

$$\|\mathbf{\Phi}_S^{\#}(f)^k \psi\|_{\#} \leq 2^{k/2}((n+k)!)^{1/2} \|f\|_{\#}^{k} \|\psi\|_{\#} \tag{17}$$

Since $Ext\text{-}\sum_{k=0}^{^*\infty} t^k 2^{k/2}((n+k)!)^{1/2}\|f\|_{\#}^{k}\|\psi\|_{\#} < {}^*\infty$ for all t, ψ is an #-analytic vector for $\mathbf{\Phi}_S^{\#}(f)$. Since F_0 is #-dense in $\mathcal{F}_s(\mathbf{H}^{\#})$ and is left invariant by $\mathbf{\Phi}_S^{\#}(f)$ is essentially self-adjoint on F_0 by generalized Nelson's analytic vector theorem (Theorem).
The proof of (b) is obviously.
To prove (c) one first computes that if $\psi \in F_0$, then

$$a^{-}(f)a^{-}(g)^{*}\psi - a^{-}(g)^{*}a^{-}(f)\psi = (f,g)\psi \tag{18}$$

Eq.(11) follows immediately. Although Eq.(11) and Eq.(12) are formally equivalent, Eq.(11) by itself does not imply Eq.(12) We sketch a proof of Eq.(12) which uses special properties of the vectors in F_0. Let $\psi \in \mathbf{H}_s^{\#(p)}$. Then

$$\|\mathbf{\Phi}_S^{\#}(f)^n \mathbf{\Phi}_S^{\#}(g)^m \psi\|_{\#} \leq 2^{(n+m)/2} \left(Ext\text{-}\prod_{i=1}^{n+m} \sqrt{p+i} \right) \|f\|_{\#}^{n} \|g\|_{\#}^{m} \|\psi\|_{\#} \tag{19}$$

which implies that hyper infinite series $Ext\text{-}\sum_{n=0,m=0}^{\infty^{\#}} \left(\|\mathbf{\Phi}_S^{\#}(f)^n \mathbf{\Phi}_S^{\#}(g)^m \psi\|_{\#}/n!m! \right)$ #-converges for all $t \in {}^*\mathbb{R}_c^{\#}$. Since ψ is an #-analytic vector for $\mathbf{\Phi}_S^{\#}(g)$, $Ext\text{-}\sum_{m=0}^{\infty^{\#}}((i\mathbf{\Phi}_S^{\#}(g)^m)/m!)\psi = (Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(g)])\psi$. Further, for each $n \in \mathbb{N}^{\#}$, $(Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(g)])\psi$ is in the domain of $\left(\overline{\mathbf{\Phi}_S^{\#}(f)}\right)^n$ since any finite and hyperfinite sum

$$Ext\text{-}\exp \sum_{m=0}^{M} \frac{(i\mathbf{\Phi}_S^{\#}(g)^m)}{m!}\psi$$

with $M \in \mathbb{N}^{\#}$ is in it and $\mathbf{\Phi}_S^{\#}(f)^n \left(Ext\text{-}\sum_{m=0}^{M}((i\mathbf{\Phi}_S^{\#}(g)^m)/m!)\psi \right)$ #-converges as $M \to \infty^{\#}$.
Thus the estimate $Ext\text{-}\sum_{n=0,m=0}^{\infty^{\#},\infty^{\#}} \left(\|\mathbf{\Phi}_S^{\#}(f)^n \mathbf{\Phi}_S^{\#}(g)^m \psi\|_{\#}/n!m! \right) t^n t^m \leq \infty^{\#}$ shows that $(Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(g)])\psi$ is an #-analytic vector for $\mathbf{\Phi}_S^{\#}(f)$ and therefore can be computed by the external hyper infinite power series. Thus

$$(Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(f)])(Ext\text{-}\exp[i\mathbf{\Phi}_S^{\#}(g)])\psi = Ext\text{-}\sum_{n=0,m=0}^{\infty^{\#},\infty^{\#}} \frac{(i\mathbf{\Phi}_S^{\#}(f))^n (i\mathbf{\Phi}_S^{\#}(g))^m}{n!m!}\psi. \tag{20}$$

Similarly one obtains

$$\begin{gathered}\left(Ext\text{-}\exp\left[-\frac{it^2}{2}\operatorname{Im}(f,g)_{\mathbf{H}^{\#}} \right] \right)(Ext\text{-}\exp[it\mathbf{\Phi}_S^{\#}(f+g)])\psi = \\ Ext\text{-}\sum_{n=0,m=0}^{\infty^{\#},\infty^{\#}} \frac{1}{n!m!}\left[\left(-\frac{it^2}{2}\operatorname{Im}(f,g)_{\mathbf{H}^{\#}} \right)^m (it\mathbf{\Phi}_S^{\#}(f+g))^n \right]\psi\end{gathered} \tag{21}$$

where the hyper infinite series in RHS of Eq.(21) #-converges absolutely. Direct computations using Eq.(11) now show that Eq.(12) holds by a term-by-term comparison of the #-convergent external hyper infinite power series.
To prove (d) let $\psi \in \mathbf{H}_s^{\#(k)}$ and suppose that $\#\text{-}\lim_{n\to\infty^{\#}} f_n = f$ in $\mathbf{H}^{\#}$. Then

$$\|\mathbf{\Phi}_S^{\#}(f_n)\psi - \mathbf{\Phi}_S^{\#}(f)\psi\| \leq \sqrt{2(k+1)}\,\|f_n - f\|\,\|\psi\| \tag{22}$$

so $\#\text{-}\lim_{n\to\infty^{\#}} \mathbf{\Phi}_S^{\#}(f_n) = \mathbf{\Phi}_S^{\#}(f)$. Thus, Φ5(/„) #-converges strongly to $\mathbf{\Phi}_S^{\#}(f)$ on F_0.

Since F_0 is a core for all $\mathbf{\Phi}^{\#}_S(f_n)$ and $\mathbf{\Phi}^{\#}_S(f)$, Theorems VIII.21 and VIII.25 imply that #-$\lim_{n\to\infty^{\#}}(Ext\text{-}\exp[it\mathbf{\Phi}^{\#}_S(f_n)]\psi) = Ext\text{-}\exp[it\mathbf{\Phi}^{\#}_S(f)]\psi$ for all $\psi \in \mathcal{F}_s(\mathbf{H}^{\#})$.
To prove (e), let $\eta \in \mathbf{H}^{\#(n)}$ be of the form $\eta = \psi_1 \otimes \cdots \otimes \psi_n$. Then
$\Gamma^{\#}(U)b^{-}(f)\Gamma^{\#}(U)^{-1}\eta = \Gamma^{\#}(U)b^{-}(f)(U^{-1}\psi_2 \otimes \cdots \otimes U^{-1}\psi_n) =$
$\Gamma^{\#}(U)(f, U^{-1}\psi_1)(U^{-1}\psi_2 \otimes \cdots \otimes U^{-1}\psi_n) = (Uf, \psi_1)(\psi_2 \otimes \cdots \otimes \psi_n) = b^{-}(Uf)\eta$.
Since finite linear combinations of such η are dense in $\mathbf{H}^{\#(n)}$ and $b^{-}(g)$ has norm $\|g\|$, we conclude that $\Gamma^{\#}(U)b^{-}(f)\Gamma^{\#}(U)^{-1} = b^{-}(Uf)$. But $\mathbf{N}$ and $\mathbf{S}$ commute with $\Gamma^{\#}(U)$ so this immediately implies that $\Gamma^{\#}(U)a^{-}(f)\Gamma^{\#}(U)^{-1} = a^{-}(Uf)$ on F_0. Taking adjoints and restricting to F_0 we also have $\Gamma^{\#}(U)(a^{-}(f))^{*}\Gamma^{\#}(U)^{-1} = (a^{-}(Uf))^{*}$.
Thus for $\psi \in F_0$, $\Gamma^{\#}(U)\mathbf{\Phi}^{\#}_S(f)\Gamma^{\#}(U)^{-1}\psi = \mathbf{\Phi}^{\#}_S(Uf)\psi$. Since the operators on both the right- and left-hand sides of this equality are essentially self-#-adjoint on F_0, we conclude that $\Gamma^{\#}(U)\overline{\mathbf{\Phi}^{\#}_S(f)}\Gamma^{\#}(U)^{-1} = \overline{\mathbf{\Phi}^{\#}_S(Uf)}$.

Remark 1.1. Henceforth we use $\mathbf{\Phi}^{\#}_S(f)$ to denote the #-closure of $\mathbf{\Phi}^{\#}_S(f)$.

Definition 1.1. For each $m > 0, m \in {}^{*}\mathbb{R}^{\#}_{c,\mathbf{fin}}$ let

$$H^{\#}_m = \{p \in {}^{*}\mathbb{R}^{\#4}_c p \cdot \tilde{p} = m^2, p_0 > 0\}, \tag{23}$$

where $\tilde{p} = (p^0, -p^1, -p^2, -p^3)$. The sets $H^{\#}_m$, which are called mass hyperboloids, are invariant under ${}^{\sigma}\mathcal{L}^{\uparrow}_{+}$. Let j_m be the #-homeomorphism of $H^{\#}_m$ onto ${}^{*}\mathbb{R}^{\#3}_c$ (or in the case $m = 0$ onto ${}^{*}\mathbb{R}^{\#3}_c\backslash\{0\}$) given by $j_m : \langle p_0, p_1, p_2, p_3\rangle \mapsto \langle p_1, p_2, p_3\rangle = \mathbf{p}$. Define a #-measure $\Omega^{\#}_m$ on $H^{\#}_m$ by

$$\Omega^{\#}_m(E) = Ext\text{-}\int_{j_m(E)} \frac{d^{\#3}\mathbf{p}}{\sqrt{|\mathbf{p}|^2 + m^2}} \tag{24}$$

for any measurable set $E \subset H^{\#}_m$. The measure $\Omega^{\#}_m(E)$ can easily be seen to be ${}^{\sigma}\mathcal{L}^{\uparrow}_{+}$-invariant. In fact, up to a constant multiple, $\Omega^{\#}_m$ is the only ${}^{\sigma}\mathcal{L}^{\uparrow}_{+}$-invariant measure on $H^{\#}_m$. Furthermore, every polynomially bounded ${}^{\sigma}\mathcal{L}^{\uparrow}_{+}$-invariant measure on $\overline{V}_{+}$ is the sum of a multiple of δ and an integral of the measures $\Omega^{\#}_m$. We state this fact as a theorem.

Theorem 1.2. Let $\mu^{\#}$ be a polynomially bounded #-measure with support in $\overline{V}_{+}$. If $\mu^{\#}$ is ${}^{\sigma}\mathcal{L}^{\uparrow}_{+}$-invariant, there exists a polynomially bounded #-measure ρ on $[0, \infty^{\#})$ and a constant c so that for any $f \in S^{\#}({}^{*}\mathbb{R}^{\#4}_c)$

$$Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#4}_c} f\, d^{\#}\mu^{\#} = cf(0) + Ext\text{-}\int_0^{\infty^{\#}} d^{\#}\rho^{\#}(m)\left(Ext\text{-}\int_{H^{\#}_m} f d^{\#}\Omega^{\#}_m\right). \tag{25}$$

Theorem 1.3.
We can now use the Segal quantization to define the free Hermitian scalar field of mass m. We take $\mathbf{H}^{\#} = \mathcal{L}^{\#}_2(H^{\#}_m, d^{\#}\Omega^{\#}_{m,x})$, where $H^{\#}_m, m > 0$, is the mass hyperboloid in ${}^{*}\mathbb{R}^{\#4}_c$ consisting of those $p \in {}^{*}\mathbb{R}^{\#4}_c$ satisfying $p \cdot \tilde{p} - m^2 = 0$ and $p_0 > 0$, and $d^{\#}\Omega^{\#}_m$ is the Lorentz invariant #-measure.
For each $f \in S^{\#}({}^{*}\mathbb{R}^{\#4}_c)$ we define $Ef \in \mathbf{H}^{\#}$ by $Ef = 2\pi_{\#}\widehat{f} \upharpoonright H^{\#}_m$ where the Fourier transform

$$(2\pi_{\#})^{-2}\left(Ext\text{-}\int\left(Exp\text{-}\exp\left[i\left(p \cdot \tilde{x}\right)\right]\right)f(x)d^{\#4}x\right) \tag{26}$$

is defined in terms of the Lorentz invariant inner product $p \cdot \tilde{x}$. The reason for the extra $\sqrt{2\pi_{\#}}$ in our definition of E and the plus sign in the definition of Fourier transform is that if f is the distribution $f(x) = g(\mathbf{x})\delta^{\#}(t)$, then $\sqrt{2\pi_{\#}}\widehat{f}$ is the ordinary

three-dimensional
Fourier transform of g. If $\mathbf{\Phi}_S^{\#}(\cdot)$ is the Segal quantization over $\mathcal{L}_2^{\#}(H_m^{\#}, d^{\#}\Omega_{m,x}^{\#})$, we define
for each ${}^*\mathbb{R}_c^{\#}$-valued $f \in S^{\#}({}^*\mathbb{R}_c^{\#4})$

$$\Phi_{m,x}^{\#}(f) = \mathbf{\Phi}_S^{\#}(Ef). \tag{27}$$

For ${}^*\mathbb{C}_c^{\#}$-valued function $f \in S^{\#}({}^*\mathbb{R}_c^{\#4})$ we define

$$\Phi_{m,x}^{\#}(f) = \Phi_{m,x}^{\#}(\mathrm{Re}\, f) + i\Phi_{m,x}^{\#}(\mathrm{Im}\, f) \tag{28}$$

The mapping $f \mapsto \Phi_m^{\#}(f)$ is called the free Hermitian scalar field of mass m.
On $\mathcal{L}_2^{\#}(H_m^{\#}, d^{\#}\Omega_m)$ we define the following unitary representation of the restricted Poincare group:

$$(U_m(a,\Lambda)\psi)(p) = \left(Exp\text{-}\exp\left[i\left(p \cdot \tilde{a}\right)\right]\right)\psi(\Lambda^{-1}p) \tag{29}$$

where we are using Λ to denote both an element of the abstract restricted Lorentz group
and the corresponding element in the standard representation on ${}^*\mathbb{R}_{\mathbf{st}}^4 = \mathbb{R}^4$.

Remark 1.3. Recall that a #-conjugation on a Hilbert space $\mathbf{H}^{\#}$ is an antilinear #-isometry $\mathbf{C}^{\#}$ so that $\mathbf{C}^{\#2} = \mathbf{I}$.

Definition 1.2. Let $\mathbf{H}^{\#}$ be a ${}^*\mathbb{C}_c^{\#}$-complex Hilbert space, $\mathbf{\Phi}_S^{\#}(\cdot)$ the associated Segal quantization. Let $\mathbf{C}^{\#}$ be a #-conjugation on $\mathbf{H}^{\#}$ and define $\mathbf{H}_{\mathbf{C}^{\#}}^{\#} = \{|\mathbf{C}^{\#}f = f\}$. For each $f \in \mathbf{H}_{\mathbf{C}^{\#}}^{\#}$ we define $\varphi^{\#}(f) = \mathbf{\Phi}_S^{\#}(f)$ and $\pi^{\#}(f) = \mathbf{\Phi}_S^{\#}(if)$. The map $f \mapsto \varphi^{\#}(f)$ is called the canonical free field over the doublet $\langle \mathbf{H}^{\#}, \mathbf{C}^{\#}\rangle$ and the map $f \mapsto \pi^{\#}(f)$ is called the canonical conjugate momentum. We often drop the $\langle \mathbf{H}^{\#}, \mathbf{C}^{\#}\rangle$ and just write $\mathbf{H}^{\#}$ if the intended #-conjugation is clear.

Remark 1.4.Note that the set of elements of $\mathbf{H}^{\#}$ for which the maps $f \mapsto \varphi^{\#}(f)$ and $f \mapsto \pi^{\#}(f)$ are defined depends on the #-conjugation $\mathbf{C}^{\#}$.

Theorem 1.4. Let $\mathbf{H}^{\#}$ be a ${}^*\mathbb{C}_c^{\#}$-complex Hilbert space with #-conjugation $\mathbf{C}^{\#}$. Let $\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ be the corresponding canonical fields. Then:
(i) For each $f \in \mathbf{H}_{\mathbf{C}^{\#}}^{\#}, \varphi^{\#}(f)$ is essentially self-adjoint on F_0.
(ii) $\{\varphi^{\#}(f) | f \in \mathbf{H}_{\mathbf{C}^{\#}}^{\#}\}$ is a commuting family of self-adjoint operators.
(iii) $\mathbf{\Omega}_0$ is a #-cyclic vector for the family $\{\varphi^{\#}(f) | f \in \mathbf{H}_{\mathbf{C}^{\#}}^{\#}\}$.
(iv) If $\#\text{-}\lim_{n\to\infty^{\#}} f_n = f$ in $\mathbf{H}_{\mathbf{C}^{\#}}^{\#}$, then

$$\#\text{-}\lim_{n\to\infty^{\#}} \varphi^{\#}(f_n)\psi = \varphi^{\#}(f)\psi \text{ for all } \psi \in F_0$$

and

$$\#\text{-}\lim_{n\to\infty^{\#}}(Exp\text{-}\exp[i\varphi^{\#}(f_n)]\psi) = Exp\text{-}\exp[i\varphi^{\#}(f)]\psi \text{ for all } \psi \in \mathcal{F}_s(\mathbf{H}^{\#})$$

(v) Properties (i)-(iv) hold with $\varphi^{\#}(f)$ replaced by $\pi^{\#}(f)$.
(vi) If $f,g \in \mathbf{H}_{\mathbf{C}^{\#}}^{\#}$, then

$$\varphi^{\#}(f)\pi^{\#}(g)\psi - \pi^{\#}(g)\varphi^{\#}(f)\psi = i(f,g)\psi \tag{30}$$

for all $\psi \in F_0$ and

$$\begin{gathered}\left(Exp\text{-}\exp\left[\ i\varphi^{\#}(f)\right]\right)\left(Exp\text{-}\exp\left[\ i\pi^{\#}(g)\right]\right) = \\ (Exp\text{-}\exp[i(f,g)])\left(Exp\text{-}\exp\left[\ i\pi^{\#}(g)\right]\right)\left(Exp\text{-}\exp\left[\ i\varphi^{\#}(f)\right]\right).\end{gathered} \tag{31}$$

Proof. (i) and (iv) follow immediately from the corresponding properties of $\mathbf{\Phi}_S^{\#}(\cdot)$ proven in Theorem 1.1. To see that $\{\varphi^{\#}(f) | f \in \mathbf{H}_{\mathbf{C}^{\#}}^{\#}\}$ is a commuting family, notice that (12) implies

$$\begin{aligned}&\left(Exp\text{-}\exp\left[\,it\varphi^{\#}(f)\,\right]\right)\left(Exp\text{-}\exp\left[\,is\varphi^{\#}(g)\,\right]\right)=\\&(Exp\text{-}\exp[-its\,\mathrm{Im}(f,g)])\left(Exp\text{-}\exp\left[\,is\varphi^{\#}(g)\,\right]\right)\left(Exp\text{-}\exp\left[\,it\varphi^{\#}(f)\,\right]\right)\end{aligned}\tag{32}$$

where we have used the fact that $\varphi^{\#}(\cdot)$ is real linear. If $f,g\in\mathbf{H}^{\#}_{\mathbf{C}^{\#}}$, then it follows from polarization that $(f,g)=(C^{\#}f,C^{\#}g)=(g,f)$, so $\mathrm{Im}(f,g)=0$. Thus

$$\begin{aligned}&\left(Exp\text{-}\exp\left[\,it\varphi^{\#}(f)\,\right]\right)\left(Exp\text{-}\exp\left[\,is\varphi^{\#}(g)\,\right]\right)=\\&\left(Exp\text{-}\exp\left[\,is\varphi^{\#}(g)\,\right]\right)\left(Exp\text{-}\exp\left[\,it\varphi^{\#}(f)\,\right]\right)\end{aligned}\tag{33}$$

for s and t. Therefore, by Theorem VIII. 13, $\varphi^{\#}(g)$ and $\varphi^{\#}(f)$ commute.
The proof of (b) is similar to the proof of (a). (X.70) and (X.71) follow immediately from (X.64), (X.65), and the fact that if $f,g\in\mathbf{H}^{\#}_{\mathbf{C}^{\#}}$, then $\mathrm{Im}(f,ig)=\mathrm{Re}(f,g)=(f,g)$.

Definition 1.3.We write $f\in\mathcal{L}^{\#}_{2}(H^{\#}_{m},d^{\#}\Omega^{\#}_{m,x})$ as $f(p_0,\mathbf{p})$ and define now the #-conjugation by $(\mathbf{C}^{\#}f)(p_0,\mathbf{p})=\overline{f(p_0,-\mathbf{p})}$.

Remark 1.4.Note that $\mathbf{C}^{\#}$ is well-defined on $\mathcal{L}^{\#}_{2}(H^{\#}_{m},d^{\#}\Omega^{\#}_{m,x})$ since $\langle p_0,\mathbf{p}\rangle\in H^{\#}_{m}$ if and only if $\langle p_0,-\mathbf{p}\rangle\in H^{\#}_{m}$. $\mathbf{C}^{\#}$ is clearly a #-conjugation.

Definition 1.4.We denote the canonical fields corresponding to $\mathbf{C}^{\#}$ by $\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ and define $\varphi^{\#}_{m}(f)=\varphi^{\#}(Ef)$ and $\pi^{\#}_{m}(f)=\pi^{\#}(\mu Ef),\mu=\sqrt{\mathbf{p}^2+m^2}$ for ${}^*\mathbb{R}^{\#}_{c}$-valued $f\in\mathcal{L}({}^*\mathbb{R}^{\#4}_{c})$, extending to all of $\mathcal{L}({}^*\mathbb{R}^{\#4}_{c})$ by linearity. In terms of $a^{-}(f)$,

$$\begin{aligned}\varphi^{\#}_{m}(f)&=\{(a^{-}(Ef))^{*}+a^{-}(\mathbf{C}^{\#}Ef)\}/\sqrt{2},\\\pi^{\#}_{m}(f)&=i\{(a^{-}(Ef))^{*}+a^{-}(\mathbf{C}^{\#}\mu Ef)\}/\sqrt{2}.\end{aligned}\tag{34}$$

Remark 1.5.Note that the a's in these last formulas differ from those most often used in discussing the free field and that the correct space-time free field is $\mathbf{\Phi}^{\#}_{m}$ and not $\varphi^{\#}_{m}$ as we will discuss below, $\varphi^{\#}_{m}$ and $\pi^{\#}_{m}$ are useful for discussing the time-zero field. The maps $f\mapsto\varphi^{\#}_{m}(f)$ and $f\mapsto\pi^{\#}_{m}(f)$ are complex linear and $\varphi^{\#}_{m}(f),\pi^{\#}_{m}(f)$ are self-adjoint if and only if $Ef\in\mathbf{H}^{\#}_{\mathbf{C}^{\#}}$.
Because of the projection E we can extend the class of functions on which $\varphi^{\#}_{m}(\cdot)$ and $\pi^{\#}_{m}(\cdot)$ are defined to include distributions of the form $\delta(t-t_0)g(x_1,x_2,x_3)$ where $g\in{}^*\mathbb{R}^{\#3}_{c}$. In particular, if $t_0=0,g$ is ${}^*\mathbb{R}^{\#}_{c}$-lvalued, and $Ext\text{-}\widehat{g}$ is the usual Fourier transform on ${}^*\mathbb{R}^{\#3}_{c}$, then

$$\left(\mathbf{C}^{\#}E\widehat{\delta g}\right)(p_0,-\mathbf{p})=(2\pi_{\#})^{-1/2}\overline{\widehat{g}(-\mathbf{p})}=(2\pi_{\#})^{-1/2}\widehat{g}(-\mathbf{p})=E\widehat{\delta g}.\tag{35}$$

Thus $E(\delta g)$ and $\mu E(\delta g)$ are in $\mathbf{H}^{\#}_{\mathbf{C}^{\#}}$. Therefore $\varphi^{\#}_{m}(\delta g)$ and $\pi^{\#}_{m}(\delta g)$ are self-adjoint if $g\in\mathcal{L}({}^*\mathbb{R}^{\#3}_{c})$ is real. For obvious reasons, the maps $g\mapsto\varphi^{\#}_{m}(\delta g),g\mapsto\pi^{\#}_{m}(\delta g)$ are called the time-zero fields. From now on we will only use test functions of the form δg in $\varphi^{\#}_{m}(\cdot)$ and $\pi^{\#}_{m}(\cdot)$ and write $\varphi^{\#}_{m}(g)$ and $\pi^{\#}_{m}(g)$ if $g\in S^{\#*}\mathbb{R}^{\#3}_{c}$ instead of $\varphi^{\#}_{m}(\delta g)$ and $\pi^{\#}_{m}(\delta g)$.
If f and g are ${}^*\mathbb{R}^{\#}_{c}$-valued functions in $\mathcal{L}({}^*\mathbb{R}^{\#3}_{c})$, then
(X.70) implies that for $\psi\in F_0$,

$$[\varphi^{\#}_{m}(f),\pi^{\#}_{m}(g)]\psi=i\left(Ext\text{-}\int_{H_m}\overline{\widehat{f}(p)}\widehat{g}(p)\mu(p)\psi d\Omega^{\#}_{m,\mathbf{x}}\right).\tag{36}$$

For convenience and also so that our notation coincides with the standard terminology, we now transfer the fields we have constructed from the Fock space built up from $\mathcal{L}_2^\#(H_m^\#, d\Omega_{m,\mathbf{x}}^\#)$ to the Fock space built up from $\mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#3})$. For notational simplicity, we define for $f \in \mathcal{L}_2^\#(H_m^\#, d\Omega_{m,\mathbf{x}}^\#)$

$$a^\dagger(f) = (a^-(f))^*, a(f) = a^-(\mathbf{C}^\# f). \tag{37}$$

First notice that each function $f(p) \in \mathcal{L}_2^\#(H_m^\#, d\Omega_{m,\mathbf{x}}^\#)$ is in a natural way a function $f(\mathbf{p}) = f(\mu(\mathbf{p}), \mathbf{p})$ on ${}^*\mathbb{R}_c^{\#3}$. For each $f \in \mathcal{L}_2^\#(H_m^\#, d\Omega_{m,\mathbf{x}}^\#)$, we define

$$(Jf)(\mathbf{p}) = f(\mu(\mathbf{p}), \mathbf{p})/\sqrt{\mu(\mathbf{p})}\,. \tag{38}$$

J is a unitary map of $\mathcal{L}_2^\#(H_m^\#, d\Omega_{m,\mathbf{x}}^\#)$ onto $\mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#3})$, so $\Gamma^\#(J)$ is a unitary map of $\mathcal{F}_s(\mathcal{L}_2^\#(H_m^\#, d\Omega_{m,\mathbf{x}}^\#))$ onto $\mathcal{F}_s(\mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#3}))$. The annihilation and creation operators on $\mathcal{F}_s(\mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#3}))$, $\widetilde{a}(\cdot)$, $\widetilde{a}^\dagger(\cdot)$, are related to $a(\cdot)$ and $a^\dagger(\cdot)$ by the formulas

$$\begin{aligned} \widetilde{a}\left(\frac{f(\mathbf{p})}{\sqrt{\mu(\mathbf{p})}}\right) &= \Gamma^\#(J)a(f)\Gamma^\#(J)^{-1} \\ \widetilde{a}^\dagger\left(\frac{f(\mathbf{p})}{\sqrt{\mu(\mathbf{p})}}\right) &= \Gamma^\#(J)a^\dagger(f)\Gamma^\#(J)^{-1} \end{aligned} \tag{39}$$

We use the unitary map $\Gamma^\#(J)$ to carry the Wightman fields over to $\mathcal{F}_s(\mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#3}))$ by defining:(i) for ${}^*\mathbb{R}_{c,\mathbf{fin}}^\#$-valued $f \in \mathcal{L}_{\mathbf{fin}}^\#({}^*\mathbb{R}_c^{\#4})$

$$\begin{aligned} &\widetilde{\Phi}_{m,x}(f) = \Gamma^\#(J)\Phi_{m,x}(f)\Gamma^\#(J)^{-1} = \\ &\frac{1}{\sqrt{2}}\left\{\widetilde{a}\left(\tilde{\mathbf{C}}^\# \frac{Ef}{\sqrt{\mu}}\right) + \tilde{a}^\dagger\left(\frac{Ef}{\sqrt{\mu}}\right)\right\} \end{aligned} \tag{40}$$

(ii) for ${}^*\mathbb{R}_{c,\mathbf{fin}}^\#$-valued $f \in \mathcal{L}_{\mathbf{fin}}^\#({}^*\mathbb{R}_c^{\#3})$

$$\begin{aligned} &\widetilde{\varphi}_{m,x}(f) = \Gamma^\#(J)\varphi_{m,x}(f)\Gamma^\#(J)^{-1} = \\ &\frac{1}{\sqrt{2}}\left\{\widetilde{a}\left(\tilde{\mathbf{C}}^\# \frac{E(f\delta)}{\sqrt{\mu}}\right) + \tilde{a}^\dagger\left(\frac{E(f\delta)}{\sqrt{\mu}}\right)\right\} \end{aligned} \tag{41}$$

where $\tilde{\mathbf{C}}^\# = J\mathbf{C}^\# J^{-1}$ acts by $\left(\tilde{\mathbf{C}}^\# g\right)(\mathbf{p}) = \overline{g(-\mathbf{p})}$. Having established this correspondence,
we now drop the ~ and the bold face letters; from now on we will only deal with the fields
on $\mathcal{F}_s(\mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#3}))$ and three-dimensional momenta. Further, we recall that the restriction of
the four-dimensional Fourier transform that we have been using in this section to functions of the form $\delta(x_0)g(x_1,x_2,x_3)$ the usual three-dimensional Fourier transform. Notice that

$$\widetilde{f} = Ext\text{-}\check{h}, h = \left(\mathbf{C}^\# \hat{f}\right) \tag{42}$$

so $\mathbf{C}^\# \hat{f} = \hat{f}$ if and only if f is ${}^*\mathbb{R}_c^\#$-valued.
For f and g ${}^*\mathbb{R}_c^\#$-valued, (36) becomes

$$[\varphi_m^\#(f), \pi_m^\#(g)] \approx i\left(Ext\text{-}\int f(x)g(x)\right)d^{\#3}x. \tag{43}$$

(43) is the space form of the canonical commutation relations (CCR).
In the Appendix to this section we prove that for each $m > 0$, this representation of the CCR is irreducible and for different m, the representations are inequivalent. Thus, the time-zero fields in the free scalar field theories give rise to different representation of the
CCR.
As a final topic before turning to interacting fields we will show how the structures developed above are related to the “fields” and “annihilation and creation operators” introduced in physics texts. We let now

$$D_{S^{\#}_{\mathbf{fin}}} = \left\{\psi|\psi \in F_0, \psi^{(n)} \in S^{\#}_{\mathbf{fin}}({}^*\mathbb{R}_c^{\#3n}), n \in \mathbb{N}\right\} \tag{44}$$

and for each $p \in {}^*\mathbb{R}_c^{\#3}$ we define an operator $a(p)$ on $\mathcal{F}_s(\mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3}))$ with domain $D_{S^{\#}_{\mathbf{fin}}}$ by

$$(a(p)\psi)^{(n)}(k_1,\ldots,k_n) = \sqrt{n+1}\,\psi^{(n+1)}(p,k_1,\ldots,k_n). \tag{45}$$

The adjoint of the operator $a(p)$ is not a #-densely defined operator since it is given formally by

$$(a^{\dagger}(p)\psi)^{(n)}(k_1,\ldots,k_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta(p-k_i)\psi^{(n+1)}(p,k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n). \tag{46}$$

However, $a^{\dagger}(p)$ is a well-defined quadratic form on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \times D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$. For example, if $\psi_1 = \{0,\psi^{(1)},0,\ldots\}$, and $\psi_2 = \{0,0,\psi^{(2)},0,\ldots\}$, then

$$(\psi_2, a^{\dagger}(p)\psi_1) = \frac{1}{\sqrt{2}}\left\{Ext\text{-}\int\left[\overline{\psi^{(2)}(k_1,p)}\psi^{(1)}(k_1) + \overline{\psi^{(2)}(p,k_1)}\psi^{(1)}(k_1)\right]d^{\#}k_1\right\}. \tag{47}$$

Remark 1.1.Note that the formulas

$$a(g) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} a(p)g(-p)d^{\#}p \tag{48}$$

and

$$a^{\dagger}(g) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} a^{\dagger}(p)g(p)d^{\#3}p \tag{49}$$

hold for all $g \in S^{\#}_{\mathbf{fin}}({}^*\mathbb{R}_c^{\#3})$ if the equalities are understood in the sense of quadratic forms. That is, (48) means that for $\psi_1,\psi_2 \in D_{S^{\#}_{\mathbf{fin}}}$ we have

$$(\psi_1, a(g)\psi_2) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}(\psi_1, a(p)\psi_2)g(-p)d^{\#3}p \tag{50}$$

and similarly for (X.76b).
Since $a(p) : D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \to D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ the powers of $a(p)$ are well-defined operators on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$.
As before we can write down a formal expression for $(a^{\dagger}(p))^n$, but it does not make sense as operator, only as ${}^*\mathbb{C}_c^{\#}$-valued quadratic form on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \times D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$.
Notice that

$$(\psi_1,(a^{\dagger}(p))^n\psi_2) = ((a(p))^n\psi_1,\psi_2) \tag{51}$$

so for each n, $(a^{\dagger}(p))^n$ and $(a(p))^n$ are formally adjoints in the sense of ${}^*\mathbb{C}_c^{\#}$-valued quadratic forms. We could of course have defined the quadratic form $(a^{\dagger}(p))^n$ by (50) and then calculated that it arises by taking the n-th power of the formal object given by (45). Since $a(p_1) : D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \to D_{\mathcal{L}^{\#}_{\mathbf{fin}}}, (\psi_1, a^{\dagger}(p_2)a(p_1)\psi_2)$ is a well-defined ${}^*\mathbb{C}_c^{\#}$-valued

quadratic form for all $\langle p_1, p_2\rangle \in {}^*\mathbb{R}_c^{\#3} \times {}^*\mathbb{R}_c^{\#3}$. Notice, however, that $(\psi_1, a(p_1)a^\dagger(p_2)\psi_2)$ does not make sense since $a^\dagger(p_2)$ is only a quadratic form. In general any product $\prod_{i=1}^{N_1} a(f_i)$ is a
well-defined operator from $D_{\mathcal{L}^\#_{\mathbf{fin}}}$ to $D_{\mathcal{L}^\#_{\mathbf{fin}}}$ and $\prod_{i=1}^{N_1} a^\dagger(f_i)$ is a well-defined quadratic form on $D_{\mathcal{L}^\#_{\mathbf{fin}}} \times D_{\mathcal{L}^\#_{\mathbf{fin}}}$. Thus

$$\left(\psi_1, \left(\prod_{i=N_1+1}^{N_2} a^\dagger(p_i)\right)\left(\prod_{i=1}^{N_1} a^\dagger(-p_i)\right)\psi_2\right) \tag{52}$$

is also well-defined ${}^*\mathbb{C}_c^\#$-valued quadratic form on $D_{\mathcal{L}^\#_{\mathbf{fin}}} \times D_{\mathcal{L}^\#_{\mathbf{fin}}}$. One can check directly that if $f \in \mathcal{L}^\#_{\mathbf{fin}}({}^*\mathbb{R}_c^{\#3})$ then as ${}^*\mathbb{C}_c^\#$-valued quadratic forms

$$\begin{gathered}\left(\prod_{i=N_1+1}^{N_2} a^\dagger(f_i)\right)\left(\prod_{i=1}^{N_1} a^\dagger(f_i)\right) = \\ Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3N_2}} \left(\prod_{i=N_1+1}^{N_2} a^\dagger(p_i)\right)\left(\prod_{i=1}^{N_1} a^\dagger(-p_i)\right)\left(\prod_{i=1}^{N_2} f_i(p_i)\right) d^\# p_1 \ldots d^\# p_{N_2}\end{gathered} \tag{53}$$

and

$$N = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} a^\dagger(p)a(p)d^\# p \tag{54}$$

The generator of time translations in the free scalar field theory of mass m is given by

$$\mathbf{H}_0 = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \mu(p)a^\dagger(p)a(p)d^\# p \tag{54}$$

$\mathbf{H}_0$ is called the free Hamiltonian of mass m. (52), (53), and (54) involve no formal manipulations, but are mathematical statements about quadratic forms.

Theorem X.44 Let n_1 and n_2 be nonnegative integers and suppose that $W \in \mathcal{L}_2^\#\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$. Then there is a unique operator T_W on $\mathcal{F}_s(\mathcal{L}_2^\#({}^*\mathbb{R}_c^{\#3}))$ so that $D_{\mathcal{L}^\#_{\mathbf{fin}}} \subset D(T_W)$ is a core for T_W and

(a) as ${}^*\mathbb{C}_c^\#$-valued quadratic forms on $D_{\mathcal{L}^\#_{\mathbf{fin}}} \times D_{\mathcal{L}^\#_{\mathbf{fin}}}$

$$T_W = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3(n_1+n_2)}} W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2})\left(\prod_{i=1}^{n_1} a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} a(p_i)\right) d^{\#n_1}k d^{\#n_2}p \tag{55}$$

(b) If m_1 and m_2 are nonnegative integers so that $m_1 + m_2 = n_1 + n_2$, then $(1+N)^{-m_1/2}T_W(1+N)^{-m2/2}$ is a bounded operator with

$$\|(1+N)^{-m_1/2}T_W(1+N)^{-m_2/2}\| < C(m_1,m_2)\|W\|_{\mathcal{L}_2^\#}. \tag{56}$$

In particular, if $m_1 = n_1$ and $m_2 = n_2$, then

$$\|(1+N)^{-n_1/2}T_W(1+N)^{-n_2/2}\| < C(m_1,m_2)\|W\|_{\mathcal{L}_2^\#}. \tag{57}$$

(c) As ${}^*\mathbb{C}_c^\#$-valued quadratic forms on $D_{\mathcal{L}^\#_{\mathbf{fin}}} \times D_{\mathcal{L}^\#_{\mathbf{fin}}}$

$$T_W^* = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3(n_1+n_2)}} \overline{W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2})}\left(\prod_{i=1}^{n_2} a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_1} a(p_i)\right) d^{\#n_1}k d^{\#n_2}p \tag{58}$$

(d) If $W_n \to_\# W$ in $\mathcal{L}_2^\#\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$, then $T_{W_n} \to_\# T_W$ strongly on $D_{\mathcal{L}^\#_{\mathbf{fin}}}$.

(e) F_0 is contained in $D(T_W)$ and $D(T_W^*)$, and on vectors in F_0, T_W and T_W^* are given by the explicit formulas

$$(T_W\psi)^{(l-n_2+n_1)} = K(l,n_1,n_2)\mathbf{S}\times$$
$$\left[Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3n_2}} W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2})\psi^{(l)}(p_1,\ldots,p_{n_2},k_{n_1+1},\ldots,k_{n_1+l-n_2})d^{\#n_2}p\right] \quad (59)$$

$(T_W\psi)^n = 0$ if $n < n_1 - n_2$

$$(T_W^*\psi)^{(l-n_1+n_2)} = K(l,n_2,n_1)\mathbf{S}\times$$
$$\left[Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3(n_1)}} \overline{W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2})}\psi^{(l)}(k_1,\ldots,k_{n_1},p_{n_2+1},\ldots,p_{n_2+l-n_1})d^{\#n_1}k\right] \quad (60)$$

$(T_W^*\psi)^n = 0$ if $n < n_2 - n_1$ where $\mathbf{S}$ is the symmetrization operator and

$$K(l,n_1,n_2) = \left[\frac{l!(l+n_1-n_2)!}{((l-n_2)!)^2}\right]^{1/2}. \quad (61)$$

Proof. For vectors in $D_{\mathcal{L}^\#_{\mathbf{fin}}}$, we define $T_W\psi$ by the formula (X.82a). By the Schwarz inequality and the fact that $\mathbf{S}$ is a projection,

$$\left\|(T_W\psi)^{(l-n_2+n_1)}\right\|^2 \leq K(l,n_1,n_2)\|\psi^{(l)}\|^2\|W\|^2. \quad (62)$$

If we now define an operator $T_W^*\psi$, on $D_{\mathcal{L}^\#_{\mathbf{fin}}}$ by using the formula in (62),
then for all φ and ψ in $D_{\mathcal{L}^\#_{\mathbf{fin}}}$ one easily verifies that $(\varphi, T_W\psi) = (T_W^*\varphi,\psi)$.
Thus, T_W is #-closable and T_W^* is the restriction of the adjoint of T_W to $D_{\mathcal{L}^\#_{\mathbf{fin}}}$.
From now on we will use T_W to denote $\overline{T}_W$ and T_W^* to denote the adjoint of T_W.
By the definition of $T_W, D_{\mathcal{L}^\#_{\mathbf{fin}}}$ is a #-core and further, since T_W is bounded on the l-particle vectors in $D_{\mathcal{L}^\#_{\mathbf{fin}}}$, we have $F_0 \subset D(T_W)$. Since the right-hand side of (59) is also bounded on the l-particle vectors, (X.82a) represents T_W on all l-particle vectors.
The proof of the statements in (e) about T_W^* are the same.
To prove (b), let $\psi \in D_{\mathcal{L}^\#_{\mathbf{fin}}}$. Then by the above computation

$$\left\|((1+N)^{-m_1/2}T_W(1+N)^{-m_2/2}\psi)^{(l-n_2+n_1)}\right\|^2 \leq$$
$$\left[\frac{K(l,n_1,n_2)}{(1+l-n_2+n_1)^{m_1/2}(1+l)^{m_2/2}}\right]^2\|\psi^{(l)}\|^2\|W\|^2 \quad (63)$$

so that

$$\left\|((1+N)^{-m_1/2}T_W(1+N)^{-m_2/2}\psi)^{(l-n_2+n_1)}\right\| \leq$$
$$\left[\sup_{l\in\mathbb{N}}\frac{K(l,n_1,n_2)}{(1+l-n_2+n_1)^{m_1/2}(1+l)^{m_2/2}}\right]\|\psi^{(l)}\|\|W\| \leq C(m_1,m_2)\|\psi^{(l)}\|\|W\| \quad (64)$$

where

$$C(m_1,m_2) = \sup_{l\in\mathbb{N}}\frac{K(l,n_1,n_2)}{(1+l-n_2+n_1)^{m_1/2}(1+l)^{m_2/2}} < \infty^\# \quad (65)$$

since $m_1+m_2 = n_1+n_2$. In all the sup's only l so that $l-n_2+n_1 > 0$ occur since the other terms are annihilated by the action of T_W. Thus, $(1+N)^{-m_1/2}T_W(1+N)^{-m_2/2}$ extends to a hyper bounded operator on $\mathcal{F}_s(\mathbf{H}^\#)$ with norm less than or equal to $C(m_1,m_2)$. If $m_1 = n_1$ and $m_2 = n_2$, then $C(m_1,m_2) = 1$.

To prove (d) we need only note that if $\psi = (0,\ldots,\psi^{(l)},0,\ldots) \in D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ and $W_n \to_{\#} W$ in $\mathcal{L}^{\#}_2$, then

$$\|T_{W_n}\psi - T_W\psi\| = \|(T_{W_n-W})\psi\| \leq K(l,n_1,n_2)\|W_n - W\|\|\psi\|, \tag{66}$$

where $\#\text{-}\lim_{n\to\infty^{\#}} K(l,n_1,n_2)\|W_n - W\|\|\psi\| = 0$.

Since $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ consists of finite linear combinations of such vectors, we have shown that T_{W_n} #-converges strongly on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ to T_W if $W_n \to_{\#} W$ in $\mathcal{L}^{\#}_2$.

To prove (a) let $\psi_1,\psi_2 \in D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ with $\psi_1 = (0,\ldots,\psi^{(l-n_2+n_1)},0,\ldots)$ and $\psi_1 = (0,\ldots,\psi^{(l)},0,\ldots)$.

Then, if $W = \left(\prod_{i=1}^{n_1} f_i(k_i)\right)\left(\prod_{i=1}^{n_2} g_i(k_i)\right)$ the definition of the form $\left(\prod_{i=1}^{n_1} a^{\dagger}(k_i)\right)\left(\prod_{i=1}^{n_2} a_i(k_i)\right)$ shows that

$$\begin{gathered}(\psi_1, T_W\psi_2) = Ext\text{-}\int_{^*\mathbb{R}_c^{\#3n_2}} W(k_1,\ldots,k_{n_1},p_1,\ldots,p_{n_2}) \times \\ \left(\psi_1, \left(\prod_{i=1}^{n_1} a^{\dagger}(k_i)\right)\left(\prod_{i=1}^{n_2} a_i(k_i)\right)\psi_2\right) d^{\#n_1}k d^{\#n_2}p\end{gathered} \tag{67}$$

Since both sides of (X.83) are linear in W, the relationship continues to hold for the all such W's that are hyperfinite linear combinations of such products. Since

$$\left(\psi_1, \left(\prod_{i=1}^{n_1} a^{\dagger}(k_i)\right)\left(\prod_{i=1}^{n_2} a_i(k_i)\right)\psi_2\right) \in \mathcal{L}^{\#}_2\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right) \tag{68}$$

and since (d) holds, both the right- and left-hand sides of (X.83) are continuous linear functionals on ${}^*\mathbb{R}_c^{\#3(n_1+n_2)}$. Since they agree on a #-dense set, they agree everywhere. Finally, (68) extends by linearity to all of $D_{\mathcal{L}^{\#}_{\mathbf{fin}}} \times D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$.

This proves (a); the proof of (c) is similar. |

Finally, we note that as quadratic forms on $D_{\mathcal{L}^{\#}_{\mathbf{fin}}}$ we can express the free scalar field and the time zero fields in terms of $a^{\dagger}(k)$ and $a(k)$:

$$\Phi_{m,x}(x,t) =$$

$$\frac{1}{(2\pi_{\#})^{3/2}} \int_{|p|\leq x} \{[Ext\text{-}\exp(\mu(p)t - ipx)]a^{\dagger}(p) + [Ext\text{-}\exp(-\mu(p)t + ipx)]a(p)\} \frac{d^{\#3}p}{\sqrt{2\mu(p)}} \tag{69}$$

$$\varphi^{\#}_{m,x}(x) = \frac{1}{(2\pi_{\#})^{3/2}} \int_{|p|\leq x} \{[Ext\text{-}\exp(-ipx)]a^{\dagger}(p) + [Ext\text{-}\exp(ipx)]a(p)\} \frac{d^{\#3}p}{\sqrt{2\mu(p)}} \tag{70}$$

$$\pi^{\#}_{m,x}(x) = \frac{1}{(2\pi_{\#})^{3/2}} \int_{|p|\leq x} \{[Ext\text{-}\exp(-ipx)]a^{\dagger}(p) - [Ext\text{-}\exp(ipx)]a(p)\} \sqrt{\frac{\mu(p)}{2}}\, d^{\#3}p. \tag{71}$$

5.2. $Q^{\#}$-space representation of the Fock space structures

In this section the construction of $Q^{\#}$-space and $L^{\#}_2(Q^{\#}, d^{\#}\mu)$, another representation of the Fock space structures are presented. In analogy with the one degree of freedom case where $\mathcal{F}^{\#}({}^*\mathbb{R}^{\#}_c)$ is isomorphic to $L^{\#}_2({}^*\mathbb{R}^{\#}_c, d^{\#}x)$ in such a way that $\Phi_S(1)$ becomes multiplication by x, we will construct a #-measure space $\langle Q^{\#}, \mu^{\#}\rangle$, with $\mu(Q^{\#}) = 1$, and a unitary map $S : \mathcal{F}^{\#}_s({}^*\mathbb{R}^{\#}_c) \to L^{\#}_2(Q^{\#}, d^{\#}\mu)$ so that for each $f \in \mathbf{H}^{\#}_{\mathbf{C}^{\#}}$, $S\varphi^{\#}(f)S^{-1}$ acts on $L^{\#}_2(Q, d^{\#}\mu^{\#})$ by multiplication by a #-measurable function. We can then

show that in the case of the free scalar field of mass m in 4-dimensional space-time,

$V = SH_{\mathbf{I}}(g)S^{-1}$ is just multiplication by a function $V(q)$ which is in $L_p^{\#}(Q, d^{\#}\mu)$ for each $p \in \mathbb{N}^{\#}$. Let $\{f_n\}_{n=1}^{\infty^{\#}}$ be an orthonormal basis for $\mathbf{H}^{\#}$ so that each $f_n \in \mathbf{H}_{\mathbb{C}^{\#}}^{\#}$ and let $\{g_k\}_{k=1}^{N}, N \in \mathbb{N}^{\#}$ be a finite or hyperfinite subcollection of the $\{f_n\}_{n=1}^{\infty^{\#}}$. Let $\mathbf{P}_N$ be a set of the all external hyperfinite polynomials $Ext\text{-}P[u_1, \ldots, u_N]$ and $\mathcal{F}_N^{\#}$ be the #-closure of the set

$$\{Ext\text{-}P[\varphi^{\#}(g_1), \ldots, \varphi^{\#}(g_N)] | P \in \mathbf{P}_N\} \tag{1}$$

in $\mathcal{F}_s^{\#}(\mathbf{H}^{\#})$ and define $F_0^N = \mathcal{F}_N^{\#} \cap F_0$ From Theorem X.43 (and its proof) it follows that $\varphi^{\#}(g_k)$ and $\pi^{\#}(g_l)$, for all $1 \leq k, l \leq N$ are essentially self-adjoint on F_0^N and that

$$\begin{gathered}(Ext\text{-}\exp[it\varphi^{\#}(g_k)])(Ext\text{-}\exp[is\pi^{\#}(g_l)]) = \\ (Ext\text{-}\exp[-ist\delta_{kl}])(Ext\text{-}\exp[is\pi^{\#}(g_l)])(Ext\text{-}\exp[it\varphi^{\#}(g_k)]).\end{gathered} \tag{2}$$

Thus we have a representation of the generalized Weyl relations in which the vector $\mathbf{\Omega}_0$ satisfies $\left([\varphi^{\#}(g_k)]^2 + [\pi^{\#}(g_k)]^2 - 1\right)\mathbf{\Omega}_0 = 0$ and is #-cyclic for the operators $\{\varphi^{\#}(g_k)\}_{k=1}^{N}, N \in \mathbb{N}^{\#}$. Therefore there is a unitary map $\widetilde{\mathbf{S}}^{(N)} : \mathcal{F}_N^{\#} \to L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ so that

$$\begin{gathered}\widetilde{\mathbf{S}}^{(N)}\varphi^{\#}(g_k)\left(\widetilde{\mathbf{S}}^{(N)}\right)^{-1} = x_k \\ \widetilde{\mathbf{S}}^{(N)}\pi^{\#}(g_k)\left(\widetilde{\mathbf{S}}^{(N)}\right)^{-1} = \frac{1}{i}\frac{d^{\#}}{dx_k^{\#}}\end{gathered} \tag{3}$$

and

$$\widetilde{\mathbf{S}}^{(N)}\mathbf{Q}_0 = \pi_{\#}^{-N/4}\left\{Ext\text{-}\exp\left[-\left(Ext\text{-}\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right]\right\}. \tag{4}$$

It is convenient to use the Hilbert space

$$L_2^{\#}\left({}^*\mathbb{R}_c^{\#N}, \pi_{\#}^{-N/2}d^{\#N}x\left\{Ext\text{-}\exp\left[-\left(Ext\text{-}\textstyle\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right]\right\}\right)$$

instead of $L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ so let $d^{\#}\mu_k = \pi_{\#}^{-1/2}\exp(-x_k^2/2)\, d^{\#}x_k$ and define

$$(Tf)(x) = \pi_{\#}^{N/4}\left[Ext\text{-}\exp\left(Ext\text{-}\textstyle\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right]f(x). \tag{5}$$

Then T is a unitary map of $L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ onto $L_2^{\#}\left({}^*\mathbb{R}_c^{\#N}, Ext\text{-}\prod_{k=1}^{N} d^{\#}\mu_k^{\#}\right)$ and if we let $\mathbf{S}^{(N)} = T\widetilde{\mathbf{S}}^{(N)}$ we get

$$\begin{gathered}\mathbf{S}^{(N)} : \mathcal{F}_N^{\#} \to L_2^{\#}\left({}^*\mathbb{R}_c^{\#N}, Ext\text{-}\textstyle\prod_{k=1}^{N} d^{\#}\mu_k^{\#}\right), \\ \mathbf{S}^{(N)}\varphi^{\#}(g_k)(\mathbf{S}^{(N)})^{-1} = x_k, \\ \mathbf{S}^{(N)}\pi^{\#}(g_k)(\mathbf{S}^{(N)})^{-1} = -\frac{x_k}{i} + \frac{1}{i}\frac{d^{\#}}{d^{\#}x_k}, \\ \mathbf{S}^{(N)}\mathbf{Q}_0 = 1,\end{gathered} \tag{6}$$

where 1 is the function identically one.Note that each $\mu_k^{\#}$ has mass one, which implies that

$$\left\langle \mathbf{Q}_0, \left(Ext\text{-}\prod_{k=1}^{N} P_k[\varphi^{\#}(g_k)]\right)\mathbf{Q}_0 \right\rangle =$$
$$\int\limits_{{}^*\mathbb{R}_c^{\#N}} \left(Ext\text{-}\prod_{k=1}^{N} P_k[x_k]\right)\left(Ext\text{-}\prod_{k=1}^{N} d^{\#}\mu_k^{\#}\right) = \tag{7}$$
$$Ext\text{-}\prod_{k=1}^{N} \int\limits_{{}^*\mathbb{R}_c^{\#}} P[x_k] d^{\#}\mu_k^{\#} = Ext\text{-}\prod_{k=1}^{N} \int\limits_{{}^*\mathbb{R}_c^{\#}} \langle \mathbf{Q}_0, P_k[\varphi^{\#}(g_k)]\mathbf{Q}_0 \rangle,$$

where $P_1,\ldots,P_N$ are external hyperfinite polynomials. This formula (7) can also be proven by direct computations on $\mathcal{F}_s^{\#}(\mathbf{H}^{\#})$.
Now it is easy to see how to construct $\langle Q^{\#}, d^{\#}\mu^{\#}\rangle$. We define $Q^{\#} = \times_{k=1}^{\infty^{\#}} {}^*\mathbb{R}_c^{\#}$. Take the $\sigma^{\#}$-algebra generated by hyper infinite products of #-measurable sets in ${}^*\mathbb{R}_c^{\#}$ and set $\mu^{\#} = \otimes_{k=1}^{\infty^{\#}} \mu_k^{\#}$. We denote the points of $Q^{\#}$ by $q = \langle q_1, q_2, \ldots \rangle$. Then $\langle Q^{\#}, d^{\#}\mu^{\#}\rangle$ is a #-measure space and the set of functions of the form $P(q_1, q_2, \ldots)$, where P is a polynomial and $n \in \mathbb{N}^{\#}$ is arbitrary, is #-dense in $\mathcal{L}_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$. Let P be a polynomial in $N \in \mathbb{N}^{\#}$ variables

$$P(x_{k_1},\ldots,x_{k_N}) = Ext\text{-}\sum_{l_1,\ldots,l_N} c_{l_1,\ldots,l_N} x_{k_1}^{l_1},\ldots,x_{k_N}^{l_N} \tag{8}$$

and define

$$\mathbf{S} : P(\varphi^{\#}(f_{k_1}),\ldots,\varphi^{\#}(f_{k_N}))\mathbf{Q}_0 \to P(q_{k_1},\ldots,q_{k_N}). \tag{9}$$

Then

$$P(\varphi^{\#}(f_{k_1}),\ldots,\varphi^{\#}(f_{k_N}))\mathbf{Q}_0 = Ext\text{-}\sum_{l,\mathbf{m}} c_l \bar{c}_{\mathbf{m}} (\mathbf{Q}_0, \varphi^{\#}(f_{k_1})^{l_1+m_1},\ldots,\varphi^{\#}(f_{k_N})^{l_N+m_N}\mathbf{Q}_0) =$$
$$Ext\text{-}\sum_{l,\mathbf{m}} c_l \bar{c}_{\mathbf{m}} \int\limits_{{}^*\mathbb{R}_c^{\#N}} q_{k_1}^{l_1+m_1}\bullet\bullet\bullet q_{k_N}^{l_N+m_N}\left(Ext\text{-}\prod_{i=1}^{N} d^{\#}\mu_{k_i}^{\#}\right) = \int\limits_{Q^{\#}} |P(x_{k_1},\ldots,x_{k_N})|^2 d^{\#}\mu^{\#} \tag{10}$$

by (**X.92**) and the fact that each $\mu_k^{\#}$ has mass one. Since $\mathbf{Q}_0$ is cyclic for polynomials in the fields (Theorem **X.42**), $\mathbf{S}$ extends to a unitary map of $\mathcal{F}_s^{\#}(\mathbf{H}^{\#})$ onto $\mathcal{L}_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$. Clearly

$$\mathbf{S}\varphi^{\#}(f_k)\mathbf{S}^{-1} = q_k \text{ and } \mathbf{S}\mathbf{Q}_0 = 1. \tag{11}$$

Theorem 1. Let $\varphi_{m,x}^{\#}(f), x \in {}^*\mathbb{R}_c^{\#} \backslash {}^*\mathbb{R}_{c,\mathbf{fin}}^{\#}$ be the free scalar field of mass m (in 4-dimensional space-time) at time zero. Let $g \in \mathcal{L}_1^{\#}({}^*\mathbb{R}_c^{\#3}) \cap \mathcal{L}_2^{\#}({}^*\mathbb{R}_c^{\#3})$ and define

$$H_{I,x,\lambda}(g) = \lambda(x)\int g(x) \colon \varphi_{m,x}^{\#}(x)^4 \colon d^{\#3}x, \tag{12}$$

where $\lambda(x) \in {}^*\mathbb{R}_c^{\#}, \lambda(x) \approx 0$. Let $\mathbf{S}$ denote the unitary map of $\mathcal{F}_s^{\#}(\mathbf{H}^{\#})$ onto $\mathcal{L}_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ constructed above. Then $V = \mathbf{S}H_{I,x,\lambda}(g)\mathbf{S}^{-1}$ is multiplication by a function $V_{x,\lambda}(q)$ which satisfies:

(a) $V_{x,\lambda}(q) \in \mathcal{L}_p^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ for all $p \in \mathbb{N}^{\#}$.

(b) $Ext\text{-}\exp(-tV_{x,\lambda}(q)) \in \mathcal{L}_1^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ for all $t \in [0,\infty)$.

Proof. We will prove (a). By Eq.() we get

$$\varphi_{m,x}^{\#}(x) = \frac{1}{(2\pi_{\#})^{3/2}} \int\limits_{|p|\le x} \{[Ext\text{-}\exp(-ipx)]a^{\dagger}(p) + [Ext\text{-}\exp(ipx)]a(p)\} \frac{d^3p}{\sqrt{2\mu(p)}}. \tag{13}$$

Then $\varphi_{m,x}^{\#}(x)$ is a well-defined operator-valued function of $x \in {}^*\mathbb{R}_c^{\#3}$. We define

$:\varphi^{\#}_{m,x}(x)^4:$ by moving all the $a^{\dagger}$'s to the left in the formal expression for $\varphi^{\#}_{m,x}(x)^4$.
By Theorem **X.44** $:\varphi^{\#}_{m,x}(x)^4:$ is also a well-defined operator for each $x \in {}^*\mathbb{R}^{\#3}_c$ and $:\varphi^{\#}_{m,x}(x)^4:$ takes F_0 into itself. Thus for each $x \in {}^*\mathbb{R}^{\#3}_c$,

$$:\varphi^{\#}_{m,x}(x)^4: = \varphi^{\#}_{m,x}(x)^4 + d_2(x)\varphi^{\#}_{m,x}(x)^2 + d_0(x) \tag{14}$$

where the coefficients $d_2(x)$ and $d_0(x)$ are independent of x. For each $x \in {}^*\mathbb{R}^{\#3}_c$, $\mathbf{S}\varphi^{\#}_{m,x}(x)\mathbf{S}^{-1}$ is just the operator on #-measurable space $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ which operates by multiplying by the function

$$Ext\text{-}\sum_{k=1}^{\infty^{\#}} c_k(x,x)q_k \tag{15}$$

where

$$c_k(x,x) = (2\pi_{\#})^{-3/2}\left(f_k, Ext\text{-}\exp(ipx)(\mu(p))^{-1/2}\right). \tag{16}$$

Furthermore,

$$Ext\text{-}\sum_{k=1}^{\infty^{\#}} |c_k(x,x)|^2 = (2\pi_{\#})^{-3/2}\left\|(\mu(p))^{-1/2}\right\|_2^2, \tag{17}$$

so $\mathbf{S}\varphi^{\#}_{m,x}(x)^4\mathbf{S}^{-1}$and $\mathbf{S}\varphi^{\#}_{m,x}(x)^2\mathbf{S}^{-1}$ are in $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ and the $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ norms are uniformly bounded in x. Therefore, since $g \in \mathcal{L}^{\#}_1({}^*\mathbb{R}^{\#3}_c)$, $\mathbf{S}H_{I,x,\lambda}(g)\mathbf{S}^{-1}$ operates on $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ by multiplication by an $\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})$ function which we denote by $V_{x,\lambda}(q)$.
Consider now the expression for $H_{I,x}(g)\mathbf{Q}_0$. This is a vector $(0,0,0,0,\psi^{(4)},0,\ldots)$

$$\begin{aligned}\psi^{(4)}(p_1,p_2,p_3,p_4) &= Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \frac{\lambda g(x)\left[Ext\text{-}\exp\left(-ix\sum_{i=1}^4 p_i\right)\right]d^{\#3}x}{(2\pi_{\#})^{3/2}\prod_{i=1}^4(2\mu(p_i))^{1/2}} = \\ &= \frac{\lambda\widehat{g}\left(\sum_{i=1}^4 k_i\right)}{(2\pi_{\#})^{9/2}\prod_{i=1}^4(2\mu(p_i))^{1/2}}\end{aligned} \tag{18}$$

where $|p_i| \leq x, 1 \leq i \leq 4$. We choose now the parameter $\lambda = \lambda(x) \approx 0$ such that $\|\psi^{(4)}\|_2 \in \mathbb{R}$, thus

$$\|H_{I,x,\lambda(x)}(g)\mathbf{Q}_0\|_2 \in \mathbb{R}, \tag{19}$$

since $\|H_{I,x,\lambda(x)}(g)\mathbf{Q}_0\|_2 = \|\psi^{(4)}\|_2$. But, since $\mathbf{SQ}_0 = 1$, we get

$$\|H_{I,x,\lambda(x)}(g)\mathbf{Q}_0\|_2 = \|\mathbf{S}H_{I,x,\lambda(x)}(g)\mathbf{S}^{-1}\|_{\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})} = \|V_{x,\lambda(x)}(q)\|_{\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})} \tag{20}$$

From (19) and Eq.(20) we get that $\|V_{x,\lambda(x)}(q)\|_{\mathcal{L}^{\#}_2(Q^{\#},d^{\#}\mu^{\#})}$ is finite. It is easily verify that each $P(q_1,q_2,\ldots,q_n), n \in \mathbb{N}^{\#}$ is in the domain of $V_{x,\lambda(x)}(q)$ and $\mathbf{S}H_{I,x,\lambda(x)}(g)\mathbf{S}^{-1} = V_{x,\lambda(x)}(q)$ on that domain. Since $\mathbf{Q}_0$ is in the domain of $[H_{I,x,\lambda(x)}(g)]^p$ for all $n \in \mathbb{N}^{\#}$, 1 is in the domain of $[V_{x,\lambda(x)}(q)]^n$ for all $n \in \mathbb{N}^{\#}$. Thus, for all $n \in \mathbb{N}^{\#}, V_{x,\lambda(x)} \in \mathcal{L}^{\#}_{2n}(Q^{\#},d^{\#}\mu^{\#})$. Since $\mu^{\#}(Q^{\#}) < \infty^{\#}$, $V_{x,\lambda(x)} \in \mathcal{L}^{\#}_p(Q^{\#},d^{\#}\mu^{\#})$ for all $p < \infty^{\#}$.

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References.

[1] J. Foukzon,Set Theory $NC^{\#}_{\infty^{\#}}$ Based on Bivalent Infinitary Logic with Restricted Modus Ponens Rule. Basic Analysis on External Non-Archimedean Field ${}^{*}\mathbb{R}^{\#}_{c}$ https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3989960

[2] J. Foukzon,The Solution of the Invariant Subspace Problem. Part I. Complex Hilbert space. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4039068

[3] Foukzon, J. (2022). The Solution of the Invariant Subspace Problem. Complex Hilbert Space. External Countable Dimensional Linear spaces Over Field ${}^{*}\mathbb{R}^{\#}_{c}$. Part II. Journal of Advances in Mathematics and Computer Science, 37(11), 31-69. https://doi.org/10.9734/jamcs/2022/v37i111721

[4] J. GLIMM and A. JAFFE,Singular Perturbations of Selfadjoint Operators COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XXII, 401-414 (1969)

[5] Foukzon, Jaykov, Model $P(\varphi)_4$ Quantum Field Theory. A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields (July 14, 2022). Available at SSRN: https://ssrn.com/abstract=4163159 or http://dx.doi.org/10.2139/ssrn.4163159