Basic non-Archimedean functional analysis over non-Archimedean field ${}^*\mathbb{R}_c^\#$

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Abstract. Definitions and theorems related to non-Archimedean functional analysis on non-Archemedean field ${}^*\mathbb{R}_c^\#$ and on complex field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$are considered.Definitions and theorems appropriate to analysis on non-Archemedean field ${}^*\mathbb{R}_c^\#$ and on complex field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$are given in [1]-[2]

Content.

Chapter I. ${}^*\mathbb{R}_c^\#$-Valued abstract measures

1.σ#-algebras

Definition 1.1 ($\sigma^{\#}$-algebra). Let X be any set. We denote by $2^X = P(X) = \{A : A \subset X\}$ the set of all subsets of X. A family $\mathcal{F} \subset 2^X$ is called a $\sigma^{\#}$-algebra (on X) if:
(i) $\varnothing \in \mathcal{F}$;
(ii) $\mathcal{F}$ is closed under complements, i.e. $A \in \mathcal{F}$ implies $X \backslash A \in \mathcal{F}$;
(iii) $\mathcal{F}$ is closed under hypercountable unions, i.e. if $(A_n)_{n \in \mathbb{N}^{\#}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcup_{n \in \mathbb{N}^{\#}} A_n \in \mathcal{F}$.
Proposition 1.1.If $\mathcal{F}$ is a $\sigma^{\#}$-algebra on X then:
1. $\mathcal{F}$ is closed under hypercountable intersections, i.e. if $(A_n)_{n \in \mathbb{N}^{\#}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcap\limits_{n \in \mathbb{N}^{\#}} A_n \in \mathcal{F}$.
2. $X \in \mathcal{F}$.
3. $\mathcal{F}$ is closed under hyperfinite unions and hyperfinite intersections.
4. $\mathcal{F}$ is closed under set differences.
5. $\mathcal{F}$ is closed under symmetric differences.
Proposition 1.2.Suppose $\mathcal{F} \subset 2^X$ is a family of subsets satisfying the following:
1. $\varnothing \in \mathcal{F}$;
2. $\mathcal{F}$ is closed under complements;
3. $\mathcal{F}$ is closed under hyperinfinite intersections.
Then $\mathcal{F}$ is a $\sigma^{\#}$-algebra.
Proposition 1.3.If $(\mathcal{F}_\alpha)_{\alpha \in I}$ is a collection of $\sigma^{\#}$-algebras on X, then $\bigcap\limits_{\alpha} \mathcal{F}_\alpha$ is also a $\sigma^{\#}$-algebra on X.
Proposition 1.4.($\sigma^{\#}$-algebra generated by subsets). Let K be a collection of subsets of X. There exists a $\sigma^{\#}$-algebra, denoted $\sigma^{\#}(K)$ such that $K \subset \sigma^{\#}(K)$ and for every other $\sigma^{\#}$ algebra $\mathcal{F}$ such that $K \subset \mathcal{F}$ we have that $\sigma^{\#}(K) \subset \mathcal{F}$
We call $\sigma^{\#}(K)$ the $\sigma^{\#}$-algebra generated by K.
Proof. Define $\sigma^{\#}(K) \triangleq \bigcap\{\mathcal{F} | \mathcal{F}$ is a $\sigma^{\#}$-algebra on $X, K \subset \mathcal{F}\}$.
This is a $\sigma^{\#}$-algebra with the required properties.
Proposition 1.5.If $K \subset \mathcal{L}$ then $\sigma^{\#}(K) \subset \sigma^{\#}(\mathcal{L})$. Also, if $K \subset \mathcal{F}$ and $\mathcal{F}$ is a $\sigma^{\#}$-algebra, then $\sigma^{\#}(K) \subset \mathcal{F}$.
Definition 1.2. (Borel $\sigma^{\#}$-algebra). Given a topological space X, the Borel $\sigma^{\#}$-algebra is the $\sigma^{\#}$-algebra generated by the open sets. It is denoted $B^{\#}(X)$.
Specifically in the case $X = {}^*\mathbb{R}_c^{\#d}, d \in \mathbb{N}^{\#}$ we have that
$B_d^{\#} \triangleq B^{\#}({}^*\mathbb{R}_c^{\#d}) = \sigma^{\#}(U | U$ is an #-open set).
A Borel-#-measurable set, i.e. a set in $B^{\#}(X)$, is called a #-Borel set.
Measurable functions. Let f be a ${}^*\mathbb{R}_c^{\#}$-valued function defined on a set X. We suppose that some $\sigma^{\#}$-algebra $\Omega \subseteq P(X)$ is fixed.
Definition 1.3. We say that f is #-measurable, if $f^{-1}([a,b]) \in \Omega$ for any hyperreals $a, b \in {}^*\mathbb{R}_c^{\#}$ such that $a < b$.
The following three propositions are obvious.
Proposition 1.7. Let $f : X \to {}^*\mathbb{R}_c^{\#}$ be a function. Then the following conditions are equivalent:
(a) f is #-measurable;
(b) $f^{-1}([0,b)) \in \Omega$ for any hyperreal $b \in {}^*\mathbb{R}_c^{\#}$;
(c) $f^{-1}((b,\infty)) \in \Omega$ for any hyperreal $b \in {}^*\mathbb{R}_c^{\#}$;

(d) $f^{-1}(B) \in \Omega$ for any $B \in B(R)$.

Proposition 1.8 Let f and g be #-measurable functions, then

(a) $\alpha \times f + \beta \times g$ is #-measurable for any $\alpha, \beta \in {}^*\mathbb{R}_c^\#$;

(b) functions $\max\{f, g\}$ and $f \times g$ are #-measurable.

In particular, functions $f^+ := \max\{f, 0\}, f^- := (-f)^+$, and $|f| := f^+ + f^-$ are #-measurable. .

§2. #-Measures

Definition 2.1. A pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a $\sigma^\#$-algebra on X is call a #-measurable space. Elements of $\mathcal{F}$ are called #-measurable sets.

Given a #-measurable space $(X, \mathcal{F})$, a function $\mu^\# : \mathcal{F} \to [0, \infty^\#]$ is called a #-measure on $(X, \mathcal{F})$ if

1. $\mu^\#(\emptyset) = 0$;
2. (Hyper infinite additivity) For all hyper infinite sequences $(A_n)_{n \in \mathbb{N}^\#} \subset \mathcal{F}$ of pairwise disjoint sets in $\mathcal{F}$, we have that $\mu^\#\left(\bigcup_{n \in \mathbb{N}^\#} A_n\right) = Ext\text{-}\sum_{n \in \mathbb{N}^\#} \mu^\#(A_n)$.

$(X, \mathcal{F}, \mu^\#)$ is called a #-measure space.

Definition 2.2. A measure space $(X, \mathcal{F}, \mu^\#)$ is called: (a) hyperfinite if $\mu^\#(X) < \infty^\#$.

(b)It is called $\sigma^\#$-hyperfinite if $X = \bigcup_{n \in \mathbb{N}^\#} A_n$ where $A_n \in \mathcal{F}$ and $\mu^\#(A_n) < \infty^\#$ for all $n \in \mathbb{N}^\#$.

Definition 2.3. Let Σ be a $\sigma^\#$-algebra of subsets of a set X, and let $E = (E, \|\cdot\|_\#)$ be a non Archimedean Banach space.A function $\mu^\# : \Sigma \to E \cup \{{}^*\infty\}$ is called a vector-valued #-measure (or E-valued measure) if

(a) $\mu^\#(\emptyset) = 0$;

(b) $\mu^\#\left(\bigcup_{n \in \mathbb{N}^\#} A_n\right) = Ext\text{-}\sum_{n \in \mathbb{N}^\#} \mu^\#(A_n)$ for any pairwise disjoint sequence $A_n, n \in \mathbb{N}^\#$, $A_n \subseteq \Sigma$;

(c) for any $S \in \Sigma$, $\mu^\#(S) = \infty$, there exists $B \in \Sigma$ such that $B \subseteq S$ and $0 < \|\mu^\#(B)\|_\# < {}^*\infty$.

Definition 2.4.(a) A function $\mu^\# : \mathcal{F} \to {}^*\mathbb{C}_c^\# \cup \{{}^*\infty\}$ is called a complex #-measure if

1.$\mu^\#(\emptyset) = 0$,

2.$\mu^\#\left(\bigcup_{n \in \mathbb{N}^\#} A_n\right) = Ext\text{-}\sum_{n \in \mathbb{N}^\#} \mu^\#(A_n)$ for any sequence $A_n, n \in \mathbb{N}^\#$ of pairwise disjoint sets from $\mathcal{F}$, and, for any $A \in \mathcal{F}, \mu^\#(A) = {}^*\infty$, there exists $B \in \mathcal{F}$ such that $B \subseteq A$ and $0 < |\mu^\#(B)|_\# < {}^*\infty$.

(b) A function $\mu^\# : \mathcal{F} \to {}^*\mathbb{R}_c^\# \cup \{{}^*\infty\}$ is called a signed #-measure if

$\mu^\#(\emptyset) = 0$

$\mu^\#\left(\bigcup_{n \in \mathbb{N}^\#} A_n\right) = Ext\text{-}\sum_{n \in \mathbb{N}^\#} \mu^\#(A_n)$ for any sequence $A_n, n \in \mathbb{N}^\#$ of pairwise disjoint sets from $\mathcal{F}$, , and, for any $A \in \mathcal{F}, \mu^\#(A) = {}^*\infty$, there exists $B \in \mathcal{F}$ such that $B \subseteq A$ and $0 < |\mu^\#(B)| < {}^*\infty$.

Definition 2.5. If a certain property involving the points of #-measure space is true, except a subset having #-measure zero, then we say that this property is true

#-almost everywhere (abbreviated as #-a.e.).

Definition 2.6. Let $f_n, n \in \mathbb{N}^{\#}$ be a hyper infinite sequence of ${}^*\mathbb{R}_c^{\#}$-valued functions defined on X. We say that:

1. $f_n \to_{\#} f$ pointwise, if $f_n(x) \to_{\#} f(x)$ for all $x \in X$;
2. $f_n \to_{\#} f$ almost #-everywhere (#-a.e.), if $f_n(x) \to_{\#} f(x)$ for all $x \in X$ except a set of #-measure 0;
3. $f_n \to_{\#} f$ uniformly, if for any $\varepsilon > 0, \varepsilon \approx 0$ there is $n(\varepsilon)$ such that $\sup\{|f_n(x) - f(x)|: x \in X\} \leq \varepsilon$ for all $n \geq n(\varepsilon)$.

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§3. The Lebesgue #-Integral

In the following consideration, we fix a $\sigma^{\#}$-finite #-measure space $(X, \mathcal{F}, \mu^{\#})$.

Definition 3.1.Let $A_i \in \mathcal{F}, i = 1,\ldots,n \in {}^*\mathbb{N},$ be such that $\mu^{\#}(A_i) < {}^*\infty$ for all i, and $A_i \cap A_j = \emptyset$ for all $i \neq j$. The external function

$$f(x) = Ext\text{-}\sum_{i=1}^{n} \lambda_i \chi_{A_i}(x), \tag{3.1}$$

$\lambda_i \in {}^*\mathbb{R}_c^{\#},$ is called a simple external function. The Lebesgue external integral (Lebesgue #-integral) of a simple external function $f(x)$ is defined as

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} = Ext\text{-}\sum_{i=1}^{n} \lambda_i \mu^{\#}(A_i). \tag{3.2}$$

The Lebesgue external integral of a simple function is well defined.

Definition 3.2. Suppose that $\mu^{\#}$ is hyperfinite. Let $f : X \to {}^*\mathbb{R}_c^{\#}$ be an arbitrary nonnegative bounded in ${}^*\mathbb{R}_c^{\#}$ #-measurable external function and let $f_n,$ be a hyper infinite sequence of simple functions which #-converges uniformly to $f.$ Then the Lebesgue integral of f is

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} = \#\text{-}\lim_{n \to {}^*\infty}\left(Ext\text{-}\int_X f_n(x)d^{\#}\mu^{\#}\right). \tag{3.3}$$

Definition 3.3. Let $f : X \to {}^*\mathbb{R}_c^{\#}$ be a #-measurable function. Then the Lebesgue #-integral of f is defined by

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} = Ext\text{-}\int_X f^{+}(x)d^{\#}\mu^{\#} - Ext\text{-}\int_X f^{-}(x)d^{\#}\mu^{\#}. \tag{3.4}$$

If both of these terms are finite or hyperfinite then the function f is called #-integrable. In this case we write $f \in L_1^{\#} = L_1^{\#}(X, \mathcal{F}, \mu^{\#})$.

Notation 3.1.We will use the following notation. For any $A \in \mathcal{F}$:

$$Ext\text{-}\int_A f(x)d^{\#}\mu^{\#} = Ext\text{-}\int_X f(x)\chi_A(x)d^{\#}\mu^{\#}. \tag{3.5}$$

Lemma 3.1.(1) Let $f : X \to {}^*\mathbb{R}_c^{\#}$ be an arbitrary nonnegative #-measurable function then

$$Ext\text{-}\int_X f(x)d^{\#}\mu^{\#} = \sup\left\{Ext\text{-}\int_X \varphi(x)d^{\#}\mu^{\#} \,\middle|\, \varphi \text{ is a simple function such that } 0 \leq \varphi(x) \leq f(x)\right\}. \tag{3.6}$$

(2) If $f,g : X \to {}^*\mathbb{R}_c^{\#}$ are #-measurable, g is #-integrable, and $|f(x)| \leq g(x),$ then f is #-integrable and

$$\left|Ext\text{-}\int_X f(x)d^\#\mu^\#\right| \leq Ext\text{-}\int_X g(x)d^\#\mu^\#. \qquad (3.7)$$

(3) $Ext\text{-}\int_X |f(x)|d^\#\mu^\# = 0$ if and only if $f(x) = 0$ #-a.e.

(4) If $f_1, f_2, \ldots f_n : X \to {}^*\mathbb{R}_c^\#, n \in {}^*\mathbb{N}$ are integrable then, for $\lambda_1, \lambda_2, \ldots, \lambda_n \in {}^*\mathbb{R}_c^\#$, the linear combination $Ext\text{-}\sum_{i=1}^n \lambda_i f_i$ is #-integrable and

$$Ext\text{-}\int_X \left(Ext\text{-}\sum_{i=1}^n \lambda_i f_i\right) d^\#\mu^\# = Ext\text{-}\sum_{i=1}^n \left(Ext\text{-}\int_X \lambda_i f_i d^\#\mu^\#\right). \qquad (3.8)$$

(5) Let $f \in L_1^\#(X, \mathcal{F}, \mu^\#)$, then the formula

$$\nu^\#(A) = Ext\text{-}\int_A f(x)d^\#\mu^\# = Ext\text{-}\int_X f(x)\chi_A(x)d^\#\mu^\# \qquad (3.9)$$

defines a signed #-measure on the $\sigma^\#$-algebra $\mathcal{F}$.

Notation 3.1. Assume that $f, g : X \to {}^*\mathbb{R}_c^\#$ are #- integrable functions and such that $f \leq g$ #-a.e. If

$$Ext\text{-}\int_X f(x)d^\#\mu^\# \leq Ext\text{-}\int_X g(x)d^\#\mu^\#$$

we abbreviate $f \leq_s g$.

Definition 3.4 We say that a hyper infinite sequence $\{f_n\}_{n=1}^{^*\infty}$ of #-integrable functions $L_1^\#$-#-converges to f (or #-converges in $L_1^\#(X, \mathcal{F}, \mu^\#)$) if

$$Ext\text{-}\int_X |f_n - f|d^\#\mu^\# \to_\# 0 \text{ as } n \to {}^*\infty. \qquad (3.11)$$

Theorem 3.1 (The monotone #-convergence theorem) If $\{f_n\}_{n=1}^{^*\infty}$ is a hyper infinite sequence in $L_1^{\#+}(X, \mathcal{F}, \mu^\#)$ such that $f_j \leq_s f_{j+1}$ for all j and $f(x) = \sup_{n \in {}^*\mathbb{N}} f_n(x)$ then

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = \#\text{-}\lim_{n \to {}^*\infty} Ext\text{-}\int_X f_n(x)d^\#\mu^\#. \qquad (3.12)$$

Theorem 3.2 (The dominated #-convergence theorem) Let f and g be #-measurable, let f_n be #-measurable for any $n \in {}^*\mathbb{N}$ such that $|f_n(x)| \leq g(x)$ #-a.e., and $f_n \to_\# f$ #-a.e. If g is #-integrable then f and f_n are also #-integrable and

$$Ext\text{-}\int_X f(x)d^\#\mu^\# = \#\text{-}\lim_{n \to {}^*\infty} Ext\text{-}\int_X f_n(x)d^\#\mu^\#. \qquad (3.12)$$

Definition 3.5.Let $\{(X_\alpha, \mathcal{F}_\alpha, \mu_\alpha^\#)\}_{\alpha \in \Delta}$ be a nonempty family of #-measure spaces. We define the family Ω of blocks:

$A(A\alpha_1, A\alpha_2, \ldots, A\alpha_n) := A\alpha_1 \times A\alpha_2 \times \cdots \times A\alpha_n \times \prod$

and define a function: $\mu^\# : \Omega \to {}^*\mathbb{R}_c^\# \cup \{{}^*\infty\}$:

This function possesses an extension (by additivity) on the algebra A generated by Ω. It is an exercise to show that μ is a premeasure on A.

Definition 3.5.If $E \subseteq X_1 \times X_2$ and $x_1 \in X_1, x_2 \in X_2$, we define

$$E_{x_1} = \{x \in X_2 : (x_1, x) \in E\} \text{ and } E^{x_2} = \{x \in X_1 : (x, x_2) \in E\}.$$

If $f : X_1 \times X_2 \to {}^*\mathbb{R}_c^\#$is a function, we define $f_{x_1} : X_2 \to {}^*\mathbb{R}_c^\#$ and $f^{x_2} : X_1 \to {}^*\mathbb{R}_c^\#$ by $f_{x_1}(x) = f(x_1, x)$ and $f^{x_2}(x) = f(x, x_2)$.

Theorem 3.3 (The generalized Fubini's theorem) Let $\mu_1^\#, \mu_2^\#$ be $\sigma^\#$-hyperfinite #-measures on $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$,

$$(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1^{\#} \otimes \mu_2^{\#}) = (X_1, \mathcal{F}_1, \mu_1^{\#}) \times (X_2, \mathcal{F}_2, \mu_2^{\#}), \quad (3.)$$

and let $f \in L_1^{\#}(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1^{\#} \otimes \mu_2^{\#})$. Then $f_{x_1} \in L_1^{\#}(X_2, \mathcal{F}_2, \mu_2^{\#})$ $\mu_1^{\#}$-#-a.e., and $f^{x_2} \in L_1^{\#}(X_1, \mathcal{F}_1, \mu_1^{\#})\mu_2^{\#}$-#-a.e., and

$$\int_{X_1 \times X_2} f d^{\#}(\mu_1^{\#} \otimes \mu_2^{\#}) = \int_{X_2}\left[\int_{X_1} f^{x_2} d^{\#}\mu_1^{\#}\right] d^{\#}\mu_2^{\#} = \int_{X_1}\left[\int_{X_2} f_{x_1} d^{\#}\mu_2^{\#}\right] d^{\#}\mu_1^{\#} \quad (3.)$$

Chapter III.Hilbert Spaces over field ${}^{*}\mathbb{C}_c^{\#}$.

1. Hilbert Spaces over field ${}^{*}\mathbb{C}_c^{\#}$ Basics.

The sequence space l consists of all hyper infinite sequences

3.#-Analytic vectors.Generalized Nelson's #-analytic vector theorem.

Let $\mathbf{H}^{\#}$ be a #-complex Hilbert space over field ${}^{*}\mathbb{C}_c^{\#}$. The most natural way to construct a #-continuous one-parameter unitary group $U(t) : \mathbf{H}^{\#} \to \mathbf{H}^{\#}$ is to try to make sense of the power series $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(itA)^n$ on a #-dense set of vectors. Notice that this can certainly be done if A is self-adjoint. For let E_{Ω} be the family of spectral projections for A. Then on each of the spaces $E_{[-M,M]}$, A is a bounded operator and $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(itA)^n/n!$ #-converges to $Ext\text{-}\exp(itA)$ in norm. In particular, for any $\varphi \in \bigcup_{M \geq 0} E_{[-M,M]}$,

$$\#\text{-}\lim_{N \to \infty^{\#}}\left(Ext\text{-}\sum_{n=0}^{N}\frac{(itA)^n}{n!}\right) = Ext\text{-}\exp(itA). \quad (3.1)$$

Since $\bigcup_{M \geq 0} E_{[-M,M]}$ is #-dense in $\mathbf{H}^{\#}$, we see that the group generated by a self-adjoint operator A is completely determined by the well-defined action of the hyper infinite series $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(itA)^n/n!$ on a #-dense set. We will prove the #-converse: namely, if A is symmetric and has a #-dense set of vectors to which $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(itA)^n/n!$ can be applied, then A is essentially self-#-adjoint. We need several definitions.

Definition1.1. Let A be an operator on a non-Archimedean Hilbert space $\mathbf{H}^{\#}$. The set $\mathbf{C}^{\infty^{\#}}(A) = \bigcap_{n=0}^{\infty^{\#}} D(A^n)$ is called the $\mathbf{C}^{\infty^{\#}}$-vectors for A. A vector $\varphi \in \mathbf{C}^{\infty^{\#}}(A)$ is called an #-analytic vector for A if

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}}\frac{\|A^n\varphi\| t^n}{n!} < {}^{*}\infty \quad (3.2)$$

for some $t > 0$. If A is self-adjoint, then $\mathbf{C}^{\infty^{\#}}(A)$ will be #-dense in $D(A)$. However, in general, a symmetric operator may have no $\mathbf{C}^{\infty^{\#}}$-vectors at all even if A is essentially self-#-adjoint. We caution the reader to remember that #-analytic vectors and vectors of

uniqueness (defined below) must be $\mathbf{C}^{\infty^{\#}}$- vectors for A. A vector $\varphi \in D(A)$ can be an #-analytic vector for an extension of A but fail to be an #-analytic vector for A because it is not in $\mathbf{C}^{\infty^{\#}}(A)$.

Definition1.2.Suppose that A is symmetric. For each $\varphi \in \mathbf{C}^{\infty^{\#}}(A)$, define

$$D_{\varphi} = \left\{Ext\text{-}\sum_{n=0}^{N} \alpha_n A^n \varphi \middle| N \in {}^*\mathbb{N}, \alpha_n \in {}^*\mathbb{C}_c^{\#}\right\}. \quad (3.3)$$

Let $\mathbf{H}_{\varphi}^{\#} = \#\text{-}\overline{D_{\varphi}}$ and define $A_{\varphi} : D_{\varphi} \to D_{\varphi}$ by $A_{\varphi}\left(Ext\text{-}\sum_{n=0}^{N} \alpha_n A^n \varphi\right) = Ext\text{-}\sum_{n=0}^{N} \alpha_n A^{n+1} \varphi$.
φ is called a vector of #-uniqueness if and only if A_{φ} is essentially self-#-adjoint on D_{φ} as an operator on $\mathbf{H}_{\varphi}^{\#}$.
Finally, a subset $S \subset \mathbf{H}^{\#}$ is called #-total if the set of hyperfinite linear combinations of elements of S is #-dense in $\mathbf{H}^{\#}$.

Lemma (Generalized Nussbaum's lemma) Let A be a symmetric operator and suppose that $D(A)$ contains a #-total set of vectors of #-uniqueness. Then A is essentially self-#-adjoint.

Proof We will show that $\mathbf{Ran}(A \pm i)$ are #-dense in $\mathbf{H}^{\#}$. By the fundamental criterion this will show that A is essentially self-#-adjoint. Suppose $\psi \in \mathbf{H}^{\#}$ and $\varepsilon > 0$ are given and let S denote the set of vectors of #-uniqueness. Since S is #-total we can find $(\alpha_n)_{n=1}^{N}$ and $(\psi_n)_{n=1}^{N}$ with $\psi_n \in S$ so that

$$\left\| \psi - Ext\text{-}\sum_{n=1}^{N} \alpha_n \psi_n \right\|_{\#} \leq \varepsilon/2. \quad (3.4)$$

Since ψ_n is a vector of #-uniqueness, there is a $\varphi_n \in D_{\psi_n}$ so that

$$\|\psi_n - (A+i)\varphi_n\|_{\#} \leq \tfrac{\varepsilon}{2}\left(Ext\text{-}\sum_{n=1}^{N} |\alpha_n|\right)^{-1}. \quad (3.5)$$

Setting $\varphi = Ext\text{-}\sum_{n=1}^{N} \alpha_n \varphi_n$ we have $\varphi \in D(A)$ and $\|\psi - (A+i)\varphi\|_{\#} < \varepsilon$.
Thus $\mathbf{Ran}(A+i)$ is #-dense. The proof for $(A-i)$ is the same.

Theorem 3.1. (Generalized Nelson's #-analytic vector theorem) Let A be a symmetric operator on a non-Archimedean Hilbert space $\mathbf{H}^{\#}$. If $D(A)$ contains a #-total set of #-analytic vectors, then A is essentially self-#-adjoint.

Proof By Generalized Nussbaum's lemma, it is enough to show that each #-analytic vector ψ is a vector of #-uniqueness. First notice that A_{ψ} always has self-#-adjoint extensions, since the operator

$$C : Ext\text{-}\sum_{n=0}^{N} \alpha_n A^n \psi \quad (3.6)$$

extends to a conjugation on $\mathbf{H}_{\psi}^{\#}$ which commutes with A_{ψ}. Suppose that B is a self-#-adjoint extension of A_{ψ} on $\mathbf{H}_{\psi}^{\#}$ and let $\mu^{\#}$ be the spectral #-measure for B associated to ψ. Since ψ is an #-analytic vector for A,

$$Ext\text{-}\sum_{n=0}^{N} \|A^n \psi\|_{\#}/n! < {}^*\infty \quad (3.7)$$

for some $t > 0$. Let $0 < s < t$. Then

$$\begin{gathered} Ext\text{-}\sum_{n=0}^{^*\infty} \frac{s^n}{n!}\left(Ext\text{-}\int_{^*\mathbb{R}_c^{\#}} |x|^n d^{\#}\mu^{\#}\right) \leq \\ \leq Ext\text{-}\sum_{n=0}^{^*\infty} \frac{s^n}{n!}\left(Ext\text{-}\int_{^*\mathbb{R}_c^{\#}} x^{2n} d^{\#}\mu^{\#}\right)^{1/2}\left(Ext\text{-}\int_{^*\mathbb{R}_c^{\#}} d^{\#}\mu^{\#}\right)^{1/2} = \\ \|\psi\|_{\#} Ext\text{-}\sum_{n=0}^{^*\infty} \frac{s^n}{n!}\|A^n\psi\|_{\#} < {}^*\infty. \end{gathered} \quad (3.8)$$

Therefore by generalized Fibini's theorem

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}\left(Ext\text{-}\sum_{n=0}^{{}^*\infty}\frac{s^n}{n!}|x|^n\right)d^\#\mu^\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} Ext\text{-}(s|x|)d^\#\mu^\# < {}^*\infty. \qquad (3.9)$$

As a result, the function

$$\langle\psi,[Ext\text{-}\exp(itB)]\psi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}[Ext\text{-}\exp(itx)]d^\#\mu^\# \qquad (3.10)$$

has an #-analytic continuation

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}[Ext\text{-}\exp(izx)]d^\#\mu^\# \qquad (3.11)$$

to the region $|\operatorname{Im} z| < t$. Since

$$\left[\left(\frac{d^\#}{d^\# z}\right)^k\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}[Ext\text{-}\exp(izx)]d^\#\mu^\#\right)\right]_{z=0} = \\ = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}\left[Ext\text{-}\exp(ix)^k\right]d^\#\mu^\# = \left\langle\psi,(iA)^k\psi\right\rangle_\#, \qquad (3.12)$$

we obtain

$$\langle\psi,[Ext\text{-}\exp(isB)]\psi\rangle_\# = Ext\text{-}\sum_{n=0}^{{}^*\infty}\frac{(is)^n}{n!} = \left\langle\psi,(iA)^k\psi\right\rangle_\# \qquad (3.13)$$

for $|s| < t$. Thus, for $|s| < t$ (and therefore for all s), the function $\langle\psi_1,[Ext\text{-}\exp(isB)]\psi_2\rangle_\#$ is completely determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_\#, n \in {}^*\mathbb{N}$.
Similar proof shows that $\langle\psi_1,[Ext\text{-}\exp(isB)]\psi_2\rangle_\#$ is determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_\#, n \in {}^*\mathbb{N}$for any $\psi_1,\psi_2 \in D_\psi$. Since D_ψ is #-dense in $\mathbf{H}_\psi^\#$ and $Ext\text{-}\exp(isB)$ is unitary, $Ext\text{-}\exp(isB)$ is completely determined by the numbers $\langle\psi_1,A^n\psi_2\rangle_\#, n \in {}^*\mathbb{N}$ for any $\psi_1,\psi_2 \in D_\psi$.Thus, all self-#-adjoint extensions of A_ψ generate the same unitary group, so by generalized Stone's theorem A_ψ has at most one self-#-adjoint extension. As we have already remarked, A_ψ has at least one self-#-adjoint extension. Thus A_ψ is essentially self-#-adjoint and ψ is a vector of uniqueness.

Corollary 1 A #-closed symmetric operator A is self-#-adjoint if and only if $D(A)$ contains a #-dense set of #-analytic vectors.

The statement of Corollary 1 is not true if "self-#-adjoint" is replaced by "essentially self-#-adjoint." A self-#-adjoint operator A may be essentially self-#-adjoint on a domain
$D \subset D(A)$ and D may not even contain any #-vectors.

Corollary 2 Suppose that A is a symmetric operator and let D be a #-dense linear set contained in $D(A)$. Then, if D contains a #-dense set of #-analytic vectors and if D is invariant under A, then A is essentially self-#-adjoint on D.

Proof Since D is invariant under A, each #-analytic vector for A in D is also an #-analytic vector for $A \upharpoonright D$. Thus, by Theorem 3.1 $A \upharpoonright D$ is essentially self-#-aadjoint.

The reason that one needs the invariance condition in Corollary 2 is that for a vector $\psi \in D$ to be #-analytic for $A \upharpoonright D$, it must first be $C^{{}^*\infty}$ for $A \upharpoonright D$. This requires that $A^n \in D$ for all $n \in {}^*\mathbb{N}$.

§4.The generalized Spectral Theorem

§ 4.1.The #-continuous functional calculus

In this section, we will discuss the generalized spectral theorem in its many guises. This structure theorem is a concrete description of all self-#-adjoint operators. There are several apparently distinct formulations of the spectral theorem. In some sense they are all equivalent.

The form we prefer says that every bounded self-#-adjoint operator is a multiplication operator. (We emphasize the word bounded since we will deal extensively with unbounded self-#-adjoint operators in the next chapter; there is a spectral theorem for unbounded operators which we discuss in Section § 4.3)

This means that given a bounded self-#-adjoint operator A on a non-Archimedean Hilbert space $\mathbf{H}^{\#}$ over field ${}^{*}\mathbb{R}_{c}^{\#}$ or ${}^{*}\mathbb{C}_{c}^{\#}$, we can always find a #-measure $\mu^{\#}$ on a #-measure space M and a unitary operator $U : \mathbf{H}^{\#} \to L_{2}^{\#}(M, d^{\#}\mu^{\#})$ so that

$$(UAU^{-1}f)(x) = F(x)f(x) \tag{4.1.1}$$

for some bounded real-valued #-measurable function F on M.

In practice, M will be a union of copies of ${}^{*}\mathbb{R}_{c}^{\#}$ and F will be x so the core of the proof of the theorem will be the construction of certain #-measures. This will be done in Section

§ 4.2 by using the generalized Riesz-Markov theorem. Our goal in this section will be to

make sense out of $f(A)$, for f a #-continuous function.

In the next section, we will consider the #-measures defined by the functionals

$$f \mapsto \langle \psi, f(A)\psi \rangle_{\#} \tag{4.1.2}$$

for fixed $\psi \in \mathbf{H}^{\#}$.

Given a fixed operator A, for which f can we define $f(A)$? First, suppose that A is an arbitrary bounded in${}^{*}\mathbb{R}_{c}^{\#}$ operator. If $f(x) = Ext\text{-}\sum_{n=1}^{N} c_n x^n$, $N \in {}^{*}\mathbb{N}$is a polynomial, we let $f(A) = Ext\text{-}\sum_{n=1}^{N} c_n A^n$. Suppose that $f(x) = Ext\text{-}\sum_{n=1}^{{}^{*}\infty} c_n x^n$ is a hyper infinite power series with radius of #-convergence R. If $\|A\|_{\#} < R$ then hyper infinite power series $Ext\text{-}\sum_{n=1}^{{}^{*}\infty} c_n A^n$ #-converges in $\mathcal{L}(H^{\#})$ so it is natural to set

$$f(A) = Ext\text{-}\sum_{n=1}^{{}^{*}\infty} c_n A^n \tag{4.1.3}$$

In this last case, f was a function #-analytic in a domain including all of $\sigma(A)$.

The functional calculus we have talked about thus far works for any operator in any Banach space. The special property of self-adjoint operators or more generally normal operators is that $\|P(A)\|_{\#} = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$ for any polynomial P, so that one can use the B.L.T. theorem to extend the functional calculus to #-continuous functions. Our major goal in this section is the proof of:

Theorem 4.1.1. (#-continuous functional calculus) Let A be a self-#-adjoint operator on a Hilbert space $H^{\#}$. Then there is a unique map $\phi : C^{\#}(\sigma(A)) \to \mathcal{L}(H^{\#})$ with the following properties:

(a) ϕ is an algebraic $*$-homomorphism, that is,

$\phi(fg) = \phi(f)\phi(g), \phi(\lambda f) = \lambda\phi(f), \phi(1) = I, \phi(\bar{f}) = \phi(f)^*$.

(b) ϕ is #-continuous, that is, $\|\phi(f)\|_{\mathcal{L}(H^\#)} \leq C\|f\|_{^*\infty}$.

(c) Let f be the function $f(x) = x$; then $\phi(f) = A$.

Moreover, ϕ has the additional properties:

(d) If $A\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$.

(e) $\sigma[\phi(f)] = \{f(\lambda)|\lambda \in \sigma(A)\}$ [spectral mapping theorem].

(f) If $f \geq 0$, then $\phi(f) \geq 0$.

(g) $\|\phi(f)\|_\# = \|f\|_{^*\infty}$. [this strengthens (b)].

The proof which we give below is quite simple, (a) and (c) uniquely determine $\phi(P)$ for any hyperfinite polynomial $P(x)$. By the generalized Weierstrass theorem, the set of polynomials is #-dense in $C^\#(\sigma(A))$ so the main part of the proof is showing that

$$\|P(A)\|_{\#op} = \|P(x)\|_{C^\#(\sigma(A))} = \sup_{\lambda\in\sigma(A)}|P(\lambda)|. \quad (4.1.4)$$

The existence and uniqueness of ϕ then follow from the generalized B.L.T. theorem. To prove the crucial equality, we first prove a special case of (e) (which holds for arbitrary bounded operators):

Lemma 4.1.1.Let $P(x) = Ext\text{-}\sum_{n=1}^N c_n x^n$, $N \in {}^*\mathbb{N}$. Let $P(A) = Ext\text{-}\sum_{n=1}^N c_n A^n$. Then

$$\sigma(P(A)) = \{P(\lambda)|\lambda \in \sigma(A)\}. \quad (4.1.5)$$

Proof Let $\lambda \in \sigma(A)$. Since $x = \lambda$ is a root of $P(x) - P(\lambda)$, we have $P(x) - P(\lambda) = (x-\lambda)Q(x)$, so $P(A) - P(\lambda) = (A-\lambda)Q(A)$. Since $(A-\lambda)$ has no inverse neither does $P(A) - P(\lambda)$ that is, $P(\lambda) \in \sigma(P(A))$.

Conversely, let $\mu \in \sigma(P(A))$ and let $\lambda_1,\ldots,\lambda_n$ be the roots of $P(x) - \mu$, that is, $P(x) - \mu = a\left(Ext\text{-}\prod_{i=1}^n (x-\lambda_i)\right)$. If $\lambda_1,\ldots,\lambda_n \notin \sigma(A)$, then

$$(P(A)-\mu)^{-1} = a^{-1}\left(Ext\text{-}\prod_{i=1}^n (A-\lambda_i)^{-1}\right) \quad (\ \ )$$

so we conclude that some $\lambda_i \in \sigma(A)$ that is, $\mu = P(\lambda)$ for some $\lambda \in \sigma(A)$.

Definition Let $r(A) = \sup_{\lambda\in\sigma(A)} |\lambda|$. Then $r(A)$ is called the spectral radius of A.

Theorem 4.1.2. Let X be a Banach space, $A \in \mathcal{L}(X)$ Then $\lim_{n\to^*\infty} \sqrt[n]{\|A^n\|_{\#op}}$ exists and is equal to $r(A)$. If X is a Hilbert space and A is self-#-adjoint, then $r(A) = \|A\|_{\#op}$.

.

Proof The reader can check that lim^00 ||ГИ||1/Я exists by following the clever subadditivity argument outlined in Problem IL The crux of the proof of the theorem is to establish that the radius of convergence of the Laurent series of Rx(T) about oo is just r(T)'1. First notice that the radius of convergence cannot be smaller than r(T)-1 since we have proven that Rx(T) is analytic on p(T)and{A | |A| > r(T)} cr p(T). On the other hand, (VI.2) is just the Laurent series about oo and we have seen that where it converges absolutely, Rx(T) exists. Since a Laurent series converges absolutely inside the circle of convergence, we conclude that the radius of convergence cannot be larger than r(T)~\ That r(T) = limfl^00 ||7^||1/л follows from the vector-valued version of Hadamard's theorem which says that the radius of convergence of (VI.2) is just the inverse of

Hm WTnWlfn = lim ||Г11/Л

Я И-* OO

Finally, if X is a Hilbert space and A is self-adjoint, then MH2 — M2H by part (/) of

Theorem VI.3. This implies that M2" || = \\A ||2” so
r(A) = Iim WAkWIfk = Iim M2T~n - MH |
k~* oc л-*oo
The following theorem is sometimes useful in determining spectra.
Theorem VI. 7 (Phillips) Let X be a Banach space, Te SP(X). Then o(T) = cr(T’) and Rx(T) = Rx(T)’. If Ж is a Hilbert space, then <r(T*) — {A IA e a(T)} and RX(T*) - Rx(T)*.
We note that the Hilbert space case follows from (d) of Theorem VI.3. We now work out in some detail an example which illustrates the various kinds of spectra.
Example Let Tbe the operator on ^f1 which acts by
TKi,Ca...-)-«a.Cj.—)
Lemma 4.1.2. Let A be a bounded self-#-adjoint operator. Then

$$\|P(A)\|_{\#} = \sup_{\lambda\in\sigma(A)}|P(\lambda)|. \tag{4.1.6}$$

Proof (by Theorem **4.1.2**.)
(by **Lemma 4.1.1**)
Proof of **Theorem 4.1.1**. Let $\phi(P) = P(A)$. Then $\|\phi(P)\|_{\mathcal{L}(H^{\#})} = \|P\|_{C^{\#}(\sigma(A))}$ so ϕ has a unique linear extension to the #-closure of the polynomials in $C^{\#}(\sigma(A))$. Since the polynomials are an algebra containing $\mathbf{I}$, containing complex conjugates, and separating points, this #-closure is all of $C^{\#}(\sigma(A))$. Properties (a), (b), (c), (g) are obvious and if $\widetilde{\phi}$ obeys (a), (b), (c) it agrees with ϕ on polynomials and thus by #-continuity on $C^{\#}(\sigma(A))$ To prove (d), note that $\phi(P)\psi = P(\lambda)\psi$ and apply #-continuity. To prove (f), notice that if $f \geq 0$, then $f = g^2$ with g ${}^*\mathbb{R}_c^{\#}$-valued and $g \in C^{\#}(\sigma(A))$. Thus $\phi(f) = \phi(g)^2$ with $\phi(g)$ self-#-adjoint, so $\phi(f) \geq 0$.
Remark 4.1.1. In addition:
(1) $\phi(f) \geq 0$ if and only if $f \geq 0$.
(2) Since $fg = gf$ for all f,g, $\{f(A)|f \in C^{\#}(\sigma(A))\}$ forms an abelian algebra closed under adjoints. Since $\|\phi(f)\|_{\#} = \|f\|_{{}^*\infty}$ and $C^{\#}(\sigma(A))$ is #-complete, $\{f(A)|f \in C^{\#}(\sigma(A))\}$ is #-norm-#-closed. It is thus an non-Archimedean abelian $\mathbf{C}^*$ algebra of operators.
(3) $\mathbf{Ran}(\phi)$ is actually the non-Archimedean $\mathbf{C}^*$ algebra generated by A that is, the smallest $\mathbf{C}^*$-algebra containing A.
(4) This result, that $C^{\#}(\sigma(A))$ and the non-Archimedean $\mathbf{C}^*$-algebra generated by A are #-isometrically isomorphic
(5) (b) actually follows from (a) and Proposition 4.1.1. Thus (a) and (c) alone determine ϕ uniquely.
Proposition 4.1.1. Suppose that ϕ: $C^{\#}(X) \to \mathcal{L}(H^{\#})$ is an algebraic $*$-homomorphism, X a #-compact metric space. Then
(a) If $f \geq 0$, then $\phi(f) \geq 0$.
(b) $\|\phi(f)\|_{\#} \leq \|f\|_{{}^*\infty}$.
Theorem 4.1.2. (Generalized Weierstrass Approximation Theorem). Let $f \in C^{\#}([a,b],{}^*\mathbb{R}_c^{\#})$. Then there is a hyper infinite sequence of polynomials $p_n(x), n \in {}^*\mathbb{N}$ that #-converges uniformly to $f(x)$ on $[a,b]$.
Definition 4.1.1 (Hyperfinite Bernstein Polynomials). For each $n \in {}^*\mathbb{N}$, the n-th Bernstein Polynomial $B_n^{\#}(x,f)$ of a function $f \in C^{\#}([a,b],{}^*\mathbb{R}_c^{\#})$ is defined as

$$B_n^{\#}(x,f) = Ext\text{-}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^k. \tag{4.1.3}$$

Theorem 4.1.3.(Generalized B.L.T.theorem) Suppose that Z is a normed space, Y is a non-Archimedean Banach space, and $S \subset Z$ is a #-dense linear subspace of Z. If $T : S \to Y$ is a bounded linear transformation (i.e. there exists $C < {}^*\infty$ such that $\|Tz\|_\# \leq C\,\|z\|_\#$ for all $z \in S$), then T has a unique extension to an element of $\mathcal{L}(Z,Y)$.

§ 4.2.The spectral #-measures

Theorem 4.2.1.(Generalized Riesz-Markov theorem) Let X be a locally #-compact non-Archimedean metric space endowed with ${}^*\mathbb{R}_c^\#$-valued metric.Let $C_c^\#(X)$ be the space of #-continuous #-compactly supported ${}^*\mathbb{C}_c^\#$-valued functions on X.
For any positive linear functional Φ on $C_c^\#(X)$, there is a unique #-measure $\mu^\#$ on X such that

$$\forall f \in C_c^\#(X) : \Phi(f) = Ext\text{-}\int_X f(x) d^\# \mu^\#(x).$$

Theorem 4.2.2.(Generalized Riesz lemma) Let Y be a #-closed proper vector subspace of a normed space $(X, \|\cdot\|_\#)$ and let $\alpha \in {}^*\mathbb{R}_c^\#$ be any real number satisfying $0 < \alpha < 1$.Then there exists a vector $u \in X$ of unit #-norm $\|u\|_\# = 1$ such that $\|u - y\|_\# \geq \alpha$ for all $y \in Y$.
We are now introduce the #-measures corresponding to bounded in${}^*\mathbb{R}_c^\#$ self-#-adjoint operators. Let A be an bounded in${}^*\mathbb{R}_c^\#$ self-#-adjoint operator. Let $\psi \in \mathbf{H}^\#$. Then

$$f \mapsto \langle \psi, f(A)\psi \rangle_\# \qquad (4.2.1)$$

is a positive linear functional on $C^\#(\sigma(A))$. Thus, by the generalized Riesz-Markov theorem, there is a unique #-measure $\mu_\psi^\#(\cdot)$ on the #-compact set $\sigma(A)$ with the property

$$\langle \psi, f(A)\psi \rangle_\# = Ext\text{-}\int_{\sigma(A)} f(\lambda) d^\# \mu_\psi^\#. \qquad (4.2.2)$$

Definition 4.2.1. The #-measure $\mu_\psi^\#(\cdot)$ is called the spectral #-measure associated with the vector $\psi \in \mathbf{H}^\#$.
The first and simplest application of the $\mu_\psi^\#(\cdot)$ is to allow us to extend the functional calculus to $B^\#({}^*\mathbb{R}_c^\#)$, the bounded in${}^*\mathbb{R}_c^\#$ #-Borel functions on ${}^*\mathbb{R}_c^\#$. Let $g \in B^\#({}^*\mathbb{R}_c^\#)$. It is natural to define $g(A)$ so that $\langle \psi, g(A)\psi \rangle_\# = Ext\text{-}\int_{\sigma(A)} g(\lambda) d^\# \mu_\psi^\#$. The polarization identity lets us recover $\langle \psi, g(A)\varphi \rangle_\#$ from the proposed $\langle \psi, g(A)\psi \rangle_\#$ and then the Generalized Riesz lemma lets us construct $g(A)$.
Theorem 4.2.1.(spectral theorem-functional calculus form) Let A be a bounded in${}^*\mathbb{R}_c^\#$ self-#-adjoint operator on $\mathbf{H}^\#$. There is a unique map $\widehat{\phi} : B^\#({}^*\mathbb{R}_c^\#) \to \mathcal{L}(\mathbf{H}^\#)$ so that
(a) $\widehat{\phi}$ is an algebraic $*$-homomorphism.
(b) $\widehat{\phi}$ is #-norm #-continuous: $\left\|\widehat{\phi}(f)\right\|_{\mathcal{L}(\mathbf{H}^\#)} \leq \|f\|_{{}^*\infty}$.
(c) Let f be the function $f(x) = x$; then $\widehat{\phi}(f) = A$.
(d) Suppose $f_n(x) \to_\# f(x)$ for each x as $n \to {}^*\infty$and hyper infinite sequence $\|f_n\|_{{}^*\infty}, n \in {}^*\infty$ is bounded in${}^*\mathbb{R}_c^\#$. Then $\widehat{\phi}(f_n) \to_\# \widehat{\phi}(f)$ as $n \to {}^*\infty$strongly.
Moreover $\widehat{\phi}(\cdot)$ has the properties:
(e) If $A\psi = \lambda\psi$, then $\widehat{\phi}(f) = f(\lambda)\psi$.
(f) If $f \geq 0$,then $\widehat{\phi}(f) \geq 0$.

(g) If $BA = AB$ then $\widehat{\phi}(f)B = B\widehat{\phi}(f)$.

Theorem 4.2.1 can be proven directly by extending Theorem 4.1.1.; part (d) requires the dominated #-convergence theorem. Or, Theorem VII.2 can be proven by an easy corollary of Theorem VII.3 below. The proof of Theorem VII.3 uses only the continuous functional calculus, ф extends ф and as before we write ф(П - I(A)t As in the continuous functional calculus, one has f(A)g(A) = g(A)f(A),

Since ^(IR) is the smallest family closed under limits of form (d) containing all of C(iR), we know that any ф(f) is in the Smallest C*-a!gebra containing A which is also strongly closed; such an algebra is called a von Neumann or W*-a!gebra. When we study von Neumann algebras in Chapter XVIII we will see that this follows from (g).

The norm equality of Theorem VILl carries over if we define HZiroo to be the L°°-norm with respect to a suitable notion of “almost everywhere.” Namely, pick an orthonormal basis {фп} and say that a property is true a.e. if it is true a.e. with respect to each дфг Then \\ф(1)\\&(#) = WfWr00 •

In the next section, we will return to the operators Xa(A) where Xa Is a characteristic function; this is the most important set of operators in the

.

§ 4.3. Spectral projections

In the last section, we constructed a functional calculus, f\-^f{A) for any Borel function/and any bounded self-adjoint operator A. The most important functions gained in passing from the continuous functional calculus to the Borel functional calculus are the characteristic functions of sets.

Definition 4.3.1. Let A be a bounded self-#-adjoint operator and Ω a #-Borel set of ${}^*\mathbb{R}_c^\#$. $P_\Omega = \chi_\Omega(A)$ is called a spectral projection of A.

As the definition suggests, P_Ω is an orthogonal projection since $\chi_\Omega = \chi_\Omega^2 = 1$ pointwise. The properties of the family of projections $\{P_\Omega | \Omega \text{ an arbitrary #-Borel set}\}$ is given by the following elementary translation of the functional calculus.

Proposition 4.3.1. The family $\{P_\Omega\}$ of spectral projections of a bounded self-#-adjoint operator A, has the following properties:

(a) Each P_Ω is an orthogonal projection.

(b) $P_\varnothing = 0;\ P_{(-a,a)} = I$ for some $a \in {}^*\mathbb{R}_c^\#$.

(c) If $\Omega = Ext\text{-}\bigcup_{n=1}^{*\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \varnothing$ for all $n \neq m$ then

$$P_\Omega = s\text{-}\#\text{-}\lim_{N \to {}^*\infty}\left(Ext\text{-}\sum_{n=1}^{N}\right). \tag{4.3.1}$$

(d) $P_{\Omega_1}P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Definition 4.3.2. A family of projections obeying (a)-(c) is called a projection-valued #-measure (p.v.#-m.).

We remark that (d) follows from (a) and (c) by abstract considerations.

As one might guess, one can integrate with respect to a p.v.m. If P_Ω is a p.v.m., then $\langle \phi, P_\Omega \phi \rangle_\#$ is an ordinary #-measure for any ϕ. We will use the symbol $d^\#\langle \phi, P_\lambda \phi \rangle_\#$ to mean integration with respect to this #-measure. By generalized Riesz lemma methods, there is a unique operator B with $\langle\phi, B\phi\rangle_{\#} = Ext\text{-}\int f(\lambda)d^{\#}\langle\phi, P_{\lambda}\phi\rangle_{\#}$.

Theorem 4.3.1. If P_{Ω} is a p.v.#-m. and f a bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ #-Borel function on $\mathbf{supp}(P_{\Omega})$, then there is a unique operator B which we denote $Ext\text{-}\int f(\lambda)d^{\#}P_{\lambda}$ so that

$$\langle\phi, B\phi\rangle_{\#} = Ext - \int f(\lambda)d^{\#}\langle\phi, P_{\lambda}\phi\rangle_{\#}.$$

Theorem 4.3.2.(spectral theorem-p.v.#-m. form) There is a one-one correspondence between (bounded) self-#-adjoint operators A and (bounded) projection valued #-measures $\{P_{\Omega}\}$ given by:

$$A \mapsto \{P_{\Omega}\} = \{\chi_{\Omega}(A)\}$$

$$\{P_{\Omega}\} \mapsto A = Ext - \int \lambda d^{\#}P_{\lambda}$$

.

§ 4.4.The #-continuous functional calculus related to unbounded in ${}^{*}\mathbb{R}_{c}^{\#}$ self-#-adjoint operators

In this section we will show how the spectral theorem for bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ self-#-adjoint operators which we developed in § 4.3 can be extended to unbounded in ${}^{*}\mathbb{R}_{c}^{\#}$ self-#-adjoint operators. To indicate what we are aiming for, we first prove the following:

Proposition 4.4.1. Let $\langle M, \mu^{\#}\rangle$ be a #-measure space with $\mu^{\#}$ a hyperfinite #-measure. Suppose that f is a #-measurable, ${}^{*}\mathbb{R}_{c}^{\#}$-valued function on M which is finite or hyperfinite a.e.$\mu^{\#}$. Then the operator $T_f : \varphi \to f\varphi$ on $L_2^{\#}(M, d^{\#}\mu^{\#})$ with domain

$$D(T_f) = \{\varphi | f\varphi \in L_2^{\#}(M, d^{\#}\mu^{\#})\} \qquad (4.4.1)$$

is self-#-adjoint and $\sigma(T_f)$ is the essential range of T_f.

Proof Tf is clearly symmetric. Suppose that ф s D(Tf) and let

fl if №)l^N O otherwise

Then, using the monotone convergence theorem,

IirrVli= lim IIziVrrVII

Thus,/ф € 1}(M, fi), so фе D(Tf) and therefore Tf is self-adjoint. That a(Tf) is the

essential range of/follows as in the bounded case (Problem 17 of Chapter VII). |

With more information about /, one can say something about the domains on which Tf is essentially self-adjoint:

Proposition 4.4.2. Let f and T_f obey the conditions in Proposition 4.4.1. Suppose in addition that $f \in L_p^{\#}(M, d^{\#}\mu^{\#})$ for $2 < p < {}^{*}\infty$. Let D be any #-dense set in $L_q^{\#}(M, d^{\#}\mu^{\#})$ where $q^{-1} + p^{-1} = 1/2$. Then D is a #-core for T_f.

Proof Let us first show that D is a core for Tft By Holder's inequality Iigll2 =S IHiIp IM,and ILfcIU < Iirilp Ыщ so Ucz D(Tf)t Moreover, IfgeD(Tf)9 let gn be that function which is zero where \g(m)\ > n and equal to g otherwise. By the dominated convergence theorem, gn-+g and fgn -*fg in L2. Since each gn is in 13, we conclude that 13 is a core for Tf,

Now let D be dense in 13 and let g e 13. Find gne D with g„-+g in 13, Since Ilgn -g||2 ^Ill Ilp Ilga - gll« and \\Tf(gn - g)\\2 < ||/||p ||gn - g||,, g e D(Tf \ D). Thus Lq a D(Tf f D) so D is a core. |

Unless/eL°°(M, pi) the operator Tf described in Propositions I and 2 will be unbounded.

Thus, we have found a large class of unbounded self-adjoint operators. In fact, we have

found them all.

Theorem 4.4.1. (spectral theorem-multiplication operator form) Let A be a self-adjoint operator on a $^*\infty$-dimensional a non-Archimedean Hilbert space $\mathbf{H}^\#$ with domain $D(A)$. Then there is a #-measure space $\langle M,\mu^\#\rangle$ with $\mu^\#$ a hyperfinite #-measure, a unitary operator $U:\mathbf{H}^\#\to L_2^\#(M,d^\#\mu^\#)$, and a $^*\mathbb{R}_c^\#$-valued function f on M which is finite or hyperfinite $\mu^\#$-a.e. so that

(a) $\psi\in D(A)$ if and only if $f(\cdot)(U\psi)(\cdot)\in L_2^\#(M,d^\#\mu^\#)$.

(b) If $\varphi\in U[D(A)]$, then $(UAU^{-1}\varphi)(m)=f(m)\varphi(m)$.

Proof In the proof of Theorem VIII.3 it was shown that A + i and A — i are one to one and Ran(y4 ± /) = Ж. Since A ± i are closed, (A ± /)“* are closed and therefore bounded (Theorem III.12). By Theorem VIIL2, (A + i)-1 and (A — i)”1 commute. The equality

((A - i)ф, (A + i)~l(A + i)<p) = ((A - i)~x(A - 1)ф$ (A + i)<p)

and the fact that Ran(/4 ± i) = Ж shows that ((A + 0”1)* = (A-i)"1. Thus (A + /)"1 is normal.

We now use the easy extension of the spectral theorem for bounded self-adjoint operators to bounded normal operators. The proof of this extension is outlined in Problems 3,4, and 5 of Chapter VII. We conclude that there is a measure space <M, p} with p a finite measure, a unitary operator U: Ж L2(M, p), and a measurable, bounded, complex-valued function g(m) so that U(A 4- O-1CZ-1Kw) — g(m)q>(m) for all <p € L2(M, dp).

Since Кег(Л + /)"1 is empty, д(т)Ф0 a.e.J/t], so the function f(m) = g(fti\~I — i is finite a.e.[д]. Now, suppose ф e D(A). Then ^= + 0”V

for some <p e Ж and I/ф — gU<p. Since fg is bounded, we conclude that /(иф) e L2(M, dp). Conversely, if/(иф) e L2(M, dp), then there is a (p e Ж so that U(p — (f + 1)11ф. Thus, gIl<p = g(f + i) I/ф = иф, so ф = (Л + 0” V* which shows that ф e £>/4). This proves (a).

To prove (b) notice that if ф e D(A)i then ф — (A 4- /)“ I<p for some ере Ж and Аф — <p — гф. Therefore,

(1/Аф)(т) = (U(p)(m) —1(11ф)(т)

= ” 0(Щ)(т)

=Ат)(иф)(т)

Finally, if Im(/) > 0 on a set of nonzero measure, there is a bounded set B in the upper half plane so that S — {x|/(x) e B} has nonzero measure. If x is the characteristic function of Sy then fx e L2(My dp) and Im(x>fx) > & This contradicts the fact that multiplication by /is self-adjoint (since it is unitarily equivalent to A). Thus/is real-valued.

There is a natural way to define functions of a self-#-adjoint operator by using the above theorem. Given a bounded #-Borel function h on $^*\mathbb{R}_c^\#$ we define

$$h(A)=UT_{h(f)}U^{-1} \tag{4.4.2}$$

where $T_{h(f)}$ is the operator on $L_2^\#(M,d^\#\mu^\#)$ which acts by multiplication by the function

$h(f(m))$. Using this definition the following theorem follows easily from Theorem 4.4.1.
Theorem 4.4.2. (spectral theorem-functional calculus form) Let A be a self-#-adjoint operator on $\mathbf{H}^{\#}$. Then there is a unique map $\widehat{\phi}$ from the bounded #-Borel functions on ${}^{*}\mathbb{R}_{c}^{\#}$ into $\mathcal{L}(\mathbf{H}^{\#})$ so that
(a) $\widehat{\phi}$ is an algebraic $*$-homomorphism.
(b) $\widehat{\phi}$ is #-norm #-continuous, that is, $\left\|\widehat{\phi}(h)\right\|_{\mathcal{L}(\mathbf{H}^{\#})} \leq \|h\|_{{}^{*}\infty}$
(c) Let $h_n(x), n \in {}^{*}\mathbb{N}$ be a hyper infinite sequence of bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ #-Borel functions with $\#\text{-}\lim_{n\to{}^{*}\infty} h_n(x) = x$
for each x and $|h_n(x)| \leq |x|$ for all x and $n \in {}^{*}\mathbb{N}$. Then, for any $\psi \in D(A)$, $\#\text{-}\lim_{n\to{}^{*}\infty} \widehat{\phi}(h_n)\psi = A\psi$.
(d) If $h_n(x) \to_{\#} h(x)$ pointwise and if the hyper infinite sequence $\|h_n\|_{{}^{*}\infty}, n \in {}^{*}\mathbb{N}$ is bounded in ${}^{*}\mathbb{R}_{c}^{\#}$, then $\widehat{\phi}(h_n) \to_{\#} \widehat{\phi}(h)$ strongly.
In addition:
(e) If $A\psi = \lambda\psi$ then $\widehat{\phi}(h) = h(\lambda)\psi$.
(f) If $h \geq 0$, then $\widehat{\phi}(h) \geq 0$.

The functional calculus is very useful. For example, it allows us to define the exponential eitA and prove easily many of its properties as a function of t (see the next section). In the case where A is bounded we do not need the functional calculus to define the exponential since we can define eitA by the power series which converges in norm.

The functional calculus is also used to construct spectral measures and can be used to develop a multiplicity theory similar to that for bounded self-adjoint operators. A vector ф is said to be cyclic for A if {д(А)ф\g е C00(R)) is dense in Ж. If ф is a cyclic vector, then it is possible to represent Ж as L2(R, d^) where fi# is the measure satisfying
f g(x)diit(x)^ty9g(A^)
in such a way that A becomes multiplication by x. In general, Ж decomposes into a direct sum of cyclic subspaces so the measure space, M9 in Theorem VIII.4 can be realized as a union of copies of U. As in the case of bounded operators we can define (Tac(A)9 (Tpp(A)9 (Tsing(A)9 and decompose Ж accordingly.

Finally, the spectral theorem in its projection-valued measure form follows easily from the functional calculus. Let P_Ω be the operator $\chi_\Omega(A)$ where χ_Ω is the characteristic function of the measurable set $\Omega \subset {}^{*}\mathbb{R}_{c}^{\#}$. The family of operators $\{P_\Omega\}$ has the following properties:
(a) Each P_Ω is an orthogonal projection.
(b) $P_\varnothing = 0$; $P_{(-{}^{*}\infty,{}^{*}\infty)} = I$.
(c) If $\Omega = Ext\text{-}\bigcup_{n=1}^{{}^{*}\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \varnothing$ for all $n \neq m$ then

$$P_\Omega = s\text{-}\#\text{-}\lim_{N\to{}^{*}\infty}\left(Ext\text{-}\sum_{n=1}^{N} P_{\Omega_n}\right). \tag{4.4.3}$$

(d) $P_{\Omega_1}P_{\Omega_2} = P_{\Omega_1\cap\Omega_2}$.
l
Definition 4.4.1. Such a family is called a projection-valued #-measure (p.v.#-m.).
Remark 4.4.1. This is a generalization of the notion of bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ projection-valued #-measure introduced in § 4.3.In that we only require $P_{(-{}^{*}\infty,{}^{*}\infty)} = I$ rather than $P_{(-a,a)} = I$ for some $a \in {}^{*}\mathbb{R}_{c}^{\#}$. For $\varphi \in \mathbf{H}^{\#}, \langle\varphi, P_\Omega\varphi\rangle_{\#}$ is a well-defined Borel #-measure on ${}^{*}\mathbb{R}_{c}^{\#}$ which we denote by $d^{\#}\langle\varphi, P_\lambda\varphi\rangle_{\#}$ as in § 4.3.

The complex ${}^*\mathbb{C}_c^\#$-valued #-measure $d^\#\langle\varphi, P_\lambda\psi\rangle_\#$ is defined by polarization. Thus, given a bounded in ${}^*\mathbb{R}_c^\#$ #-Borel function g we can define $g(A)$ by

$$\langle\varphi, g(A)\varphi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi, P_\lambda\varphi\rangle_\# \qquad (4.4.4)$$

It is not difficult to show that this map $g \mapsto g(A)$ has the properties (a)-(d) of Theorem 4.4.1, so $g(A)$ as defined by (4.4.4) coincides with the definition of $g(A)$ given by Theorem 4.4.1. Now, suppose g is an unbounded ${}^*\mathbb{C}_c^\#$-valued #-Borel function and let

$$D_g = \left\{\varphi | Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi, P_\lambda\varphi\rangle_\# < {}^*\infty\right\}. \qquad (4.4.5)$$

Then, D_g is #-dense in $H^\#$ and an operator $g(A)$ is defined on D_g by

$$\langle\varphi, g(A)\varphi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi, P_\lambda\varphi\rangle_\#. \qquad (4.4.6)$$

As in § 4.3, we write symbolically

$$g(A) = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\# P_\lambda. \qquad (4.4.7)$$

In particular, for $\varphi, \psi \in D(A)$,

$$\langle\varphi, A)\psi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda)d^\#\langle\varphi, P_\lambda\varphi\rangle_\#. \qquad (4.4.8)$$

if g is ${}^*\mathbb{R}_c^\#$-valued, then $g(A)$ is self-#-adjoint on D_g. We summarize:

Theorem 4.4.3. (spectral theorem-projection valued measure form) There is a one-to-one correspondence between self-#-adjoint operators A and projection-valued #-measures $\{P_\Omega\}$ on $\mathbf{H}^\#$ the correspondence being given by

$$A = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} \lambda d^\# P_\lambda. \qquad (4.4.9)$$

We use the functional calculus developed above to define $Ext\text{-}\exp(itA)$.

Theorem 4.4.4. Let A be a self-#-adjoint operator and define $U(t) = Ext\text{-}\exp(itA)$. Then

(a) For each $t \in {}^*\mathbb{R}_c^\#, U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s,t \in {}^*\mathbb{R}_c^\#$.

(b) If $\varphi \in \mathbf{H}^\#$ and $t \to_\# t_0$, then $U(t)\varphi \to_\# U(t_0)\varphi$.

(c) For any $\psi \in D(A)$: $\dfrac{U(t)\psi - \psi}{t} \to_\# iA\psi$ as $t \to_\# 0$.

(d) If $\#\text{-}\lim_{t\to_\# 0} \dfrac{U(t)\psi - \psi}{t}$ exists, then $\psi \in D(A)$.

Proof (a) follows immediately from the functional calculus and the corresponding statements for the complex-valued function $Ext\text{-}\exp(it\lambda)$. To prove (b) observe that

$$\|Ext\text{-}\exp(itA)\varphi - \varphi\|_\#^2 = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} |Ext\text{-}\exp(it\lambda) - 1|^2 d^\#\langle P_\lambda\varphi, \varphi\rangle_\#. \qquad (4.4.10)$$

Since $|Ext\text{-}\exp(it\lambda) - 1|^2$ is dominated by the #-integrable function $g(\lambda) = 2$ and since for each $\lambda \in {}^*\mathbb{R}_c^\#$: $|Ext\text{-}\exp(it\lambda) - 1|^2 \to_\# 0$ as $t \to_\# 0$ we conclude that $\|U(t)\varphi - \varphi\|_\#^2 \to_\# 0$ as $t \to_\# 0$, by the generalized Lebesgue dominated-#-convergence theorem. Thus $t \mapsto U(t)$ is strongly #-continuous at $t = 0$, which by the group property proves $t \mapsto U(t)$ is strongly #-continuous everywhere. The proof of (c), which again uses the dominated #-convergence theorem and the estimate $|Ext\text{-}\exp(ix) - 1|^2 \leq |x|$. To prove (d), we define

$$D(B) = \left\{\psi \middle| \#\text{-}\lim_{t\to_\# 0} \frac{U(t)\psi - \psi}{t} \text{ exists}\right\}$$

and let $iB\psi = \#\text{-}\lim_{t\to_\# 0} \frac{U(t)\psi - \psi}{t}$. A simple computation shows that B is symmetric. By (c), $B \supset A$, so $B = A$.

Definition 4.4.2. An operator-valued function $U(t)$ satisfying (a) and (b) is called a strongly #-continuous one-parameter unitary group.

Definition 4.4.3. If $U(t)$ is a strongly #-continuous one-parameter unitary group, then the

self-#-adjoint operator A with $U(t) = Ext\text{-}\exp(itA)$ is called the infinitesimal generator of $U(t)$.

Suppose that $U(t)$ is a weakly continuous one-parameter unitary group. Then

I U(t)<p - ipH2 = '\u(t)(p\\2 - (U(t)cp, ip) - (ip, u(t)(p) + Ii^ii2 ^2MI2 -2MI2 =0

as f-»0. Thus (J(t) is actually strongly continuous. As a matter of fact, to conclude that U(t) is strongly continuous one need only show that U(t) is weakly measurable, that is, that (U{i)(p, ф) is measurable for each <p and ф. This startling result, proven by von Neumann, is sometimes useful since in applications one can often show that (U(t)cpy ф) is the limit of a sequence of continuous functions; (U(t)ip, ф) is therefore measurable and by von Neumann's theorem $U(t)$ is then strongly #-continuous.

Theorem 4.4.5. Let $U(t)$ be a one-parameter group of unitary operators on a hyper

infinite dimensional Hilbert space $\mathbf{H}^{\#}$. Suppose that for all $\varphi, \psi \in \mathbf{H}^{\#}$, $\langle U(t)\psi, \varphi\rangle_{\#}$ is #-measurable. Then $U(t)$ is strongly #-continuous.

Proof Let ф G Ж. Then for all <p e Ж, (£/(f)^, cp) is a bounded measurable function and

is a linear functional on Ж of norm less than or equal to a\\ф\\. Thus, by the Riesz lemma

there is a фa e Ж so that

lim (и(Ь)фa, <p) = (ф„, <p) o

so that U(b) is weakly and therefore strongly continuous on the set of vectors of the form {фa\ф 6 Ж}. It remains only to show that this set is dense, since by by an e/3 argument we can then conclude that t *-> U(t) is strongly continuous on Ж. Suppose that <pe{фa\ф e Жyae U}1 and let {ф{п)} be an orthonormal basis for Ж. Then for each л,

o=(*w (p) = Cт)Ф{п\ <p) dt

jO

for ail a which implies that (C/(0*A(n>> <p) = 0 except for t e Sn, a set of measure zero. Choose t0 ф U^Li Then (и(10)ф(*\ cp) — 0 for all n which implies that q> ~0, since £/(f0) is unitary. |

The proof of essentia] self-adjointness in Theorem VIII.8 directly implies the following self-adjointness criterion.

Theorem 4.4.6.Suppose that $U(t)$ is a strongly continuous one-parameter unitary group. Let D be a #-dense domain which is invariant under $U(t)$ and on which $U(t)$ is strongly #-differentiable. Then i^{-1} times the strong #-derivative of $U(t)$ is essentially self-#-adjoint on D and its #-closure is the #-infinitesimal generator of $U(t)$.

This theorem has a reformulation which is sufficiently important that we state it as a theorem.

Theorem 4.4.7. Let A be a self-adjoint operator on $\mathbf{H}^{\#}$ and D be a #-dense linear set contained in $D(A)$. If for all t, $Ext\text{-}\exp(itA) : D \to D$ then D is a #-core for A.

Theorem 4.4.8.Let $U(t)$ be a strongly #-continuous one-parameter unitary group on a Hilbert space $\mathbf{H}^{\#}$. Then, there is a self-#-adjoint operator A on $\mathbf{H}^{\#}$ so that $U(t) = Ext\text{-}\exp(itA)$.

Proof Part (d) of Theorem VI11.7 suggests that we obtain A by differentiating U(t) at t = 0. We will show that this can be done on a dense set of especially nice vectors and then show that the limiting operator is essentially self-adjoint by using the basic criterion. Finally, we show that the exponential of this limiting operator is just U(t).

Let/e Cq(U) and for each <p e Ж define

* f(t)U(t)q>dt

- OO

Since U(t) is strongly continuous the integral can be taken to be a Riemann integral. Let D be the set of finite linear combinations of all such <pf for <P e Ж and /e C“(R). If jt(x) is the approximate identity introduced in Section VIII,I, then

Remark 4.4.2.Finally, we have the following generalization of Stone’s theorem 4.4.8. If g is a real-valued BoreI function on ${}^{*}\mathbb{R}_c^{\#}$, then

$$g(A) = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#}} g(\lambda)d^{\#}P_{\lambda}$$

defined on D_g (4.4.5) is self-#-adjoint. If g is bounded, $g(A)$ coincides with $\widehat{\phi}(g)$ in Theorem 4.4.2.

We conclude with several remarks. First, Stone’s formula, given in Theorem VII. 13, relates the resolvent and the projection-valued measure associated with any self-adjoint operator. The proof is the same as in the bounded case.

The spectrum of an unbounded self-adjoint operator is an unbounded subset of the real axis. One can define discrete and essential spectrum; they are still characterized by Theorems VII.9, VII. 10, and VITI I. Theorem VII.12 (Weyl’s criterion) still holds if one adds the criterion that the vectors {фп} must be in the domain of A.

Finally, we note that the measure space of Theorem VIIT4 can always be chosen so that Proposition 2 is applicable:

The following theorem says that every strongly continuous unitary group arises as the exponential of a self-#-adjoint operator.

Theorem 4.4.8. Let $U(\mathbf{t}) = U(t_1,\ldots,t_n)$ be a strongly continuous map of ${}^{*}\mathbb{R}_c^{\#n}$ into the unitary operators on a hyper infinite dimensional Hilbert space $\mathbf{H}^{\#}$ satisfying $U(\mathbf{t}+\mathbf{s}) = U(\mathbf{t})U(\mathbf{s})$ Let D be the set of hyperfinite linear combinations of vectors of the form

$$\varphi_f = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#n}} f(\mathbf{t})U(\mathbf{t})d^{\#n}t$$

where $\varphi \in \mathbf{H}^{\#}, f \in C_0^{\#^{*}\infty}({}^{*}\mathbb{R}_c^{\#n})$.Then D is a domain of essential self-#-adjointness for each of the generators A_j of the one-parameter subgroups $U(0,0,\ldots,t_j,\ldots,0)$, each $A_j : D \to D$ and the A_j commute, $j = 1,\ldots,n$. Furthermore, there is a projection-valued #-measure P_{Ω} on ${}^{*}\mathbb{R}_c^{\#n}$ so that

$$\langle\varphi, U(\mathbf{t})\psi\rangle_{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} Ext\text{-}\exp(it\lambda)d^{\#}\langle\varphi, P_{\lambda}\psi\rangle_{\#}$$

for all $\varphi, \psi \in \mathbf{H}^{\#}$.

Proof Let Aj be the infinitesimal generator of Uj(Ii) = U(0,..., Ij,..,, 0). The procedure used in the proof of Theorem VIII.8 shows that D c D(Aj)9 Aj: D-+D, and Ujitj): D-+ D. Theorem VIILll shows that Aj is essentially self-adjoint on D. Because of the relation U(t + s)= U(t)U(s), UjUj) commutes with Ui(Ii) for all Ii, Xj eU. Therefore, it follows from Theorem V11I.13, that Ai and Aj commute in the sense defined in the next section; that is, their spectral projections commute.

Let Plk be the projection-valued measure on R corresponding to Aj. Define a projection valued measure Psi on R" by defining it first on rectangles r = (a„b,)x ■ ■ x (a„, bn) by Pr = Pla,,bl)PL.b2) and then let-

ting Psi be the unique extension to the smallest <r-algebra containing the rectangles, namely the Borel sets. Notice that, by Theorem VIII. 13, the Plkj commute since the groups Uj commute. For each <p9 ф e Ж, (<p, Рцф) is a complex-valued measure of finite mass which we denote by d(q>9 Pхф). Applying Fubini's theorem we easily conclude that

f e'*>x d(<p, Px ф) = (<p, UlHx)..., и„П„)Ф)

= (<p, U(t)^) I

§ 4.3.

Suppose that A and B are two unbounded self-#-adjoint operators on a non-Archimedean Hilbert space $H^{\#}$. We would like to find a reasonable meaning for the statement: "A and B commute."

This cannot be done in the straightforward way since $AB - BA$ may not make sense on any vector $\psi \in H^{\#}$ for example, one might have $(\mathbf{Ran}(A)) \cap D(B) = \varnothing$ in which case BA does not have a meaning. This suggests that we find an equivalent formulation of commutativity for bounded self-#-adjoint operators. The spectral theorem for bounded self-#-adjoint operators A and B shows that in that case $AB - BA = 0$ if and only if all their projections, $\{P_{\Omega}^{A}\}$ and $\{P_{\Omega}^{B}\}$, commute, We take this as our definition in the unbounded case.

Definition 4.3.1. Two (possibly unbounded in ${}^*\mathbb{R}_c^{\#}$) self-#-adjoint operators A and B are said to commute if and only if all the projections in their associated projection-valued #-measures commute.

Remark 4.3.1. The spectral theorem shows that if A and B commute, then all the bounded in ${}^*\mathbb{R}_c^{\#}$ #-Borel functions of A and B also commute. In particular, the resolvents $R_{\lambda}(A)$ and $R_{\mu}(B)$ commute and the unitary groups $Ext\text{-}\exp(itA)$ and $Ext\text{-}\exp(isA)$ commute.

The converse statement is also true and this shows that the above definition of "commute" is reasonable:

Theorem 4.3.1. Let A and B be self-#-adjoint operators on a non-Archimedean

Hilbert spaceHilbert space $H^{\#}$.
Then the following three statements are equivalent:

(a) Spectral projections $P^A_{(a,b)}$ and $P^B_{(c,d)}$, commute.

(b) If $\operatorname{Im}\lambda$ and $\operatorname{Im}\mu$ are nonzero, then $R_\lambda(A)R_\mu(B) - R_\mu(B)R_\lambda(A) = 0$.

(c) For all $s,t \in {}^*\mathbb{R}^\#_c$,$[Ext\text{-}\exp(itA)][Ext\text{-}\exp(isB)] = [Ext\text{-}\exp(isB)][Ext\text{-}\exp(itA)]$.

Proof The fact that (a) implies (b) and (c) follows from the functional calculus. The fact that (b) implies (a) easily follows from the formula which expresses the spectral projections of A and B as strong #-limits of the resolvents (generalized Stone's formula) together with the fact that

$$s\text{-}\#\text{-}\lim_{\varepsilon\to_\#0}[i\varepsilon R_{a+i\varepsilon}(A)] = P^A_{\{a\}}. \qquad (4.2.1)$$

To prove that (c) implies (a), we use some simple facts about the Fourier transform. Let $f \in S^\#({}^*\mathbb{R}^\#_c)$. Then, by generalized Fubini's theorem,

$$\begin{aligned} &Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} f(t)\langle[Ext\text{-}\exp(itA)]\varphi,\psi\rangle_\# d^\#t = \\ &= Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} f(t)\Big(Ext\text{-}\int_{{}^*\mathbb{R}^\#_c}\big([Ext\text{-}\exp(-it\lambda)]d^\#_\lambda\langle P^A_\lambda\varphi,\psi\rangle_\#\big)\Big)d^\#t = \\ &= \sqrt{2\pi_\#}\left(Ext\text{-}\int_{{}^*\mathbb{R}^\#_c}\widehat{f}(\lambda)d^\#_\lambda\langle P^A_\lambda\varphi,\psi\rangle_\#\right) = \sqrt{2\pi_\#}\left\langle\varphi,\widehat{f}(A)\psi\right\rangle_\#. \end{aligned} \qquad (4.2.2)$$

Thus, using (c) and generalized Fubini's theorem again,

$$\begin{aligned} &\left\langle\varphi,\widehat{f}(A)\widehat{g}(B)\psi\right\rangle_\# = \\ &Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} f(t)g(s)\langle\varphi,[Ext\text{-}\exp(-itA)][Ext\text{-}\exp(-isB)]\psi\rangle_\# d^\#s d^\#t = \\ &= \left\langle\varphi,\widehat{g}(B)\widehat{f}(A)\psi\right\rangle_\# \end{aligned} \qquad (4.2.3)$$

so, for all $f,g \in S^\#({}^*\mathbb{R}^\#_c)$, $\widehat{f}(A)\widehat{g}(B) - \widehat{g}(B)\widehat{f}(A) = 0$.
Since the Fourier transform maps $S^\#({}^*\mathbb{R}^\#_c)$ onto $S^\#({}^*\mathbb{R}^\#_c)$ we conclude that $f(A)g(B) = g(B)f(A)$ for all $f,g \in S^\#({}^*\mathbb{R}^\#_c)$. But, the characteristic function, $\chi_{(a,b)}$ can be expressed as the pointwise #-limit of a hyperinfinite sequence $f_n, n \in {}^*\mathbb{N}$ of uniformly bounded functions in $S^\#({}^*\mathbb{R}^\#_c)$. By the functional calculus,

$$s\text{-}\#\text{-}\lim_{n\to{}^*\infty} f_n(A) = P^A_{(a,b)}.$$

Similarly,we find uniformly bounded $g_n \in S^\#({}^*\mathbb{R}^\#_c)$ #-converging pointwise to $\chi_{(c,d)}$ and

$$s\text{-}\#\text{-}\lim_{n\to{}^*\infty} g_n(B) = P^B_{(c,d)}.$$

Since the f_n and g$_n$ are uniformly bounded in${}^*\mathbb{R}^\#_c$ and

$$f_n(A)g_n(B) = g_n(B)f_n(A)$$

for each $n \in {}^*\mathbb{N}$, we conclude that $P^A_{(a,b)}$ and $P^B_{(c,d)}$, commute which proves (a).

.

Chapter IV.Non-Archimedean Banach spaces endroved

with${}^{*}\mathbb{R}_c^{\#}$-valued norm.

1.Definitions and examples

A non-Archimedean normed space with${}^{*}\mathbb{R}_c^{\#}$-valued norm (#-norm) is a pair $(X, \|\cdot\|_{\#})$ consisting of a vector space X over a non-Archimedean scalar field ${}^{*}\mathbb{R}_c^{\#}$or complex field ${}^{*}\mathbb{C}_c^{\#}$ together with a distinguished norm $\|\cdot\|_{\#} : X \to {}^{*}\mathbb{R}_c^{\#}$. Like any norms, this #-norm induces a translation invariant distance function, called the canonical or (norm) induced non-Archimedean ${}^{*}\mathbb{R}_c^{\#}$-valued metric for all vectors $x, y \in X$, defined by

$$d^{\#}(x,y) = \|x-y\|_{\#} = \|y-x\|_{\#}. \qquad (1.1)$$

Thus (1.1) makes X into a metric space $(X, d^{\#})$. A hyper infinite sequence $(x_n)_{n=1}^{\infty^{\#}}$ is called $d^{\#}$-Cauchy or Cauchy in $(X, d^{\#})$ or $\|\cdot\|_{\#}$ -Cauchy if for every hyperreal $r \in {}^{*}\mathbb{R}_c^{\#}$, $r > 0$, there exists some $N \in \mathbb{N}^{\#}$ such that

$$d^{\#}(x_n, x_m) = \|x_n - x_m\|_{\#} < r, \qquad (1.2)$$

where m and n are greater than N. The canonical metric $d^{\#}$ is called a #-complete metric if the pair $(X, d^{\#})$ is a #-complete metric space, which by definition means for every $d^{\#}$-Cauchy sequence $(x_n)_{n=1}^{\infty^{\#}}$ in $(X, d^{\#})$, there exists some $x \in X$ such that

$$\#\text{-}\lim_{n\to\infty^{\#}} \|x_n - x\|_{\#} = 0 \qquad (1.3)$$

where because $\|x_n - x\|_{\#} = d^{\#}(x_n, x)$, this hyper infinite sequence's #-convergence to x can equivalently be expressed as: #-$\lim_{n\to\infty^{\#}} x_n = x$ in $(X, d^{\#})$.

Definition 1.1. The normed space $(X, \|\cdot\|_{\#})$ is a non-Archimedean Banach space endroved with${}^{*}\mathbb{R}_c^{\#}$-valued norm if the #-norm induced metric $d^{\#}$ is a #-complete metric, or said differently, if $(X, d^{\#})$ is a #-complete metric space. The #-norm $\|\cdot\|_{\#}$ of a #-normed space $(X, \|\cdot\|_{\#})$ is called a #-complete #-norm if $(X, \|\cdot\|_{\#})$ is a non-Archimedean Banach space endroved with${}^{*}\mathbb{R}_c^{\#}$-valued #-norm.

Remark 1.1.For any #-normed space $(X, \|\cdot\|_{\#})$, there exists an L-semi-inner product $\langle\cdot,\cdot\rangle_{\#} : X \times X \to {}^{*}\mathbb{R}_c^{\#}$ such that $\|x\|_{\#} = \sqrt{\langle x, x\rangle_{\#}}$ for all $x \in X$; in general, there may be infinitely many L-semi-inner products that satisfy this condition. L-semi-inner products are a generalization of inner products, which are what fundamentally distinguish non-Archimedean Hilbert spaces from all other non-Archimedean Banach spaces. Characterization in terms of hyper infinite series,see ref. [1].

The vector space structure allows one to relate the behavior of hyper infinite Cauchy sequences to that of #-converging hyper infinite series of vectors.

Remark 1.2.A #-normed space X is a non-Archimedean Banach space if and only if each absolutely #-convergent hyper infinite series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} v_n$ in X #-converges in X,i.e., $Ext\text{-}\sum_{n=1}^{\infty^{\#}} \|v_n\| < \infty^{\#}$ implies that $Ext\text{-}\sum_{n=1}^{\infty^{\#}} v_n$ #-converges in X.

2.Linear operators,isomorphisms

If X and Y are #-normed spaces over the same ground field ${}^{*}\mathbb{R}_c^{\#}$, the set of all #-continuous ${}^{*}\mathbb{R}_c^{\#}$-linear maps $T : X \to Y$ is denoted by $B^{\#}(X, Y)$. In hyper infinite-dimensional spaces, not all linear maps are #-continuous. A linear mapping from a #-normed space X to another normed space is #-continuous if and only if it is bounded or hyper bounded on the #-closed unit ball of X. Thus, the vector space

$B^{\#}(X,Y)$ can be endroved with the operator norm

$$\|T\| = \sup\{\|Tx\|_{\#Y} \mid x \in X, \|x\|_{\#X} \leq 1\}. \qquad (2.1)$$

For Y a non-Archimedean Banach space, the space $B^{\#}(X,Y)$ is a Banach space with respect to this #-norm.

If X is a non-Archimedean Banach space, the space $B^{\#}(X) = B^{\#}(X,X)$ forms a unital Banach algebra; the multiplication operation is given by the composition of linear maps.

Definition 2.1.If X and Y are #-normed spaces, they are #-isomorphic #-normed spaces

if there exists a linear bijection $T : X \to Y$ such that T and its inverse T^{-1} are #-continuous. If one of the two spaces X or Y is #-complete then so is the other space. Two #-normed spaces X and Y are #-isometrically isomorphic if in addition, T is an #-isometry, that is, $\|T(x)\| = \|x\|$ for every $x \in X$.

Definition 2.2.Let $\{X, \|\cdot\|\}$ be standard Banach space.For $x \in {}^*X$ and $\varepsilon > 0, \varepsilon \approx 0$ we define the open $\approx$-ball about x of radius ε to be the set

$B_\varepsilon(x) = \{y \in {}^*X | {}^*\|x-y\| < \varepsilon\}$.

Definition 2.3.Let $\{X, \|\cdot\|\}$ be standard Banach space, $Y \subset X$ thus ${}^*Y \subseteq {}^*X$ and let $x \in {}^*X$.Then x is an $*$-accumulation point of ${}^*X$ if for every $\varepsilon > 0, \varepsilon \approx 0, Y \cap (B_\varepsilon(x) \backslash \{x\}) \neq \varnothing$.

Definition 2.4.Let $\{X, \|\cdot\|\}$ be a standard Banach space, let $Y \subseteq {}^*X, Y$ is $*$-closed if every $*$-accumulation point of Y is an element of Y.

Definition 2.5.Let $\{X, \|\cdot\|\}$ be standard Banach space.We shall say that internal hyper infinite sequence $\{x_n\}_{n=1}^{n={}^*\infty}$ in ${}^*X$ $*$-converges to $x \in {}^*X$ as $n \to {}^*\infty$if for any $\varepsilon > 0, \varepsilon \approx 0$ there is $N \in {}^*\mathbb{N}$ such that for any $n > N : {}^*\|x_n - x\| < \varepsilon$.

Definition 2.6.Let $\{X, \|\cdot\|\}, \{Y, \|\cdot\|\}$ be a standard Banach spaces. A linear internal operator $A : D(A) \subseteq {}^*X \to {}^*Y$is $*$-closed if for every internal hyper infinite sequence $\{x_n\}_{n=1}^{n={}^*\infty}$ in $D(A)$ $*$-converging to $x \in {}^*X$ such that $Ax_n \to y \in {}^*Y$ as $n \to {}^*\infty$ one has $x \in D(A)$ and $Ax = y$. Equivalently, A is $*$-closed if its graph is $*$-closed

in the direct sum ${}^*X \oplus {}^*Y$.

Given a linear operator $A : {}^*X \to {}^*Y$, not necessarily $*$-closed, if the $*$-closure of its graph in ${}^*X \oplus {}^*Y$ happens to be the graph of some operator, that operator is called the $*$-closure of A, and we say that A is $*$-closable. Denote the $*$-closure of A by $*$-$\bar{A}$. It follows that A is the restriction of $*$-$\bar{A}$ to $D(A)$.

A $*$-core (or $*$-essential domain) of a $*$-$\bar{A}$closable operator is a subset $C \subset D(A)$ such that the $*$-closure of the restriction of A to C is $*$-$\bar{A}$.

Definition 2.7. The graph of the linear transformation $T : H \to H$ is the set of pairs $\{\langle\varphi, T\varphi\rangle | (\varphi \in D(T))\}$.

The graph of T, denoted by $\Gamma(T)$, is thus a subset of $H \times H$ which is a non-Archimedean

Hilbert space with inner product $(\langle\varphi_1,\psi_1\rangle, \langle\varphi_2,\psi_2\rangle)$.

T is called a #-closed operator if $\Gamma(T)$ is a #-closed subset of $H \times H$.

Definition 2.8. Let T_1 and T be operators on H. If $\Gamma(T_1) \supset \Gamma(T)$, then T_1 is said to be an

extension of T and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$

and $T_1\varphi = T\varphi$ for all $\varphi \in D(T)$.

Definition 2.9. An operator T is #-closable if it has a #-closed extension. Every #-closable

operator has a smallest #-closed extension, called its #-closure, which we denote by #-$\overline{T}$.

Theorem 2.1.If T is #-closable, then $\Gamma(\#\text{-}\overline{T}) = \#\text{-}\overline{\Gamma(T)}$.

Definition 2.10.Let T be a #-densely defined linear operator on a non-Archimedean Hilbert space H. Let $D(T^*)$ be the set of $\varphi \in H$ for which there is an $\xi \in H$ with $(T\psi,\varphi) = (\psi,\xi)$ for all $\psi \in D(T)$.

For each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$. T^* is called the #-adjoint of T. Note that $\varphi \in D(T^*)$ if and only if $|(T\psi,\varphi)| \leq C\|\psi\|$ for all $\psi \in D(T)$. We note that $S \subset T$ implies $T^* \subset S^*$.

Theorem 2.2. Let T be a #-densely defined operator on a non-Archimedean Hilbert space H.

Then:(i) T^* is #-closed.

(ii) T is #-closabie if and only if $D(T^*)$ is #-dense in which case $T = T^{**}$.

(iii) If T is #-closabie, then $(\# - \overline{T})^* = T^*$.

Definition 2.11. Let T be a #-closed operator on a Hilbert space H. A complex number $\lambda \in {}^*\mathbb{C}_c^\#$ is in the resolvent set,$\rho(T)$,if $\lambda I - T$ is a bijection of $D(T)$ onto H with a a finitely or hyper finitely bounded inverse. If $\lambda \in \rho(T)$, $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.

The definitions of spectrum, point spectrum, and residual spectrum are the same for unbounded operators as they are for bounded operators. We will sometimes refer to the spectrum of nonclosed, but closabie operators. In this case we always mean the spectrum of the closure.

3. Symmetric and self-#-adjoint operators: the basic criterion for self-#-adjointness.

Definition 3.1. A #-densely defined operator T on a non-Archimedean Hilbert space is called symmetric (or Hermitian) if $T \subset T^*$, that is, if $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$.

Equivalently, T is symmetric if and only if $(T\varphi,\psi) = (\varphi,T\psi)$ for all $\varphi,\psi \in D(T)$

Definition 3.2. T is called self-adjoint if $T = T^*$, that is, if and only if T is symmetric and $D(T) = D(T^*)$.

A symmetric operator is always #-closable, since $D(T^*) \supset D(T)$ is #-dense in H. If T is symmetric, T^* is a closed extension of T so the smallest #-closed extension T^{**} of T must be contained in T^*. Thus for symmetric operators, we have $T \subset T^{**} \subset T^*$.For #-closed symmetric operators,$T = T^{**} \subset T^*$ and, for self-adjoint operators,$T = T^{**} = T^*$

From this one can easily see that a #-closed symmetric operator T is self-adjoint if and only if T^* is symmetric.

The distinction between #-closed symmetric operators and self-adjoint operators is very

important. It is only for self-adjoint operators that the spectral theorem holds and it is only self-adjoint operators that may be #-exponentiated to give the one-parameter unitary groups which give the dynamics in

QFT. Chapter X is mainly devoted to studying methods for proving that operators are self-adjoint. We content ourselves here with proving the basic criterion for selfadjointness.

First, we introduce the useful notion of essential self-adjointness.

Definition 3.3 A symmetric operator T is called essentially self- #-adjoint if its #-closure #-$\overline{T}$ is self- #-adjoint. If T is #-closed, a subset $D \subset D(T)$ is called a core for T if

$\overline{\# - T \upharpoonright D} = T.$

If T is essentially self-#-adjoint, then it has one and only one self-#-adjoint extension. The importance of essential self-#-adjointness is that one is often given a nonclosed symmetric operator T. If T can be shown to be essentially self-#-adjoint, then there is uniquely associated to Ta self-adjoint operator $T = T^{**}$. Another way of saying this is that if A is a self-#-adjoint operator, then to specify A uniquely one need not give the exact domain of A (which is often difficult), but just some #-core for A

Chapter V. Semigroups of operators on a non-Archimedean Banach spaces.

§1.Semigroups on non-Archimedean Banach spaces and their generators.

A family of #-bounded operators $\{T(t)|0 < t < \infty^{\#}\}$ on external hyper infinite dimensional non-Archimedean Banach space X endoved with ${}^{*}\mathbb{R}^{\#}_{c,+}$ - valued norm $\|\cdot\|_{\#}$ is called a strongly #-continuous semigroup if:

(a) $T(0) = I$

(b) $T(s)T(t) = T(s+t)$ for all $s,t \in {}^{*}\mathbb{R}^{\#}_{c,+}$

(c) For each $\varphi \in X, t \mapsto T(t)$ is #-continuous mapping.

We will see that strongly continuous semigroups are the “exponentials,” $T(t) = Ext\text{-}\exp(-tA)$, of a certain class of operators. .

We begin by studying a special class of semigroups:

Definition 1.1. A family $\{T(t)|0 < t < \infty^{\#}\}$ of bounded or hyper bounded operators on external hyper infinite dimensional Banach space X is called a contraction semigroup if it is a strongly #-continuous semigroup and moreover $\|T(t)\|_{\#} < 1$ for all $t \in [0,\infty^{\#})$. Note that the all theorems about general strongly #-continuous semigroups are easy generalizations of the corresponding theorems for #-contraction semigroups. Thus, we study the special case first. We then briefly discuss the general theory and conclude the section by studying another special class, #-holomorphic semigroups.

Proposition 1.1. Let $T(t)$ be a strongly #-continuous semigroup on a non-Archimedean Banach space X and set $A\varphi = \#\text{-}\lim_{r\to_{\#}0} A_r\varphi$ where $D(A) = \{\varphi|\ \#\text{-}\lim_{r\to_{\#}0} A_r\varphi \text{ exists}\}$. Then A is #-closed and #-densely defined. A is called the infinitesimal generator of $T(t)$. We will also say that A generates $T(t)$ and write $T(t) = Ext\text{-}\exp(-tA)$.

Proof.Let $T(t)$ be a contraction semigroup on a Banach space X. We obtain the generator of $T(t)$ by #-differentiation. Set $A_t = t^{-1}(I - T(t))$ and define

$$D(A) = \{\varphi|\ \#\text{-}\lim_{t\to_{\#}0} A_t\varphi \text{ exists}\}.$$

For $\varphi \in D(A)$, we define $A\varphi = \#\text{-}\lim_{t\to_{\#}0} A_t\varphi$. Our first goal is to show that $D(A)$ is

#-dense. For $\varphi \in X$, we set

$$\varphi_s = Ext\text{-}\int_0^s T(t)\varphi d^\# t. \tag{2.1}$$

For any $r > 0$, we get

$$T(r)\varphi_s = Ext\text{-}\int_0^s T(t+r)\varphi d^\# t \tag{2.2}$$

thus

$$\begin{gathered} A_r\varphi_s = -\frac{1}{r}\left(Ext\text{-}\int_0^s [T(t+r)\varphi - T(t)\varphi] d^\# t\right) = \\ -\frac{1}{r}\left(Ext\text{-}\int_s^{r+s} T(t)\varphi d^\# t\right) + \frac{1}{r}\left(Ext\text{-}\int_s^r T(t)\varphi d^\# t\right). \end{gathered} \tag{2.3}$$

From Eq.(2.3) one obtains $\#\text{-}\lim_{r\to_\# 0} A_r\varphi_s = -T(s)\varphi + \varphi$. Therefore, for each $\varphi \in X$

and $s > 0$, $\varphi_s \in D(A)$. Since $s^{-1}\varphi_s \to_\# \varphi$ as $\to_\# 0$, A is #-densely defined. Furthermore, if $\varphi \in D(A)$, then $A_r T(t)\varphi = T(t)A_r\varphi$, so $T(t) : D(A) \to D(A)$ and

$$\frac{d^\#}{d^\# t} T(t)\varphi = -AT(t)\varphi = -T(t)A\varphi \tag{2.4}$$

A is also #-closed, for if $\varphi_n \in D(A)$, $\#\text{-}\lim_{n\to\infty^\#}\varphi_n = \varphi$, and $\#\text{-}\lim_{n\to\infty^\#} A\varphi_n = \psi$, then

$$\begin{gathered} \#\text{-}\lim_{r\to_\# 0} A_r\varphi = \#\text{-}\lim_{r\to_\# 0}\#\text{-}\lim_{n\to\infty^\#}\left[-\frac{1}{r}(T(r)\varphi_n - \varphi_n)\right] = \\ \#\text{-}\lim_{r\to_\# 0}\#\text{-}\lim_{n\to\infty^\#}\frac{1}{r}\left(Ext\text{-}\int_s^r T(t)A\varphi_n d^\# t\right) = \\ \#\text{-}\lim_{r\to_\# 0}\frac{1}{r}\left(Ext\text{-}\int_s^r T(t)\psi d^\# t\right) \end{gathered} \tag{2.5}$$

so $\varphi \in D(A)$ and $A\varphi = \psi$.

The formal Laplace transform

$$\frac{1}{\lambda + A} = -\left(Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))d^\# t\right) \tag{2.6}$$

suggests that all $\mu \in {}^*\mathbb{C}_c^\#$ with $\operatorname{Re}\mu < 0$ are in $\rho(A)$. This is in fact true and the formula (2.6) holds in the strong sense. For suppose that $\operatorname{Re}\lambda > 0$. Then, since $\|Ext\text{-}\exp(-tA)\| < 1$, the formula (2.7)

$$R\varphi = Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA)\varphi)d^\# t \tag{2.7}$$

defines a hyper bounded linear operator of #-norm less than or equal to $(\operatorname{Re}\lambda)^{-1}$.

Moreover, for $r > 0$,

$$A_r R\varphi = -\frac{1}{r}\left(Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-(t+r)A) - Ext\text{-}\exp(-tA))\varphi d^\# t \right) =$$

$$\frac{1 - Ext\text{-}\exp(\lambda r)}{r}\left(Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^\# t \right) + \tag{2.8}$$

$$\frac{Ext\text{-}\exp(\lambda r)}{r}\left(Ext\text{-}\int_0^{r} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^\# t \right)$$

so as $r \to_\# 0, A_r R\varphi \to_\# (\varphi - \lambda R\varphi)$. Thus $R\varphi \in D(A)$ and $AR\varphi = \varphi - \lambda R\varphi$ which implies $(\lambda + A)R\varphi = \varphi$. In addition, for $\varphi \in D(A)$ we have $AR\varphi = RA\varphi$ since

$$A\left(Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi d^\# t \right) =$$

$$Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))A(Ext\text{-}\exp(-tA))\varphi d^\# t = \tag{2.9}$$

$$Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))A\varphi d^\# t.$$

The first equality follows by approximation with external hyperfinite Riemann sums (see [1]) from the facts that $(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))\varphi$ and $A(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))$ are #-integrable, A is #-closed. Thus, for $\varphi \in D(A)$, $R(\lambda + A)\varphi = \varphi = (\lambda + A)R\varphi$ which implies that

$$R = (\lambda + A)^{-1}. \tag{2.10}$$

The properties of A which we have derived are also sufficient to guarantee that A generates a contraction semigroup. In fact, we only need information about real positive A.

Theorem 1.1. (Generalized Hille-Yosida theorem) A necessary and sufficient condition that a #-closed
linear operator A on a Banach space X generate a contraction semigroup is that
(i) $(-\infty^\#, 0) \subset \rho(A)$
(ii) $\|(\lambda + A)^{-1}\|_\#$ for all $\lambda > 0$.
Furthermore, if A satisfies (i) and (ii), then the entire #-open left half-plane is contained in $\rho(A)$ and

$$(\lambda + A)^{-1}\varphi = -Ext\text{-}\int_0^{\infty^\#} (Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(-tA))d^\# t \tag{2.11}$$

for all $\varphi \in X$ and λ with $\operatorname{Re}\lambda > 0$. Finally, if $T_1(t)$ and $T_2(t)$ are contraction semigroups generated by A_1 and A_2 respectively, then $T_2(t) \neq T_1(t)$ for some t implies that $A_1 \neq A_2$.

Proof. Since we showed above that conditions (i) and (ii) are necessary and that (2.11) holds, we need only show sufficiency. So, suppose that A is a #-closed operator on X satisfying (i) and (ii). For $\lambda > 0$, define $A^{(\lambda)} = \lambda - \lambda^2(\lambda + A)^{-1}$. We will show that as $\lambda \to \infty^{\#}$, $A^{(\lambda)} \to_{\#} A$ strongly on $D(A)$ and then construct $Ext\text{-}\exp(-tA)$ as the strong #-limit of the semigroups $Ext\text{-}\exp(-tA^{(\lambda)})$. For $\varphi \in D(A)$, $A^{(\lambda)}\varphi = \lambda(\lambda + A)^{-1}A\varphi$. Moreover, by (ii),

$$\#\text{-}\lim_{\lambda\to\infty^{\#}}[\lambda(\lambda + A)^{-1}\varphi - \varphi] = \#\text{-}\lim_{\lambda\to\infty^{\#}}[-(\lambda + A)^{-1}A\varphi] = 0. \quad (2.12)$$

By condition (ii) the family $\{\lambda(\lambda + A)^{-1} | \lambda > 0\}$ is #-uniformly hyperfinitely bounded in #-norm, so since $D(A)$ is #-dense, $\#\text{-}\lim_{\lambda\to\infty^{\#}}[\lambda(\lambda + A)^{-1}\psi] = \psi$ for all $\psi \in X$. Thus $\#\text{-}\lim_{\lambda\to\infty^{\#}} A^{(\lambda)}\varphi = A\varphi$ for all $\varphi \in D(A)$. Since A is hyperfinitely bounded, the semigroups $Ext\text{-}\exp(-tA^{(\lambda)})$ can be defined by hyper infinite power series. Since

$$\begin{gathered}\|Ext\text{-}\exp(-tA^{(\lambda)})\|_{\#} = \|(Ext\text{-}\exp(-\lambda t))(Ext\text{-}\exp(t\lambda^2(\lambda + A)^{-1}))\|_{\#} \leq \\ \leq (Ext\text{-}\exp(-\lambda t))\left(Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{t^n\lambda^{2n}}{n!}\|(\lambda + A)^{-1}\|_{\#}^{n}\right) \leq 1\end{gathered} \quad (2.13)$$

they are contraction semigroups. For all $\mu, \lambda, t > 0$, and all $\varphi \in D(A)$, we have

$$\begin{gathered}[Ext\text{-}\exp(-tA^{(\lambda)})]\varphi - [Ext\text{-}\exp(-tA^{(\mu)})]\varphi = \\ Ext\text{-}\int_0^t \frac{d^{\#}}{d^{\#}s}(Ext\text{-}\exp(-sA^{(\lambda)}))((Ext\text{-}\exp(-(t-s)A^{(\lambda)}))\varphi)d^{\#}s\end{gathered} \quad (2.14)$$

so,

$$\begin{gathered}\|[Ext\text{-}\exp(-tA^{(\lambda)})]\varphi - [Ext\text{-}\exp(-tA^{(\mu)})]\varphi\|_{\#} \leq \\ Ext\text{-}\int_0^t \|(Ext\text{-}\exp(-sA^{(\lambda)}))((Ext\text{-}\exp(-(t-s)A^{(\lambda)})))\|_{\#}\|A^{(\mu)}\varphi - A^{(\lambda)}\varphi\|_{\#}d^{\#}s \leq \quad (2.15) \\ \leq t\|A^{(\mu)}\varphi - A^{(\lambda)}\varphi\|_{\#}.\end{gathered}$$

We have used the fact that $Ext\text{-}\exp(-tA^{(\lambda)})$ and $[Ext\text{-}\exp(-(t-s)A^{(\mu)})]$ commute since $\{A^{(\lambda)} | \lambda > 0\}$ is a commuting family. Since we have proven above that $\#\text{-}\lim_{\lambda\to\infty^{\#}} A^{(\lambda)}\varphi = A\varphi$, $\{Ext\text{-}\exp(-tA^{(\lambda)})\}$ is Cauchy as $\lambda \to \infty^{\#}$ for each $t > 0$ and $\varphi \in D(A)$. Since $D(A)$ is #-dense and the $Ext\text{-}\exp(-tA^{(\lambda)})$ are uniformly hyperfinitely bounded, the same statement holds for all $\varphi \in X$. Now, define

$$T(t)\varphi = \#\text{-}\lim_{\lambda\to\infty^{\#}}[Ext\text{-}\exp(-tA^{(\lambda)})\varphi]. \quad (2.16)$$

$T(t)$ is a semigroup of contraction operators since these properties are preserved under strong #-limits. The above inequality shows that the #-convergence in Eq.(2.16) is uniform for t restricted to a hyperfinite interval, so $T(t)$ is strongly #-continuous since $Ext\text{-}\exp(-tA^{(\lambda)})$ is. Thus, $T(t)$ is a contraction semigroup.It remains to show that the infinitesimal generator of $T(t)$, call it $\widetilde{A}$, is equal to A. For all t and $\varphi \in D(A)$,

$$[Ext\text{-}\exp(-tA^{(\lambda)})\varphi] - \varphi == -\left[Ext\text{-}\int_0^t\right] \quad (2.17)$$

so, since $\#\text{-}\lim_{\lambda\to\infty^{\#}} A^{(\lambda)}\varphi = A\varphi$, we have

$$T(t)\varphi - \varphi = -\left[Ext\text{-}\int_0^t T(s)A\varphi d^{\#}s \right]. \tag{2.18}$$

Thus, $\widetilde{A}_t\varphi \to_{\#} A\varphi$ as $t \to_{\#} 0.$ Therefore $D\left(\widetilde{A}\right) \supset D(A)$ and $\widetilde{A} \restriction D(A) = A.$ For $\lambda > 0,$ $(\lambda + A)^{-1}$ exists by hypothesis and $\left(\lambda + \widetilde{A}\right)^{-1}$ exists by the necessity part of the theorem.

.

§2 Hypercontractive semigroups

In the previous section we discussed $\mathcal{L}^p_{\#}$-contractive semigroups. In this section we will prove a self-adjointness theorem for operators of the form $A + V$ where V is a multiplication operator and A generates an $\mathcal{L}^p_{\#}$-contractive semigroup that satisfies a strong additional property.

Definition 2.1. Let $\langle M, \mu^{\#} \rangle$ be a #-measure space with $\mu^{\#}(M) = 1$ and suppose that A is a positive self-adjoint operator on $\mathcal{L}^2_{\#}(M, d^{\#}\mu^{\#})$. We say that $Ext\text{-}\exp(-tA)$ is a hypercontractive semigroup if:

(i) $Ext\text{-}\exp(-tA)$ is $\mathcal{L}^p_{\#}$-contractive;

(ii) for some $b > 2$ and some constant $C_b,$ there is a $T > 0$ so that $\|Ext\text{-}\exp(-tA)\varphi\|_b \leq C_b\|\varphi\|_2$ for all $\varphi \in \mathcal{L}^2_{\#}(M, d^{\#}\mu^{\#}).$

By Theorem X.55, condition (i) implies that $Ext\text{-}\exp(-tA)$ is a strongly #-continuous contraction semigroup for all $p < \infty^{\#}.$ Holder's inequality shows that

$$\|\bullet\|_q \leq \|\bullet\|_p \tag{1}$$

if $p \geq q.$ Thus the $\mathcal{L}^p_{\#}$-Spaces are a nested family of spaces which get smaller as p gets larger; this suggests that (ii) is a very strong condition. The following proposition shows that b plays no special role.

Proposition 2.1. Let $Ext\text{-}\exp(-tA)$ be a hypercontractive semigroup on $\mathcal{L}^2_{\#}(M, d^{\#}\mu^{\#}).$ Then for all $p, q \in (1, \infty^{\#}),$ there is a constant C_{pq} and a $t_{pq} > 0$ so that if $t > t_{pq}$ then $\|Ext\text{-}\exp(-tA)\varphi\|_p \leq C_{pq}\|\varphi\|_q$ for all $\varphi \in \mathcal{L}^q_{\#}.$

Proof. The case where $p < q$ follows immediately from (i) and (1). So suppose that $p > q.$ Since $Ext\text{-}\exp(-tA) : \mathcal{L}^2_{\#} \to \mathcal{L}^b_{\#}$ and $Ext\text{-}\exp(-tA) : \mathcal{L}^{\infty^{\#}}_{\#} \to \mathcal{L}^{\infty^{\#}}_{\#},$ the generalized Riesz-Thorin theorem implies that there is a constant C so that for all $r \geq 2,$ $\|Ext\text{-}\exp(-tA)\varphi\|_r \leq C\|\varphi\|_{br/2}.$ We now consider two cases. First, if $q \geq 2$ we choose n large enough so that $2(b/2)^n > p.$ Then $\|Ext\text{-}\exp(-nTA)\varphi\|_{2(b/2)^n} \leq C\|\varphi\|_2$ so the conclusion follows if $2 < q, p > 2(b/2)^n,$ by using (1), and hypothesis (i). If $1 < q < 2,$ then we choose n large enough so that $2(b/2)^n > p$ and $q > c$ where $c^{-1} + (2(b/2)^n)^{-1} = 1.$ Since A is self-adjoint and $Ext\text{-}\exp(-nTA)\varphi$ is a bounded or hyper bounded map from $\mathcal{L}^2_{\#}$ to $\mathcal{L}^{2(b/2)^n}_{\#}, (Ext\text{-}\exp(-nTA))^* = Ext\text{-}\exp(-nTA)$ is a bounded or hyper bounded map from $\mathcal{L}^c_{\#}$ to $\mathcal{L}^2_{\#}.$ Thus $Ext\text{-}\exp(-2nTA)$ is a bounded or hyper bounded map from $\mathcal{L}^c_{\#}$ to $\mathcal{L}^{2(b/2)^n}_{\#}.$ Since $c < q < p < 2(b/2)^n$, (1) implies the proposition.

Theorem 2.1. The operator $-\frac{1}{2}d^{\#2}/d^{\#}x^2 + xd^{\#}/d^{\#}x$ on $\mathcal{L}^2_{\#}\left({}^*\mathbb{R}^{\#}_c, \pi^{-1/2}_{\#}Ext\text{-}\exp(-x^2)d^{\#}x\right)$ is positive and essentially self-adjoint on the set of hyperfinite linear combinations of Hermite polynomials, and generates a hypercontractive semigroup.

As a preparation for our main theorem, we prove the following result.
Theorem 2.2 Let $\langle M,\mu\rangle$ be a #-measure space with $\mu(M) = 1$ and let H_0 be the generator of a hypercontractive semigroup on $\mathcal{L}^2_{\#}(M,d\mu)$. Let V be a real-valued measurable function on $\langle M,\mu^{\#}\rangle$ such that $V \in \mathcal{L}^p_{\#}(M,d^{\#}\mu^{\#})$ for all $p \in [1,\infty^{\#})$ and $Ext\text{-}e^{-tV} \in \mathcal{L}^1_{\#}(M,d^{\#}\mu^{\#})$ for all $t > 0$. Then $H_0 + V$ is essentially self-#-adjoint on $C^{\infty^{\#}}(H_0) \cap D(V)$ and is bounded below. $C^{\infty^{\#}}(H_0) = \bigcap_{p\in\mathbb{N}^{\#}} D(H_0^p)$
.

Chapter VI. Singular Perturbations of Selfadjoint Operators on a non-Archimedean Hilbert space.

§1. Introduction

We study the sum $A+B$ of two #-selfadjoint operators on a non-Archimedean Banach spaces, and we find sufficient conditions for $C = A+B$ to be #-selfadjoint. Our technique is to approximate B by a hyperinfinite sequence of bounded #-selfadjoint
operators $B_n, n \in {}^*\mathbb{N}$ and so to approximate C by #-selfadjoint operators $C_n = A+B_n$.
We answer three questions separately:
1.When do the operators C_n have a #-$\lim C$? 2.When is C a #-selfadjoint operator?
3.When is $C = A+B$?
In Theorem 8 we give a set of estimates on the relative size of A and B which ensure a positive answer to all three questions. Hence these estimates show that $A+B = C$ is #-selfadjoint. In another paper [5], we use Theorem 2.8 to prove the existence of a self-interacting, causal quantum field in 4-dimensional space-time. Formally this field theory is Lorentz covariant and has non-trivial scattering; this application was the motivation for the present work.
In order to investigate the meaning of #-$\lim_{n\to{}^*\infty} C_n$, we give a new definition for the strong #-convergence of a hyperinfinite sequence of operators. Consequences of this definition
are worked out in Section 2. In Section 3 we give estimates on operators C_n which are sufficient to ensure that the #-$\lim_{n\to{}^*\infty} C_n = C$ exists and that C is maximal symmetric or #-selfadjoint. This result is given in Theorem 5 and Corollary 6.
In Section 4 we investigate whether #-$\lim_{n\to{}^*\infty} C_n = C$ is equal to $A+B$.
We combine this work in Theorem 8, our second main theorem, where B is a singular, but nearly positive #-selfadjoint perturbation of a positive #-selfadjoint operator A. To illustrate this theorem, let $A \geq I$ and let B be essentially #-selfadjoint on

$$D^{\#} = \bigcap_{n\in{}^*\mathbb{N}} D(A^n). \qquad (1.0)$$

Assume now that, for some $\beta > 0$ and some α,

$$A^{-(1-\beta)}BA^{-(1-\beta)} \text{ and } A^{\beta}BA^{\alpha} \qquad (1.1)$$

are #-densely defined, bounded operators. Also, for some positive $a,\varepsilon \in {}^*\mathbb{R}^{\#}_{c+}$ satisfying $2a+\varepsilon < 1$, suppose that there is a constant $b \in {}^*\mathbb{R}^{\#}_{c}$ such that, as bilinear forms on $D \times D$,

$$0 \leq aA + B + b \qquad (1.2)$$

and

$$0 \leq \varepsilon A^2 + [A^{1/2}, [A^{1/2}, B]] + b. \qquad (1.3)$$

Then $A + B$ is #-selfadjoint.
We see from this example that neither the operator B nor the bilinear form B need be bounded relative to A.
While it may not appear evident, the conditions (1.1)-(1.3) are closely related to a more easily understandable estimate on $D^{\#} \times D^{\#}$,

$$A^2 + B^2 c(A+B)^2 + c. \qquad (1.4)$$

In fact, estimates (1.1)-(1.3) are chosen because they allow us not only to prove (1.4), but also the similar inequality where B is replaced by B_n.
Let us now see that if $A + B$ is #-selfadjoint, then (1.4) must hold for every vector in $D(A+B) = D(A) \cap D(B)$.
Proposition 1.1. Let A and B be #-closed operators. Then $A + B$ is #-closed if and only if there is a constant $c \in {}^*\mathbb{R}_c^{\#}$ such that for all $\psi \in D(A+B)$

$$\|A\psi\|_{\#} + \|B\psi\|_{\#} \leq \|(A+B)\psi\|_{\#} + c\|\psi\|_{\#} \qquad (1.5)$$

and (1.5) is equivalent to (1.4) on $D(A+B) \times D(A+B)$.
Proof: Certainly (1.5) implies that $A + B$ is #-closed. Conversely, assume that $A + B$ is #-closed and introduce the #-norms on $D(A+B) = D(A) \cap D(B)$,

$$\|\psi\|_{\#1} \triangleq \|\psi\|_{\#} + \|A\psi\|_{\#} + \|B\psi\|_{\#} \qquad (1.6)$$

and

$$\|\psi\|_{\#2} \triangleq \|\psi\|_{\#} + \|(A+B)\psi\|_{\#} \qquad (1.7)$$

Then $D(A+B), \|\cdot\|_{\#2}$ is a non-Archimedean Banach space because $A + B$ is #-closed.
The identity map from $D(A+B), \|\cdot\|_{\#2}$ to $D(A+B), \|\cdot\|_{\#1}$ has a #-closed graph because $A, B,$ and $A + B$ are c#-losed. By the #-closed graph theorem, the identity map is #-continuous; hence

$$\|\psi\|_{\#1} \leq c\|\psi\|_{\#2}. \qquad (1.7')$$

Proposition 1.2.Let $A \geq I, B$ be #-selfadjoint operators with $D^{\#} \subset D$(B) and suppose (1.2) and (1.3) hold. Then (1.4) is valid on $D^{\#} \times D^{\#}$.
Proof The operators $A^2, B^2, AB, BA,$ and $A^{1/2}BA^{1/2}$ define bilinear forms on $D^{\#} \times D^{\#}$. Using (1.2) and (1.3), we have the inequality:
$A^2 + B^2 = (A+B)^2 - 2A^{1/2}BA^{1/2} - [A^{1/2}, [A^{1/2}, B]] \;\leq (A+B)^2 + (2a+\varepsilon)A^2 + 2Ab + b$
which establishes (1.4).

§2. Strong #-Convergence of Operators

Let $\mathcal{L}(C)$ be the graph of the operator C. For any hyperinfinite sequence $\{C_n\}, n \in {}^*\mathbb{N}$ of #-densely defined operators we define

$$\mathcal{L}_{^*\infty}(C) = \{\phi, \chi | \phi = \#\text{-}\lim_{n\to{}^*\infty} \phi_n, \phi_n \in D(C_n), \chi = \#\text{-}\lim_{n\to{}^*\infty} C_n\phi_n\}. \qquad (8)$$

In general, $\mathcal{L}_{^*\infty}$ will not be the graph of an operator. If the hyperinfinite sequence $\{C_n^*\}, \quad n \in {}^*\mathbb{N}$#-converges strongly on a #-dense domain D to an operator C^*, namely,

$$C^*\psi = \#\text{-}\lim_{n\to{}^*\infty} C_n^*\psi, \psi \in D,$$

then $\mathcal{L}_{^*\infty}$ is the graph of some operator C^*. In particular, if each C_n is self #-adjoint, and if the C_n #-converge on a #-dense set D to an operator C defined on D, then $\mathcal{L}_{^*\infty} = \mathcal{L}_{^*\infty}(C_{^*\infty})$ and $C_{^*\infty}$ is a symmetric extension of C.

Definition 2.1. G #-CONVERGENCE. The hyperinfinite sequence of operators $C_n, n \in {}^*\mathbb{N}$ #-converge strongly to $C_{^*\infty}$ in the sense of graphs, written

$$C_n \to_{\#G} C_{^*\infty} \tag{8'}$$

if $\mathcal{L}_{^*\infty}$ is the graph of a #-densely defined operator $C_{^*\infty}$.

Remark 2.1.Note that for a hyperinfinite sequence of uniformly bounded operators $\{C_n^*\}_{n\in {}^*\mathbb{N}}$ such that $C_n \to_{\#G} C_{^*\infty}$, $C_{^*\infty}$ is the usual strong #-limit of the operators $C_n, n \in {}^*\mathbb{N}$ and is everywhere defined.

Definition 2.2.R #-CONVERGENCE. Let the resolvents $R_n(z) = (C_n - z)^{-1}, n \in {}^*\mathbb{N}$ exist for some $z \in {}^*\mathbb{C}_c^\#$, and be uniformly bounded in n. The operators C_n #-converge strongly to $C_{^*\infty}$ in the sense of resolvents, written

$$C_n \to_{\#R} C_{^*\infty} \tag{8''}$$

if the resolvents $R_n(z)$ #-converge strongly to an operator $R(z)$, which has a #-densely defined inverse.

Remark 2.2.Note thatIn that case, the operator $C_{^*\infty} = R^{-1}(z) + z$ exists for all $z \in {}^*\mathbb{C}_c^\#$ for which the strong #-limit of the $R_n(z)$ exists, and $R^{-1}(z) + z$ is independent of z.

Remark 2.3.Note that G #-convergence is weaker than R #-convergence, in the case $C_n = C_n^*$ at least, because, as we shall show, in this case $C_n \to_{\#R} C_{^*\infty}$ implies $C_n \to_{\#G} C_{^*\infty}$. It seems likely that G #-convergence is strictly weaker than R #-convergence; this could be established by giving an example for which $C_n^* = C_n \to_{\#G} C_{^*\infty}$ with $C_{^*\infty}$ not maximal symmetric. The importance of G #-convergence is that it is technically easier to verify-and gives less information about the #-limit-than R #-convergence, while automatically selecting the correct domain in the case that R #-convergence also holds. The most familiar examples of G #-convergence occur where there is C_n strong #-convergence on a #-dense domain. A less trivial example occurs where there is $D(C_n)$ is independent of n, but apparently

$$D(C) \cap D(C_n) = \{0\}.$$

We have the following connection between G and R #-convergence for a hyperinfinite sequence of #-selfadjoint operators.

Proposition 3.Let $C_n, n \in {}^*\mathbb{N}$ be #-selfadjoint.

(a) The domain $D_{^*\infty} = \left\{\phi | \{\phi, \chi\} \in \mathcal{L}_{^*\infty} \text{ for some } \chi\right\}$ is #-dense in H and oniy if $C_n \to_{\#G} C_{^*\infty}$, and in this case $C_{^*\infty}$ is necessarily symmetric.

(b) If $R_n(z) = (C_n - z)^{-1}, n \in {}^*\mathbb{N}$ #-converges to a bounded operator $R(z)$ for an unbounded set of z's with $\|zR,(z)\|_\#$ bounded uniformly in $z \in {}^*\mathbb{C}_c^\#$ and $n \in {}^*\mathbb{N}$ and if $C_n \to_{\#G} C_{^*\infty}$, then each $R(z)$ is invertible.

(c) If $R_n(z)$ #-converges to an invertible $R(z)$, then $C_n \to_{\#R} C$.

(d) If $C_n \to_{\#R} C$, then $C_n \to_{\#G} C_{^*\infty}, \mathcal{L}_{^*\infty} = \mathcal{L}(C)$, and C is maximal symmetric.

(e) Conversely, if $C_n \to_{\#G} C$, where C is maximal symmetric, then $C_n \to_{\#R} C$.

In case the #-limit of the $C_n, n \in {}^*\mathbb{N}$ is actually selfadjoint, there are further connections between G and R #-convergence.

Theorem 4.

(a) $C_n \to_{\#G} C$, and $C = C^*$.

(b) $C_n \to_{\#R} C$, and $C = C^*$.
(c) The hyper infinite sequences $\{R_n(z)\}$ and $\{[R_n(z)]^*\}, n \in {}^*\mathbb{N}$ #-converge strongly and $\#\text{-}\lim_{n\to{}^*\infty} R_n(z)$ is invertible for some z.
(d) Statement (c) holds for all non-real $z \in {}^*\mathbb{C}_c^{\#}$

§3.Estimates on a G #-convergent hyper infinite sequence

In this section we give estimates which are sufficient to assure that it G #-convergent sequence of operators is R #-convergent, and that the limit is maximal symmetric or selfadjoint. In order to measure the rate of #-convergence, we introduce a selfadjoint operator $N \geq I$ and the associated non-Archimedean Hilbert spaces H_λ with the scalar product

$$\langle \psi, \psi \rangle_{\#\lambda} = \langle N^{\lambda/2}\psi, N^{\lambda/2}\psi \rangle_{\#}. \tag{3.1}$$

By standard identifications we have for $\lambda \geq 0 : H_\lambda \subset H_0 \subset H_{-1}$ and $H_0 = H$.
If $D : H_\alpha \to H_\beta$ is a #-densely defined, bounded operator from H_α to H_β, we let $\|D\|_{\#\alpha,\beta}$ denote its #-norm. Setting $\|D\|_{\#} = \|D\|_{\#0,0}$ we obtain

$$\|D\|_{\#\alpha,\beta} = \|N^{\beta/2} D N^{-\alpha/2}\|. \tag{3.2}$$

Let $C_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of selfadjoint operators, and consider the following three conditions.
(i) Suppose that $C_n - C_m$ is a #-densely defined, bounded operator from $H_\lambda$to $H_{-\lambda}$, for some λ, and that as $n,m \to {}^*\infty$

$$\|C_n - C_m\|_{\#\lambda,-\lambda} \to_{\#} 0. \tag{3.3}$$

(ii) Suppose that, for some p and for an unbounded set of $z = x + iy \in {}^*\mathbb{C}_c^{\#}$ in the sector $|x| \leq const \times |y|$,

$$\|R_n(z)\|_{\#\mu,\lambda} \leq M(z), \tag{3.4}$$

where the bound $M(z)$ is uniform in $n \in {}^*\mathbb{N}$.
(iii) Suppose that, for the above z's,

$$\|R_n(\bar{z})\|_{\#\mu,\lambda} \leq M(z). \tag{3.5}$$

Theorem 5. Let $C_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of #-selfadjoint operators with a common domain, such that

$$C_n \to_{\#G} C.$$

If conditions (i) and (ii) hold, then

$$C_n \to_{\#R} C$$

and C is maximal symmetric.
Corollary 6. If in addition to the hypothesis of Theorem 5, condition (iii) also holds, then C is #-selfajoint.
Remark 3.1.(1) If $\mu = 0$ in (ii), then the resolvents #-converge uniformly.
(2) If the C_n are uniformly semibounded from below, then we may choose the z in condition (ii) to be infinite large negative numbers. In that case the conclusion of Theorem 5 is that $C_n \to_{\#R} C = C^*$.

§ 4.Estimates for singular perturbations

In this section we consider a singular perturbation B of a #-selfadjoint operator A. We give estimates on B which ensure that the sum $A+B$ is #-selfadjoint.

Abbreviation 4.1.We abbreviate $A^{\#-}$ instead #-$\overline{A}$.

Definition 4.1. A #-core of an operator C is a domain D contained in $D(C)$ such that $C = (C \upharpoonright D)^{\#-}$.

Lemma 7. Let $A, A_n, n \in {}^*\mathbb{N}, B, B_n, n \in {}^*\mathbb{N}$ and $C_n = A, +B_n, n \in {}^*\mathbb{N}$ be #-selfadjoint operators with a common #-core D. Assume the hypotheses of Theorem 5 and Corollary 6 for $C_n, n \in {}^*\mathbb{N}$ and suppose also that, for $\theta \in D$,

$$\|(A-A_n)\theta\|_{\#} + \|(B-B_n)\theta\|_{\#} \to_{\#} 0 \text{ as } n \to {}^*\infty \quad (4.9)$$

and

$$\|A_n\theta\|_{\#}^2 + \|B_n\theta\|_{\#}^2 \leq const. \times \|\theta\|_{\#}^2 + const. \times \|C_n\theta\|_{\#}^2, \quad (4.10)$$

with constants independent of n. Then $A+B$ is #-selfadjoint and $C_n \to_{\#R} A+B$.

Remark 4.1.As hypothesis for our next theorem, our second main result, we assume that $N \leq A$ and that N and A commute. Let

$$D^{*\infty}(A) = \bigcap_{n \in {}^*\mathbb{N}} A(A^n) \quad (4.11)$$

the elements of $D^{*\infty}(A)$ are called $C^{*\infty}$ vectors for A. Assume that $D^{*\infty}(A)$ is a #-core for the #-selfadjoint operator B. Also assume that, as bilinear forms on $D^{*\infty} \times D^{*\infty}$, and for some α and ε in the indicated ranges,

$$0 \leq \alpha N + B + const., 0 \leq \alpha < 1/2 \quad (4.12)$$

and

$$0 \leq \varepsilon A^2 + const \times B + [A^{1/2}, [A^{1/2}, B]] + const., 2\alpha + \varepsilon < 1. \quad (4.13)$$

Let B be a bounded operator from H_v to H_{-v} and from H_α to H_β for some α, β and $v, \beta > 0$ (H_α is defined following Theorem 4.) If $v \geq 2$, assume that for all $\varepsilon > 0$

$$0 \leq \varepsilon N^{\mu+2} + [N^{(\mu+1)/2}, [N^{(\mu+1)/2}, B]] + const. \quad (4.14)$$

as bilinear forms on $D^{*\infty} \times D^{*\infty}$, for some $\mu > v-2$.

Theorem 8. Under the above hypothesis, $A+B$ is #-selfadjoint.

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