# UNEXPECTED CONNECTION BETWEEN TRIANGULAR NUMBERS AND THE GOLDEN RATIO

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Abstract. We find out that when a sum of five consecutive triangular numbers, $S_5(n) = T(n) + ... + T(n+4)$, is also a triangular number $T(k)$, the ratios of consecutive terms of $a(i)$ that represent values of $n$ for which this happens, tend to $\phi^2$ or $\phi^4$ as $i$ tends to infinity, where $\phi$ is the Golden Ratio. At the same time, the ratios of consecutive terms $S_5(a(i))$ tend to $\phi^4$ or $\phi^8$. We also note that such ratios that are the powers of $\phi$ can appear in the sequences of triangular numbers that are also higher polygonal numbers, one case of which are the heptagonal triangular numbers.

## 1. Introduction

This expository paper is about an unexpected finding revealing a connection between triangular numbers and the Golden Ratio, $\phi = (1 + \sqrt{5})/2 = 1.61803...$

Both the Golden Ratio, intimately related to the famous and ubiquitous Fibonacci sequence as well as to a number of problems one encounters in science, engineering or mathematics, and the triangular numbers belong to elementary mathematical objects that have been studied for centuries if not millennia now.

The former enjoys a near-cult following with a large and growing number of papers (see, e.g,. [1] for a recent review) and many books dedicated to it, ranging from more scholarly [2] to more popular [4], [3]. To be fair, the cult debunkers also exist (e.g., [5], [6]).

The latter object, while perhaps less popular, too has produced plenty of both research and expository work (see, e.g., [7], [8], [9]). It is often studied within a broader context of figurate [10] or polygonal numbers [11], which form connections between number theory and geometry, and, if only because of that, serve as a source of entertainment for the aficionados of recreational mathematics [12].

It is the triangular numbers, in particular, that should be very familiar to most high school students, who may even not be the fans of recreational mathematics. The $n$-th triangular number, often denoted by $T(n)$, is simply the sum of all natural numbers from 1 up to $n$: 1, 1+2, 1+2+3, ... and so on, which works out to 1, 3, 6, ... and, in general, $T(n) = n(n + 1)/2$.

The ratio of two consecutive triangular numbers is easy to calculate, but even in its asymptotic form, with $n$ tending to infinity, when one can say something definitive about it, it is just a boring 1.

To spice things up a bit, let us consider more complex structures. Let us take two consecutive triangular numbers and add them up. The result of this would be

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$S_2(n) = T(n) + T(n+1) = (n+1)^2$, which, incidentally, is a well-known formula that informs us that the sum of two consecutive triangular numbers is square.

We can proceed further this way, constructing $S_3$, $S_4$, and so on, where $S_k(n)$ is the sum of $k$ consecutive triangular numbers starting from $T(n)$. One can work out formulas for $S_k(n)$ in terms of $n$ and $k$, but that's hardly exciting, and rather easy to do too: $S_k(n) = k(3n(n+k) + k^2 - 1)/6$ (1).

However, things get more interesting when we investigate under what circumstances these new sequences can also give rise to triangular numbers. This leads to a condition on $n$ that singles out those $n$'s for which this is possible for any $S_k(n)$ that we may want to study.

## 2. Unexpected connection to the Golden Ratio

The most interesting situation takes place when $k = 5$, that is, when we are dealing with the sum of 5 consecutive triangular numbers. In this case, (1) generates $S_5(n) = 5(n^2 + 5n + 8)/2$.

If we stipulate that this sum is also a triangular number we get a sequence of $n$'s that meet this condition. Using PARI/GP or Mathematica (see the next section), we can work out the first of several $n$'s for which this happens. Here are the first 12 of them, all that one can find up to $10^8$:

2, 29, 80, 563, 1478, 10145, 26564, 182087, 476714, 3267461, 8554328, 58632251.

Now, to the untrained eye, these numbers may not look particularly interesting. However, what actually is interesting lies in their ratios, the ratios of consecutive terms of this sequence. If we call this sequence $a(i)$, it is $a(i+1)/a(i)$ that we are after here.

The list of these ratios - let us call this sequence $r_a(i)$ - follows below:

14.5, 2.75862, 7.0375, 2.62522, 6.86401, 2.61843, 6.85465, 2.61806, 6.85413, 2.61804, 6.8541.

What this list reveals is that some of the terms of $r_a(i)$ are very close to 2.61803..., which is $1 + \phi = \phi^2$. The other ratios too can be recognized as having something in common with $\phi$ for they turn out to be very close to the fourth power of that number, $\phi^4 = 3\phi + 2 = 6.85410...$ The further down the sequence, the closer these terms are to the values of $\phi^2$ and $\phi^4$.

In other words, the ratios $a(i+1)/a(i)$ asymptotically tend to either $\phi^2$ or $\phi^4$, as $i$ tends to infinity, or, in practical terms, as it gets larger and larger. More formally and precisely, the limits of $a(2k+1)/a(2k)$ and $a(2k)/a(2k-1)$ as $k$ tends to infinity are $\phi^2$ and $\phi^4$, respectively.

Using Mathematica we can figure out a recurrence relation for these numbers. To this end, Mathematica's function FindLinearRecurrence is of tremendous help. With its aid one finds out that $a(i)$ satisfies the following recurrence formula:

$a(i) = a(i-1) + 18a(i-2) - 18a(i-3) - a(i-4) + a(i-5)$, with initial conditions given by $a(1) = 2, a(2) = 29, a(3) = 80, a(4) = 563, a(5) = 1478$.

Once we have $a(i)$, we can also find $b(i) = S_5(a(i))$, which, unlike the former, are triangular numbers. Here is their list corresponding to the 12 first values of $a(i)$ listed above:

55, 2485, 17020, 799480, 5479705, 257429395, 1764447310, 82891465030, 568146553435, 26690794309585, 182941425758080, 8594352876220660.

As before, we can find the ratios - $r_b(i)$ - for the consecutive terms of this sequence (also as before, we have limited ourselves to the first 6 significant digits rounded off):

45.1818, 6.84909, 46.973, 6.85409, 46.9787, 6.8541, 46.9787, 6.8541, 46.9787, 6.8541, 46.9787.

It is easy to see that they tend to $\phi^4$, or its square, $\phi^8 = 46.97871...$ More precisely, the limits of $b(2k+1)/b(2k)$ and $b(2k)/b(2k-1)$ as $k$ tends to infinity are $\phi^4$ and $\phi^8$, respectively.

Using again Mathematica's FindLinearRecurrence function, one finds out that $b(i)$ satisfies the following recurrence formula:

$b(i) = b(i-1)+322b(i-2)-322b(i-3)-b(i-4)+b(i-5)$, with initial conditions given by $b(1) = 55, b(2) = 2485, b(3) = 17020, b(4) = 799480, b(5) = 5479705$.

## 3. PARI/GP and Mathematica code

The following simple but efficient PARI/GP code was used to print the first 12 terms of $a(i)$:

```
for(n=1, 10^8, s=5*(8+5*n+n^2)/2; ispolygonal(s, 3)&&print1(n, ", "))
```

The following Mathematica code can be used to accomplish the same:

```
Select[Range[10^8], IntegerQ[Sqrt[20(8+5#+#^2)+1]]&]
```

The following PARI/GP code was used to print the first 12 terms of $S_5(a(i))$:

```
for(n=1, 10^8, s=5*(8+5*n+n^2)/2; ispolygonal(s, 3)&&print1(s, ", "))
```

With Mathematica, the same numbers can obtained with this piece of code:

```
Select[Range[10^8], IntegerQ[Sqrt[20(8+5#+#^2)+1]]&]//5(8+5#+#^2)/2&
```

However, it makes more sense to use Mathematica's LinearRecurrence function for that:

```
LinearRecurrence[{1,322,-322,-1,1}, {55,2485,17020,799480,5479705}, 12]
```

We can use the same very efficient function to generate the sequence $a(i)$, even up to 100 terms (or more if we wish so):

```
LinearRecurrence[{1,18,-18,-1,1}, {2,29,80,563,1478}, 100]
```

## 4. Conclusion

This expository essay presented an unexpected connection between the famous Golden Ratio and also well-known, though less glamorous, triangular numbers. To the best of our knowledge, this connection was not known before. Considering that both the Golden Ratio and triangular numbers are so elementary and have been studied for so long now one would think that everything should already be known about them. As this paper demonstrates, this may not necessarily be so.

For the sake of completeness, let us also very briefly discuss the case of $k = 2$, which we have already alluded to above, and that of $k = 3$.

The former case has quite a bit of history that involves Euler himself, who was the first to work out a general solution to this problem, the problem of square triangular numbers, in the form of a Binet-like formula. It turns out that the ratios of the consecutive terms of $a(i)$ in this case tend to the square of the Silver Ratio [1]

[1] For more on the family of metallic means (or ratios) that both the Golden Ratio and the Silver Ratio are part of see [13].

(equal $1+\sqrt{2}$), that is, to $3+2\sqrt{2}=5.82842...$ The ratios of $b(i)=S_2(a(i))$ tend to the fourth power of the Silver Ratio, or $17+12\sqrt{2}=33.97056...$

The problem of the sum of three consecutive triangular numbers is much less prominent than that of the two, arguably also because the ratio of the consecutive terms of its $a(i)$ is not given by a very recognizable number. It just happens to be a humble[2] $2+\sqrt{3}=3.73205...$ Predictably enough, its $S_3((a(i))$ exhibits the ratios of consecutive terms that tend to the square of this number, $7+4\sqrt{3}=13.92820...$

In closing, let us also note that the problem of the sums of 2, 3, and 5 consecutive triangular numbers that are also triangular features the square roots of 2, 3, and 5, respectively, which suggests that it is rather unlikely to find the Golden Ratio in other sums of consecutive triangular numbers.

However, this does not mean that the powers of $\phi$ do not appear in other contexts involving triangular numbers. As a matter a fact, they do. For instance, in the sequence of heptagonal triangular numbers, although this fact too appears to be largely unknown.

This is the sequence of the triangular numbers that are also heptagonal, the first few terms of which are 1, 55, 121771, 5720653, 12625478965, ..., where the ratios of consecutive terms tend to $\phi^8$ and $\phi^{16}=2206.99954...$ Not surprisingly, the ratios of indices of the triangular numbers corresponding to this sequence, 1, 10, 493, 3382, 158905, ..., tend to $\phi^4$ and $\phi^8$.

It is conceivable that this is not a unique case, i.e., that there exist other sequences of triangular numbers that are also higher polygonal numbers, where the powers of the Golden Ratio make their appearance in the ratios we have investigated throughout this paper.

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[2] One can see this number as the square of $(1+\sqrt{3})/\sqrt{2}$=1.93185.., and treating this smaller number as the base ratio, not unlike the Golden or Silver ratios in the other cases, leads to a full analogy with these cases.

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