# New series and concise algorithm for "Lambert W function"

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ABSTRACT: We give a new series expression, and code up a concise algorithm, for the "Lambert W function" W(X) such that $We^W=X$ with $W\geq-1$.

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By the "Lambert W function" W(X), I mean the greatest solution of $We^W=X$. A real solution exists if and only if $X\geq-1/e\approx-0.367879$. An equivalent equation (if we regard W instead as a function of L=lnX) when X>0 is W+lnW=L.

Some special values: W(-1/e)=-1, W(0)=0, W(e)=1, $W(e^{1+e})=e$.

Derivatives: $W'(x)=W/(x+xW)$ if $x\neq0$, 1 if x=0; $W''(x)=-(W+2)W^2x^{-2}/(W+1)^3$ if $x\neq0$, -2 if x=0; dW/dL=W/(W+1); $d^2W/dL^2=W/(W+1)^3$.

W(x) monotonically increases from -1 to ∞.

Indefinite integrals: $\int W(x)\,dx = [W(x)-1+1/W(x)]x+C$, $\int W(x)/x\,dx = [W(x)/2+1]\,W(x) + C$,

Large-X asymptotic: $W(X)=L-\ln L+G/L+O(G/L)^2$, where L=lnX and G=lnL.

Euler's series, convergent for $|z|\leq1/e$: $W(z) = \sum_{n\geq1} -n^{n-1} (-z)^n / n!$.

My new series, also convergent for $|z|\leq1/e$: $[1+W(z)]^2 = 1-2\sum_{n\geq1} n^{n-3} (-z)^n / (n-1)!$.

The new series is superior to Euler's in the sense that on the circle |z|=1/e, my *n*th |summand| falls asymptotically proportionally to $n^{-5/2}$, while Euler's falls asymptotically proportionally to $n^{-3/2}$. Also the function $[1+W(z)]^2$ has better numerical behavior than W(z) in the sense that W(z) has infinite "condition number," i.e. derivative, at z=-1/e, causing accurate evaluation of W(z) to be impossible when $z\approx-1/e$ using standard approximate-real arithmetic; but $(d/dz)[1+W(z)]^2=2W(z)/z$ is a member of the interval [0,2e] for all $z\geq-1/e$ so no obstacle inherently prevents precisely evaluating $[1+W(z)]^2$ anywhere.

Proof sketch: The new series may be shown to follow from the old one by a known combinatorial argument: -W(-z) is the exponential generating function $\sum_{n\geq1} t_n z^n/n!$ for the number $t_n=n^{n-1}$ of rooted trees with n labeled vertices (and demand $t_0=0$). (One of the vertices is special and is called "root." This formula for $t_n$ usually has been attributed to A.Cayley in 1889, but others also derived it in other ways, including some before Cayley, with one nice derivation being by Heinz Prüfer in 1918.) Then $[1+W(-z)]^2$ is the exponential generating function for *twice* the number $u_n=n^{n-2}$ of *un*rooted trees with n labeled vertices, *except* we insist the constant term of this series be 1, i.e. artificially insist on regarding the zero-vertex tree as "half a tree." Essentially, the equivalence of Euler's and my new generating function identities then is simply expressing the fact that all n-vertex unrooted trees with labeled vertices, can be got by gluing A-vertex and B-vertex trees (A+B=n+1) at their common root – provided we do the right things to the constant terms of the series (i.e. insist on the right arbitrary artificial conventions about how to count "zero-vertex trees") to make it work. Q.E.D.

Asymptotics for X near -1/e: These are best expressed in terms of $Q=(-2[1+\ln(-X)])^{1/2}$. Then Q=0 when X=-1/e, and Q>0 if -1/e<X<0, and Q=∞ when X=0-, and $X=-\exp(-Q^2/2-1)$. Then

$$[1+W(X)]^2 = Q^2 - 2Q^3/3 + Q^4/6 - Q^5/90 - Q^6/810 - Q^7/15120 + Q^8/68040 + 139Q^9/24494400 + Q^{10}/1020600 \pm O(Q^{11})$$

and

$$W(X) = -1 + Q - Q^2/3 + Q^3/36 + Q^4/270 + Q^5/4320 - Q^6/17010 - 139Q^7/5443200 - Q^8/204120 - 571Q^9/2351462400 + 281Q^{10}/1515591000 \pm O(Q^{11})$$

Algorithm: C code, "real" means 64-bit IEEE 754 floating point reals. Tested by, for W=-1..100 in steps of 0.001, computing $X=We^W$ then computing V=`LambertW`(X) and assessing the error V-W. The maximum |error| found was $3.39\times10^{-14}$ at W=-0.996 and X≈-0.36787649027603347:

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real LambertW(real x){ //Warren D. Smith algorithm, assumes x ≥ -1/e:
  real W,y,z,q;  //Compute initial guess W with max |error| < 0.103:
  if(x < -0.29856){ //asymptotic valid for x near -1/e
    z = -2*(1+ln(-x));  q = sqrt(z);  W = q-1+z*(q/36-1/3.0);
  }else{
    if(x > 2.57){ W = ln(x); y = ln(W); W += y/W - y; } //large-x asymptotic
    else{ W = 0.9*sqrt(x+0.3)-0.52; } //fast ad hoc approx if -0.3<x<2.6
  }
  do{ //Cubic iterations: loop-body executes at most 3 times
    q = exp(W); y = W*q-x;  z = W+1;  y /= q*z - (W+2)*y/(2*z);  W -= y;
  }while( fabs(y) > 0.000001 );
  return(W);
}
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## References

NIST Digital Library of Mathematical Functions, F.W.J.Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I.Schneider, R.F.Boisvert, C.W.Clark, B.R.Miller, B.V.Saunders, H.S.Cohl, and M.A.McClain, eds, 2023 release.

Heinz Prüfer: Neuer Beweis eines Satzes über Permutationen, Arch. Math. Phys. 27 (1918) 742-744.