sines and cosines of ANY angles may be determined to ANY degree of accuracy
and a relativistic non-Doppler effect

The unit circle yields an exact half-angle formulas for sines, cosines, tangents, etc. of ANY angles, with examples.

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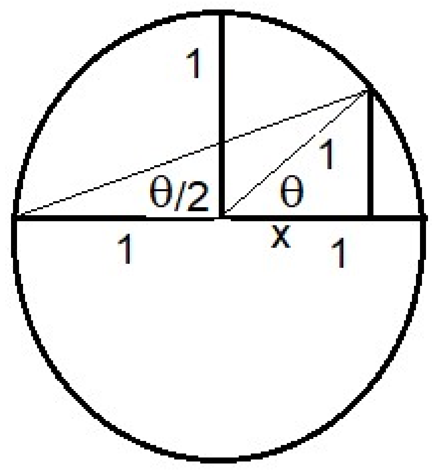
theorem 1: if $0 < \theta < \frac{\pi}{2}$: $\cos\theta \Rightarrow \cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}}$



if $\theta < \frac{\pi}{2}$:

$$\cos\theta = x \Rightarrow \cos(\theta/2) = \frac{1+x}{\sqrt{(1+x)^2 + \left(\sqrt{1-x^2}\right)^2}} = \frac{1+x}{\sqrt{(1+2x+x^2)+(1-x^2)}}$$

$$= \frac{1+x}{\sqrt{2+2x}} = \frac{1+x}{\sqrt{2(1+x)}} = \sqrt{\frac{1+x}{2}} = \sqrt{\frac{1+\cos\theta}{2}}$$

also:

$$\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1 \Rightarrow \cos\theta = \sqrt{\frac{1+\cos(2\theta)}{2}}$$

$$\Rightarrow \cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}}$$

□

corollary 1.1: if $0 < \theta \le \frac{\pi}{4}$ &' $(n \in \mathbb{N} \ge 1)$:

$$\Rightarrow \cos\frac{\pi}{2^{(1+n)}} = \frac{\overbrace{\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \ge 1)$$

proof:

$$\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \cos\frac{\pi}{8} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\frac{1+\sqrt{2}}{\sqrt{2}}}{2}} = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2} \quad ✓$$

$$\Rightarrow \cos\frac{\pi}{16} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2+\sqrt{2}}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{4}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \quad ✓ \quad ,(11.25^\circ)$$

$$\Rightarrow \cos\frac{\pi}{32} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \quad ✓$$

$$\Rightarrow \cos\frac{\pi}{64} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$$

$$= \cos\frac{\pi}{2^{(1+4)}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$$

$$\cos\left(\frac{\pi}{2^{(1+N)}}\right) = \frac{\overbrace{\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{N\,2's}}{2} \quad ' \quad (N \in \mathbb{N} \ge 1)$$

$$\Rightarrow \cos\left(\frac{\pi}{2^{(1+(N+1))}}\right) = \sqrt{\frac{1+\cos\left(\frac{\pi}{2^{(1+N)}}\right)}{2}} = \sqrt{\frac{1+\frac{\overbrace{\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{N\,2's}}{2}}{2}}$$

$$= \frac{\overbrace{\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{(N+1)\,2's}}{2} \quad ' \quad (N \in \mathbb{N} \geq 1)$$

so, by induction:

$$\Rightarrow \cos\frac{\pi}{2^{(1+n)}} = \frac{\overbrace{\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \geq 1)$$

□

and

corollary 1.2: if $0 < \theta \leq \frac{\pi}{4} \;\;\&' \;\; (n \in \mathbb{N} \geq 1)$:

$$\Rightarrow \cos\frac{\pi}{3 \cdot 2^{(1+n)}} = \frac{\overbrace{\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \geq 1)$$

proof:

$$\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \Rightarrow \cos\frac{\pi}{12} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2} \quad ✓$$

$$\Rightarrow \cos\frac{\pi}{24} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{3}}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2+\sqrt{3}}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2} \quad ✓$$

$$\cos\left(\frac{\pi}{3 \cdot 2^{(1+N)}}\right) = \frac{\overbrace{\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{N\,2's}}{2} \quad ' \quad (N \in \mathbb{N} \geq 1)$$

$$\Rightarrow \cos\left(\frac{\pi}{3 \cdot 2^{(1+(N+1))}}\right) = \sqrt{\frac{1+\cos\left(\frac{\pi}{3 \cdot 2^{(1+N)}}\right)}{2}} = \sqrt{\frac{1+\frac{\overbrace{\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{N\,2's}}{2}}{2}}$$

$$= \frac{\overbrace{\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{(N+1)\,2's}}{2} \quad ' \quad (N \in \mathbb{N} \geq 1)$$

so, by induction:

$$\Rightarrow \cos\frac{\pi}{3 \cdot 2^{(1+n)}} = \frac{\overbrace{\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \geq 1)$$

□

AND

theorem 2: if $0 < \theta \leq \frac{\pi}{4} \;\;\&' \;\; (n \in \mathbb{N} \geq 1)$:

$$\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow \sin(\theta/2) = \sqrt{1-\frac{1+\cos\theta}{2}} = \sqrt{\frac{1-\cos\theta}{2}}$$

□

so:

corollary 2.1: if $0 < \theta \leq \frac{\pi}{4} \;\;\&' \;\; (n \in \mathbb{N} \geq 1)$:

$$\Rightarrow \sin\frac{\pi}{2^{(1+n)}} = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{n\,2's}}{2}\ ', \quad (n \in \mathbb{N} \succeq 1)$$

proof:

$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \Rightarrow \sin\frac{\pi}{8} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2} \quad \checkmark\checkmark$$

$$\Rightarrow \sin\frac{\pi}{16} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2+\sqrt{2}}}{2}}{2}} = \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2} \quad \checkmark\checkmark$$

$$\Rightarrow \sin\frac{\pi}{32} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}{2}} = \sqrt{\frac{\frac{\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}{2}}{2}} = \frac{\sqrt{\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}}{2} \quad \checkmark\checkmark$$

$$\Rightarrow \sin\frac{\pi}{2^6} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$$

$$= \sin\frac{\pi}{2^{(1+5)}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2} \quad \checkmark$$

$$\sin\left(\frac{\pi}{2^{(1+N)}}\right) = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{N\,2's}}{2}\ ', \quad (n \in \mathbb{N} \succeq 1)$$

$$\Rightarrow \sin\left(\frac{\pi}{2^{(1+(N+1))}}\right) = \sqrt{\frac{1-\cos\left(\frac{\pi}{2^{(1+(N+1))}}\right)}{2}} = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{N\,2's}}{2}$$

$$\Rightarrow \sin\frac{\pi}{2^{(1+n)}} = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{(N+1)\,2's}}{2}\ ', \quad (n \in \mathbb{N} \succeq 1)$$

so, by induction:

$$\Rightarrow \sin\frac{\pi}{2^{(1+n)}} = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{n\,2's}}{2}\ ', \quad (n \in \mathbb{N} \succeq 1)$$

□

and

corollary 2.2: if $0 < \theta \le \frac{\pi}{4}$ &′ $(n \in \mathbb{N} \succeq 1)$:

$$\Rightarrow \sin\frac{\pi}{3\cdot 2^{(1+n)}} = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{n\,2's}}{2}\ ', \quad (n \in \mathbb{N} \succeq 1)$$

proof:

$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin\frac{\pi}{12} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\approx 0.25881904510252076234889883762405 \quad \checkmark\checkmark$$

$$\Rightarrow \sin\frac{\pi}{24} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{3}}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2+\sqrt{3}}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2}$$

$$\approx 0.13052619222005159154840622789549 \quad \checkmark\checkmark$$

$$\Rightarrow \sin\frac{\pi}{48} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2}$$

$$\approx 0.06540312923014306681531555877518 \quad ✓✓$$

$$\sin\left(\frac{\pi}{3\cdot 2^{(1+N)}}\right) = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{N\,2's}}{2} \quad ,\quad (N \in \mathbb{N} \succeq 1)$$

$$\Rightarrow \sin\left(\frac{\pi}{3\cdot 2^{(1+(N+1))}}\right) = \sqrt{\frac{1-\cos\left(\frac{\pi}{3\cdot 2^{(1+N)}}\right)}{2}} = \sqrt{\frac{1+\frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{N\,2's}}{2}}{2}}$$

$$= \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{(N+1)\,2's}}{2} \quad ,\quad (N \in \mathbb{N} \succeq 1)$$

so, by induction:

$$\Rightarrow \sin\frac{\pi}{3\cdot 2^{(1+n)}} = \frac{\overbrace{\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{3}}}}}}}}^{n\,2's}}{2} \quad ,\quad (n \in \mathbb{N} \succeq 1)$$

$$\Rightarrow \sin\frac{\pi}{3\cdot 2^{(1+n)}} = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{n\,2's}}{2} \quad ,\quad (n \in \mathbb{N} \succeq 1)$$

□

and so:

theorem 3: if $0 < \theta < \frac{\pi}{2}$:

$$\tan(\theta/2) = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{\sqrt{\frac{1-\cos\theta}{2}}}{\sqrt{\frac{1+\cos\theta}{2}}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\Rightarrow \cot(\theta/2) = \frac{1}{\tan(\theta/2)} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

□

NOTE:

$$\frac{v}{c} = \cos\left(\theta-\frac{\pi}{2}\right) = \sin\theta \ \ \& \ \ \lambda = \cot\left(\left[\theta-\frac{\pi}{2}\right]/2\right) \Rightarrow \lambda = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

is an expression for a "relativistic Doppler effect", but not really Doppler at all.

AND

since:

$$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\Rightarrow \cos A\sin(A\pm B) = \cos A\sin A\cos B \pm \cos^2 A\sin B$$
$$\Rightarrow \sin A\cos(A\pm B) = \sin A\cos A\cos B \mp \sin^2 A\sin B$$
$$\Rightarrow \cos A\sin(A\pm B) - \sin A\cos(A\pm B) = \pm\sin B$$

for example:

$$(A,B) = \left(\frac{\pi}{6},\frac{\pi}{4}\right) \Rightarrow \cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$$

$$\Rightarrow \cos\left(\frac{5\pi}{12}\right) = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4} \quad ,(30^\circ+45^\circ = 75^\circ)$$

$$\Rightarrow \cos\left(\frac{5\pi}{24}\right) = \sqrt{\frac{1+\frac{\sqrt{6}-2}{4}}{2}} = \sqrt{\frac{2+\sqrt{6}}{8}} \quad ,(37.5^\circ)$$

$$(A,B) = \left(\frac{5\pi}{12},\frac{\pi}{16}\right) \Rightarrow \cos\left(\frac{5\pi}{12}+\frac{\pi}{16}\right) = \cos\frac{5\pi}{12}\cos\frac{\pi}{16} - \sin\frac{5\pi}{12}\sin\frac{\pi}{16}$$

$$\Rightarrow \cos\left(\frac{23\pi}{48}\right) = \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right) +$$

$$-\left(\sqrt{1-\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2}\right)\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}\right) \quad ,(86.5^\circ)$$

etc.

SO:

sines and cosines of ANY angles may be determined to ANY degree of accuracy.
(even without the aid of a computer)