Pricing of European Options using GBM and Heston Models in C++

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Abstract

The valuation of financial derivatives, particularly options, has long been a topic of interest in finance. Among the various methods developed for option pricing, the Monte Carlo simulation stands out due to its versatility and capability to model complex financial instruments. In this article, we apply the Monte Carlo method to price European options using two prominent models: the Geometric Brownian Motion (GBM) and the Heston model. While the GBM model assumes constant volatility and offers simplicity, it often falls short in capturing real market dynamics. Conversely, the Heston model introduces stochastic volatility, providing a more nuanced representation of market behaviors. Leveraging the computational efficiency of C++, our simulations reveal distinct price paths for each model. The GBM paths exhibit smooth trajectories, while the Heston paths are more varied, reflecting its allowance for stochastic volatility. Statistical analyses further underscore a significant difference in the final stock prices generated by the two models. The Heston model's prices display a broader distribution, capturing the model's inherent variability. Additionally, autocorrelation analyses suggest a more intricate autoregressive structure for the Heston model. In conclusion, while the GBM model provides simplicity and predictability, the Heston model offers a richer, albeit more complex, representation, especially in volatile market scenarios. This article offers a comparative study of the GBM and Heston models, shedding light on their utility under varying market conditions.

1 Introduction

The Monte Carlo simulation is a powerful method for option pricing [12]. It enables the modeling of complex financial instruments that may be challenging to solve using closed-form or numerical solutions. In this project, we use the Monte Carlo method to price a European option, specifically, a call option.

The Geometric Brownian Motion (GBM) and Heston model [5] are two models commonly used in financial mathematics for the modeling of stock prices and other financial derivatives.

The GBM model assumes that the logarithmic returns of a stock price are normally distributed [9] and that the volatility of the returns is constant over time. This model has been very popular because of its simplicity and the existence of an analytical solution for European options.

On the other hand, the Heston model, named after Steve Heston [8] , is a type of stochastic volatility model, which directly addresses one of the limitations of the GBM model. In reality, financial markets often exhibit periods of low and high volatility [6]. That is, volatility itself can be volatile and is not constant over time. The Heston model takes this into account by modeling volatility as a stochastic process, separate from the process under consideration (such as a stock's price).

The results from these models can potentially be quite different. Under the GBM model [2], where volatility is assumed to be constant, the price paths of an asset tend to follow a "smooth" path. In contrast, under the Heston model, the paths could show more erratic behavior, mimicking periods of low and high volatility.

The differences between these models would be particularly noticeable in the pricing of options and other derivatives where the payoff is dependent on the path of the underlying asset. Here, the additional variability in the price paths modeled by the Heston model could lead to higher option prices compared to the GBM model. It is also worth noting that GBM tends to underestimate the prices of out-of-the-money options, a fact that is known as the volatility smile or smirk, a pattern that the Heston model can reproduce.

In general, the GBM model might be a good choice if you believe the market is relatively stable and will remain so, while the Heston model might be a better choice in more volatile markets.

2 Method

Our project leverages the Monte Carlo method for pricing European options [7] by simulating the paths of the underlying asset price following a Geometric Brownian Motion (GBM) model and the Heston model.

The GBM model is one of the simplest and most widely used models in finance for generating future price scenarios. It assumes that the log-returns of asset prices follow a normal distribution and are independent of each other. The initial price (S0 = 100.0), strike price (K = 100.0), risk-free rate (r = 0.05), volatility (sigma = 0.2), and the time to expiry (T = 1.0) are the main parameters for this model. The payoff for each simulation at expiry is calculated and averaged over a large number of scenarios (N = 10000).

On the other hand, the Heston model is a more sophisticated option pricing model that considers volatility as a random process, dependent on the asset price itself. It is especially useful when dealing with options that have a significant amount of time until expiry. The parameters for this model include the initial variance (v0 = 0.06), rate of reversion (kappa = 2.0), long-run variance (theta = 0.02), volatility of volatility (xi = 0.1), and the correlation coefficient (rho =

-0.7).

By comparing and contrasting these models in the context of European options [1], which can only be exercised at expiry, we can better understand their strengths and weaknesses, and their suitability under different market conditions.

3 Implementation

C++ was chosen for its performance capabilities [10], which are critical in the context of Monte Carlo simulations due to the substantial computational requirement. With its lower-level system access and efficient memory management, C++ delivers the performance needed for such heavy computations.

The GBM function in our implementation is a representation of a stochastic process called Geometric Brownian Motion (GBM). This process is often used in finance to model the evolution of stock prices or other market variables over time. The GBM model assumes that the logarithmic returns of the price are normally distributed and that these returns are independent of each other.

The equation for the Geometric Brownian Motion is given by:

$$S = S_0 \exp((r - 0.5\sigma^2)T + \sigma\sqrt{T}Z) \tag{1}$$

where:

- S_0 is the initial asset price,
- r is the risk-free rate,
- σ is the standard deviation of the asset's returns (volatility),
- Z is a standard normal random variable,
- T is the length of the time step.

The function returns the simulated asset price at the end of the time step.

The Heston model is a stochastic volatility model that describes two stochastic processes: one for the underlying asset price and another for the variance. Here are the two stochastic differential equations that make up the Heston model:

The price dynamics are governed by:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1 \tag{2}$$

And the variance dynamics are given by:

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^2 \tag{3}$$

In the above equations:

- S_t is the asset price.

- v_t is the variance of the asset price.
- μ is the expected return of the asset.
- dW_t^1 and dW_t^2 are two Wiener processes (i.e., sequences of random variables) that have a correlation of ρ.
- κ is the rate at which v_t reverts to the mean.
- θ is the long-term average price variance.
- ξ is the volatility of the volatility, or how much the volatility itself varies.

4 Results and Discussion

The implementation successfully prices the European option, with the output being the calculated price. It was observed that the final option price is influenced by various factors such as the volatility, the risk-free rate, and the strike price. Given the random nature of the Monte Carlo simulation, the output value varies slightly on each run.

Figure 1 showcases the simulated price paths for the Geometric Brownian Motion (GBM) and the Heston model. The GBM paths exhibit smooth and continuous movements, characteristic of its assumption of constant volatility. In contrast, the Heston paths appear more erratic due to the model's allowance for stochastic volatility.

The distribution of the final prices from both models is depicted in Figure 2. The GBM distribution appears log-normal, while the Heston distribution showcases greater variability due to its incorporation of stochastic volatility.

The autocorrelation and partial autocorrelation for the first simulated price paths from the GBM and Heston models are displayed in Figure 3. The ACF for the GBM path shows gradual decay, while the Heston path's ACF is more erratic, reflecting the model's stochastic volatility.

Figure 4 illustrates the rolling volatilities of the simulated price paths. The volatility for the GBM path remains relatively stable, while the Heston's shows pronounced fluctuations, underscoring the impact of stochastic volatility in the Heston model.

The boxplots in Figure 5 provide a comparative view of the distributions of the final prices for both models. The Heston model's final prices exhibit a wider distribution, with a more pronounced spread between the 25th and 75th percentiles compared to the GBM.

Table 1 tabulates the descriptive statistics for the final prices from the GBM and Heston models. A statistically significant difference is noted between the means of the final stock prices generated by the two models.

In conclusion, the choice between the GBM and Heston models depends on the specific asset and market conditions. While the Heston model may provide a more realistic representation in certain scenarios due to its allowance for stochastic volatility, it does come with added complexity in terms of parameter estimation and computational demands.

5 Future Work

For future projects, there are several pathways to further enhance the performance and realism of the Monte Carlo simulations. Here are a few potential directions:

5.1 Parallel Computing Implementation

Monte Carlo simulations are inherently parallelizable, as each simulation is independent of the others. Implementing parallel computing techniques, such as multithreading or distributed computing, can significantly reduce the computational time. This will be particularly beneficial for handling larger datasets or increasing the number of simulations for higher accuracy.

5.2 Extensions to American and Exotic Options

The current implementation only prices European options, which can only be exercised at expiry. Extending the implementation to handle American options [3], which can be exercised at any time before expiry, would require the development of early exercise strategies. Additionally, more complex derivatives, like exotic options (e.g., Asian or Barrier options), could be incorporated to provide additional insights.

5.3 Inclusion of More Realistic Models

The current simulations use the Geometric Brownian Motion model and the Heston model. Although these models offer a reasonable degree of realism, there are more sophisticated models that can account for the complexities of financial markets more accurately. For instance, stochastic volatility models (e.g., the SABR model [11]) or jump-diffusion models (e.g., the Merton's jump-diffusion model [4]) can be incorporated. These models allow for more complex behaviours such as sudden jumps in price and changes in volatility, offering a more accurate estimation of option prices.

References

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6 Appendix

6.1 Code

The C++ code used in this project is available on GitHub at the following link:
`https://github.com/FaridSoroush/Options_Pricing_in_CPP`

The pseudocode representing the main steps of the Monte Carlo method used for pricing a European option in project is shown below:

Algorithm 1 European Option Pricing Using GBM and Heston Models

```
procedure OptionPricing
    Initialize parameters for GBM: S0, K, r, T, sigma, M, N
    Initialize parameters for Heston: v0, kappa, theta, xi, rho
    Define GBM, Heston and Payoff functions
    Open files for recording simulation results and paths
    dt ← T/M
    for i ∈ range(1, N+1) do
        S_gbm ← S0
        S_heston ← S0
        for j ∈ range(1, M+1) do
            Z1, Z2 ← two correlated random normal variables
            S_gbm ← GBM(S_gbm, r, sigma, Z1, dt)
            S_heston ← Heston(S_heston, v0, r, rho, kappa, theta, xi, Z1, Z2, dt)
            Write S_gbm and S_heston to path files
        end for
        sum_payoff ← sum_payoff + payoff(S_gbm, K)
        Write S_gbm and S_heston to simulation results files
    end for
    Close files
    option_price ← exp(-r * T) * (sum_payoff / N)
    print(option_price)
end procedure
```

6.2 Figures and Tables

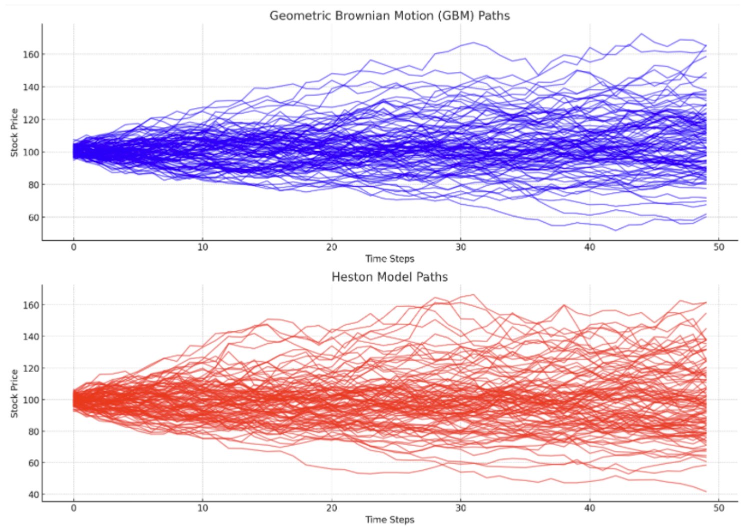

Figure 1: Simulated price paths for GBM (top) and Heston model (bottom).

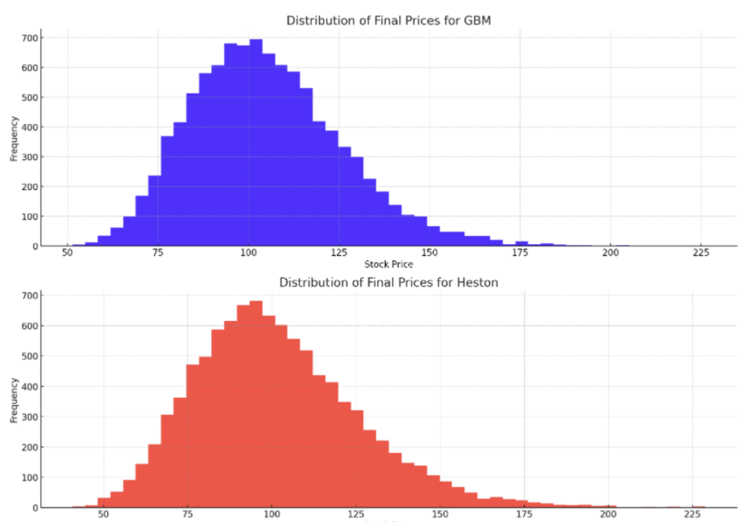

Figure 2: Distribution of final prices for GBM (top) and Heston model (bottom).

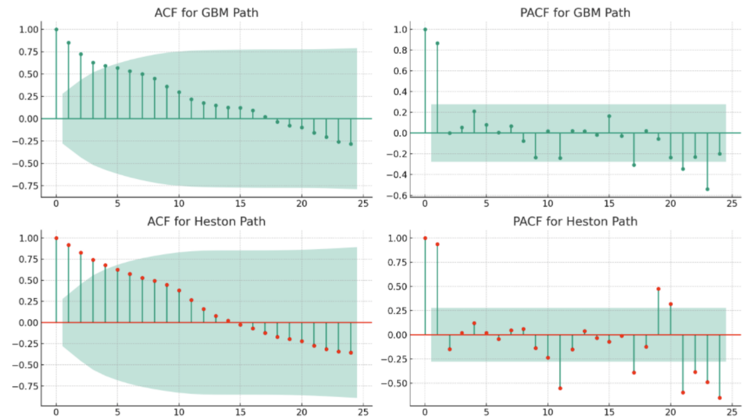

Figure 3: ACF and PACF for GBM (top) and Heston model (bottom).

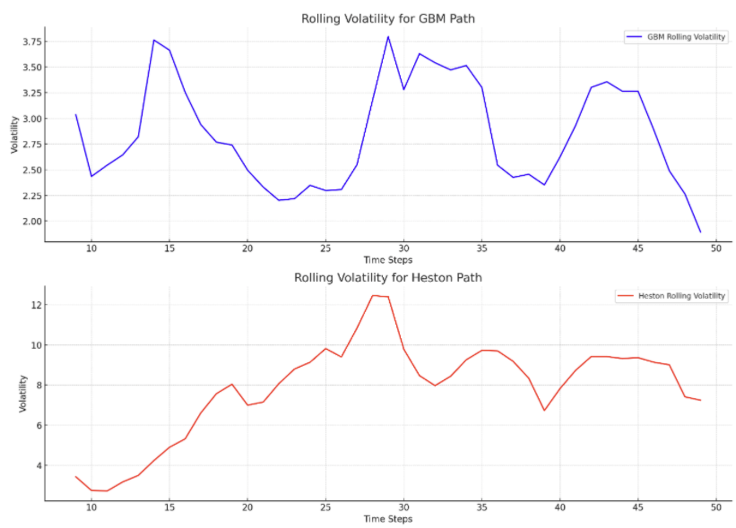

Figure 4: Rolling volatility for GBM (top) and Heston model (bottom).

Statistic	**GBM**	**Heston**
Mean	105.02	101.66
Standard Deviation	21.00	24.65
Median	102.97	98.91
25th Percentile	89.92	84.31
75th Percentile	117.72	116.24
Minimum	51.41	40.76
Maximum	226.13	228.80

Table 1: Descriptive statistics for GBM and Heston final prices.

Figure 5: Boxplots of final prices for GBM and Heston models.