HIGHER RANK SUBSTITUTIONS FOR TENSOR DECOMPOSITIONS. I. DIRECT SUM CONJECTURES

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Abstract. The *substitution method* of tensor rank computation is a higher dimensional analogue of Gaussian elimination, and it builds on the fact that the removal of a rank one slice s and a subsequent addition of arbitrary scalar multiples of s to all other slices of the same direction decreases the minimum rank exactly by one. We explain how to embed an initial tensor T to a larger linear space and replace its higher rank slice σ by a family φ of rank one slices in the new space so that the substitutions performed with respect to σ in every direction of T have the same effect on the minimum rank as the corresponding substitutions with respect to φ. We present several applications, which include a resolution of the well known and widely studied direct sum conjecture for Waring ranks and a strong form of counterexamples to Strassen's conjecture.

1. Introduction

Let f be a homogeneous polynomial of the degree d with coefficients in $\mathbb{C}$. The *Waring rank* $\mathrm{WR}(f)$ is the smallest integer r for which the equality

$$f = c_1 \left(\ell_1\right)^d + \ldots + c_r \left(\ell_r\right)^d \tag{1.1}$$

is satisfied with appropriate complex numbers $c_1, \ldots, c_r$ and linear forms $\ell_1, \ldots, \ell_r$ with coefficients in $\mathbb{C}$. For $d = 1$, this concept is trivial because the homogeneous polynomials of the degree one are linear forms themselves. The assumption $d = 2$ makes f a *quadratic form*, and the equality of its Waring rank to the rank of the corresponding matrix is a basic result of linear algebra. For $d \geqslant 3$, the situation is much more complicated, but still the existence of a decomposition of the form (1.1) is guaranteed by the choice of $\mathbb{C}$ as the ground field, so the Waring rank of a given polynomial is a well defined number [65]. The initial motivation of this work is the following *additivity conjecture*, which relates to an old research paper of Strassen [100] and is being extensively discussed in current literature.

Conjecture 1.1 ([9, 20, 26, 28, 32, 34, 35, 44, 48, 51, 54, 59, 82, 89, 104])**.** *Let f and g be homogeneous polynomials of the degree $d \geqslant 3$ with coefficients in $\mathbb{C}$. Then*

$$\mathrm{WR}(f+g) = \mathrm{WR}(f) + \mathrm{WR}(g) \tag{1.2}$$

if the variable families corresponding to f and g are disjoint.

This problem arises from studies in computational complexity [18, 64, 80, 102, 108], algebraic geometry [4, 55, 62, 69, 73] and linear algebra [3, 38, 52, 83, 110], and it can be thought of as an algebraic viewpoint on the celebrated conjecture of Strassen on the additivity of tensor ranks [5, 15, 17, 56, 71, 88, 90, 100]. Indeed,

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2010 *Mathematics Subject Classification.* 15A03, 15A69, 68Q17.

Key words and phrases. Waring rank, tensor rank, direct sum conjecture, Comon's conjecture.

the symmetric version of the tensor decomposition problem and Waring rank can be naturally expressed in terms of secant varieties, and many researchers with the expertise in algebraic geometry actively work on these concepts [1, 6, 12, 13, 14, 39, 49, 77, 84, 85, 91]. Also, apart from algebraic geometry, these notions arise in the complexity of matrix multiplication [36], parametrized algorithms [82] and other theoretical studies [70]. As explained in the foundational paper [41], the symmetric decompositions appear in various contexts of applied mathematics, which include speech, mobile communications, machine learning, factor analysis of k-way arrays, biomedical engineering, psychometrics, and chemometrics [40, 42, 74, 97, 99].

Subsequently, Conjecture 1.1 has attracted a great deal of independent interest and survived multiple solution attempts, as seen from many recent papers [26, 27, 28, 29, 32, 34, 51, 104, 111] motivated by this problem. It is also remarkable that, apart from its deep connections to multiple contexts in mathematics, Conjecture 1.1 has a very accessible and natural formulation, so it has been an attractive topic to be discussed during a research talk [23, 25, 67, 98, 105] and to be proposed to an academic research group or a graduate student [63, 68, 106, 107], and, indeed, the problem appeared in several degree theses [33, 79, 81, 86, 87]. For a more detailed review of the origins of the problem and related work, we refer the reader to Section 3 below and to further research papers [9, 20, 21, 22, 24, 35, 44, 48, 54, 59, 82, 89, 94, 103] discussing Conjecture 1.1 and its equivalent versions.

The purpose of this work is to build a framework that allows one to adapt the well known substitution method of tensor rank computation [2, 60, 71] to the cases when the slices to be used in the substitutions are not rank one. Indeed, assume that we are given an $n \times n \times n$ tensor T and a finite family σ of $n \times n$ matrices over a field $\mathbb{F}$, and the question is to find the minimal possible rank of a tensor that can appear after the addition of arbitrary $\mathbb{F}$-linear combinations of σ to every slice in every direction of T. If all matrices in σ are rank one, then the substitution method gives an immediate answer, see [2, 60, 71] or Lemma 4.9 below. However, the study of various problems would benefit from the ability of using this method in general case, and, in fact, this paper explains how to embed T to a larger linear space and replace the family σ that may potentially contain matrices of large rank by another family φ of rank one matrices so that the substitutions performed with respect to σ in every direction of T have the same effect on the minimum rank as the corresponding substitutions with respect to φ. We present several applications of this technique, and one of our main results is the refutation of Conjecture 1.1.

Theorem 1.2. *For some integer $n \geqslant 1$, there exist homogeneous polynomials*

$$f \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } g \in \mathbb{C}[y_1, \ldots, y_n]$$

of the degree three such that $\mathrm{WR}(f+g) < \mathrm{WR}(f) + \mathrm{WR}(g)$.

In fact, we give a strong form of a counterexample to the classical direct sum conjecture of Strassen [100]. Namely, for any infinite field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2, 3$, we build a pair of three-dimensional symmetric tensors such that the symmetric rank of their direct sum is less than the sum of their ranks, even if the corresponding symmetric rank is computed with respect to $\mathbb{F}$ while the conventional ranks are taken with respect to the algebraic closure or any other field containing $\mathbb{F}$.

Remark 1.3. The equality of the rank and symmetric rank of a symmetric tensor was posited in a well known conjecture that was refuted earlier [41, 53, 93, 96].

2. Prerequisites

The material presented in this paper is almost self-contained. In fact, no specific expertise is required for a reader except that

- the substitution method of tensor rank computation as in [2, Appendix B], [60, Lemma 2] or [70, Proposition 5.3.1.1] is used throughout, and
- several results in algebraic geometry are needed in Section 7, namely, the existence of large generic subspaces for the row spaces of matrices that are close to generic ones as in [94, Lemma 19] and the evaluation of the rank of a generic $n\times n\times n$ tensor over an algebraically closed field [75, Theorem 4.4].

The forthcoming Sections 3–5 collect several standard notational conventions, basic results, the formulations of the main results and the overview of our approach. The technical part begins in Sections 6 and 7, which can be read separately from each other, but the material of each of the remaining Sections 8–12 builds on the preceding parts of the paper, so these sections should be read in sequence.

3. Tensors and decompositions

We proceed with several notational conventions valid over an arbitrary field $\mathbb{F}$. We define a *tensor* as a three-dimensional array T with elements $T(i|j|k)$ in $\mathbb{F}$, where i, j, k run over the corresponding finite *indexing sets* I, J, K. We say that this tensor is $I \times J \times K$, and the expression $|I| \times |J| \times |K|$ is called the *size* of T.

Remark 3.1. Our discussion is restricted to three-dimensional tensors for simplicity of the notation and because they are sufficient for the purpose of this paper.

Remark 3.2. We write $[T]_{ijk}$ or $T(i|j|k)$ to denote the (i, j, k) entry of a tensor T.

An $I \times J \times K$ tensor T is called *decomposable* if there exists

$$(a, b, c) \in \mathbb{F}^I \times \mathbb{F}^J \times \mathbb{F}^K$$

such that $T = a \otimes b \otimes c$, which means that

$$T(i|j|k) = a_i b_j c_k$$

for all $i \in I$, $j \in J$, $k \in K$. Further, an $I\times J\times K$ tensor T is *symmetric* if $I = J = K$ and the equality $T(i|j|k) = T(i'|j'|k')$ holds whenever (i, j, k) is a permutation of (i', j', k'). Two standard notions of the rank of a symmetric tensor are as follows.

Definition 3.3. Let $\mathbb{F} \subseteq \mathbb{K}$ be a pair of fields, and let T be a tensor over $\mathbb{F}$. The *rank* of T *with respect to* $\mathbb{K}$ is the smallest integer r such that T can be written as a sum of r tensors decomposable over $\mathbb{K}$. This quantity is denoted by $\operatorname{rk}_{\mathbb{K}} T$.

Definition 3.4. Let $\mathbb{F} \subseteq \mathbb{K}$ be a pair of fields, and let T be a symmetric tensor with entries in $\mathbb{F}$. The *symmetric rank* of T *with respect to* $\mathbb{K}$ is the smallest integer r such that T can be written as a $\mathbb{K}$-linear combination of r symmetric tensors decomposable over $\mathbb{K}$. This quantity is denoted by $\operatorname{srk}_{\mathbb{K}} T$.

Remark 3.5. The values of rk and srk may depend on the choice of $\mathbb{K}$, see [8, 45].

Let $S^3\mathbb{F}^n \subset \mathbb{F}[x_1, \ldots, x_n]$ be the space of homogeneous polynomials of the degree three, and let V be an $\mathbb{F}$-linear space with basis $(e_1, \ldots, e_n)$. If $\operatorname{char} \mathbb{F} \neq 2, 3$, then we can define the $\mathbb{F}$-linear mapping $\omega : S^3\mathbb{F}^n \to V \otimes V \otimes V$ by the formula

$$x_i x_j x_k \to \frac{1}{6} \sum_{\pi\in \mathrm{Sym}_3} e_{\pi_i} \otimes e_{\pi_j} \otimes e_{\pi_k}$$

in which Sym_3 is the group of all permutations of the letters (i, j, k). It is easy to see that ω is a bijection between $S^3\mathbb{F}^n$ and symmetric $n \times n \times n$ tensors, and it gives a one-to-one correspondence between powers of linear forms and decomposable tensors $a \otimes a \otimes a$ with $a \in V$. In particular, the Waring rank of a polynomial equals the symmetric tensor rank of its image under ω, see also [10, 73]. We are almost ready to reformulate Conjecture 1.1 in the language of tensor decompositions.

Definition 3.6. For $q = 1, 2$, we assume that T_q is an $I_q \times J_q \times K_q$ tensor. If the indexing sets $I_1, I_2, J_1, J_2, K_1, K_2$ are disjoint, then the *direct sum* $T_1 \oplus T_2$ is the

$$(I_1 \cup I_2) \times (J_1 \cup J_2) \times (K_1 \cup K_2)$$

tensor with T_1 at the $(I_1|J_1|K_1)$ block and T_2 at the $(I_2|J_2|K_2)$ block, and with all entries outside these blocks zero. If the indexing sets of T_1 and T_2 are not disjoint, then we can still define $T_1 \oplus T_2$ as the direct sum of the corresponding copies of T_1 and T_2 obtained by an appropriate relabeling of the indexing sets.

Now we recall the symmetric version of Strassen's direct sum conjecture, which is equivalent to Conjecture 1.1 and which we restrict to the case $d = 3$.

Conjecture 3.7. *If A, B are symmetric tensors over $\mathbb{C}$, then*

$$\mathrm{srk}_{\mathbb{C}}(A \oplus B) = \mathrm{srk}_{\mathbb{C}}\, A + \mathrm{srk}_{\mathbb{C}}\, B. \tag{3.1}$$

We proceed with a short survey of the related work. The equality

$$\mathrm{rk}_{\mathbb{C}}(A \oplus B) = \mathrm{rk}_{\mathbb{C}}\, A + \mathrm{rk}_{\mathbb{C}}\, B \tag{3.2}$$

was conjectured by Strassen [100] for all pairs of tensors (A, B) without assuming that the tensors are symmetric. The border rank analog of the direct sum conjecture was disproved in a seminal paper of Schönhage [90], and the corresponding method has lead to important progress towards understanding the algorithmic complexity of matrix multiplication [43, 80, 101]. Several further very recent works studied the possible additivity of the direct sums for other rank functions of tensors, and the analytic rank [76], geometric rank [66], G-stable rank [46] and slice rank [57] were shown to be additive, while the subrank turned out to be not additive [47].

Concerning the property (3.2) itself and its similarity to Conjecture 3.7, the positive results on (3.2) can sometimes be translated to the setting of Conjecture 3.7 despite the failure of Comon's conjecture [41, 93, 96], which posited the equality of the rank and symmetric rank. More precisely, the equality (3.1) follows from its non-symmetric counterpart (3.2) whenever the tensors A and B are both symmetric and satisfy $\mathrm{srk}_{\mathbb{C}}\, A = \mathrm{rk}_{\mathbb{C}}\, A$ and $\mathrm{srk}_{\mathbb{C}}\, B = \mathrm{rk}_{\mathbb{C}}\, B$, and hence the conditions sufficient for the validity of Conjecture 1.1 can be obtained from those positive results on the original additivity conjecture of Strassen [3, 17, 18, 15, 20, 52, 56, 64, 71, 88, 90, 100, 110] which belong to the range where Comon's conjecture is known to be true [4, 19, 37, 41, 53, 61, 89, 112, 113]. In particular, Buczyński, Postinghel and Rupniewski [20] prove the equality (3.2) for those tensors A and B such that either A or B has the rank not exceeding six, and this positive result remains best known in terms of the ranks of (A, B). Using this result together with the known sufficient conditions of the validity of Comon's conjecture, one can confirm the equality (1.2) for all polynomials $f \in \mathbb{C}[x_1, \ldots, x_m]$ and $g \in \mathbb{C}[y_1, \ldots, y_n]$ of the degree three such that $\mathrm{WR}(f) \leqslant 6$ and at least one of the following conditions is satisfied:

- g has border rank at most two [4],

- g has Waring rank at most seven [89],
- g has Waring rank at most its flattening rank plus two [53],
- g is a generic polynomial of Waring rank at most $1.5n - 1$ [37],
- g is one of the Coppersmith–Winograd tensors [71, 72],
- g is the permanent of a generic 3×3 matrix [39, 95].

Several further results can be seen as sufficient conditions of both the equality of the ranks and the non-symmetric additivity conjecture at the same time [5], and hence they give additional special cases of the validity of Conjecture 3.7. However, since the main objective of our paper is the symmetric version of Strassen's conjecture, the remaining part of this short survey of the related work has an emphasis on those results on Conjectures 1.1 and 3.7 that do not appear by combining the work on their non-symmetric version and Comon's conjecture, and we refer to [20, 38, 34, 70, 94] for some further recent information on the non-symmetric case.

Indeed, in contrast to its non-symmetric counterpart, Conjecture 3.7 is not trivial even if one of the tensors is $1 \times \ldots \times 1$, which corresponds to the polynomial

$$x^d + g(y_1, \ldots, y_n)$$

in the equivalent notation of Conjecture 1.1. Several examples of this kind are collected in [59] and [73], and a full treatment of this case was given by Carlini, Catalisano and Chiantini in [26]. More generally, the paper [26] confirms the conclusion of Conjecture 1.1 whenever one of the following conditions is satisfied:

- either f or g is a power of a linear form,
- both f and g depend on two variables.

A similar result was proved by García-Marco, Koiran and Pecatte [54], but they worked with an analogue of Conjecture 1.1 corresponding to a certain model of computation more general than the Waring rank. Other major progress came in 2012, when Carlini, Catalisano and Geramita computed the Waring ranks of the sums of several pairwise coprime monomials [29]. More precisely, they showed

$$\mathrm{WR}\,(x_1^{a_1} x_2^{a_2} \ldots x_t^{a_t}) = (a_2 + 1)(a_3 + 1) \ldots (a_t + 1) \tag{3.3}$$

with $1 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_t$ and also

$$\mathrm{WR}(F_1 + \ldots + F_k) = \mathrm{WR}(F_1) + \ldots + \mathrm{WR}(F_k) \tag{3.4}$$

if $F_1, \ldots, F_k$ are monomials which have equal total degrees and depend on pairwise disjoint variable families, which means that Conjecture 1.1 holds in the class of the sums of such monomials. Carlini, Kummer, Oneto and Ventura [31] proved that the equality (3.3) holds for the real Waring rank if $a_1 = 1$, which shows that the real versions of Conjectures 1.1 and 3.7 are valid for the corresponding class of monomial sums. A subsequent result of Brustenga i Moncusí and Masuti [22] agrees to the special case of Conjecture 1.1 in which both f and g are binomials, and, moreover, the authors of [22] obtain exact formulas for the Waring rank of the sum of two binomials even if they are not coprime. Further bounds on the sums of not necessarily coprime monomials were given by Carlini and Ventura [32] as a part of the study of the *simultaneous Waring rank* $\mathrm{WR}_{\mathrm{sim}}\{F_1, \ldots, F_n\}$. This quantity is known to satisfy the inequalities

$$\mathrm{WR}(F_1 + \ldots + F_k) \leqslant \mathrm{WR}_{\mathrm{sim}}\{F_1, \ldots, F_n\} \leqslant \mathrm{WR}(F_1) + \ldots + \mathrm{WR}(F_k),$$

which demonstrate the relation to the direct sum conjecture, because, whenever the simultaneous Waring rank of a family is strictly less than the sum of the Waring ranks of its elements, the Waring rank is strictly subadditive [32]. In a different line of research, Casarotti [33] and Casarotti, Massarenti and Mella [34] recall another sufficient condition to the validity of Conjecture 1.1 that arises to an earlier textbook by Iarrobino and Kanev [62] and give a development of this condition, namely, the result of Conjecture 1.1 is true whenever there exists an integer s such that the Waring ranks of f and g are equal to the corresponding dimensions of the linear spaces spanned by the partial derivatives of f and g of the order s. In fact, Teitler [104] used a similar assumption to deal with a natural stronger version of the problem, which is given a more detailed discussion and appears as Conjecture 3.8 below. Later on, Carlini, Catalisano, Chiantini, Geramita and Woo [27] introduced the notion of *e-computablilty* and proved the equality (3.4) for various families of homogeneous polynomials (F_i) of equal degrees with disjoint variable families, including those in which every F_i has one of the following forms:

- a monomial,
- a form with at most two variables,
- a Vandermonde determinant,
- $x^a(y^b + z^b)$ with arbitrary a and b.

In a subsequent work [28], Carlini, Catalisano and Oneto introduced the notion of a *Waring locus* motivated by Conjecture 1.1, which turned out to be of a considerable independent interest [7, 30, 78, 109]. Also, the authors of [28] developed several of the sufficient conditions above to satisfy a natural stronger version of Conjecture 1.1.

Conjecture 3.8 ([28, 79, 104])**.** *Any decomposition of the form $F_1 + \ldots + F_n$ into the sum of the minimal possible number of the powers of linear forms is the sum of the corresponding minimal decompositions of given homogeneous polynomials $(F_1, \ldots, F_n)$ with coefficients in $\mathbb{C}$ and with the same degree $d \geqslant 3$ each, provided that the variable families involved in $(F_1, \ldots, F_n)$ are pairwise disjoint.*

Although a potential generalization of Conjecture 3.8 for $d = 2$ is false,

$$x^2 - 2yz = (x + y)^2 + (x + z)^2 - (x + y + z)^2$$

while $\mathrm{WR}(x^2) = 1$ and $\mathrm{WR}(yz) = 2$, the conjecture itself has not been invalidated until this date [28, 79, 104]. Nevertheless, several sufficient conditions of its validity were known, including the above mentioned case of the polynomials whose Waring rank is certified by the catalecticant bound [104] and the sums of those pairs of monomials one of which has the lowest exponent equal to one [28, 79].

We disprove Conjectures 1.1, 3.7 and 3.8. Our main result is as follows.

Theorem 3.9. *If $\mathbb{F}$ is an infinite field with $\operatorname{char}\mathbb{F} \neq 2, 3$, then there exist symmetric tensors $\mathfrak{A}$, $\mathfrak{B}$ with entries in $\mathbb{F}$ such that, for any field extension $\mathbb{K} \supseteq \mathbb{F}$, one has*

$$\operatorname{srk}_{\mathbb{F}}(\mathfrak{A} \oplus \mathfrak{B}) < \operatorname{rk}_{\mathbb{K}} \mathfrak{A} + \operatorname{rk}_{\mathbb{K}} \mathfrak{B}.$$

In view of the basic fact that $\mathrm{rk} \leqslant \mathrm{srk}$, this theorem invalidates the equality (3.1) in Conjecture 3.7 even if $\mathbb{C}$ is replaced by an infinite field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2, 3$. Also, this result disproves Conjecture 1.1, which corresponds to the case $\mathbb{F} = \mathbb{K} = \mathbb{C}$.

4. Basic techniques and some further notation

This section collects several standard results and notational conventions that we require in our approach to Theorem 3.9. In fact, these notations are also needed in the detailed overview of our approach, which we give in Section 5 below.

Remark 4.1. All results given in this section are indeed standard, and several of them are well known. All of them are probably clear to the experts in the topic of the paper, but, in some cases, we give detailed proofs for completeness.

One particular goal of this section is to recall the basic *substitution method* for lower bounds on the ranks of tensors. The corresponding statements are standard, and we refer an interested reader to Lemma 2 in [60] for an old appearance of this technique and to Proposition 3.1 in [71] for a more recent account.

Definition 4.2. If T is an $I \times J \times K$ tensor and $k \in K$, then we define the k-th 3-*slice* of T as the $I \times J$ matrix whose (i, j) entry equals $T(i|j|k)$. For all $i \in I$ and $j \in J$, we define the ith 1-slice of T and the jth 2-slice of T in a similar way.

Definition 4.3. Let $I \subseteq I'$, $J \subseteq J'$, $K \subseteq K'$ be indexing sets. We say that an $I' \times J' \times K'$ tensor T' is the *padding* of its $I \times J \times K$ block T if all entries of T' outside the $I \times J \times K$ block are zero. In this situation, we also write $T' = T(I' \times J' \times K')$.

The following important construction appeared in [93] and [94].

Definition 4.4. Let T be an $I \times J \times K$ tensor over a field $\mathbb{F}$, and assume

- (M1) $\mathcal{M}_1$ is a finite family of $J \times K$ matrices over $\mathbb{F}$,
- (M2) $\mathcal{M}_2$ is a finite family of $I \times K$ matrices over $\mathbb{F}$,
- (M3) $\mathcal{M}_3$ is a finite family of $I \times J$ matrices over $\mathbb{F}$.

Assuming that the names of the matrices in $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ have no overlap with $I \cup J \cup K$, we define the tensor $\mathcal{T}$ of the format $(I \cup \mathcal{M}_1) \times (J \cup \mathcal{M}_2) \times (K \cup \mathcal{M}_3)$ as follows. Indeed, for any $m_i \in \mathcal{M}_i$ and $(\alpha, \beta, \gamma) \in I \times J \times K$, we declare that

- (0) $\mathcal{T}(\alpha|\beta|\gamma) = T(\alpha|\beta|\gamma)$,
- (1) the m_1-th 1-slice of $\mathcal{T}$ is the padding $m_1((J \cup \mathcal{M}_2) \times (K \cup \mathcal{M}_3))$,
- (2) the m_2-th 2-slice of $\mathcal{T}$ is the padding $m_2((I \cup \mathcal{M}_1) \times (K \cup \mathcal{M}_3))$,
- (3) the m_3-th 3-slice of $\mathcal{T}$ is the padding $m_3((I \cup \mathcal{M}_1) \times (J \cup \mathcal{M}_2))$.

We say that $\mathcal{T}$ is obtained from T by *adjoining* $\mathcal{M}_1$ as the 1-slices, $\mathcal{M}_2$ as the 2-slices, and $\mathcal{M}_3$ as the 3-slices, or, simply, by adjoining $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ to T.

Definition 4.5. If we have $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3$ in the setting of Definition 4.4, and if, additionally, the matrices in $\mathcal{M}_1$ are symmetric, and the tensor T is symmetric, then the operation of taking $\mathcal{T}$ is called the *symmetrical adjoining* of $\mathcal{M}_1$ to T.

Remark 4.6. As in [93, 94], the construction of Definition 4.4 can be easily explained without the use of the notational conventions of Definition 4.3. Indeed, in particular, the item (1) of Definition 4.4 states that the m-th 1-slice of $\mathcal{T}$ becomes equal to m when restricted to $J \times K$, and it has zeros everywhere outside $J \times K$.

Notation 4.7. Let V be a linear space over a field $\mathbb{F}$. If S is an arbitrary subset of V, then the notation $S\,\mathbb{F}$ stands for the $\mathbb{F}$-linear span of S.

Definition 4.8. For any field $\mathbb{F}$ and the sets $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ as in the items (M1)–(M3) of Definition 4.4, we take $V_i = \mathcal{M}_i\,\mathbb{F}$, for all $i \in \{1, 2, 3\}$, and we define

$$T \bmod (V_1, V_2, V_3) \tag{4.1}$$

as the set of all tensors that can be obtained from T by adding elements of V_1 to the 1-slices of T, followed by the addition of elements of V_2 to the 2-slices of the resulting tensor, and, finally, by the addition of elements of V_3 to the 3-slices of what was obtained after the transformation of the 2-slices. Also, we write

$$T \operatorname{mod}_{\mathbb{F}} (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) \tag{4.2}$$

with the same meaning as (4.1), and, similarly, the notation

$$\min \operatorname{rk}_{\mathbb{F}} T \operatorname{mod} (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$$

stands for the smallest rank, computed with respect to $\mathbb{F}$, over the family (4.2).

The following is a well known special case of the substitution method.

Lemma 4.9 ([60, 71, 93, 94]). *Let T be an $I \times J \times K$ tensor over a field $\mathbb{F}$, assume*

- *$\mathcal{M}_1$ is a finite family of $J \times K$ matrices over $\mathbb{F}$,*
- *$\mathcal{M}_2$ is a finite family of $I \times K$ matrices over $\mathbb{F}$,*
- *$\mathcal{M}_3$ is a finite family of $I \times J$ matrices over $\mathbb{F}$,*

and let $\mathcal{T}$ be the result of the adjoining of $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ to T. Then

$$\operatorname{rk}_{\mathbb{F}} \mathcal{T} \geqslant \min \operatorname{rk}_{\mathbb{F}} T \operatorname{mod} (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) + \dim_{\mathbb{F}}(\mathcal{M}_1 \mathbb{F}) + \dim_{\mathbb{F}}(\mathcal{M}_2 \mathbb{F}) + \dim_{\mathbb{F}}(\mathcal{M}_3 \mathbb{F}),$$

and the equality holds if all matrices in $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ are rank-one.

We proceed with a restricted analogue of the substitution method for symmetric tensors. One particular relevant and well known example is as follows [4, 29, 41, 92].

Example 4.10. Let $|\mathbb{F}| \geqslant 4$ and $x \in \mathbb{F}$. If A is the $2 \times 2 \times 2$ symmetric tensor such that $A(1|1|1) = x$, $A(1|1|2) = 1$, $A(1|2|2) = A(2|2|2) = 0$, then $\operatorname{srk}_{\mathbb{F}} A = 3$.

We close the section with several standard observations and their corollaries.

Observation 4.11. *Let T be an $I_1 \times I_2 \times I_3$ tensor over a field $\mathbb{F}$, and let M_χ be a $J_\chi \times I_\chi$ matrix over $\mathbb{F}$, for any $\chi \in \{1, 2, 3\}$. Then $\operatorname{rk}_{\mathbb{F}} T \geqslant \operatorname{rk}_{\mathbb{F}}(M_1 \otimes M_2 \otimes M_3)\, T$.*

Observation 4.12. *If T is a symmetric $I \times I \times I$ tensor and M is a $J \times I$ matrix of the rank $|I|$, both over some field $\mathbb{F}$, then $\operatorname{srk}_{\mathbb{F}} T = \operatorname{srk}_{\mathbb{F}}(M \otimes M \otimes M)T$.*

Corollary 4.13. *For an $I \times I \times I$ tensor T with entries in a field $\mathbb{F}$, assume that*

$$T \in O \operatorname{mod}_{\mathbb{F}} (u \otimes u,\, u \otimes u,\, u \otimes u),$$

where $u \in \mathbb{F}^I$ is a vector. If $|\mathbb{F}| \geqslant 4$ and T is symmetric, then $\operatorname{srk}_{\mathbb{F}} T \leqslant 3$.

Proof. Follows from Example 4.10 due to a transformation in Observation 4.12. □

Corollary 4.14. *Let T be a symmetric $I \times I \times I$ tensor over a field $\mathbb{F}$, and let m be an $\mathbb{F}$-linear combination of the slices of T with the indexes different from some fixed $i \in I$. Then the tensor T' obtained by the successive addition of m to the i-th 1-slice, i-th 2-slice, and i-th 3-slice of T satisfies $\operatorname{srk}_{\mathbb{F}} T' = \operatorname{srk}_{\mathbb{F}} T$.*

Proof. The transformation $T \to T'$ corresponds to Observation 4.12 as well. □

5. An overview of our approach

As said above, the main result of this paper is Theorem 3.9, and, here, we outline the strategy of our proof. In general, since the computation of both the usual rank and symmetric rank are hard for a given tensor in general [58, 92], our consideration needs to be switched to more tractable instances. In particular, the *substitution method* of the previous section allows a more tractable evaluation of the ranks of tensors with many rank one slices, which explains the relevance of this method in our approach to the direct sum conjecture. Another family that allows a relatively easy rank computation comes from *generic* tensors, and one particular result relevant to our proof is the computation of the rank of a generic $n \times n \times n$ tensor by Lickteig [75]. In fact, the idea of our work on the non-symmetric version of Strassen's conjecture in [94] was a mixture of these two approaches as we combined the use of the substitution method with the lower bounds on the ranks coming from the counting of the transcendence degrees of the families of tensors close to generic ones. The first foundational result of the current paper is also based on this methodology and develops its non-symmetric counterpart in [94, Claim 6].

Claim 5.1. *Take an integer $n \geqslant 5\,104$ and an integer r for which the inequalities*

$$\frac{n^3}{3n-2} + 3n^{1.5} < r < 2 \cdot \frac{n^3 - 18n^2\sqrt{3n} - 3n^2}{3n-2}$$

are satisfied. Let $\mathbb{F}$ be an infinite field, and let

$$A_1,\ A_2,\ A_3,\ B_1,\ B_2,\ B_3$$

be pairwise disjoint sets of cardinality n each and

$$A = A_1 \cup A_2 \cup A_3,\ \ B = B_1 \cup B_2 \cup B_3,\ \ I = A \cup B.$$

Then there exists a symmetric $I \times I \times I$ tensor $\mathcal{S}$ with entries in $\mathbb{F}$ such that

$$\operatorname{srk}_{\mathbb{F}} \mathcal{S} \leqslant r < \min \operatorname{rk}_{\mathbb{K}} \mathcal{A}^\circ \bmod (U_1, U_2, U_3) + \min \operatorname{rk}_{\mathbb{K}} \mathcal{B}^\circ \bmod (V_1, V_2, V_3) \tag{5.1}$$

holds over any field $\mathbb{K}$ containing $\mathbb{F}$, where

(U1) U_1 is the $\mathbb{K}$-linear span of the 1-slices of the $B \times A_2 \times A_3$ block of $\mathcal{S}$,
(U2) U_2 is the $\mathbb{K}$-linear span of the 2-slices of the $A_1 \times B \times A_3$ block of $\mathcal{S}$,
(U3) U_3 is the $\mathbb{K}$-linear span of the 3-slices of the $A_1 \times A_2 \times B$ block of $\mathcal{S}$,
(V1) V_1 is the $\mathbb{K}$-linear span of the 1-slices of the $A \times B_2 \times B_3$ block of $\mathcal{S}$,
(V2) V_2 is the $\mathbb{K}$-linear span of the 2-slices of the $B_1 \times A \times B_3$ block of $\mathcal{S}$,
(V3) V_3 is the $\mathbb{K}$-linear span of the 3-slices of the $B_1 \times B_2 \times A$ block of $\mathcal{S}$,

and also $\mathcal{A}^\circ$ is the $A_1 \times A_2 \times A_3$ block of $\mathcal{S}$, and $\mathcal{B}^\circ$ is the $B_1 \times B_2 \times B_3$ block of $\mathcal{S}$.

In fact, Claim 5.1 appears to be a combination of the above mentioned ideas of [94] with the constructions of the symmetric tensors in which effective lower bounds on the corresponding symmetric ranks can come from the computation of the more tractable values of the ranks of their non-symmetric blocks, which has been previously used in our determination of the algorithmic complexity of the symmetric tensor rank [92]. Indeed, the right hand side of the inequality (5.1) in the formulation of Claim 5.1 above is the lower bound of the rank of a symmetric tensor in the form of the rank of its non-symmetric $A_1 \times A_2 \times A_3$ block. As we will see later, the purpose of Claim 5.1 is not to only give a symmetric adaptation of [94, Claim 6] but also address the criticism raised in [87] and in the abstract of [20], which states that the counterexamples in [94] *are not very explicit, and they are only known to exist asymptotically for very large tensor spaces.* Namely, the lower bound on the size of the tensors in Claim 5.1 is quite small, and, later on, this fact will allow us to deduce an explicit and not absolutely unreasonably huge upper bound on the size of the smallest counterexample to the direct sum conjecture.

However, the main source of the complexity of our approach and counterexamples in the non-symmetric version of the direct sum conjecture was [94, Claim 5]. We proceed with several auxiliary definitions needed to formulate the main technical result of this paper, and, when this is done, we return to a more detailed comparison of Claim 5 in [94] to the technique presented here, see Remark 5.10 below.

Definition 5.2. A symmetric matrix A is said to be a *skew projector* over a field $\mathbb{F}$ if there exists a non-singular square matrix C over $\mathbb{F}$ such that $C^\top AC$ is a block diagonal matrix with every diagonal block equal either to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.2}$$

or to the zero 1×1 block.

Remark 5.3. We recall that the direct sum of an identity matrix and a zero matrix represents the projection as a linear operator. Instead, we consider a matrix that can be seen as the direct sum of the skew identity matrix and a zero matrix, which fact motivated the choice of the term '*skew projector*' in Definition 5.2.

As we will see in Claim 5.6, we are particularly focused on skew projectors of the ranks which are the powers of two, so we also adopt the following convention.

Notation 5.4. For a field $\mathbb{F}$, a positive integer k, and an indexing set I, we use $\operatorname{skp}(\mathbb{F}, k, I)$ to denote the set of all $I \times I$ skew projectors of the rank 2^k over $\mathbb{F}$. Also, $\operatorname{skp}(\mathbb{F}, 0, I)$ denotes the set of all symmetric rank one $I \times I$ matrices over $\mathbb{F}$.

As said above, this paper gives an effective bound on the size of the smallest counterexample to Strassen's conjecture, apart from giving an existence proof. In order to discuss and formulate this bound, the following functions are useful.

Notation 5.5. We use the abbreviation

$$H(k,\rho) = (132\,000\,000)^{k-1} \cdot \left(\prod_{t=2}^{k} \left\lceil \frac{\rho}{2^{t-2}} \right\rceil \right)^4$$

to define the functions

$$\mu(k,\rho,w) = w \cdot \left(5 \cdot 10^{194} \cdot (\rho+18)^{40} + 3k \cdot 10^7 \cdot 2^k \cdot \rho^4\right) \cdot H(k,\rho),$$
$$\sigma(k,\rho,w) = w \cdot 15 \cdot 10^{194} \cdot (\rho+18)^{40} \cdot H(k,\rho)$$

each of which maps a tuple of three positive integers (k, ρ, w) to a positive integer.

The second main stepping stone towards Theorem 3.9 is as follows.

Claim 5.6. *Let $k \geqslant 0$ and $\rho \geqslant 1$ be integers, let I be a finite indexing set, let $\mathbb{F}$ be a field with $\operatorname{char}\mathbb{F} \neq 2, 3$, and let W be a finite subset in $\operatorname{skp}(\mathbb{F}, k, I)$. Then there exist a set $I' \supseteq I$ and a family $\mathcal{M}$ of symmetric rank one $I' \times I'$ matrices with*

$$|I'| - |I| \leqslant \mu\left(k, \rho, |W|\right), \quad |\mathcal{M}| \leqslant \sigma\left(k, \rho, |W|\right),$$

where μ and σ are as in Notation 5.5, and also

(i) the $\mathbb{F}$-linear span of $\mathcal{M}$ contains the padding $w(I' \times I')$ of every $w \in W$, and
(ii) for any field $\mathbb{K} \supseteq \mathbb{F}$ and any $I \times I \times I$ tensor T with $\operatorname{rk}_{\mathbb{F}} T \leqslant \rho$, we have

$$\min \operatorname{rk}_{\mathbb{K}} T(I' \times I' \times I') \operatorname{mod} (\mathcal{M}, \mathcal{M}, \mathcal{M}) = \min \operatorname{rk}_{\mathbb{K}} T \operatorname{mod} (W, W, W). \tag{5.3}$$

Remark 5.7. For a fixed ρ, the construction in Claim 5.6 is a polynomial reduction, so our results might also be of a particular independent interest in the study of the algorithmic complexity of tensor ranks in the direct sum problems and in general.

As we can see, Claim 5.6 does not allow the matrices in W to be arbitrary, but rather it requires an additional technical assumption that every matrix in W is a skew projector. We require W to be of this form because it allows a proof of Claim 5.6 that we managed to come up with, and also this assumption does not add any significant complications to our argument in Section 6 below.

Remark 5.8. Claim 5.6 may look similar to the results in Section 2 in [94] and its main conclusion [94, Corollary 14]. For the sake of a further discussion, we give a sightly more general formulation of the latter result matching the notation in the current study. Here, the notation $\overline{\mathbb{F}}$ stands for the algebraic closure of a field $\mathbb{F}$.

Proposition 5.9 (see Corollary 14 in [94])**.** *If W_1, W_2, W_3 are finite sets of $J \times K$, $I \times K$, $I \times J$ matrices over a field $\mathbb{F}$, respectively, then there are finite indexing sets $I' \supseteq I$, $J' \supseteq J$, $K' \supseteq K$ and finite sets $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ of the $J' \times K'$, $I' \times K'$, $I' \times J'$ rank one matrices over $\mathbb{F}$ so that, for all $\chi \in \{1, 2, 3\}$, every matrix in W_χ has the padding in $\mathcal{M}_\chi \mathbb{F}$, and, for any $I \times J \times K$ tensor T over $\mathbb{F}$, one has*

$$\min \operatorname{rk}_{\overline{\mathbb{F}}} T(I \times J' \times K') \bmod (\mathcal{M}_1, 0, 0) = \min \operatorname{rk}_{\overline{\mathbb{F}}} T \bmod (W_1, 0, 0), \tag{5.4}$$

$$\min \operatorname{rk}_{\overline{\mathbb{F}}} T(I' \times J \times K') \bmod (0, \mathcal{M}_2, 0) = \min \operatorname{rk}_{\overline{\mathbb{F}}} T \bmod (0, W_2, 0), \tag{5.5}$$

$$\min \operatorname{rk}_{\overline{\mathbb{F}}} T(I' \times J' \times K) \bmod (0, 0, \mathcal{M}_3) = \min \operatorname{rk}_{\overline{\mathbb{F}}} T \bmod (0, 0, W_3). \tag{5.6}$$

The further considerations of this paper show that Claim 5.6 gives independent solutions to Strassen's direct sum problem in both the classical and symmetrical cases. In contrast, although we used Proposition 5.9 in the initial take on the classical version [94], it does not seem to be applicable to the symmetric counterpart of the problem. In general, a potential use of the approach of Proposition 5.9 faces at least two critical obstructions in the current study which are as follows.

Remark 5.10. Claim 5 in [94] is stated as in Proposition 5.9 but with

$$\min \operatorname{rk}_{\overline{\mathbb{F}}} T(I' \times J' \times K') \bmod (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) = \min \operatorname{rk}_{\overline{\mathbb{F}}} T \bmod (W_1, W_2, W_3) \tag{5.7}$$

replacing (5.4)–(5.6), and the example of [94] used the adjoining of $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ to $T(I' \times J' \times K')$. This construction can only give a symmetric tensor if

$$\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3, \tag{5.8}$$

and one of the obstructions lies in the fact that (5.8) is not guaranteed even if

$$I' = J' = K' \text{ and } W_1 = W_2 = W_3 \tag{5.9}$$

in the formula (5.7). In particular, one natural approach to a proof of [94, Claim 5] requires three subsequent applications of Proposition 5.9 for each of the directions corresponding to the formulas (5.4), (5.5) and (5.6), respectively, where, at the χ-th such application with $\chi \in \{1, 2, 3\}$, one defines the tensor T_χ by the adjoining of the resulting set $\mathcal{M}_\chi$ as the χ-slices to the tensor $T_{\chi-1}$ appearing after the $\chi - 1$ previous applications. However, the construction of $\mathcal{M}_\chi$ depends on the size of the initial tensor in Proposition 5.9, and, due to the adjoining operation, the sizes and ranks should grow as the value of χ progresses to further steps. Therefore, this approach can never return a symmetric tensor except several trivial cases.

Remark 5.11. Of course, the inductive approach taken in the last paragraph of Section 2 in [94] does not preserve the symmetry either. Indeed, if we have

$$(T, I, J, K, W_1, W_2, W_3)$$

as in Proposition 5.9, then, in order to prove [94, Claim 5], we need to construct the corresponding families $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ of rank one matrices which satisfy (5.7). To this end, since the case $W_1 = W_2 = W_3 = \varnothing$ is trivial, we pick $u_3 \in W_3$ without loss of generality, and we write $U_3 = W_3 \setminus \{u_3\}$. As suggested in [94], we further

(S1) apply Proposition 5.9 to the family $(\varnothing, \varnothing, \{u_3\})$ and
(S2) use the inductive assumption with the paddings of (W_1, W_2, U_3),

and the step (S1) gives two indexing sets $I'' \supseteq I$, $J'' \supseteq J$ and a family $\mathcal{U}_3$ of several $I'' \times J''$ rank one matrices such that the padding of u_3 is in $\mathcal{U}_3\,\mathbb{F}$, and the condition

$$\min \operatorname{rk}_{\overline{\mathbb{F}}}\, \tau(I'' \times J'' \times K) \bmod (0,\, 0,\, \mathcal{U}_3) = \min \operatorname{rk}_{\overline{\mathbb{F}}}\, \tau \bmod (0,\, 0,\, u_3) \tag{5.10}$$

holds for any $I \times J \times K$ tensor τ with entries in $\mathbb{F}$. The corresponding step (S2) returns three families $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ of rank one matrices of the corresponding formats

$$J' \times K',\ I' \times K',\ I' \times J' \ \text{ with } \ I' \supseteq I'',\ J' \supseteq J'',\ K' \supseteq K \cup \mathcal{U}_3$$

so that every matrix in W_1 has the padding in $\mathcal{W}_1\,\mathbb{F}$, every matrix in W_2 has the padding in $\mathcal{W}_2\,\mathbb{F}$, every matrix in U_3 has the padding in $\mathcal{W}_3\,\mathbb{F}$, and the condition

$$\min \operatorname{rk}_{\overline{\mathbb{F}}}\, \widehat{\tau}(I' \times J' \times K') \bmod (\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) = \min \operatorname{rk}_{\overline{\mathbb{F}}}\, \widehat{\tau} \bmod (\omega_1, \omega_2, \upsilon_3) \tag{5.11}$$

is satisfied by any $I'' \times J'' \times (K \cup \mathcal{U}_3)$ tensor $\widehat{\tau}$ with entries in $\mathbb{F}$, where $(\omega_1, \omega_2, \upsilon_3)$ are the corresponding paddings of (W_1, W_2, U_3). Here, one could define

$$\mathcal{M}_1 = \mathcal{W}_1,\ \ \mathcal{M}_2 = \mathcal{W}_2,\ \ \mathcal{M}_3 = \mathcal{W}_3 \cup \mathcal{U}_3\,(I' \times J')\,,$$

but the proof of (5.7) might depend on a particular choice of $(\mathcal{W}_1,\, \mathcal{W}_2,\, \mathcal{W}_3,\, \mathcal{U}_3)$. In one potential attempt of showing (5.7), one could write the left hand side as

$$\min \operatorname{rk}_{\overline{\mathbb{F}}}\, \widetilde{\tau} \bmod (\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \tag{5.12}$$

with

$$\widetilde{\tau} \in T(I' \times J' \times K') \operatorname{mod}_{\overline{\mathbb{F}}} (0,\, 0,\, \mathcal{U}_3(I' \times J')),$$

which, under the further assumption that

$$\widetilde{\tau} \ \text{ is the padding of an } \ I'' \times J'' \times (K \cup \mathcal{U}_3) \ \text{ tensor,} \tag{5.13}$$

is further transformed to

$$\min \operatorname{rk}_{\overline{\mathbb{F}}}\, \widehat{\tau} \bmod (\omega_1, \omega_2, \upsilon_3) \tag{5.14}$$

with

$$\widehat{\tau} \in T(I'' \times J'' \times (K \cup \mathcal{U}_3)) \operatorname{mod}_{\overline{\mathbb{F}}} (0,\, 0,\, \mathcal{U}_3)$$

after the application of the property (5.11). Clearly, the quantity (5.14) equals

$$\min \operatorname{rk}_{\overline{\mathbb{F}}}\, \tau' \bmod (0,\, 0,\, \mathcal{U}_3) \ \text{ with } \ \tau' \in T(I'' \times J'' \times (K \cup \mathcal{U}_3)) \operatorname{mod}_{\overline{\mathbb{F}}} (\omega_1, \omega_2, \upsilon_3),$$

and, since ω_1, ω_2 are the paddings of $J \times K$, $I \times K$ matrix families, this gives

$$\min \operatorname{rk}_{\overline{\mathbb{F}}}\, \tau'' \bmod (0,\, 0,\, \mathcal{U}_3) \tag{5.15}$$

with

$$\tau'' \in T(I'' \times J'' \times K) \ \operatorname{mod}_{\overline{\mathbb{F}}} \ (W_1(J'' \times K),\, W_2(I'' \times K),\, U_3(I'' \times J'')).$$

Finally, if

$$\tau'' \ \text{ is the padding of an } \ I \times J \times K \ \text{ tensor,} \tag{5.16}$$

then we could further apply the condition (5.10) to transform (5.15) to

$$\min \operatorname{rk}_{\overline{\mathbb{F}}}\, \tau \bmod (0, 0, u_3) \ \text{ with } \ \tau \in T \ \operatorname{mod}_{\overline{\mathbb{F}}} \ (W_1, W_2, U_3), \tag{5.17}$$

which equals the right hand side of (5.7). As to the intermediate condition (5.13), we have that, indeed, $\widetilde{\tau}$ is the padding of an $I'' \times J'' \times K'$ tensor because the matrices in $\mathcal{U}_3$ are the paddings of the $I'' \times J''$ matrices. If one takes some nonzero elements of $\mathcal{U}_3\,\overline{\mathbb{F}}$ at the 3-slices of $\widetilde{\tau}$ with the indexes in $K' \setminus (\mathcal{U}_3 \cup K)$, then, at the cost of a potential increase of the rank of $\widetilde{\tau}$, this operation invalidates the use of the condition (5.11), so the desired lower bound is not immediately applicable in this case. A situation similar to the one with the condition (5.13) appears with (5.16) as well, and hence, as said above, the derivation of (5.7) might require some information on a particular construction of $(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{U}_3)$.

In view of examples discussed in Remarks 5.10 and 5.11, which show that the technique of [94] cannot be sufficient to deal with the symmetric version of Strassen's conjecture, the resolution of both the symmetric and classical cases simultaneously and independently from previous studies serves as another demonstration of the power of the framework developed in the current work. In addition, our approach has a methodological advantage over the technique in the paper [94], which was focused on the coordinate descriptions of the tensors involved in the proof, as, in Section 8, we select several properties required to construct the families in Claim 5.6 and, after having proved the sufficiency of these properties, we proceed with a particular definition of the families. This approach allows us to deal with a much more complicated construction required in the current study as, in particular, in Section 11, the proof of the validity of our construction requires the language and several techniques in combinatorial matrix theory to prove the appropriate lower bounds on the ranks of sparse tensors, which can also be of independent interest.

The technical part of the paper is structured as follows. In Section 6 below, we assume the validity of Claims 5.1 and 5.6 to give a conditional proof of Theorem 3.9 and an effective upper bound on the sizes of the smallest counterexamples to the direct sum conjecture and its symmetric version. Therefore, the main results reduce to Claims 5.1 and 5.6, and, in Section 7, we confirm the validity of Claim 5.1 with the use of the combination of several ideas, which include the substitution method of rank computation, the dimension counting approach as in [94], and the framework used in the determination of the algorithmic complexity of the symmetric rank in [92], which gives examples of symmetric tensors of the format $A \times A \times A$ which admit strong lower bounds on their ranks in terms of the non-symmetric $A_1 \times A_2 \times A_3$ blocks, where (A_1, A_2, A_3) is a partition of the initial indexing set A.

The main obstacle of our approach is Claim 5.6. We start working with it in Section 8, which treats the problem from a somewhat generalized perspective in which, namely, we do not require the matrices corresponding to W to be skew projectors, and, in addition, we do not assume that the matrices as in $\mathcal{M}$ are rank one. We define the class of matrix families, which we call *eliminating families*, by imposing several conditions that guarantee that a given family satisfies the conclusions similar to (i) and (ii) in Claim 5.6. In Section 9, we continue to study the eliminating families, and we give a relevant construction of how to get a new eliminating family from a given one. Namely, we observe that the mapping

$$X \to \begin{pmatrix} -2X & O \\ O & 2X \end{pmatrix}$$

doubles the rank of a matrix, and, also, whenever an initial matrix X is a skew projector, it turns out that the resulting matrix is a skew projector as well. This allows us to construct an eliminating family of matrices in $\operatorname{skp}(\mathbb{F}, k-1, I')$ for an arbitrary matrix in $\operatorname{skp}(\mathbb{F}, k, I)$ and for any k, provided that we are given an appropriate set Φ of symmetric rank one matrices that also appears to be an eliminating family for some rank two skew projector, and, indeed, we propose an inductive argument that reduces Claim 5.6 to the existence of such an eliminating family Φ. Indeed, an explicit construction of a sequence of families $\Phi(q)$ depending on a positive integer q is given in Section 10, and we also prove a part of the statements needed to confirm that the desired families are eliminating. In Section 11, we switch to the hardest part of the argument, which is the confirmation that $\Phi(q)$ satisfies the point (v) in Definition 8.3 below with the appropriate parameters. This is done with the extensive use of the coordinate description of the matrices in $\Phi(q)$ and some tools in combinatorial matrix theory. Finally, the remaining Section 12 completes the proof of our main result and collects several further comments on the topic.

6. A conditional proof of Theorem 3.9

The aim of this section is to deduce Theorem 3.9 from Claims 5.1 and 5.6.

Definition 6.1. For any positive integer k, we say that a $\{1,\ldots,2k\}\times\{1,\ldots,2k\}$ matrix A is *two-block diagonal* if the union of its 2×2 submatrices

$$\{1,2\}\times\{1,2\},\ \{3,4\}\times\{3,4\},\ \ldots,\ \{2k-1,2k\}\times\{2k-1,2k\}$$

covers all the non-zero entries of A.

Lemma 6.2. *Let $k\in\mathbb{Z}$, and let A be a symmetric $\{1,\ldots,2k\}\times\{1,\ldots,2k\}$ matrix over a field $\mathbb{F}$. Then there exists a pair (P,Q) of two-diagonal matrices such that*

(i) $A-P$ is a full rank skew projector over $\mathbb{F}$,

(ii) $A-Q$ is a skew projector of the rank $2k-2$ over $\mathbb{F}$.

Proof. Both statements (i) and (ii) are trivial for $k=1$, so we assume $k>1$ and proceed with the proof by the induction. We consider the matrices

$$A=\left(\begin{array}{c|c} A_0 & B^\top \\ \hline B & A_1\end{array}\right),\ \mathcal{C}=\left(\begin{array}{c|c} I_2 & O_{2\times(2k-2)} \\ \hline O_{(2k-2)\times 2} & C\end{array}\right)\left(\begin{array}{c|c} I_2 & X^\top \\ \hline O_{(2k-2)\times 2} & I_{2k-2}\end{array}\right)$$

with A being the initial matrix, where the notations I_t and $O_{p\times q}$ stand for the identity and zero matrices of the corresponding sizes, respectively, and the partitions into the blocks are such that all upper left blocks are 2×2, and all bottom right blocks are $(2k-2)\times(2k-2)$. Also, we define D as an unknown $2k\times 2k$ two-block diagonal symmetric matrix, and we see that the matrix $\mathcal{A}=\mathcal{C}(A-D)\mathcal{C}^\top$ is

$$\left(\begin{array}{c|c} A_0-D_0+X^\top B+B^\top X+X^\top(A_1-D_1)X & B^\top C^\top+X^\top(A_1-D_1)C^\top \\ \hline CB+C(A_1-D_1)X & C(A_1-D_1)C^\top\end{array}\right)$$

with D_0 and D_1 being the upper left and bottom right blocks of D, respectively.

Using the part (i) of the inductive assumption, we find a two-block diagonal matrix D_1 and a non-singular matrix C such that the bottom-right block of $\mathcal{A}$ has the form of the block diagonal matrix with all blocks equal to (5.2). In particular, the matrix A_1-D_1 is non-singular, which allows one to choose

$$X=-(A_1-D_1)^{-1}B$$

to ensure that the off-diagonal blocks of $\mathcal{A}$ are zero. Since D_0 is an arbitrary symmetric matrix, we can complete the part (i) of the inductive step by taking the upper left block of $\mathcal{A}$ to the form (5.2), and the part (ii) is proved if, instead, we constrain the upper left block of $\mathcal{A}$ to be the zero matrix in a similar way. □

In order to proceed, we need one more technical definition.

Definition 6.3. Let $J \subseteq I$ be indexing sets, and assume $|J| = 2k$ with $k \in \mathbb{Z}$. A family S of $3k$ matrices of the size $I \times I$ is called a *two-diagonal spanning set of J in I* if, for some enumeration $J = \{j_1, \ldots, j_{2k}\}$, the matrices in S are

$$e_p \otimes e_p \ \text{ and } (e_{2q-1} + e_{2q}) \otimes (e_{2q-1} + e_{2q}), \ \text{ for } p \in \{1, \ldots, 2k\},\ q \in \{1, \ldots, k\},$$

where e_t is the vector with the one at the entry j_t and zeros at the places in $I \setminus \{j_t\}$.

Now, assuming the validity of Claims 5.1 and 5.6, we propose a construction which leads to the proof of Theorem 3.9, as we will see later in this section.

Definition 6.4 (Counterexamples to the direct sum conjecture for Waring rank)**.** As suggested in Claims 5.1 and 5.6, we consider an infinite field $\mathbb{F}$ with char $\mathbb{F} \neq 2, 3$, and we construct the tensors $\mathfrak{A}$ and $\mathfrak{B}$ in several steps. Namely, we declare that

(1) $\mathcal{S}$ is the tensor as in Claim 5.1 for some n of the form $(2^k+2)/3$ with $k \in \mathbb{Z}$ (due to the requirement $n \geqslant 5\,104$ in Claim 5.1, we can take $k = 14$),
(2A) $\mathcal{A}$ is the $A \times A \times A$ block of $\mathcal{S}$,
(2B) $\mathcal{B}$ is the $B \times B \times B$ block of $\mathcal{S}$,
(3A) $\mathcal{A}'$ is obtained from $\mathcal{A}$ by the symmetrical adjoining of three two-diagonal spanning sets $(\alpha_1, \alpha_2, \alpha_3)$ of the subfamilies (A_1, A_2, A_3) in A,
(3B) $\mathcal{B}'$ is obtained from $\mathcal{B}$ by the symmetrical adjoining of three two-diagonal spanning sets $(\beta_1, \beta_2, \beta_3)$ of the subfamilies (B_1, B_2, B_3) in B,
(4A) the indexing set of $\mathcal{A}'$ is denoted by A', that is, $\mathcal{A}'$ is an $A' \times A' \times A'$ tensor,
(4B) the indexing set of $\mathcal{B}'$ is denoted by B', so that $\mathcal{B}'$ is a $B' \times B' \times B'$ tensor,
(5A) $G = (g_1, \ldots, g_s)$ is the maximal linearly independent family of the 1-slices of the $B \times A \times A$ block of $\mathcal{S}$ (since $|B| = 3n$ in Claim 5.1, we get $s \leqslant 3n$),
(5B) $\Gamma = (\gamma_1, \ldots, \gamma_\zeta)$ is the maximal linearly independent family of the 1-slices of the $A \times B \times B$ block of $\mathcal{S}$ (similarly, we have $\zeta \leqslant 3n$),
(6A) using Lemma 6.2, we take a family $\tilde{G} = (\tilde{g}_1, \ldots, \tilde{g}_s)$ of $A \times A$ skew projectors of the rank 2^k over $\mathbb{F}$ with $\tilde{g}_i - g_i \in \alpha_1\mathbb{F} + \alpha_2\mathbb{F} + \alpha_3\mathbb{F}$ for $i \in \{1, \ldots, s\}$,
(6B) similarly, we take a family $\tilde{\Gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_\zeta)$ of $B \times B$ skew projectors of the rank 2^k over $\mathbb{F}$ such that $\tilde{\gamma}_i - \gamma_i \in \beta_1\mathbb{F} + \beta_2\mathbb{F} + \beta_3\mathbb{F}$ for $i \in \{1, \ldots, \zeta\}$,
(7A) G' is the family of $A' \times A'$ matrices obtained from $\tilde{G}$ by the padding,
(7B) Γ' is the family of $B' \times B'$ matrices obtained from $\tilde{\Gamma}$ by the padding,
(8A) A'' is the indexing set containing A', and G'' is the family of $A'' \times A''$ symmetric rank one matrices resulting from the use of Claim 5.6 with

$$\rho = \left\lfloor 2n^2/3 - 20\, n^{1.5} \right\rfloor$$

and with G' in the role of W, so we get

$$|A''| \leqslant \mu(k, \rho, 3n) + 7.5n,\ \ |G''| \leqslant \sigma(k, \rho, 3n),$$

(8A.1) the $\mathbb{F}$-linear span of G'' contains $g'(A'' \times A'')$, for every $g' \in G'$, and,
(8A.2) for any field extension $\mathbb{K} \supseteq \mathbb{F}$, we have the equality

$$\min \mathrm{rk}_{\mathbb{K}}\ \mathcal{A}'(A'' \times A'' \times A'') \bmod (G'', G'', G'') = \min \mathrm{rk}_{\mathbb{K}}\ \mathcal{A}' \bmod (G', G', G'),$$

(8B) B'' is the indexing set containing B', and Γ'' is the family of symmetric rank one $B''\times B''$ matrices resulting from the application of Claim 5.6 with Γ' in the role of W and with the same ρ as above, so we get

$$|B''|\leqslant \mu(k,\rho,3n)+7.5n,\ \ |\Gamma''|\leqslant \sigma(k,\rho,3n),$$

(8B.1) the $\mathbb{F}$-linear span of Γ'' contains $\gamma'(B''\times B'')$, for every $\gamma'\in\Gamma'$, and,
(8B.2) for any field extension $\mathbb{K}\supseteq\mathbb{F}$, we have the equality

$$\min\operatorname{rk}_{\mathbb{K}}\mathcal{B}'(B''\times B''\times B'')\operatorname{mod}(\Gamma'',\Gamma'',\Gamma'')=\min\operatorname{rk}_{\mathbb{K}}\mathcal{B}'\operatorname{mod}(\Gamma',\Gamma',\Gamma').$$

We remark that the conclusions (8A.2) and (8B.2) follow due to the inequalities $\operatorname{rk}_{\mathbb{F}}\mathcal{A}'\leqslant\rho$ and $\operatorname{rk}_{\mathbb{F}}\mathcal{B}'\leqslant\rho$, which are true because each of $\mathcal{A}'$ and $\mathcal{B}'$ is obtained by the symmetrical adjoining of $4.5n$ rank one slices, as in the steps (3A) and (3B), to each of the corresponding tensors $\mathcal{A}$ and $\mathcal{B}$, and, according to Lemma 4.9, we get

$$\operatorname{rk}_{\mathbb{F}}\mathcal{A}'\leqslant\operatorname{rk}_{\mathbb{F}}\mathcal{A}+4.5n\ \text{ and }\ \operatorname{rk}_{\mathbb{F}}\mathcal{B}'\leqslant\operatorname{rk}_{\mathbb{F}}\mathcal{B}+4.5n,$$

and, since the inequalities

$$\operatorname{rk}_{\mathbb{F}}\mathcal{A}\leqslant\operatorname{srk}_{\mathbb{F}}\mathcal{S}\leqslant 2n^2/3-20n^{1.5}-4.5n$$

and

$$\operatorname{rk}_{\mathbb{F}}\mathcal{B}\leqslant\operatorname{srk}_{\mathbb{F}}\mathcal{S}\leqslant 2n^2/3-20n^{1.5}-4.5n$$

can be deduced from Claim 5.1, we see that the corresponding ranks of $\mathcal{A}'$ and $\mathcal{B}'$ are indeed at most ρ. Therefore, finally, we are ready to define

(9A) $\mathfrak{A}$ by the symmetrical adjoining of G'' to the padding $\mathcal{A}'(A''\times A''\times A'')$,
(9B) $\mathfrak{B}$ by the symmetrical adjoining of Γ'' to the padding $\mathcal{B}'(B''\times B''\times B'')$.

Remark 6.5. As we see in Definition 6.4, the orders of $\mathfrak{A}$ and $\mathfrak{B}$ do not exceed

$$|G''|+|A''|\leqslant\mu(k,\rho,3n)+\sigma(k,\rho,3n)+7.5n,$$

where

$$\rho=\left\lfloor 2n^2/3-20\,n^{1.5}\right\rfloor \tag{6.1}$$

as defined on the step (8A).

One more general lemma is needed to work with the tensors in Definition 6.4.

Corollary 6.6. *Let $I'\supseteq I$, $J'\supseteq J$, $K'\supseteq K$ be finite indexing sets. Let T' be an $I'\times J'\times K'$ tensor over a field $\mathbb{F}$ obtained from an $I\times J\times K$ tensor T by adjoining*

- *a family $U=(u_1,\ldots,u_q)$ of $J\times K$ matrices as the 1-slices,*
- *a family $V=(v_1,\ldots,v_r)$ of $I\times K$ matrices as the 2-slices, and*
- *a family $W=(w_1,\ldots,w_s)$ of $I\times J$ matrices as the 3-slices.*

Also, we consider three further arbitrary families (U',V',W') of matrices of the respective sizes $J\times K$, $I\times K$, $I\times J$ over $\mathbb{F}$, and we define (U'',V'',W'') as their corresponding paddings $U'(J'\times K')$, $V'(I'\times K')$, $W'(I'\times J')$. Then

$$\min\operatorname{rk}_{\mathbb{F}}T'\operatorname{mod}(U'',V'',W'')\geqslant\min\operatorname{rk}_{\mathbb{F}}T\operatorname{mod}(U\mathbb{F}+U'\mathbb{F},V\mathbb{F}+V'\mathbb{F},W\mathbb{F}+W'\mathbb{F})-$$
$$-\dim_{\mathbb{F}}((U\mathbb{F})\cap(U'\mathbb{F}))-\dim_{\mathbb{F}}((V\mathbb{F})\cap(V'\mathbb{F}))-\dim_{\mathbb{F}}((W\mathbb{F})\cap(W'\mathbb{F}))+$$
$$+\dim_{\mathbb{F}}(U\mathbb{F})+\dim_{\mathbb{F}}(V\mathbb{F})+\dim_{\mathbb{F}}(W\mathbb{F}).$$

Proof. Any element S' in $T'\operatorname{mod}_{\mathbb{F}}(U'',V'',W'')$ is an $I'\times J'\times K'$ tensor obtained from its $I\times J\times K$ block $S\in T\operatorname{mod}(U'\mathbb{F},V'\mathbb{F},W'\mathbb{F})$ by adjoining

- a family $\widehat{U}=(u_1+\widehat{u}_1,\ldots,u_q+\widehat{u}_q)$ of 1-slices, for some $\{\widehat{u}_1,\ldots,\widehat{u}_q\}\subset U'$,

- a family $\widehat{V} = (v_1 + \widehat{v}_1, \dots, v_r + \widehat{v}_r)$ of 2-slices, for some $\{\widehat{v}_1, \dots, \widehat{v}_r\} \subset V'$,
- a family $\widehat{W} = (w_1+\widehat{w}_1, \dots, w_s+\widehat{w}_s)$ of 3-slices, for some $\{\widehat{w}_1, \dots, \widehat{w}_s\} \subset W'$,

and hence an application of Lemma 4.9 gives

$$\operatorname{rk}_{\mathbb{F}} S' \geqslant \min \operatorname{rk}_{\mathbb{F}} S \bmod (\widehat{U}, \widehat{V}, \widehat{W}) + \dim_{\mathbb{F}}(\widehat{U}\mathbb{F}) + \dim_{\mathbb{F}}(\widehat{V}\mathbb{F}) + \dim_{\mathbb{F}}(\widehat{W}\mathbb{F}). \tag{6.2}$$

The condition $S \bmod_{\mathbb{F}} (\widehat{U}, \widehat{V}, \widehat{W}) \subseteq T \bmod_{\mathbb{F}} (U\mathbb{F}+U'\mathbb{F}, V\mathbb{F}+V'\mathbb{F}, W\mathbb{F}+W'\mathbb{F})$ implies

$$\min \operatorname{rk}_{\mathbb{F}} S \bmod (\widehat{U}, \widehat{V}, \widehat{W}) \geqslant \min \operatorname{rk}_{\mathbb{F}} T \bmod (U\mathbb{F}+U'\mathbb{F}, V\mathbb{F}+V'\mathbb{F}, W\mathbb{F}+W'\mathbb{F}), \tag{6.3}$$

and we also have

$$\dim_{\mathbb{F}}(\widehat{U}\mathbb{F}) \geqslant \dim_{\mathbb{F}}((U\mathbb{F} + U'\mathbb{F})/(U'\mathbb{F})) = \dim_{\mathbb{F}}(U\mathbb{F}) - \dim_{\mathbb{F}}((U\mathbb{F}) \cap (U'\mathbb{F})),$$
$$\dim_{\mathbb{F}}(\widehat{V}\mathbb{F}) \geqslant \dim_{\mathbb{F}}((V\mathbb{F} + V'\mathbb{F})/(V'\mathbb{F})) = \dim_{\mathbb{F}}(V\mathbb{F}) - \dim_{\mathbb{F}}((V\mathbb{F}) \cap (V'\mathbb{F})),$$
$$\dim_{\mathbb{F}}(\widehat{W}\mathbb{F}) \geqslant \dim_{\mathbb{F}}((W\mathbb{F} + W'\mathbb{F})/(W'\mathbb{F})) = \dim_{\mathbb{F}}(W\mathbb{F}) - \dim_{\mathbb{F}}((W\mathbb{F}) \cap (W'\mathbb{F})).$$

It remains to compare these three inequalities with the conditions (6.2)–(6.3). □

Now, assuming the validity of Claims 5.1 and 5.6, we are going to prove the inequality in Theorem 3.9 for the tensors $\mathfrak{A}$ and $\mathfrak{B}$ introduced in Definition 6.4. We begin with a lower bound on the right hand side of the inequality in Theorem 3.9.

Lemma 6.7. *If Claims 5.1 and 5.6 are true, then* $\operatorname{rk}_{\mathbb{K}} \mathfrak{A}$ *is at least*

$$\min \operatorname{rk}_{\mathbb{K}} \mathcal{A}^{\circ} \bmod (U_1, U_2, U_3) + 3 \cdot (\dim_{\mathbb{F}}(G''\mathbb{F}) + \dim_{\mathbb{F}}(\alpha\mathbb{F}) - \dim_{\mathbb{F}}((\widetilde{G}\mathbb{F}) \cap (\alpha\mathbb{F}))),$$

where the meanings of $\mathbb{F}$, $\mathbb{K}$, $\mathfrak{A}$, G'' *correspond to Definition 6.4,* $\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3$ *is the union of the sets as in the step (3A) in Definition 6.4, and, as in the notation of Claim 5.1, the tensor* $\mathcal{A}^{\circ}$ *is the* $A_1 \times A_2 \times A_3$ *block of* $\mathcal{S}$*, and, similarly,*

- U_1 *is the* $\mathbb{K}$*-linear span of the* 1*-slices of the* $B \times A_2 \times A_3$ *block of* $\mathcal{S}$*,*
- U_2 *is the* $\mathbb{K}$*-linear span of the* 2*-slices of the* $A_1 \times B \times A_3$ *block of* $\mathcal{S}$*,*
- U_3 *is the* $\mathbb{K}$*-linear span of the* 3*-slices of the* $A_1 \times A_2 \times B$ *block of* $\mathcal{S}$*.*

Proof. We recall that $\mathcal{A}$ is an $A \times A \times A$ tensor, and we take the further indexing sets A' and A'' as in the steps (4A) and (8A) of Definition 6.4, respectively. Also, we denote $A_0 = A \cup (A'' \setminus A')$, and, using the steps (3A) and (9A) of Definition 6.4, we remark that the tensor $\mathfrak{A}$ is obtained from the padding $\mathcal{A}(A_0 \times A_0 \times A_0)$ by the successive application of the following two steps:

(i) the symmetrical adjoining of the paddings $\alpha(A_0 \times A_0 \times A_0)$, which returns the tensor $\mathcal{A}'(A'' \times A'' \times A'')$,

(ii) the symmetrical adjoining of G'' to this tensor $\mathcal{A}'(A'' \times A'' \times A'')$.

We apply Lemma 4.9 to the construction at the step (ii), and we get

$$\operatorname{rk}_{\mathbb{K}} \mathfrak{A} = \min \operatorname{rk}_{\mathbb{K}} \mathcal{A}'(A'' \times A'' \times A'') \bmod (G'', G'', G'') + 3 \dim_{\mathbb{F}}(G''\mathbb{F}),$$

and then we apply the step (8A.2) of Definition 6.4 to obtain

$$\operatorname{rk}_{\mathbb{K}} \mathfrak{A} = R + 3 \dim_{\mathbb{F}}(G''\mathbb{F}), \tag{6.4}$$

where R denotes the value $\min \operatorname{rk}_{\mathbb{K}} \mathcal{A}' \bmod (G', G', G')$. Further, we get

$$R \geqslant \min \operatorname{rk}_{\mathbb{K}} \mathcal{A} \bmod (\tilde{G} \cup \alpha, \tilde{G} \cup \alpha, \tilde{G} \cup \alpha) + 3 \dim_{\mathbb{F}}(\alpha\mathbb{F}) - 3 \dim_{\mathbb{F}}((\tilde{G}\mathbb{F}) \cap (\alpha\mathbb{F})) \tag{6.5}$$

as the result of the application of Corollary 6.6. Further, we have $\tilde{G}\mathbb{F}+\alpha\mathbb{F} = G\mathbb{F}+\alpha\mathbb{F}$ by the step (6A) of Definition 6.4, and hence we can replace $\tilde{G} \cup \alpha$ by $G \cup \alpha$ in (6.5):

$$R \geqslant \min \operatorname{rk}_{\mathbb{K}} \mathcal{A} \bmod (G \cup \alpha, G \cup \alpha, G \cup \alpha) + 3 \dim_{\mathbb{F}}(\alpha\mathbb{F}) - 3 \dim_{\mathbb{F}}((\widetilde{G}\mathbb{F}) \cap (\alpha\mathbb{F})). \tag{6.6}$$

Now we recall that the linear spaces $U_1\mathbb{F}$, $U_2\mathbb{F}$, $U_3\mathbb{F}$ are equal to the $\mathbb{F}$-linear spans of the restrictions of the matrices in G to their $A_2 \times A_3$, $A_1 \times A_3$, $A_1 \times A_2$ blocks, respectively, and the corresponding restrictions for α are zero. These facts imply

$$\min \operatorname{rk}_{\mathbb{K}} \mathcal{A} \bmod (G \cup \alpha, G \cup \alpha, G \cup \alpha) \geqslant \min \operatorname{rk}_{\mathbb{K}} \mathcal{A}^\circ \bmod (U_1, U_2, U_3) \tag{6.7}$$

since the rank of an $A \times A \times A$ tensor is at least the corresponding rank of its $A_1 \times A_2 \times A_3$ block, and a comparison of (6.4), (6.6), (6.7) completes the proof. □

The following statement is similar to Lemma 6.7 but deals with the $\mathfrak{B}$-block.

Lemma 6.8. *If Claims 5.1 and 5.6 are true, then* $\operatorname{rk}_{\mathbb{K}} \mathfrak{B}$ *is at least*

$$\min \operatorname{rk}_{\mathbb{K}} \mathcal{B}^\circ \bmod(V_1, V_2, V_3) + 3 \cdot (\dim_{\mathbb{F}}(\Gamma''\mathbb{F}) + \dim_{\mathbb{F}}(\beta\mathbb{F}) - \dim_{\mathbb{F}}((\widetilde{\Gamma}\mathbb{F}) \cap (\beta\mathbb{F}))),$$

where the meanings of $\mathbb{F}$, $\mathbb{K}$, $\mathfrak{B}$, Γ'' *correspond to Definition 6.4,* $\beta = \beta_1 \cup \beta_2 \cup \beta_3$ *is the union of the sets as in the step (3B) in Definition 6.4, and, as in the notation of Claim 5.1, the tensor* $\mathcal{B}^\circ$ *is the* $B_1 \times B_2 \times B_3$ *block of* $\mathcal{S}$*, and, similarly,*

- V_1 *is the* $\mathbb{K}$*-linear span of the* 1*-slices of the* $A \times B_2 \times B_3$ *block of* $\mathcal{S}$*,*
- V_2 *is the* $\mathbb{K}$*-linear span of the* 2*-slices of the* $B_1 \times A \times B_3$ *block of* $\mathcal{S}$*,*
- V_3 *is the* $\mathbb{K}$*-linear span of the* 3*-slices of the* $B_1 \times B_2 \times A$ *block of* $\mathcal{S}$*.*

Proof. Follows from Lemma 6.7 by the symmetry of the construction. □

Now we can switch to the symmetric rank of $\mathfrak{A} \oplus \mathfrak{B}$, which corresponds to the left hand side of the inequality in Theorem 3.9.

Lemma 6.9. *Assuming the validity of Claims 5.1 and 5.6, we have*

$$\operatorname{srk}_{\mathbb{F}}(\mathfrak{A} \oplus \mathfrak{B}) \leqslant \operatorname{srk}_{\mathbb{F}} \mathcal{S} + \Delta,$$

where, following the notation of Lemma 6.7 and Corollary 6.8, we write

$$\Delta = 3 \cdot (\dim_{\mathbb{F}}(G''\,\mathbb{F}) + \dim_{\mathbb{F}}(\Gamma''\,\mathbb{F}) + \dim_{\mathbb{F}}(\alpha\,\mathbb{F}) + \dim_{\mathbb{F}}(\beta\,\mathbb{F})) -$$
$$- 3 \cdot (\dim_{\mathbb{F}}((\widetilde{G}\,\mathbb{F}) \cap (\alpha\,\mathbb{F})) + \dim_{\mathbb{F}}((\widetilde{\Gamma}\,\mathbb{F}) \cap (\beta\,\mathbb{F}))).$$

Proof. In a way similar to Lemma 6.7, we define

$$A_0 = A \cup (A'' \setminus A') \ \text{ and } \ B_0 = B \cup (B'' \setminus B'),$$

and we define $C = A_0 \cup B_0$. Then, according to the steps (3A), (3B), (9A), (9B) of Definition 6.4, the tensor $\mathfrak{A} \oplus \mathfrak{B}$ is obtained from the direct sum of the paddings

$$\mathcal{A}(A_0 \times A_0 \times A_0) \oplus \mathcal{B}(B_0 \times B_0 \times B_0)$$

by the successive application of the following two steps:

(i) the symmetrical adjoining of the paddings $\alpha(C \times C \times C)$ and $\beta(C \times C \times C)$, which returns the tensor $\mathcal{A}'(A'' \times A'' \times A'') \oplus \mathcal{B}'(B'' \times B'' \times B'')$,

(ii) the symmetrical adjoining of the paddings

$$G''((A'' \cup B'') \times (A'' \cup B'') \times (A'' \cup B'')) \ \text{ and } \ \Gamma''((A'' \cup B'') \times (A'' \cup B'') \times (A'' \cup B''))$$

to this tensor $\mathcal{A}'(A'' \times A'' \times A'') \oplus \mathcal{B}'(B'' \times B'' \times B'')$.

In the rest of this proof, those slices of $\mathfrak{A} \oplus \mathfrak{B}$ that arisen at the step (i) above are called the *type (i) slices*, and those at the step (ii) are the *type (ii) slices*. Also, the type (i) slices that correspond to the paddings of α are called the *type (ia) slices*, and the type (ii) slices that come from the paddings of G'' are the *type (iia) slices*. Similarly, those type (i) slices that are not type (ia) are called the type (ib) slices, and the type (ii) slices that are not of the type (iia) are said to be of the type (iib).

Further, we take a family of matrices $(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_t)$ in α such that

$$(\widetilde{\alpha}_1 + \widetilde{G}\,\mathbb{F}, \ldots, \widetilde{\alpha}_t + \widetilde{G}\,\mathbb{F})$$

is a basis of the quotient space $(\widetilde{G}\,\mathbb{F} + \alpha\,\mathbb{F})/(\widetilde{G}\,\mathbb{F})$ over $\mathbb{F}$, where $\widetilde{G}$ is the family as in the step (6A) of Definition 6.4. In this case, according to the steps (5A) and (6A) of Definition 6.4, if m is an $A \times A$ matrix which either belongs to α or appears as a 1-slice of the $B \times A \times A$ block of $\mathcal{S}$, we can write

$$m = \widetilde{g}(m) + \widetilde{\alpha}(m) \text{ with } \widetilde{g}(m) \in \widetilde{G}\,\mathbb{F} \text{ and } \widetilde{\alpha}(m) \in \widetilde{\alpha}_1\,\mathbb{F} + \ldots + \widetilde{\alpha}_t\,\mathbb{F}.$$

We proceed with the following transformations of $\mathfrak{A} \oplus \mathfrak{B}$:

- for any $\chi \in \{1, 2, 3\}$ and for any type (ia) χ-slice σ, we subtract the padding of $\widetilde{g}(m')$ from this slice, where m' is the $A \times A$ block of σ,
- for any $\chi \in \{1, 2, 3\}$ and for any $b \in B$, we add the padding of $\widetilde{g}(m_b)$ to the b-th χ-slice, where m_b is the $A \times A$ block of the b-th 1-slice of $\mathcal{S}$.

Since the paddings of the matrices in $\widetilde{G}$ belong to $G''\,\mathbb{F}$ by the steps (7A) and (8A.1) of Definition 6.4, the above transformations do not affect the symmetric rank in view of Corollary 4.14. Now we remove all the (iia) slices of the resulting tensor, and, in view of Corollary 4.13, this cannot decrease the symmetric rank by more than $3 \dim_{\mathbb{F}}(G''\,\mathbb{F})$. We proceed with two further transformations as follows:

- for any $\chi \in \{1, 2, 3\}$ and for any type (ia) slice σ, we remove σ if it is not the padding of some matrix in $(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_t)$,
- for any $\chi \in \{1, 2, 3\}$ and for any $b \in B$, we replace the $A \times A$ block of the b-th χ-slice by the corresponding $A \times A$ block of the b-th 1-slice of $\mathcal{S}$.

In particular, the slices corresponding to the matrices $(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_t)$ are not affected, and all the remaining slices are added a matrix in $\widetilde{\alpha}_1\,\mathbb{F} + \ldots + \widetilde{\alpha}_t\,\mathbb{F}$, which shows that the symmetric rank did not change by the application of Corollary 4.14. Finally, we remove the remaining type (ia) slices, and, according to Corollary 4.13, this cannot cause a decrease of the symmetric rank larger than $3t$, which is

$$3 \cdot \dim_{\mathbb{F}}((\widetilde{G}\,\mathbb{F} + \alpha\,\mathbb{F})/(\widetilde{G}\,\mathbb{F})) = 3 \dim_{\mathbb{F}}(\alpha\,\mathbb{F}) - 3 \dim_{\mathbb{F}}((\widetilde{G}\,\mathbb{F}) \cap (\alpha\,\mathbb{F})).$$

Due to the symmetry of our construction, we can transform the part $\mathfrak{B}$ in a way similar to the above considerations, which would lead us to

- the removal of all the type (ib) and type (iib) slices,
- the replacement of the $B \times B$ block of the a-th χ-slice by the corresponding $B \times B$ block of the a-th 1-slice of $\mathcal{S}$, for all $\chi \in \{1, 2, 3\}$ and $a \in A$,

and the resulting tensor cannot have its symmetric rank decreased by more than

$$3 \dim_{\mathbb{F}}(\Gamma''\,\mathbb{F}) + 3 \dim_{\mathbb{F}}(\beta\,\mathbb{F}) - 3 \dim_{\mathbb{F}}((\widetilde{\Gamma}\,\mathbb{F}) \cap (\beta\,\mathbb{F})).$$

Therefore, the subsequent use of both series of transformations does not decrease the symmetric rank more than by Δ, and we end up with the tensor $\mathcal{S}$. $\square$

Now we have reached the main results of this section.

Theorem 6.10. *Assuming the validity of Claims 5.1 and 5.6, we have*

$$\operatorname{srk}_{\mathbb{F}}(\mathfrak{A} \oplus \mathfrak{B}) < \operatorname{rk}_{\mathbb{K}} \mathfrak{A} + \operatorname{rk}_{\mathbb{K}} \mathfrak{B},$$

for the tensors $(\mathfrak{A}, \mathfrak{B})$ *as in Definition 6.4 and for any extension* $\mathbb{K} \supseteq \mathbb{F}$.

Proof. Lemma 6.7 and Corollary 6.8 show that

$$\mathrm{rk}_{\mathbb{K}}\,\mathfrak{A} + \mathrm{rk}_{\mathbb{K}}\,\mathfrak{B} \geqslant \min \mathrm{rk}_{\mathbb{K}}\,\mathcal{A}^{\circ} \bmod (U_1, U_2, U_3) + \min \mathrm{rk}_{\mathbb{K}}\,\mathcal{B}^{\circ} \bmod (V_1, V_2, V_3) + \Delta.$$

A comparison to Lemma 6.9 shows that $\mathrm{rk}_{\mathbb{K}}\,\mathfrak{A} + \mathrm{rk}_{\mathbb{K}}\,\mathfrak{B} - \mathrm{srk}_{\mathbb{F}}(\mathfrak{A} \oplus \mathfrak{B})$ is at least

$$\min \mathrm{rk}_{\mathbb{K}}\,\mathcal{A}^{\circ} \bmod (U_1, U_2, U_3) + \min \mathrm{rk}_{\mathbb{K}}\,\mathcal{B}^{\circ} \bmod (V_1, V_2, V_3) - \mathrm{srk}_{\mathbb{F}}\,\mathcal{S},$$

which is positive due to the inequality (5.1) in Claim 5.1. □

Remark 6.11. In view of Remark 6.5, we constructed a pair $(\mathfrak{A}, \mathfrak{B})$ which delivers a counterexample to Strassen's direct sum conjecture (again assuming the validity of Claims 5.1 and 5.6) and has both tensors of the orders not exceeding

$$\mu(k, \rho, 3n) + \sigma(k, \rho, 3n) + 7.5n,$$

provided that $n = (2^k + 2)/3$ is an integer, where ρ is the integer as in (6.1). Also, the step (1) of Definition 6.4 allows us to take $k = 14$, so we get

$$\mu(14, 11\,815\,542, 16\,386) + \sigma(14, 11\,815\,542, 16\,386) + 40\,965 \leqslant 10^{888}$$

as the upper bound on the orders of the tensors in the smallest counterexample.

Corollary 6.12. *Claims 5.1 and 5.6 imply Theorem 3.9.*

Proof. Immediate from Theorem 6.10. □

7. The proof of Claim 5.1

As in [94], a family P of elements in an extension of a field $\mathbb{F}$ is said to have τ *degrees of freedom over* $\mathbb{F}$ if the transcendence degree of $\mathbb{F}(P)$ over $\mathbb{F}$ is τ. In addition, if τ equals the total number of elements in P, then P is *generic over* $\mathbb{F}$. Our argument requires a classical result of Lickteig [75] and one lemma in [94].

Theorem 7.1 (Theorem 4.4 in [75])**.** *For $k \in \mathbb{Z}$, let u be a family of $3k$ vectors of the length n each. If $n > 3$, $k > n^3/(3n-2)$ and u is generic over a field $\mathbb{F}$, then*

$$\left(u^1 \otimes u^{k+1} \otimes u^{2k+1}\right) + \left(u^2 \otimes u^{k+2} \otimes u^{2k+2}\right) + \ldots + \left(u^k \otimes u^{2k} \otimes u^{3k}\right)$$

is a tensor generic over $\mathbb{F}$.

Lemma 7.2 (Lemma 19 in [94])**.** *Let $\mathbb{F}$ be an infinite field, let $d < n$ be positive integers, and let M be an $n \times m$ matrix that has at least $mn - \delta$ degrees of freedom over $\mathbb{F}$. Then there exists an $(n-d) \times n$ matrix Q over $\mathbb{F}$ such that QM becomes generic over $\mathbb{F}$ after the removal of at most δ/d appropriately chosen columns.*

We proceed with an effective version of the technique on the page 375 in [94].

Theorem 7.3. *Let r, n, and δ be positive integers such that $n > 3$,*

$$n^3 < (3n-2)(r - \sqrt{\delta n}) \tag{7.1}$$

and $3n(n-1)^2 > \delta$. Consider the two families of the vectors of the length n,

$$\sigma = (a_1, \ldots, a_r,\, c_1, \ldots, c_r,\, g_1, \ldots, g_r) \text{ and } v = (x_1, \ldots, x_r,\, y_1, \ldots, y_r,\, z_1, \ldots, z_r),$$

and assume that the vectors in σ have entries in an infinite field $\mathbb{F}$. If the family v has at least $3rn - \delta$ degrees of freedom over $\mathbb{F}$, then the $n \times n \times n$ tensor

$$\Phi = \sum_{\alpha=1}^{r} \left(x_\alpha \otimes y_\alpha \otimes z_\alpha + a_\alpha \otimes y_\alpha \otimes z_\alpha + x_\alpha \otimes c_\alpha \otimes z_\alpha + x_\alpha \otimes y_\alpha \otimes g_\alpha\right)$$

has at least $n^3 - 6n\sqrt{3\delta n} - 3n^2$ degrees of freedom over $\mathbb{F}$.

Proof. We define the $n \times 3r$ matrix M by stating that $(x_1, y_1, z_1, \ldots, x_r, y_r, z_r)$ is the family of the columns of M. Further, we apply Lemma 7.2 with

$$d := \left\lceil \sqrt{\frac{\delta}{n}} \right\rceil \tag{7.2}$$

and, using the resulting matrix Q, we take the vector $Q\pi_\alpha$ and denote it by $\overline{\pi}_\alpha$, where π replaces any of the letters a, c, g, x, y, z. Also, we consider the tensor

$$\Phi' = (Q \otimes Q \otimes Q)\, \Phi, \tag{7.3}$$

which has the size $(n-d) \times (n-d) \times (n-d)$ and can be written as

$$\Phi' = \sum_{\alpha=1}^{r} \left(\overline{x}_\alpha \otimes \overline{y}_\alpha \otimes \overline{z}_\alpha + \overline{a}_\alpha \otimes \overline{y}_\alpha \otimes \overline{z}_\alpha + \overline{x}_\alpha \otimes \overline{c}_\alpha \otimes \overline{z}_\alpha + \overline{x}_\alpha \otimes \overline{y}_\alpha \otimes \overline{g}_\alpha \right). \tag{7.4}$$

Now we define $\mathcal{A}$ as the set of all $\alpha \in \{1, \ldots, r\}$ for which none of the three columns $(\overline{x}_\alpha, \overline{y}_\alpha, \overline{z}_\alpha)$ was removed at the application of Lemma 7.2, so we have

$$|\{1, \ldots, r\} \setminus \mathcal{A}| \leqslant \delta/d, \tag{7.5}$$

and the union $\mathcal{V}$ of all families $(\overline{x}_\alpha, \overline{y}_\alpha, \overline{z}_\alpha)$ with $\alpha \in \mathcal{A}$ is generic over $\mathbb{F}$. We get

$$|\mathcal{A}| \geqslant r - \delta/d \tag{7.6}$$

immediately from (7.5), and a further examination of (7.1), (7.2), (7.6) gives

$$|\mathcal{A}| > n^3/(3n-2). \tag{7.7}$$

Since the bound (7.7) matches the one in Theorem 7.1, we get that the tensor

$$\Phi'' = \sum_{\alpha \in \mathcal{A}} \overline{x}_\alpha \otimes \overline{y}_\alpha \otimes \overline{z}_\alpha \tag{7.8}$$

is generic over $\mathbb{F}$. Further, since the entries of the tensor

$$\Phi''' = \sum_{\alpha \in \mathcal{A}} \left(\overline{a}_\alpha \otimes \overline{y}_\alpha \otimes \overline{z}_\alpha + \overline{x}_\alpha \otimes \overline{c}_\alpha \otimes \overline{z}_\alpha + \overline{x}_\alpha \otimes \overline{y}_\alpha \otimes \overline{g}_\alpha \right) \tag{7.9}$$

all have the total degree two when considered as polynomials in $\mathbb{F}[\mathcal{V}]$, and, since the entries of Φ'' are homogeneous polynomials in $\mathbb{F}[\mathcal{V}]$ of the larger degree, and, also, due to the algebraic independence of the entries of Φ'' over $\mathbb{F}$, the tensor $\Phi'' + \Phi'''$ is still generic over $\mathbb{F}$, that is, the tensor $\Phi'' + \Phi'''$ has exactly $(n-d)^3$ degrees of freedom over $\mathbb{F}$. Using the formulas (7.4), (7.8), and (7.9), we obtain

$$\Delta\Phi = \sum_{\alpha \notin \mathcal{A}} \left(\overline{x}_\alpha \otimes \overline{y}_\alpha \otimes \overline{z}_\alpha + \overline{a}_\alpha \otimes \overline{y}_\alpha \otimes \overline{z}_\alpha + \overline{x}_\alpha \otimes \overline{c}_\alpha \otimes \overline{z}_\alpha + \overline{x}_\alpha \otimes \overline{y}_\alpha \otimes \overline{g}_\alpha \right)$$

with $\Delta\Phi = \Phi' - (\Phi'' + \Phi''')$, and we see that the tensor $\Delta\Phi$ has at most

$$\frac{3(n-d)\delta}{d}$$

degrees of freedom as each of the vectors $\overline{x}_\alpha$, $\overline{y}_\alpha$, $\overline{z}_\alpha$ has the length $n-d$. Since Φ' differs from the generic tensor $\Phi'' + \Phi'''$ by $\Delta\Phi$, the tensor Φ' should have at least

$$(n-d)^3 - \frac{3(n-d)\delta}{d} \geqslant n^3 - 6n\sqrt{3\delta n} - 3n^2$$

degrees of freedom over $\mathbb{F}$. It remains to note that the same bound applies to Φ as well because the matrix Q in (7.3) has all entries in $\mathbb{F}$. □

We are almost ready to prove Claim 5.1, and we proceed with a similar statement taken with an additional restriction on the ground field.

Theorem 7.4. *Take an integer $n \geqslant 5\,104$ and an integer r for which the inequalities*

$$\frac{n^3}{3n-2} + 3n^{1.5} < r < 2 \cdot \frac{n^3 - 18n^2\sqrt{3n} - 3n^2}{3n-2} \tag{7.10}$$

are satisfied. Let $\mathbb{F}$ be an arbitrary field, and let

$$A_1,\ A_2,\ A_3,\ B_1,\ B_2,\ B_3$$

be pairwise disjoint sets of cardinality n each and

$$A = A_1 \cup A_2 \cup A_3,\ \ B = B_1 \cup B_2 \cup B_3,\ \ I = A \cup B.$$

Then there exists a symmetric $I \times I \times I$ tensor $\mathcal{S}$ with entries in some purely transcendental extension $\mathbb{F}' \supset \mathbb{F}$ of the degree at most $4n^3$ over $\mathbb{F}$ such that

$$\operatorname{srk}_{\mathbb{F}'} \mathcal{S} \leqslant r < \min \operatorname{rk}_{\mathbb{K}} \mathcal{A}^\circ \operatorname{mod}(U_1, U_2, U_3) + \min \operatorname{rk}_{\mathbb{K}} \mathcal{B}^\circ \operatorname{mod}(V_1, V_2, V_3) \tag{7.11}$$

holds over any field $\mathbb{K}$ containing $\mathbb{F}'$, where

(U1) U_1 is the $\mathbb{K}$-linear span of the 1-slices of the $B \times A_2 \times A_3$ block of $\mathcal{S}$,
(U2) U_2 is the $\mathbb{K}$-linear span of the 2-slices of the $A_1 \times B \times A_3$ block of $\mathcal{S}$,
(U3) U_3 is the $\mathbb{K}$-linear span of the 3-slices of the $A_1 \times A_2 \times B$ block of $\mathcal{S}$,
(V1) V_1 is the $\mathbb{K}$-linear span of the 1-slices of the $A \times B_2 \times B_3$ block of $\mathcal{S}$,
(V2) V_2 is the $\mathbb{K}$-linear span of the 2-slices of the $B_1 \times A \times B_3$ block of $\mathcal{S}$,
(V3) V_3 is the $\mathbb{K}$-linear span of the 3-slices of the $B_1 \times B_2 \times A$ block of $\mathcal{S}$,

and also $\mathcal{A}^\circ$ is the $A_1 \times A_2 \times A_3$ block of $\mathcal{S}$, and $\mathcal{B}^\circ$ is the $B_1 \times B_2 \times B_3$ block of $\mathcal{S}$.

Proof. We define $\mathbb{F}' = \mathbb{F}(\mathrm{vec})$, where $\mathrm{vec} = (\mathrm{vec}_1, \ldots, \mathrm{vec}_r)$ is a generic family of r vectors of the length $6n$ each, and we assume without loss of generality that $\mathbb{K}$ is the algebraic closure of $\mathbb{F}'$. We take the $6n \times 6n \times 6n$ tensor

$$\mathcal{S} = \sum_{\alpha=1}^{r} \mathrm{vec}_\alpha \otimes \mathrm{vec}_\alpha \otimes \mathrm{vec}_\alpha \tag{7.12}$$

and write

$$\mathrm{vec}_\alpha = (x_\alpha, y_\alpha, z_\alpha, \xi_\alpha, \gamma_\alpha, \zeta_\alpha), \tag{7.13}$$

where $x_\alpha, y_\alpha, z_\alpha, \xi_\alpha, \gamma_\alpha, \zeta_\alpha$ are vectors consisting of n independent variables each. Further, we define the indexing sets as in the formulation of the current theorem with respect to the partition of the vector vec into the six tuples as in (7.13):

$$A_1 = \{1, \ldots, n\},\ \ A_2 = \{n+1, \ldots, 2n\},\ \ A_3 = \{2n+1, \ldots, 3n\},$$

$$B_1 = \{3n+1, \ldots, 4n\},\ \ B_2 = \{4n+1, \ldots, 5n\},\ \ B_3 = \{5n+1, \ldots, 6n\}.$$

We note that the equality (7.12) implies $\operatorname{srk}_{\mathbb{F}'} \mathcal{S} \leqslant r$, and, hence, in view of the symmetry and the right inequality in (7.10), a verification of the condition

$$\min \operatorname{rk}_{\mathbb{K}} \mathcal{A}^\circ \operatorname{mod}(U_1, U_2, U_3) \geqslant \frac{n^3 - 18n^2\sqrt{3n} - 3n^2}{3n-2} \tag{7.14}$$

would suffice to prove the current theorem. To this end, we build an arbitrary tensor Φ in $\mathcal{A}^\circ \operatorname{mod}_{\mathbb{K}}(U_1, U_2, U_3)$ with the use of Definition 4.8, so we get

$$\Phi = \mathcal{A}^\circ + (Q_1 \otimes \mathrm{id}_2 \otimes \mathrm{id}_3)\, S_1 + (\mathrm{id}_1 \otimes Q_2 \otimes \mathrm{id}_3)\, S_2 + (\mathrm{id}_1 \otimes \mathrm{id}_2 \otimes Q_3)\, S_3 \tag{7.15}$$

in which S_1 is the $B \times A_2 \times A_3$ block of $\mathcal{S}$, and, similarly, S_2 is the $A_1 \times B \times A_3$ block of $\mathcal{S}$, and S_3 is the $A_1 \times A_2 \times B$ block of $\mathcal{S}$, and, also, for any $\chi \in \{1,2,3\}$, the matrix Q_χ is $A_\chi \times B$, and id_χ is the $A_\chi \times A_\chi$ identity matrix. We further define the vector $\nu_\alpha = (\xi_\alpha, \gamma_\alpha, \zeta_\alpha)$ and reconstruct the tensors $\mathcal{A}^\circ$ and S_χ from (7.12):

$$\mathcal{A}^\circ = \sum_{\alpha=1}^{r} (x_\alpha \otimes y_\alpha \otimes z_\alpha)\,, \quad S_1 = \sum_{\alpha=1}^{r} (\nu_\alpha \otimes y_\alpha \otimes z_\alpha)\,,$$

$$S_2 = \sum_{\alpha=1}^{r} (x_\alpha \otimes \nu_\alpha \otimes z_\alpha)\,, \quad S_3 = \sum_{\alpha=1}^{r} (x_\alpha \otimes y_\alpha \otimes \nu_\alpha)\,.$$

In view of the formula (7.15), this gives

$$\Phi = \sum_{\alpha=1}^{r} (x_\alpha \otimes y_\alpha \otimes z_\alpha + a_\alpha \otimes y_\alpha \otimes z_\alpha + x_\alpha \otimes c_\alpha \otimes z_\alpha + x_\alpha \otimes y_\alpha \otimes g_\alpha) \tag{7.16}$$

with $(a_\alpha, c_\alpha, g_\alpha) = (Q_1\nu_\alpha, Q_2\nu_\alpha, Q_3\nu_\alpha)$ for all $\alpha \in \{1,\ldots,r\}$. Now we define $\mathcal{K}$ as the field obtained from $\mathbb{F}$ by adjoining all the entries of the matrices Q_1, Q_2, Q_3 and the entries of the vectors ν_α with all $\alpha \in \{1,\ldots,r\}$. Indeed, we note that the entries of all vectors $(a_\alpha, c_\alpha, g_\alpha)$ belong to $\mathcal{K}$ again for all $\alpha \in \{1,\ldots,r\}$, which allows us to apply Theorem 7.3 to the condition (7.16) with $\mathcal{K}$ in the role of the ground field. In fact, the definition of $\mathcal{K}$ tells it immediately that the transcendence degree of $\mathcal{K}$ over $\mathbb{F}$ is at most $9n^2 + 3rn$, and, since the $6rn$ variables appearing as the coordinates of the vectors (7.13) are independent, the vector

$$(x_1, \ldots, x_r,\, y_1, \ldots, y_r,\, z_1, \ldots, z_r)$$

has at least $6rn - (3rn + 9n^2) = 3rn - 9n^2$ degrees of freedom over $\mathcal{K}$. This gives $\delta = 9n^2$ in the notation of Theorem 7.3, and hence the tensor Φ has at least

$$n^3 - 6n\sqrt{3\delta n} - 3n^2 = n^3 - 18n^2\sqrt{3n} - 3n^2$$

degrees of freedom over $\mathcal{K}$, and, since a rank-one $n \times n \times n$ tensor cannot have more than $3n-2$ degrees of freedom, we prove the desired inequality (7.14). □

In order to complete the proof of Claim 5.1, we need to transfer the result of Theorem 7.4 to the case $\mathbb{F} = \mathbb{F}'$ whenever the field $\mathbb{F}$ is infinite.

Theorem 7.5. *If $\mathbb{F}$ is either an infinite field or a finite field with at least*

$$2^{2^{7n^3}}$$

elements, then the assertion of Theorem 7.4 is valid with $\mathbb{F}' = \mathbb{F}$.

Proof. Let v be an arbitrary vector of the length $6nr$ with the coordinates in $\mathbb{F}$, which we think of as the values assigned to the corresponding coordinates in the variable vector vec as in the proof of Theorem 7.4. We immediately have

$$\mathrm{srk}_{\mathbb{F}}\ \mathcal{S}(v) \leqslant r \tag{7.17}$$

for any tensor $\mathcal{S}(v)$ obtained from (7.12) by the substitution vec $\to v$.

In view of the inequality (7.17), the current theorem is true if some appropriate v takes the corresponding right hand side of (7.10) to the value greater than r, to which end we want to verify that the following statements are both false:

$$\min \mathrm{rk}_{\mathbb{K}}\ \mathcal{A}^\circ \bmod (U_1, U_2, U_3) \leqslant r/2, \quad \min \mathrm{rk}_{\mathbb{K}}\ \mathcal{B}^\circ \bmod (V_1, V_2, V_3) \leqslant r/2, \tag{7.18}$$

where the subspaces $U_1, U_2, U_3, V_1, V_2, V_3$ and tensors $\mathcal{A}^\circ$ and $\mathcal{B}^\circ$ are defined as in the formulation of Theorem 7.4 but with the tensor $\mathcal{S}$ replaced by $\mathcal{S}(v)$.

Indeed, in the proof of Theorem 7.4, we showed that the conditions (7.18) are false generically. On the other hand, a natural description of the tensor families

$$\mathcal{A}^\circ \operatorname{mod}_{\mathbb{K}}(U_1, U_2, U_3) \text{ and } \mathcal{B}^\circ \operatorname{mod}_{\mathbb{K}}(V_1, V_2, V_3)$$

as in the left hand sides of (7.18) requires, for each family, an addition of $9n^2$ variables corresponding to the matrices Q_1, Q_2, Q_3 in the proof of Theorem 7.4.

Also, each of the inequalities (7.18) is encoded by a system of n^3 equations with at most $3n(r/2)$ new variables which correspond to a potential certificate that a given tensor has the rank not exceeding $r/2$. Therefore, the fact that at least one of the conditions (7.18) is valid can be written by a system Φ of degree eight polynomial equations which employ v and at most $3nr + 18n^2 \leqslant 2n^3 + 18n^2$ additional variables. We need to establish the existence of an assignment of v which admits no lifting to a solution of Φ, or, in other words, if we construct the ideal I_Φ generated by the polynomials in Φ over $\mathbb{F}$, we need an assignment of v outside the intersection of the zero loci of the polynomials in the elimination ideal $I_\Phi \cap \mathbb{F}[v]$. By the result of Dubé [50], the ideal $I_\Phi \cap \mathbb{F}[v]$ contains a nonzero polynomial of the degree at most

$$2 \cdot (8^2/2 + 8)^{2^{4n^3 + \left(2n^3 + 18n^2\right) - 1}}$$

so the cardinality of $\mathbb{F}$ is sufficient to find an appropriate assignment of v. $\square$

Remark 7.6. As said above, Theorems 7.4 and 7.5 imply the validity of Claim 5.1.

Remark 7.7. Obviously, the polynomials in $I_\Phi \cap \mathbb{F}[v]$ may be too large for modern computers, so we can hardly determine an explicit assignment of v at which one of these polynomials does not vanish. Therefore, the existence of an appropriate v may be hard to certify in smaller fields, which fact turns out to be an obstruction for a potential generalization of our technique in the case of small finite fields. A further obstacle is the lack of good finite field analogues of Theorem 7.1, which can be illustrated by the fact that the maximal rank of a $3 \times 3 \times 3$ tensor over $\mathbb{Z}/2\mathbb{Z}$ is larger than the corresponding maximal rank over $\mathbb{R}$ or $\mathbb{C}$ [11, 20].

8. Eliminating families. Definitions and motivation

In this section, we discuss Claim 5.6 from a somewhat generalized perspective in which, namely, we do not require the matrices corresponding to W to be skew projectors, and, in addition, we do not assume that the matrices as in $\mathcal{M}$ are rank one. In the following definition, we collect several conditions that we need to impose on these families to guarantee the conclusions similar to (i) and (ii) in Claim 5.6.

Notation 8.1. We write rows φ to denote the set of all rows of a matrix φ.

Remark 8.2. In Theorem 8.6 below, several further specifications to Definition 8.3 are made in order to allow an actual application to the proof of Claim 5.6, which is why the concept introduced in Definition 8.3 is termed the *candidate* family.

Definition 8.3. Assume that I and J are disjoint indexing sets, m is a symmetric $I \times I$ matrix over a field $\mathbb{F}$, and $\rho, r, \pi, \sigma, \delta$ are nonnegative integers. For a family $\Phi = (\varphi_1, \ldots, \varphi_s)$ of symmetric $(I \cup J) \times (I \cup J)$ matrices over $\mathbb{F}$, we say that

$$\Phi \text{ is a } \textit{candidate family} \text{ of the } \textit{type } (\mathbb{F}, m, \rho, r, \pi, \sigma, \delta)$$

if the following conditions (i)–(v) are satisfied (here, the notation $\Psi = (\psi_1, \ldots, \psi_s)$ stands for the family of the corresponding $J \times J$ restrictions of the matrices in Φ):

(i) $\varphi_1 + \ldots + \varphi_s$ equals the padding $m((I \cup J) \times (I \cup J))$,

(ii) there are no linear dependence relations in Ψ except those that follow from the above condition (i), which means that, strictly speaking, a linear combination $\lambda_1\psi_1 + \ldots + \lambda_s\psi_s$ can be zero only if the scalars $\lambda_1, \ldots, \lambda_s \in \mathbb{F}$ are all equal,

(iii) for any $\mathbb{F}$-linear space V' that is obtained as the sum of the row spaces of all the matrices in some subset $\Phi' \subseteq \Phi$ and satisfies $\dim V' \leqslant \delta$, it holds that the zero vector is the only vector in V' in which all the J-coordinates are zero,

(iv) for any extension $\mathbb{K} \supseteq \mathbb{F}$ and any matrix $\psi_0 \in \Psi\,\mathbb{K}$ satisfying $\operatorname{rk}\psi_0 \leqslant r$, there exists a subset $D \subseteq \{1, \ldots, s\}$ such that ψ_0 belongs to the linear space

$$\sum_{d \in D} \psi_d\,\mathbb{K}$$

and

$$\dim_{\mathbb{F}} \left(\sum_{d \in D} (\operatorname{rows}\varphi_d)\,\mathbb{F} \right) \leqslant \sigma,$$

that is, the sum of the row spaces of φ_d over $d \in D$ has the dimension at most σ,

(v) for any field $\mathbb{K} \supseteq \mathbb{F}$ and any $(I \cup J) \times (I \cup J) \times (I \cup J)$ tensors Δ and

$$T \in O\operatorname{mod}_{\mathbb{K}}(\Phi, \Phi, \Phi)$$

such that all entries outside the $I \times I \times I$, $I \times I \times J$, $I \times J \times I$, $J \times I \times I$ blocks of Δ are zero, if the further condition $\operatorname{rk}_{\mathbb{K}}(T + \Delta) \leqslant \rho$ is true, then, for any $\chi \in \{1, 2, 3\}$, there exists a subset $\Phi_\chi \subseteq \Phi$ such that

$$\dim_{\mathbb{F}} \left(\sum_{\varphi \in \Phi_\chi} (\operatorname{rows}\varphi)\,\mathbb{F} \right) \leqslant \pi,$$

that is, the sum of the row spaces of all $\varphi \in \Phi_\chi$ has the dimension at most π, and

$$T \in O\operatorname{mod}_{\mathbb{K}}(\Phi_1\,\mathbb{K} + m'\,\mathbb{K}, \Phi_2\,\mathbb{K} + m'\,\mathbb{K}, \Phi_3\,\mathbb{K} + m'\,\mathbb{K}),$$

where m' is the $(I \cup J) \times (I \cup J)$ padding of m.

Remark 8.4. In Definition 8.3 above, the type of a given family is not unique. In fact, clearly, if Φ is a candidate family of some type $(\mathbb{F}, m, \rho, r, \pi, \sigma, \delta)$, then Φ is also a candidate family of any type of the form $(\mathbb{F}, m, \rho', r', \pi'', \sigma'', \delta')$ with any

$$\rho' \leqslant \rho,\ r' \leqslant r,\ \pi'' \geqslant \pi,\ \sigma'' \geqslant \sigma,\ \delta' \leqslant \delta.$$

The following is an easy observation needed in the main result of the section.

Lemma 8.5. *For any field $\mathbb{F}$, we consider an $\mathbb{F}$-linear space $\mathcal{V}$ that is a direct sum of its subspaces $U, V_1, \ldots, V_c \subseteq \mathcal{V}$. If $\mathbb{F}$-linear spaces $W_1, \ldots, W_c$ satisfy $W_i \subseteq U + V_i$ and $W_i \cap U = 0$ for all i, then there exists an $\mathbb{F}$-linear mapping $\alpha : \mathcal{V} \to U$ such that the condition $\alpha(W_i) = 0$ holds for all i, and $\alpha(u) = u$ holds for all $u \in U$.*

Proof. For all $i \in \{1, \ldots, c\}$, we take an arbitrary basis γ_i of the $\mathbb{F}$-linear space W_i and an arbitrary basis β of the $\mathbb{F}$-linear space U. The assumptions of the lemma imply that $\beta \cup \gamma_1 \cup \ldots \cup \gamma_c$ is linearly independent, so there is an $\mathbb{F}$-linear mapping $\alpha : \mathcal{V} \to U$ satisfying $\alpha(v) = 0$ for $v \in \gamma_1 \cup \ldots \cup \gamma_c$ and $\alpha(u) = u$ for $u \in \beta$. $\square$

Now we are ready to explain the relevance of the conditions for the candidate families in Definition 8.3. The following is the main result of this section.

Theorem 8.6. *Let $\mathbb{F}$ be a field, let ρ be a positive integer, let $(I, J_1, \ldots, J_c)$ be a family of pairwise disjoint indexing sets, and, for any $\tau \in \{1, \ldots, c\}$, let I_τ be an indexing family with $I_\tau \subseteq I$. Further, let $\mu = (m_1, \ldots, m_c)$ be a family of symmetric $I \times I$ matrices each of which is represented as*

$$m_\tau = g_\tau \, \overline{m}_\tau \, (g_\tau)^\top$$

with some $I \times I_\tau$ matrix g_τ satisfying $\operatorname{rk}_{\mathbb{F}} g_\tau = |I_\tau|$ and some $I_\tau \times I_\tau$ matrix $\overline{m}_\tau$ with entries in $\mathbb{F}$. Also, for any $\tau \in \{1, \ldots, c\}$, we assume that

$$\overline{\Phi}_\tau = \left(\overline{\varphi}_{\tau 1}, \ldots, \overline{\varphi}_{\tau s_\tau}\right)$$

is a candidate family of $(I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau)$ symmetric matrices of the type

$$(\mathbb{F}, \overline{m}_\tau, \rho, \rho, \pi_\tau, \sigma_\tau, \pi_\tau + \sigma_\tau),$$

where (s_τ), (π_τ), (σ_τ) are families of positive integers. Further, we take

$$\varphi_{\tau j} = (g_\tau \oplus \operatorname{id}(J_\tau)) \, \overline{\varphi}_{\tau j} \, (g_\tau \oplus \operatorname{id}(J_\tau))^\top \tag{8.1}$$

for $j \in \{1, \ldots, s_\tau\}$, where $\operatorname{id}(J_\tau)$ is the $J_\tau \times J_\tau$ identity matrix, and also we write

$$\Phi_\tau = \{\varphi_{\tau 1}, \ldots, \varphi_{\tau s_\tau}\}$$

and define

$$\Phi = \Phi_1(K \times K) \cup \ldots \cup \Phi_c(K \times K), \tag{8.2}$$

where $K = I \cup J_1 \cup \ldots \cup J_c$ and, as always, the notation $\Phi_\tau(K \times K)$ stands for the family of the $K \times K$ matrices obtained from those in Φ_τ by the padding. Then

(1) $\Phi\,\mathbb{F}$ contains the padding $m_\tau(K \times K)$ of every matrix $m_\tau \in \mu$, and

(2) for any field $\mathbb{K} \supseteq \mathbb{F}$ and any $I \times I \times I$ tensor T with $\operatorname{rk}_{\mathbb{F}} T \leqslant \rho$, we have

$$\min \operatorname{rk}_{\mathbb{K}} T(K \times K \times K) \bmod (\Phi, \Phi, \Phi) = \min \operatorname{rk}_{\mathbb{K}} T \bmod (\mu, \mu, \mu). \tag{8.3}$$

Proof. The conclusion (1) of the current theorem follows immediately from the item (i) of Definition 8.3, which implies

$$\overline{\varphi}_{\tau 1} + \ldots + \overline{\varphi}_{\tau s_\tau} = \overline{m}_\tau((I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau))$$

or $\varphi_{\tau 1} + \ldots + \varphi_{\tau s_\tau} = m_\tau((I \cup J_\tau) \times (I \cup J_\tau))$ for any τ, and hence we get

$$\varphi_{\tau 1}(K \times K) + \ldots + \varphi_{\tau s_\tau}(K \times K) = m_\tau(K \times K).$$

Also, the conclusion (1) of the current theorem confirms that the left hand side of the formula (8.3) does not exceed the corresponding right hand side. Therefore, the remaining conclusion (2) is also true whenever the inequality

$$\operatorname{rk}_{\mathbb{K}} Q \geqslant \min \operatorname{rk}_{\mathbb{K}} T \bmod (\mu, \mu, \mu) \tag{8.4}$$

holds for an arbitrary tensor

$$Q \in T(K \times K \times K) \bmod_{\mathbb{K}} (\Phi, \Phi, \Phi). \tag{8.5}$$

We are going to complete the proof by demonstrating (8.4). Since we assume $\operatorname{rk}_{\mathbb{F}} T \leqslant \rho$ in the conclusion (2), there is nothing to prove whenever $\operatorname{rk}_{\mathbb{K}} Q > \rho$, and, in the rest of the argument, we assume without loss of generality that

$$\operatorname{rk}_{\mathbb{K}} Q \leqslant \rho. \tag{8.6}$$

The condition (8.5) allows one to write

$$Q = T(K \times K \times K) + Q_1 + Q_2 + Q_3 \tag{8.7}$$

with

$$Q_1 \in O \operatorname{mod}_{\mathbb{K}} (\Phi, \varnothing, \varnothing), \;\; Q_2 \in O \operatorname{mod}_{\mathbb{K}} (\varnothing, \Phi, \varnothing), \;\; Q_3 \in O \operatorname{mod}_{\mathbb{K}} (\varnothing, \varnothing, \Phi), \tag{8.8}$$

or, in other words, this means that Q_χ is a $K \times K \times K$ tensor whose χ-slices are $\mathbb{K}$-linear combinations of Φ, where $\chi \in \{1, 2, 3\}$. Namely, there exists a family $(\lambda_{\chi g \varphi})$ of elements in $\mathbb{K}$, where $g \in K$ and $\varphi \in \Phi$, such that

$$\text{the } g\text{-th } \chi\text{-slice of } Q_\chi \text{ is } \sum_{\varphi \in \Phi} \lambda_{\chi g \varphi}\, \varphi. \tag{8.9}$$

Further, we separate the summands in (8.9) into the c families with respect to the partition (8.2), that is, for any $\tau \in \{1, \ldots, c\}$ and $\chi \in \{1, 2, 3\}$, we get the $K \times K \times K$ tensor $\mathcal{Q}_\chi(\tau)$ in which, for any $g \in K$, the g-th χ-slice equals

$$\sum_{\varphi \in \Phi_\tau} \lambda_{\chi g \varphi (K \times K)}\, \varphi(K \times K).$$

Of course, if we recall the condition (8.9), we get that

$$Q_\chi = \mathcal{Q}_\chi(1) + \ldots + \mathcal{Q}_\chi(c). \tag{8.10}$$

Also, in the rest of the proof, we write $\psi_{\tau i}$ to denote the restriction of the matrix $\varphi_{\tau i}$ onto its $J_\tau \times J_\tau$ block, and Ψ_τ stands for the family obtained from Φ_τ by the application of such a restriction, which means that $\Psi_\tau = (\psi_{\tau 1}, \ldots, \psi_{\tau s_\tau})$.

Now we fix an arbitrary $\tau \in \{1, \ldots, c\}$ and proceed to study the coefficients $(\lambda_{\chi g \varphi})$ in Steps 1–4 below. In these Steps 1–4, we additionally assume that

$$g_\tau \text{ is the } I \times I_\tau \text{ padding of the } I_\tau \times I_\tau \text{ identity matrix,} \tag{8.11}$$

and, immediately after the completion of Steps 1–4, we explain how to advocate the use of the assumption (8.11) to proceed with the general case.

Step 1. Let $Q_\chi(\tau)$ and $Q(\tau)$ be the restrictions of Q_χ and Q to the corresponding $(K \setminus (I_\tau \cup J_\tau)) \times J_\tau \times J_\tau$ blocks. The assumption (8.11) requires that the tensors $Q_2(\tau)$ and $Q_3(\tau)$ are zero, so an application of the condition (8.7) implies

$$Q(\tau) = Q_1(\tau), \tag{8.12}$$

and then the formula (8.6) gives

$$\operatorname{rk}_{\mathbb{K}} Q_1(\tau) \leqslant \rho. \tag{8.13}$$

Further, the condition (8.8) shows that every 1-slice of $Q_1(\tau)$ is the $J_\tau \times J_\tau$ restriction of some matrix in $\Phi_\tau\, \mathbb{K}$, that is, every 1-slice of $Q_1(\tau)$ is a member of $\Psi_\tau\, \mathbb{K}$. In view of the inequality (8.13), we get that a generic linear combination γ of the 1-slices of $Q_1(\tau)$ satisfies $\operatorname{rk} \gamma \leqslant \rho$, and we want to apply the point (iv) in Definition 8.3 to γ. In other words, strictly speaking, we consider the purely transcendental extension $\mathcal{K} = \mathbb{K}(\xi_1, \ldots, \xi_\omega)$ with $\omega = |K| - |I_\tau| - |J_\tau|$ so that the 1-slices of $Q_1(\tau)$ can be enumerated as $(\Xi_1, \ldots, \Xi_\omega)$, and we define

$$\gamma = \xi_1\, \Xi_1 + \ldots + \xi_\omega\, \Xi_\omega$$

to be a $J_\tau \times J_\tau$ matrix over $\mathcal{K}$. Indeed, the point (iv) in Definition 8.3 guarantees the existence of a subset $D_{1\tau} \subseteq \{1, \ldots, s_\tau\}$ such that

$$\dim_{\mathbb{F}} \left(\sum_{d \in D_{1\tau}} (\operatorname{rows} \varphi_{\tau d})\, \mathbb{F} \right) \leqslant \sigma_\tau$$

and γ belongs to the linear space

$$\sum_{d \in D_{1\tau}} \psi_{\tau d}\, \mathcal{K}. \tag{8.14}$$

Since the choice of γ is generic, every 1-slice of $Q_1(\tau)$ belongs to (8.14) as well, and, also, since a solution to an inconsistent system of linear equations cannot appear upon an extension of the ground field, every 1-slice of $Q_1(\tau)$ belongs to

$$\sum_{d \in D_{1\tau}} \psi_{\tau d}\, \mathbb{K} \tag{8.15}$$

as well. In other words, for any $g \in K \setminus (I_\tau \cup J_\tau)$, the g-th 1-slice of $Q_1(\tau)$ is

$$\sum_{d \in D_{1\tau}} \Lambda_{1g\tau d}\, \psi_{\tau d} = \sum_{d \in D_{1\tau}} \Lambda_{1g\tau d}\, \psi_{\tau d} + \sum_{d \notin D_{1\tau}} 0\, \psi_{\tau d} \tag{8.16}$$

with some family $(\Lambda_{1g\tau d})$ of scalars in $\mathbb{K}$. In addition, we define

$$\Lambda_{1g\tau d} = 0 \text{ for all } d \notin D_{1\tau} \tag{8.17}$$

and get another expression after a trivial transformation of the formula (8.16):

$$\sum_{d=1}^{s_\tau} \Lambda_{1g\tau d}\, \psi_{\tau d} \text{ is the } g\text{-th 1-slice of } Q_1(\tau), \text{ for any } g \in K \setminus (I_\tau \cup J_\tau). \tag{8.18}$$

Further, the point (ii) of Definition 8.3 guarantees that every matrix of the form

$$\alpha_1\, \psi_{\tau 1} + \ldots + \alpha_{s_\tau}\, \psi_{\tau s_\tau}$$

has the values $\alpha_1, \ldots, \alpha_{s_\tau} \in \mathbb{K}$ defined uniquely up to the addition of a multiple of

$$\psi_{\tau 1} + \ldots + \psi_{\tau s_\tau} = O,$$

which implies that, if the index $g \in K \setminus (I_\tau \cup J_\tau)$ is fixed, the differences between the coefficients in (8.18) and their corresponding counterparts in (8.9) are the same:

$$\lambda_{1g\varphi_{\tau d}(K\times K)} - \Lambda_{1g\tau d} = \Theta_{1g\tau} \text{ for all } d \in \{1, \ldots, s_\tau\}$$

with $\Theta_{1g\tau} \in \mathbb{K}$ being constant for any fixed (g, τ). In view of (8.17), we get

$$\lambda_{1g\varphi_{\tau d}(K\times K)} = \Theta_{1g\tau} \text{ with } d \in \{1, \ldots, s_\tau\} \setminus D_{1\tau} \text{ and } g \in K \setminus (I_\tau \cup J_\tau). \tag{8.19}$$

Step 2. By the symmetry, the argument of Step 1 can be applied to the blocks

$$J_\tau \times (K \setminus (I_\tau \cup J_\tau)) \times J_\tau \text{ and } J_\tau \times J_\tau \times (K \setminus (I_\tau \cup J_\tau))$$

of Q instead of its corresponding $(K \setminus (I_\tau \cup J_\tau)) \times J_\tau \times J_\tau$ block. Therefore, for any $\chi \in \{1, 2, 3\}$, there exists a subset $D_{\chi\tau} \subseteq \{1, \ldots, s_\tau\}$ such that

$$\dim_{\mathbb{F}} \left(\sum_{d \in D_{\chi\tau}} (\operatorname{rows} \varphi_{\tau d})\, \mathbb{F} \right) \leqslant \sigma_\tau \tag{8.20}$$

and also

$$\lambda_{\chi g\varphi_{\tau j}(K\times K)} = \Theta_{\chi g\tau} \text{ whenever } j \in \{1, \ldots, s_\tau\} \setminus D_{\chi\tau} \text{ and } g \in K \setminus (I_\tau \cup J_\tau) \tag{8.21}$$

with $\Theta_{\chi g\tau} \in \mathbb{K}$ being constant for any fixed (χ, g, τ).

Step 3. Now we switch to a separate treatment of those coefficients $(\lambda_{\chi g\varphi})$ that are not covered in Steps 1 and 2. To this end, we define the

$$(I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau)$$

tensors $\widetilde{Q}_\chi(\tau)$ by declaring that, for any $g \in I_\tau \cup J_\tau$ and $\chi \in \{1, 2, 3\}$,

$$\sum_{\varphi \in \overline{\Phi}_\tau} \lambda_{\chi g\varphi}\, \varphi \text{ is the } g\text{-th } \chi\text{-slice of } \widetilde{Q}_\chi(\tau), \tag{8.22}$$

where $\overline{\Phi}_\tau = \{\overline{\varphi}_{\tau 1}, \ldots, \overline{\varphi}_{\tau s_\tau}\}$ is the family of $(I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau)$ matrices defined in the formulation of the theorem. In particular, the family $\Phi_\tau = \{\varphi_{\tau 1}, \ldots, \varphi_{\tau s_\tau}\}$ consists of the $(I \cup J_\tau) \times (I \cup J_\tau)$ paddings of the matrices in $\overline{\Phi}_\tau$ as seen from the formula (8.1) and assumption (8.11). Therefore, in view of the conditions (8.9) and (8.22), for any $g \in I_\tau \cup J_\tau$ and $\chi \in \{1, 2, 3\}$, the g-th χ-slices of the difference

$$Q_\chi - \left(\widetilde{Q}_\chi(\tau)\right)(K \times K \times K)$$

are linear combinations of $\Phi \setminus \Phi_\tau(K \times K)$. The definition of Φ_t in the formulation of the theorem implies that all such linear combinations have all entries in their $J_\tau \times K$ and $K \times J_\tau$ blocks zero, and hence the block

$$(I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau)$$

in $Q_1 + Q_2 + Q_3$ agrees with $\widetilde{Q}_1(\tau) + \widetilde{Q}_2(\tau) + \widetilde{Q}_3(\tau)$ at every entry outside the

$$J_\tau \times I_\tau \times I_\tau, \quad I_\tau \times J_\tau \times I_\tau, \quad I_\tau \times I_\tau \times J_\tau \quad \text{and} \quad I_\tau \times I_\tau \times I_\tau \tag{8.23}$$

blocks. In view of the condition (8.7), this also means that $\widetilde{Q}_1(\tau) + \widetilde{Q}_2(\tau) + \widetilde{Q}_3(\tau)$ agrees with the $(I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau)$ restriction of Q at every entry not in (8.23). According to the inequality (8.6), the tensor Q is of the rank not exceeding ρ, and hence an appropriate amendment of the entries on the positions (8.23) of

$$\widetilde{Q}_1(\tau) + \widetilde{Q}_2(\tau) + \widetilde{Q}_3(\tau)$$

leads to a tensor of the corresponding rank not exceeding ρ. We apply the point (v) in Definition 8.3, and, for any $\chi \in \{1, 2, 3\}$, this gives a subset $\overline{\Phi}_{\chi\tau} \subseteq \overline{\Phi}_\tau$ such that

$$\dim_{\mathbb{F}} \left(\sum_{\varphi \in \overline{\Phi}_{\chi\tau}} (\operatorname{rows} \varphi)\, \mathbb{F} \right) \leqslant \pi_\tau \tag{8.24}$$

and

$$\sum_{\chi=1}^{3} \widetilde{Q}_\chi(\tau) \in O \bmod_{\mathbb{K}} (\overline{\Phi}_{1\tau}\, \mathbb{K} + \widetilde{m}_\tau\, \mathbb{K}, \overline{\Phi}_{2\tau}\, \mathbb{K} + \widetilde{m}_\tau\, \mathbb{K}, \overline{\Phi}_{3\tau}\, \mathbb{K} + \widetilde{m}_\tau\, \mathbb{K}), \tag{8.25}$$

where $\widetilde{m}_\tau = \overline{m}_\tau((I_\tau \cup J_\tau) \times (I_\tau \cup J_\tau))$.

Step 4. Now we return to the consideration of the tensors $\mathcal{Q}_\chi(\tau)$ defined in the discussion before Step 1. In particular, in view of the condition (8.21), we get that, for any $g \in K \setminus (I_\tau \cup J_\tau)$ and $\chi \in \{1, 2, 3\}$, the g-th χ-slice of $\mathcal{Q}_\chi(\tau)$ belongs to

$$\sum_{d \in D_{\chi\tau}} \varphi_{\tau d}(K \times K)\, \mathbb{K} + \Theta_{\chi g\tau} \left(\varphi_{\tau 1}(K \times K) + \ldots + \varphi_{\tau s_\tau}(K \times K)\right),$$

and, since we have $\varphi_{\tau 1}(K\times K)+\ldots+\varphi_{\tau s_\tau}(K\times K)=m_\tau(K\times K)$ by the item (i) of Definition 8.3, we see that the g-th χ-slice of $\mathcal{Q}_\chi(\tau)$ belongs to the space

$$\sum_{d\in D_{\chi\tau}} \varphi_{\tau d}(K\times K)\,\mathbb{K}+m_\tau(K\times K)\,\mathbb{K}.$$

Further, since the removal of all the g-th 1-slices, g-th 2-slices, and g-th 3-slices, with every $g\in K\setminus(I_\tau\cup J_\tau)$, transforms $\mathcal{Q}_1(\tau)+\mathcal{Q}_2(\tau)+\mathcal{Q}_3(\tau)$ into the tensor $\widetilde{Q}_1(\tau)+\widetilde{Q}_2(\tau)+\widetilde{Q}_3(\tau)$ as in Step 3, we apply the condition (8.25) and get

$$\mathcal{Q}_1(\tau)+\mathcal{Q}_2(\tau)+\mathcal{Q}_3(\tau)\in O\bmod_{\mathbb{K}}\left(L_{1\tau},L_{2\tau},L_{3\tau}\right), \tag{8.26}$$

where, for any $\chi\in\{1,2,3\}$, we denote $L_{\chi\tau}=\ell_{\chi\tau}+m_\tau(K\times K)\,\mathbb{K}$ and

$$\ell_{\chi\tau}=\sum_{d\in D_{\chi\tau}}\varphi_{\tau d}(K\times K)\,\mathbb{K}+\overline{\Phi}_{\chi\tau}(K\times K)\,\mathbb{K}. \tag{8.27}$$

Further, using the conditions (8.20) and (8.24), we get that

$$\dim_{\mathbb{K}} W_{\chi\tau}\leqslant \pi_\tau+\sigma_\tau \tag{8.28}$$

in which

$$W_{\chi\tau}=\sum_{\varphi\in\ell_{\chi\tau}}(\operatorname{rows}\varphi)\,\mathbb{K}, \tag{8.29}$$

or, in other words, $W_{\chi\tau}$ is the sum of all the row spaces of the matrices in (8.27). Since every matrix in $\ell_{\chi\tau}$ is the padding of an $(I_\tau\cup J_\tau)\times(I_\tau\cup J_\tau)$ matrix, we get

$$W_{\chi\tau}\subseteq V_\tau+U, \tag{8.30}$$

where V_τ and U are the families of those vectors in $\mathbb{K}^K$ which have all their non-zeros collected at the coordinates in J_τ and I, respectively. Further, the condition (8.28) allows us to apply the point (iii) in Definition 8.3, and hence we get

$$W_{\chi\tau}\cap U=0. \tag{8.31}$$

Also, since all matrices in $\ell_{\chi\tau}$ are symmetric, the formula (8.29) implies

$$\ell_{\chi\tau}\subseteq (W_{\chi\tau}\otimes W_{\chi\tau})\,\mathbb{K},$$

and hence the linear space $L_{\chi\tau}$ as in (8.26) satisfies

$$L_{\chi\tau}\subseteq (W_{\chi\tau}\otimes W_{\chi\tau})\,\mathbb{K}+m_\tau(K\times K)\,\mathbb{K}. \tag{8.32}$$

Finally, a comparison of (8.26) and (8.32) allows us to write

$$\mathcal{Q}_1(\tau)+\mathcal{Q}_2(\tau)+\mathcal{Q}_3(\tau)\in O\bmod_{\mathbb{K}}\left(\mathcal{S}_{1\tau},\mathcal{S}_{2\tau},\mathcal{S}_{3\tau}\right) \tag{8.33}$$

with $\mathcal{S}_{\chi\tau}=(W_{\chi\tau}\otimes W_{\chi\tau})\,\mathbb{K}+m_\tau(K\times K)\,\mathbb{K}$.

The consideration of Steps 1–4 is now complete, so we return to the discussion of the assumption (8.11). Namely, we are going to confirm that the existence of linear spaces $(W_{1\tau},W_{2\tau},W_{3\tau})$ satisfying all the relevant conclusions (8.30), (8.31), (8.33) is still in effect in the general case, that is, without assuming the condition (8.11).

Indeed, the condition $\operatorname{rk} g_\tau=|I_\tau|$ guarantees the existence of an invertible $I\times I$ matrix $\mathcal{C}_\tau$ over $\mathbb{F}$ for which $\mathcal{C}_\tau g_\tau$ is the $I\times I_\tau$ padding of the identity matrix $\operatorname{id}(I_\tau)$. We note that the assumptions of the current theorem remain valid if we replace

$$(g_1,\ldots,g_c)\to(\mathcal{C}_\tau g_1,\ldots,\mathcal{C}_\tau g_c)\ \text{ and }\ T\to(\mathcal{C}_\tau\otimes\mathcal{C}_\tau\otimes\mathcal{C}_\tau)\,T, \tag{8.34}$$

which implies that, for any $t\in\{1,\ldots,c\}$, we also substitute

$$m_t\to(\mathcal{C}_\tau\otimes\mathcal{C}_\tau)\,m_t$$

and make the further amendments to the tensors in the proof, namely,

$$Q \to \beta(Q),\ \ Q_\chi \to \beta(Q_\chi),\ \ \mathcal{Q}_\chi(t) \to \beta\left(\mathcal{Q}_\chi(t)\right)$$

with

$$\beta = (\mathcal{C}_\tau \oplus \mathrm{id}(K \setminus I)) \otimes (\mathcal{C}_\tau \oplus \mathrm{id}(K \setminus I)) \otimes (\mathcal{C}_\tau \oplus \mathrm{id}(K \setminus I))\,.$$

Also, the assumption (8.11) comes into effect in the new setting. Therefore, we can apply Steps 1–4 and find subsets (H_1, H_2, H_3) of $\{1, \ldots, s_\tau\}$ such that the space $\mathcal{W}_{\chi\tau}$ equal to the $\mathbb{K}$-linear span of all the rows of the $K \times K$ paddings of the matrices

$$((\mathcal{C}_\tau \oplus \mathrm{id}(J_\tau)) \otimes (\mathcal{C}_\tau \oplus \mathrm{id}(J_\tau)))\,\varphi_{\tau h}$$

with $h \in H_\chi$ satisfies the analogues of the desired conclusions (8.30), (8.31), (8.33):

$$\mathcal{W}_{\chi\tau} \subseteq V_\tau + U, \tag{8.35}$$

$$\mathcal{W}_{\chi\tau} \cap U = 0, \tag{8.36}$$

$$\beta\left(\mathcal{Q}_1(\tau) + \mathcal{Q}_2(\tau) + \mathcal{Q}_3(\tau)\right) \in O \bmod_{\mathbb{K}} (\widehat{\mathcal{S}}_{1\tau}, \widehat{\mathcal{S}}_{2\tau}, \widehat{\mathcal{S}}_{3\tau}), \tag{8.37}$$

where

$$\widehat{\mathcal{S}}_{\chi\tau} = (\mathcal{W}_{\chi\tau} \otimes \mathcal{W}_{\chi\tau})\mathbb{K} + (\mathcal{C}_\tau \oplus \mathrm{id}(K \setminus I)) \otimes (\mathcal{C}_\tau \oplus \mathrm{id}(K \setminus I))\,(m_\tau(K \times K))\,\mathbb{K}.$$

Further, the takeback of the substitution (8.34) transforms the space $\mathcal{W}_{\chi\tau}$ into the $\mathbb{K}$-linear span of all the rows of the $K \times K$ paddings of the matrices $\varphi_{\tau h}$ with $h \in H_\chi$ (and we define the desired $W_{\chi\tau}$ to be this resulting space). The conditions (8.30) and (8.31) are now obtained immediately because the mapping

$$(\mathcal{C}_\tau \oplus \mathrm{id}(K \setminus I)) \otimes (\mathcal{C}_\tau \oplus \mathrm{id}(K \setminus I)) \tag{8.38}$$

is the direct sum of the corresponding linear operators on $(U, V_1, \ldots, V_c)$, and hence the inverse of the operator (8.38) still leaves these subspaces invariant. Finally, it remains to note that the takeback of (8.34) turns $\widehat{\mathcal{S}}_{\chi\tau}$ into $\mathcal{S}_{\chi\tau}$ and removes β from the left hand side of (8.37), so we get the condition (8.33) as well.

Therefore, we have finally confirmed the existence of the spaces $(W_{\chi\tau})$ satisfying the conditions (8.30), (8.31), (8.33), for all $\chi \in \{1, 2, 3\}$ and $\tau \in \{1, \ldots, c\}$. We proceed the proof and use the conditions (8.30) and (8.31) to apply Lemma 8.5, and hence we find three $\mathbb{K}$-linear mappings $(\alpha_1, \alpha_2, \alpha_3)$ from $\mathbb{K}^K$ to $\mathbb{K}^I$ such that

$$\alpha_\chi(W_{\chi\tau}) = 0 \ \text{ holds for all } \ \chi \in \{1, 2, 3\} \ \text{ and } \ \tau \in \{1, \ldots, c\}, \tag{8.39}$$

and

$$\text{the restriction of } \alpha_\chi \text{ to } U \text{ is the identification mapping } \ U \to \mathbb{K}^I \tag{8.40}$$

or, in other words, the mapping (8.40) sends a vector $u \in U$ to the same vector but with all coordinates in $K \setminus U$ removed (of course, these removed coordinates are all zero in u by the definition of U above). In particular, we obtain

$$(\alpha_3 \otimes \alpha_1 \otimes \alpha_2)\,T(K \times K \times K) = T,$$

and then we apply $\alpha_3 \otimes \alpha_1 \otimes \alpha_2$ to both sides of the expression (8.7) to get

$$(\alpha_3 \otimes \alpha_1 \otimes \alpha_2)\,Q = (\alpha_3 \otimes \alpha_1 \otimes \alpha_2)\,(Q_1 + Q_2 + Q_3) + T. \tag{8.41}$$

Further, according to the conditions (8.10) and (8.33), we get

$$Q_1 + Q_2 + Q_3 \in O \bmod_{\mathbb{K}} \left(\mathcal{S}_{11} + \ldots + \mathcal{S}_{1c},\, \mathcal{S}_{21} + \ldots + \mathcal{S}_{2c},\, \mathcal{S}_{31} + \ldots + \mathcal{S}_{3c}\right), \tag{8.42}$$

and, whenever $\hat{\imath}, \hat{\jmath} \in \{1,2,3\}$ and $\chi \in \{\hat{\imath}, \hat{\jmath}\}$, the conditions (8.39) and (8.40) imply

$$(\alpha_{\hat{\imath}} \otimes \alpha_{\hat{\jmath}})(\mathcal{S}_{\chi 1} + \ldots + \mathcal{S}_{\chi c}) = \mu\, \mathbb{K}. \tag{8.43}$$

A further application of the conditions (8.42) and (8.43) to (8.41) implies

$$(\alpha_3 \otimes \alpha_1 \otimes \alpha_2)\, Q \in T \operatorname{mod}_{\mathbb{K}} (\mu, \mu, \mu)$$

and hence

$$\operatorname{rk}_{\mathbb{K}} (\alpha_3 \otimes \alpha_1 \otimes \alpha_2)\, Q \geqslant \min \operatorname{rk}_{\mathbb{K}} T \operatorname{mod} (\mu, \mu, \mu). \tag{8.44}$$

Finally, we also get the inequality

$$\operatorname{rk}_{\mathbb{K}} Q \geqslant \operatorname{rk}_{\mathbb{K}} (\alpha_3 \otimes \alpha_1 \otimes \alpha_2)\, Q \tag{8.45}$$

after an application of Observation 4.11, and this completes the proof because a comparison of the conditions (8.44) and (8.45) gives the desired inequality (8.4). □

The above theorem suggests the following specification of Definition 8.3.

Definition 8.7. Let $I \subseteq I'$ be indexing families, and let m be an $I \times I$ matrix over a field $\mathbb{F}$. A family Φ of $I' \times I'$ matrices is called an *eliminating family* for m *with respect to* the field $\mathbb{F}$ and rank bound ρ, or, simply, *with respect to* $(\mathbb{F}, \rho)$, if there are π and σ such that Φ is a candidate family of the type $(\mathbb{F}, m, \rho, \rho, \pi, \sigma, \pi + \sigma)$.

Remark 8.8. The conditions (i) and (ii) in Claim 5.6 can be seen as a partial generalization of the Gaussian elimination process to the case of the slices of the rank different from one in higher dimensional tensors. Since, as we can see from Theorem 8.6, the families in Definition 8.7 guarantee the validity of these conditions, we chose the name *eliminating families* for the concept in Definition 8.7.

Corollary 8.9. *Let $\rho, s_1, \ldots, s_c$ be positive integers, let $I, I_1, \ldots, I_c$ be indexing sets. For any $\tau \in \{1, \ldots, c\}$, let $\overline{m}_\tau$ be a symmetric $I_\tau \times I_\tau$ matrix over a field $\mathbb{F}$ so that $|I_\tau| \leqslant |I|$. If, for any $\tau \in \{1, \ldots, c\}$, we are given a family Φ_τ of matrices of the size $s_\tau \times s_\tau$ so that Φ_τ is eliminating for $\overline{m}_\tau$ with respect to $(\mathbb{F}, \rho)$, then, for any family $(g_1, \ldots, g_c)$ in which every g_τ is an $I \times I_\tau$ matrix with* $\operatorname{rk}_{\mathbb{F}} g_\tau = |I_\tau|$, *there exist an indexing set $K \supseteq I$ and a family Φ of $K \times K$ matrices such that*

(a) $|\Phi| = |\Phi_1| + \ldots + |\Phi_c|$,
(b) $|K| - |I| = (s_1 - |I_1|) + \ldots + (s_c - |I_c|)$,
(c) *for any $m_\tau \in \mu$, we have $m_\tau(K \times K) \in \Phi\, \mathbb{F}$, where*

$$m_\tau = g_\tau\, \overline{m}_\tau\, (g_\tau)^\top$$

and $\mu = (m_1, \ldots, m_c)$,
(d) *any extension $\mathbb{K} \supseteq \mathbb{F}$ and any $I \times I \times I$ tensor T with* $\operatorname{rk}_{\mathbb{F}} T \leqslant \rho$ *satisfy*

$$\min \operatorname{rk}_{\mathbb{K}} T(K \times K \times K) \operatorname{mod} (\Phi, \Phi, \Phi) = \min \operatorname{rk}_{\mathbb{K}} T \operatorname{mod} (\mu, \mu, \mu),$$

(e) *for any $\varphi \in \Phi$, there exist a matrix $\varphi' \in \Phi_1 \cup \ldots \cup \Phi_c$ and a matrix C_φ with full column rank and with entries in $\mathbb{F}$ such that*

$$C_\varphi \cdot \varphi' \cdot (C_\varphi)^\top = \varphi.$$

Proof. This is a reformulation of Theorem 8.6 in terms of Definition 8.7. □

9. Eliminating families. New ones from known ones

As we will see later, the construction of a particular relevant eliminating family can be a highly demanding task, and, in this section, we proceed with some general information on such families that is needed in our argument. In particular, we give one particular relevant construction that provides us with a further example of an eliminating family if we are given one such example. The ideas of this section are motivated by Theorem 9.2 below, which explains how to prove Claim 5.6 if we are given the appropriate eliminating families between the sets in Notation 5.4.

Remark 9.1. In the $t = k$ summand of the right hand side of the formula (9.2) below, the product is taken over the empty set, so we declare it to be equal to 1.

Theorem 9.2. *Assume that $k \geqslant 0$, $\rho \geqslant 1$ are integers, and $\mathbb{F}$ is a field. Assume that, for any positive integer t not exceeding k, some full rank $2^t \times 2^t$ skew projector admits an eliminating family $\Phi_t \subset \operatorname{skp}(\mathbb{F}, t-1, I_t)$ with respect to $(\mathbb{F}, \rho)$ such that*

$$|\Phi_t| = s_t(\rho) \ \text{ and } \ |I_t| = 2^t + c_t(\rho) \tag{9.1}$$

with some integers $s_t(\rho)$ and $c_t(\rho)$. Then, for any finite subset $W \subset \operatorname{skp}(\mathbb{F}, k, I)$, there exist a set $I' \supseteq I$ and a family $\mathcal{M}$ of symmetric rank one $I' \times I'$ matrices with

$$|I'| - |I| = |W| \cdot \left(\sum_{t=1}^{k} c_t(\rho) \cdot \left(\prod_{\tau=t+1}^{k} s_\tau(\rho) \right) \right) \tag{9.2}$$

and

$$|\mathcal{M}| = s_1(\rho) \cdot s_2(\rho) \cdot \ldots \cdot s_k(\rho) \cdot |W| \tag{9.3}$$

so that the conclusions (i) and (ii) in Claim 5.6 are satisfied.

Proof. In the trivial case $k = 0$, we take $I' = I$ and $\mathcal{M} = W$ and immediately check the conditions (9.2) and (9.3), in which, again, we assume that the empty product equals 1 and the empty sum is 0. Therefore, we can assume $k > 0$ and proceed by the induction on k. Indeed, we can further apply Corollary 8.9, which provides us with a family $\mathcal{F}_k \subset \operatorname{skp}(\mathbb{F}, k-1, \mathcal{I}_k)$ of $\mathcal{I}_k \times \mathcal{I}_k$ matrices over $\mathbb{F}$ for which

$$|\mathcal{F}_k| = s_k(\rho) \cdot |W|, \tag{9.4}$$

$$|\mathcal{I}_k| - |I| = c_k(\rho) \cdot |W|, \tag{9.5}$$

and, also,

$$\text{any } w \in W \text{ satisfies } w(\mathcal{I}_k \times \mathcal{I}_k) \in \mathcal{F}_k \, \mathbb{F}, \tag{9.6}$$

and, for any extension $\mathbb{K} \supseteq \mathbb{F}$ and any $I \times I \times I$ tensor T with $\operatorname{rk}_{\mathbb{F}} T \leqslant \rho$, we have

$$\min \operatorname{rk}_{\mathbb{K}} T(\mathcal{I}_k \times \mathcal{I}_k \times \mathcal{I}_k) \bmod (\mathcal{F}_k, \mathcal{F}_k, \mathcal{F}_k) = \min \operatorname{rk}_{\mathbb{K}} T \bmod (W, W, W). \tag{9.7}$$

Further, the application of the inductive assumption to the family $\mathcal{F}_k$ gives a family $\mathcal{M}$ consisting of symmetric rank one $I' \times I'$ matrices over $\mathbb{F}$ which satisfies

$$|\mathcal{M}| = s_1(\rho) \cdot \ldots \cdot s_{k-1}(\rho) \cdot |\mathcal{F}_k|, \tag{9.8}$$

$$|I'| - |\mathcal{I}_k| = |\mathcal{F}_k| \cdot \left(\sum_{t=1}^{k-1} c_t(\rho) \cdot \left(\prod_{\tau=t+1}^{k-1} s_\tau(\rho) \right) \right), \tag{9.9}$$

and, also,

$$\text{any } f \in \mathcal{F}_k \text{ satisfies } f(I' \times I') \in \mathcal{M} \, \mathbb{F}, \tag{9.10}$$

and, for any $\mathcal{I}_k \times \mathcal{I}_k \times \mathcal{I}_k$ tensor $\mathcal{T}$ with $\operatorname{rk}_{\mathbb{F}} \mathcal{T} \leqslant \rho$, we have

$$\min \operatorname{rk}_{\mathbb{K}} \mathcal{T}(I' \times I' \times I') \bmod (\mathcal{M}, \mathcal{M}, \mathcal{M}) = \min \operatorname{rk}_{\mathbb{K}} \mathcal{T} \bmod (\mathcal{F}_k, \mathcal{F}_k, \mathcal{F}_k). \tag{9.11}$$

Now we are ready to complete the proof. In fact, a comparison of the equalities (9.4) and (9.8) confirms the desired property (9.3), and, similarly, the formulas (9.5) and (9.9) confirm the equality (9.2), and the conditions (9.6) and (9.10) imply the point (i) in Claim 5.6. Finally, we apply the equality (9.11) with the tensor $T(\mathcal{I}_k \times \mathcal{I}_k \times \mathcal{I}_k)$ in the role of $\mathcal{T}$, and we use the resulting condition

$$\min \operatorname{rk}_{\mathbb{K}} T(I' \times I' \times I') \bmod (\mathcal{M}, \mathcal{M}, \mathcal{M}) = \min \operatorname{rk}_{\mathbb{K}} T(\mathcal{I}_k \times \mathcal{I}_k \times \mathcal{I}_k) \bmod (\mathcal{F}_k, \mathcal{F}_k, \mathcal{F}_k)$$

together with the equality (9.7) to confirm the remaining item (ii) in Claim 5.6. $\square$

As we can see, Theorem 9.2 suggests a potential approach to Claim 5.6 which requires the construction of a sequence of sets (Φ_k) such that, for any $k \geqslant 1$, one has $\Phi_k \subset \operatorname{skp}(\mathbb{F}, k-1, I')$, and Φ_k is an eliminating family for some matrix in $\operatorname{skp}(\mathbb{F}, k, I)$. In regard to that, we note that the transformation

$$\mathrm{LSP}: \ X \to \begin{pmatrix} -2X & O \\ O & 2X \end{pmatrix}$$

doubles the rank of a given matrix whenever the ground field is of the characteristic different from two, and, moreover, the following is true.

Observation 9.3. *If M is either a skew projector over a field $\mathbb{F}$ or a symmetric rank one matrix with entries in $\mathbb{F}$, and if, in addition, we have* $\operatorname{char} \mathbb{F} \neq 2$, *then*

$$M' = \begin{pmatrix} -2M & O \\ O & 2M \end{pmatrix}$$

is a skew projector with $\operatorname{rk} M' = 2 \operatorname{rk} M$.

Proof. The possibility $\operatorname{rk} M = 1$ immediately reduces to the case when M is a 1×1 matrix, and then the analysis is straightforward. Otherwise, using Definition 5.2, we get that $M = C^\top J C$, where C is a non-singular matrix over $\mathbb{F}$, and J is a block diagonal matrix with every diagonal block equal either to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9.12}$$

or to the 1×1 zero matrix. We note that

$$\begin{pmatrix} -2M & O \\ O & 2M \end{pmatrix} = \begin{pmatrix} -I & I \\ I & I \end{pmatrix} \begin{pmatrix} C^\top & O \\ O & C^\top \end{pmatrix} \begin{pmatrix} O & J \\ J & O \end{pmatrix} \begin{pmatrix} C & O \\ O & C \end{pmatrix} \begin{pmatrix} -I & I \\ I & I \end{pmatrix}$$

with I being the identity matrix of the appropriate size, or $M' = D^\top J' D$ with

$$D = \begin{pmatrix} C & O \\ O & C \end{pmatrix} \begin{pmatrix} -I & I \\ I & I \end{pmatrix} \text{ and } J' = \begin{pmatrix} O & J \\ J & O \end{pmatrix},$$

and now it is clear how to turn J' into the form of the padding of the direct sum of the blocks of the form (9.12) by the application of the same permutation to its rows and columns, so we have confirmed that M' is a skew projector as well. $\square$

In fact, this natural connection between $\operatorname{skp}(\mathbb{F}, k-1, I)$ and $\operatorname{skp}(\mathbb{F}, k, I)$ extends to the property of a set of matrices to be a candidate family in the sense of Definition 8.3. More precisely, Theorem 9.6 below allows one to control the type of the family constructed from a given family Φ in the way as in Observation 9.3 if we know the type of the initial family Φ. Together with Theorem 9.2, this fact will give a reduction of Claim 5.6 to its special case with $k = 1$.

Definition 9.4. Let I and J be two indexing sets which are not necessarily disjoint, and let $K = I \cup J = \{k_1, \dots, k_s\}$. We consider the two disjoint copies

$$K(1) = \{k_1(1), \dots, k_s(1)\} \text{ and } K(2) = \{k_1(2), \dots, k_s(2)\},$$

and, if we have a subset

$$P = \{k_{q_1}, \dots, k_{q_r}\} \subseteq K,$$

then, for $\tau \in \{1, 2\}$, we write $P(\tau)$ to denote the corresponding copy

$$P(\tau) = \{k_{q_1}(\tau), \dots, k_{q_r}(\tau)\} \subseteq K(\tau).$$

For any $I \times J$ matrix A, we define $\mathrm{LSP}(A)$ as the $(I(1) \cup I(2)) \times (J(1) \cup J(2))$ matrix in which, for all $i \in I$, $j \in J$ and $\tau', \tau'' \in \{1, 2\}$, the entries are

$$[\mathrm{LSP}(A)]_{i(\tau'),j(\tau'')} = \begin{cases} -2[A]_{ij} & \text{if } \tau' = \tau'' = 1, \\ 2[A]_{ij} & \text{if } \tau' = \tau'' = 2, \\ 0 & \text{otherwise (that is, if } \tau' \neq \tau''). \end{cases}$$

Remark 9.5. Clearly, Definition 9.4 is a formal recording of the mapping realizing the construction in Observation 9.3. In other words, this mapping allows one to lift a given skew projector to another one whose rank is twice larger, and hence we chose the name 'LSP' as an abbreviation of the phrase 'a *larger skew projector*'.

Let us show that LSP preserves the property of being a candidate family.

Theorem 9.6. *If $\mathbb{F}$ is a field with $\operatorname{char} \mathbb{F} \neq 2$, and Φ is a candidate family of the type $(\mathbb{F}, m, \rho, r, \pi, \sigma, \delta)$, for some nonnegative integers $(\rho, r, \pi, \sigma, \delta)$ and a matrix m, then $\mathrm{LSP}(\Phi)$ is a candidate family of the type $(\mathbb{F}, \mathrm{LSP}(m), r, 2r, 4\sigma, 2\sigma, 2\delta)$.*

Proof. In the setting of Definition 8.3, m is a symmetric $I \times I$ matrix with entries in $\mathbb{F}$. In order to prove the current theorem, we need to assume the validity of the conditions (i)–(v) in Definition 8.3 for the initial family Φ and deduce the validity of the same conditions but applied to the family $\mathrm{LSP}(\Phi)$ and its corresponding type. Indeed, as in Definition 8.3, we assume that the matrices in $\Phi = \{\varphi_1, \dots, \varphi_s\}$ are $(I \cup J) \times (I \cup J)$, where J is an indexing set disjoint with I, and then the matrices in $\mathrm{LSP}(\Phi) = \{\mathrm{LSP}(\varphi_1), \dots, \mathrm{LSP}(\varphi_s)\}$ are of the format

$$(I(1) \cup I(2) \cup J(1) \cup J(2)) \times (I(1) \cup I(2) \cup J(1) \cup J(2)).$$

In a way similar to Definition 8.3, we further write $\Psi = \{\psi_1, \dots, \psi_s\}$ for the family of the corresponding $J \times J$ blocks of the matrices in Φ, that is, the matrix ψ_τ is the $J \times J$ block of φ_τ for all $\tau \in \{1, \dots, s\}$, and we get $\mathrm{LSP}(\Psi) = \{\mathrm{LSP}(\psi_1), \dots, \mathrm{LSP}(\psi_s)\}$, where $\mathrm{LSP}(\psi_\tau)$ appears as the $(J(1) \cup J(2)) \times (J(1) \cup J(2))$ block of $\mathrm{LSP}(\varphi_\tau)$.

In this notation, we have

$$\mathrm{LSP}(\varphi_1) + \ldots + \mathrm{LSP}(\varphi_s) = \mathrm{LSP}(\varphi_1 + \ldots + \varphi_s) = \mathrm{LSP}(m((I \cup J) \times (I \cup J))),$$

so the matrix $\mathrm{LSP}(\varphi_1) + \ldots + \mathrm{LSP}(\varphi_s)$ is indeed the padding of $\mathrm{LSP}(m)$, and hence the point (i) in Definition 8.3 is confirmed. Further, we consider the equation

$$\lambda_1 \mathrm{LSP}(\psi_1) + \ldots + \lambda_s \mathrm{LSP}(\psi_s) = O$$

with $\lambda_1, \dots, \lambda_s \in \mathbb{F}$, and we rewrite it as

$$\mathrm{LSP}(\lambda_1 \psi_1 + \ldots + \lambda_s \psi_s) = O,$$

which is equivalent to

$$\lambda_1 = \ldots = \lambda_s \tag{9.13}$$

in view of the initial condition (ii) applied to the family Φ, and we remark that the equalities (9.13) prove the condition (ii) for the new family $\mathrm{LSP}(\Phi)$ as well.

We proceed with the item (iii) in Definition 8.3. If M is an arbitrary matrix with the $\mathbb{F}$-linear span of the rows denoted by V, then we remark that the row space of $\mathrm{LSP}(M)$ is the direct sum of the two copies $V(1) \oplus V(2)$ of the initial space V. Therefore, as requested in the $\mathrm{LSP}(\Phi)$ version of the item (iii), we consider a subset $\{q_1, \ldots, q_k\} \subseteq \{1, \ldots, s\}$ such that the sum of the row spaces of the matrices

$$\mathrm{LSP}(\varphi_{q_1}), \ldots, \mathrm{LSP}(\varphi_{q_k}) \tag{9.14}$$

has the dimension at most 2δ, and we get that the sum of the row spaces of

$$\varphi_{q_1}, \ldots, \varphi_{q_k} \tag{9.15}$$

is at most δ. Consequently, the subset (9.15) allows an application of the item (iii) corresponding to the initial family Φ, and we conclude that the sum of the row spaces of the matrices (9.15) does not contain any nonzero vector which has all coordinates in J equal to zero. This implies that the sum of the row spaces of (9.14) does not contain any nonzero vector with all coordinates in $J(1) \cup J(2)$ equal to zero, and hence this concludes the consideration of the item (iii).

Now we switch to the item (iv) in Definition 8.3 and consider a matrix

$$\mathrm{LSP}(\psi_0) \in \mathrm{LSP}(\psi_1)\,\mathbb{K} + \ldots + \mathrm{LSP}(\psi_s)\,\mathbb{K}$$

which satisfies $\operatorname{rk} \mathrm{LSP}(\psi_0) \leqslant 2r$, where $\mathbb{K}$ is an arbitrary field extension of $\mathbb{F}$. In view of the definition of LSP, this immediately implies $\operatorname{rk} \psi_0 \leqslant r$, and hence the Φ version of the item (iv) applies. So we find a subset $D \subseteq \{1, \ldots, s\}$ such that

$$\psi_0 \in \sum_{d \in D} \psi_d\,\mathbb{K} \tag{9.16}$$

and

$$\dim_{\mathbb{F}} \left(\sum_{d \in D} (\text{rows } \varphi_d)\,\mathbb{F} \right) \leqslant \sigma. \tag{9.17}$$

An immediate application of the LSP operator to the condition (9.16) implies

$$\mathrm{LSP}(\psi_0) \in \sum_{d \in D} \mathrm{LSP}(\psi_d)\,\mathbb{K}, \tag{9.18}$$

and, in view of the above mentioned fact that the row space of $\mathrm{LSP}(\varphi)$ is the direct sum of the two copies of the row space of φ, the formula (9.17) implies

$$\dim_{\mathbb{F}} \left(\sum_{d \in D} (\text{rows } \mathrm{LSP}(\varphi_d))\,\mathbb{F} \right) \leqslant 2\sigma, \tag{9.19}$$

and the conditions (9.18) and (9.19) prove the $\mathrm{LSP}(\Phi)$ version of the item (iv).

As to the remaining item (v) of Definition 8.3, we set

$$K = I(1) \cup I(2) \cup J(1) \cup J(2)$$

and take a pair (T, Δ) of $K \times K \times K$ tensors such that

$$T \in O \bmod_{\mathbb{K}} (\mathrm{LSP}(\Phi), \mathrm{LSP}(\Phi), \mathrm{LSP}(\Phi))$$

and

$$\operatorname{rk}_{\mathbb{K}}(T+\Delta) \leqslant r, \tag{9.20}$$

and also every nonzero entry of Δ should belong to the union of the blocks

$$(I(1)\cup I(2))\times(I(1)\cup I(2))\times(I(1)\cup I(2)),$$
$$(I(1)\cup I(2))\times(I(1)\cup I(2))\times(J(1)\cup J(2)),$$
$$(I(1)\cup I(2))\times(J(1)\cup J(2))\times(I(1)\cup I(2)),$$
$$(J(1)\cup J(2))\times(I(1)\cup I(2))\times(I(1)\cup I(2)).$$

Also, we write $T = T_1 + T_2 + T_3$ so that, for any $\chi \in \{1, 2, 3\}$, the χ-slices of T_χ belong to $\mathrm{LSP}(\Phi)\,\mathbb{K}$. In particular, we see that the $(I(1)\cup J(1))\times J(2)\times J(2)$ blocks of T_2 and T_3 are zero, and, similarly, according to the definition of Δ, the

$$(I(1)\cup J(1))\times J(2)\times J(2) \tag{9.21}$$

restriction of Δ is also zero, so the corresponding restrictions of T_1 and $T+\Delta$ to (9.21) coincide. In view of (9.20), we get that the corresponding rank of the restriction of T_1 to (9.21) is at most r, and then we want to apply the point (iv) to a generic linear combination γ of the 1-slices of this restriction with the argument similar to the one in Step 1 of the proof of Theorem 8.6. Namely, we get

$$\operatorname{rk}\gamma \leqslant r \tag{9.22}$$

for the $J(2)\times J(2)$ matrix

$$\gamma = \xi_1\,\Xi_1 + \ldots + \xi_\omega\,\Xi_\omega$$

over the purely transcendental extension

$$\mathcal{K} = \mathbb{K}(\xi_1, \ldots, \xi_\omega)$$

with $\omega = |I| + |J|$, where $(\Xi_1, \ldots, \Xi_\omega)$ is the family of all 1-slices of the restriction of T_1 to the block $(I(1)\cup J(1))\times J(2)\times J(2)$. The definition of T_1 confirms that these 1-slices are the $J(2)\times J(2)$ copies of the corresponding matrices in $\Psi\,\mathcal{K}$, so the inequality (9.22) allows an application of the Φ version of the point (iv) in Definition 8.3. Therefore, there exists a subset $D \subseteq \{1, \ldots, s\}$ such that γ is the $J(2)\times J(2)$ copy of the corresponding matrix in

$$\sum_{d\in D}\psi_d\,\mathcal{K}, \tag{9.23}$$

and the sum of the row spaces of φ_d over all $d \in D$ has the dimension at most σ. Since the choice of γ was generic, we conclude that the $J(2)\times J(2)$ block of every 1-slice of T_1 with an index in $I(1)\cup J(1)$ is also the $J(2)\times J(2)$ copy of some matrix in (9.23), and, since a solution to an inconsistent system of linear equations cannot appear upon an extension of the ground field, the $J(2)\times J(2)$ block of every 1-slice of T_1 with an index in $I(1)\cup J(1)$ is the $J(2)\times J(2)$ copy of a matrix in

$$\sum_{d\in D}\psi_d\,\mathbb{K}$$

as well. Further, the items (i) and (ii) of Definition 8.3 show that the

$$(I(2)\cup J(2))\times(I(2)\cup J(2))$$

block of every 1-slice of T_1 with an index in $I(1) \cup J(1)$ is the copy of a matrix in

$$\sum_{d \in D} \varphi_d \, \mathbb{K} + m((I \cup J) \times (I \cup J)) \, \mathbb{K},$$

and hence every 1-slice of T_1 with an index in $I(1) \cup J(1)$ belongs to

$$\sum_{d \in D} \mathrm{LSP}(\varphi_d) \, \mathbb{K} + \widetilde{m} \, \mathbb{K},$$

where $\widetilde{m}$ is the $K \times K$ padding of $\mathrm{LSP}(m)$. Further, due to the symmetry, for any $(\beta_1, \beta_2, \beta_3)$ such that $\{\beta_1, \beta_2, \beta_3\} = \{1, 2\}$, we can repeat the argument above with

$$(I(\beta_1) \cup J(\beta_1)) \times (I(\beta_2) \cup J(\beta_2)) \times (I(\beta_3) \cup J(\beta_3))$$

instead of the corresponding $(I(1) \cup J(1)) \times (I(2) \cup J(2)) \times (I(2) \cup J(2))$ restrictions as above. Indeed, for any $\chi \in \{1, 2, 3\}$ and $\beta \in \{1, 2\}$, we get a subset $D_{\chi\beta} \subseteq \{1, \ldots, s\}$ such that every χ-slice of T_χ with an index in $I(\beta) \cup J(\beta)$ is in

$$\sum_{d \in D_{\chi\beta}} \mathrm{LSP}(\varphi_d) \, \mathbb{K} + \widetilde{m} \, \mathbb{K},$$

and the sum of the row spaces of φ_d over all $d \in D_{\chi\beta}$ has the dimension at most σ. Therefore, for any $\chi \in \{1, 2, 3\}$, the choice $\mathcal{D}_\chi = D_{\chi 1} \cup D_{\chi 2}$ guarantees that

$$\dim_{\mathbb{F}} \left(\sum_{d \in \mathcal{D}_\chi} (\text{rows } \mathrm{LSP}(\varphi_d)) \, \mathbb{F} \right) \leqslant 4\sigma$$

and, in addition, every χ-slice of T_χ belongs to

$$\sum_{d \in \mathcal{D}_\chi} \mathrm{LSP}(\varphi_d) \, \mathbb{K} + \widetilde{m} \, \mathbb{K},$$

which concludes the consideration of the point (v) of Definition 8.3. □

We finalize the section with a immediate consequence of our results.

Corollary 9.7. *If $\mathbb{F}$ is a field with $\operatorname{char} \mathbb{F} \neq 2$, and Φ is a candidate family of the type $(\mathbb{F}, m, \rho, r, \pi, \sigma, \delta)$ for some $(m, \rho, r, \pi, \sigma, \delta)$ as in Definition 8.3, and, also, if*

$$\mathrm{LSP}_t = \mathrm{LSP} \circ \ldots \circ \mathrm{LSP}$$

is the t-fold application of the mapping LSP, for some positive integer $t > 0$, then $\mathrm{LSP}_t(\Phi)$ is a candidate family of the type $\big(\mathbb{F}, \mathrm{LSP}_t(m), 2^{t-1} r, 2^t r, 2^{t+1} \sigma, 2^t \sigma, 2^t \delta\big)$.

Proof. Follows from Theorem 9.6 by the induction on t. □

10. Eliminating families. The construction

The results of Section 9 are essentially a reduction of Claim 5.6 to the existence of an appropriate eliminating family Φ for the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{10.1}$$

such that the matrices in Φ are rank one. As explained earlier in Section 5, the selection of (10.1) was made in view of the complexity of the potential constructions of eliminating families for general matrices, so we decided to start with a matrix of the simplest possible form and then generalize it to larger matrices by a further inductive construction as in Section 9. Another natural way to choose the base of an initial eliminating family could be the corresponding diagonal 2×2 matrix, but we preferred (10.1) because the matrices in Φ should sum to the padding of m as in Definition 8.3, and, in view of the symmetry assumption, the diagonal blocks of the matrices in Φ can be harder to control. Therefore, indeed, it seems natural to put a zero at every diagonal entry so that the resulting matrix (10.1) corresponds to the monomial xy, and hence our technique can be seen as the development of the more restricted study of what we called the *monomial emulators* in the dimension four and any higher even dimension in [96] and an earlier construction [93, Section 2].

Remark 10.1. We fix an integer $q \geqslant 5$ to be used throughout this section.

Remark 10.2. Also, in the considerations of the current section, we write $\mathcal{Z}$ to denote the residue group $\mathbb{Z}/(1+q^2)\mathbb{Z}$. If $\mathcal{Z}$ is used as an indexing set, then the sum of the corresponding indexes in $\mathcal{Z}$ is still understood as the sum modulo q^2+1.

We proceed with the definition of the family $\Phi(q)$, which is an explicit coordinate description of its elements. Unfortunately, the use of generic matrices as in Section 7 does not seem to help now because the target rank in Claim 5.6 is large, and calculations required in the point (v) in Definition 8.3 become too demanding.

Definition 10.3. We define the family $\Phi(q)$ of the matrices of the size $2(q^2+2) \times 2(q^2+2)$ as follows. The corresponding indexing family is $\mathcal{I} \cup \mathcal{J}$, where

$$\mathcal{I} = \{(\text{♘}, 1), (\text{♘}, 2)\} \text{ and } \mathcal{J} = \mathcal{Z} \times \{1, 2\}.$$

We begin with the four families of vectors indexed with $k \in \mathcal{Z}$ each, namely,

$$u'(k) = \begin{cases} e_{k+1} - e_k - e_{\text{♘}} & \text{if } k \in \{0, -q, -2q, \dots, -q(q-1)\}, \\ e_{k+1} - e_k & \text{if } k \in \mathcal{Z} \setminus \{0, -q, -2q, \dots, -q(q-1)\}, \end{cases}$$

where e_s is the vector with the indexing set $(\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}$ which has a one at the coordinate $(s, 1)$ and zeros at all other places,

$$u''(k) = \begin{cases} \varepsilon_k - \varepsilon_{k-q} - \varepsilon_{\text{♘}} & \text{if } k \in \{0, -1, -2, \dots, -q+1\}, \\ \varepsilon_k - \varepsilon_{k-q} & \text{if } k \in \mathcal{Z} \setminus \{0, -1, -2, \dots, -q+1\}, \end{cases}$$

where ε_s is the vector with the indexing set $(\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}$ which has a one at the coordinate $(s, 2)$ and zeros at all other places,

$$v'(k) = \begin{cases} e_k - e_{k+q} - e_{\text{♘}} & \text{if } k = 1, \\ e_k - e_{k+q} & \text{if } k \in \mathcal{Z} \setminus \{1\}, \end{cases}$$

and

$$v''(k) = \begin{cases} \varepsilon_k - \varepsilon_{k-1} - \varepsilon_{\text{♘}} & \text{if } k = 1-q, \\ \varepsilon_k - \varepsilon_{k-1} & \text{if } k \in \mathcal{Z} \setminus \{1-q\}. \end{cases}$$

Finally, we define the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ symmetric rank one matrices

$$A_1(k) = (u'(k) \oplus O) \otimes (u'(k) \oplus O), \; A_2(k) = (O \oplus u''(k)) \otimes (O \oplus u''(k)),$$
$$A_3(k) = (u'(k) \oplus u''(k)) \otimes (u'(k) \oplus u''(k)),$$
$$B_1(k) = (v'(k) \oplus O) \otimes (v'(k) \oplus O), \; B_2(k) = (O \oplus v''(k)) \otimes (O \oplus v''(k)),$$
$$B_3(k) = (v'(k) \oplus v''(k)) \otimes (v'(k) \oplus v''(k)),$$

where the O's stand for the zero vectors of the appropriate sizes, and we set

$$\Phi(q) = \bigcup_{k \in \mathcal{Z}} \{-A_1(k), -A_2(k), A_3(k), -B_1(k), -B_2(k), B_3(k)\}.$$

The following symmetry relation of the $(\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}$ and $(\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}$ indexing families in $\Phi(q)$ is important for some further considerations.

Observation 10.4. *The family $\Phi(q)$ is invariant under the permutation of the indexing set $\mathcal{I} \cup \mathcal{J}$ defined by the formulas*

$$(t, 2) \leftrightarrow (q + 2 - qt, 1) \text{ and } (\text{♘}, 2) \leftrightarrow (\text{♘}, 1)$$

for all $t \in \mathcal{Z}$.

We proceed with several observations concerning the family $\Phi(q)$ with an aim of checking the conditions (i)–(iv) of Definition 8.3. Since the condition (v) is more complicated, its proof is given separately in Section 11 below.

Definition 10.5. For any $k \in \mathcal{Z}$, we define the $\mathcal{Z} \times \mathcal{Z}$ matrices $\mathcal{A}(k)$ and $\mathcal{B}(k)$ by declaring that, for any $i, j \in \mathcal{Z}$, the (i, j) entry of $\mathcal{A}(k)$ is the $((i, 1), (j, 2))$ entry of $A_3(k)$, and the (i, j) entry of $\mathcal{B}(k)$ is the $((i, 1), (j, 2))$ entry of $B_3(k)$.

Remark 10.6. In other words, we obtain $\mathcal{A}(k)$ and $\mathcal{B}(k)$ as the $\mathcal{Z} \times \mathcal{Z}$ copies of the $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{2\})$ restrictions of $u'(k) \otimes u''(k)$ and $v'(k) \otimes v''(k)$, respectively.

Observation 10.7. *If x, y are unknown vectors with coordinates in $\mathcal{Z}$ and*

$$m(x, y) = \sum_{k \in \mathcal{Z}} x_k \, \mathcal{A}(k) + \sum_{k \in \mathcal{Z}} y_k \, \mathcal{B}(k),$$

then, for all $i, j \in \mathcal{Z}$, we have

$$[m(x, y)]_{ij} = \begin{cases} -x_i + y_i & \text{if } i = j, \\ x_{i-1} - y_i & \text{if } i - j = 1, \\ x_i - y_{i-q} & \text{if } i - j = q, \\ -x_{i-1} + y_{i-q} & \text{if } i - j = q + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Follows immediately from Definitions 10.3 and 10.5. □

Corollary 10.8. *In the notation of Observation 10.7, the condition*

$$x_1 = y_1 = \ldots = x_{1+q^2} = y_{1+q^2} \tag{10.2}$$

holds if and only if $m(x, y) = 0$.

Proof. The result follows from Observation 10.7 because the equations

$$-x_i + y_i = x_{i-1} - y_i = x_i - y_{i-q} = -x_{i-1} + y_{i-q} = 0$$

are satisfied for all $i \in \mathcal{Z}$ if and only if (10.2) is true. □

We proceed with a more detailed study of the matrices $\mathcal{A}(k)$ and $\mathcal{B}(k)$. The following simple observation could have appeared earlier, but we did not manage to find an appropriate reference and give its proof for completeness.

Observation 10.9. *Let Γ be a bipartite graph on non-empty vertex families I and J, let $F(\Gamma)$ be the space of all those $I \times J$ matrices over a field $\mathbb{K}$ which have a zero at every position (i, j) such that i and j are not adjacent in Γ, and let $F'(\Gamma)$ be the set of all those matrices in $F(\Gamma)$ which have all column sums and all row sums equal to zero. If Γ is connected, then* $\dim_{\mathbb{K}} F(\Gamma) - \dim_{\mathbb{K}} F'(\Gamma) = |I| + |J| - 1$.

Proof. For any adjacent pair $(\hat{\imath}, \hat{\jmath}) \in I \times J$, we take a variable $x(\hat{\imath}, \hat{\jmath})$, and, further, for any $i \in I$ and $j \in J$, we define $R(i)$ as the sum of all those variables $x(\hat{\imath}, \hat{\jmath})$ which have $\hat{\imath} = i$, and, similarly, we declare that $C(j)$ is the sum of all variables $x(\hat{\imath}, \hat{\jmath})$ with $\hat{\jmath} = j$. The matrices in $F(\Gamma)$ correspond to an arbitrary choice of the variables defined above, and the corresponding subspace $F'(\Gamma)$ is defined within $F(\Gamma)$ by the system of the linear equations $R(i) = C(j) = 0$ with all $i \in I$ and $j \in J$. Since

$$\sum_{i \in I} \lambda_i \, R(i) - \sum_{j \in J} \mu_j \, C(j) = 0 \tag{10.3}$$

is true whenever $\lambda_i = \mu_j = 1$, for all $i \in I$ and $j \in J$, we get that

$$\dim_{\mathbb{K}} F(\Gamma) - \dim_{\mathbb{K}} F'(\Gamma) \leqslant |I| + |J| - 1. \tag{10.4}$$

Since every variable $x(\hat{\imath}, \hat{\jmath})$ appears in a unique $R(i)$ and in a unique $C(j)$, we get $\lambda_i = \mu_j$ if i and j are adjacent. Thus, if Γ is connected, one has (10.3) only if the elements λ_i and μ_j are all equal, and then the inequality (10.4) is not strict. □

Observation 10.10. *Let M be a matrix over a field $\mathbb{F}$. If (1) at least Δ entries of M are nonzero, (2) at most $\delta_1 > 0$ entries of every row of M are nonzero, (3) at most $\delta_2 > 0$ entries of every column of M are nonzero, then* $\operatorname{rk} M \geqslant \Delta/(\delta_1\delta_2)$.

Proof. The statement is easy for $\Delta = 0$, so we assume $\Delta > 0$, which means that M has a column c with least one non-zero entry. The removal of all the rows containing the nonzero entries of c returns a matrix M' with at least $\Delta - \delta_1\delta_2$ nonzero entries and $\operatorname{rk} M' \geqslant \operatorname{rk} M + 1$, so the result follows by the induction on Δ. □

We are ready to return to the study of the matrices $\mathcal{A}(k)$ and $\mathcal{B}(k)$.

Lemma 10.11. *Let $\mathbb{K}$ be a field, let*

$$\mathcal{L} = \sum_{k \in \mathcal{Z}} \mathcal{A}(k)\, \mathbb{K} + \sum_{k \in \mathcal{Z}} \mathcal{B}(k)\, \mathbb{K}, \tag{10.5}$$

and let H be the set of all $\mathcal{Z} \times \mathcal{Z}$ matrices M over a field $\mathbb{K}$ such that

- *for all $i, j \in \mathcal{Z}$ with $M(i|j) \neq 0$, one has $i - j \in \{0, 1, q, q+1\}$,*
- *every row sum of M and every column sum of M are zero.*

If γ is a matrix in $H \setminus \mathcal{L}$, then $\operatorname{rk} \gamma \geqslant (q-1)/32$.

Proof. We assume $H \setminus \mathcal{L} \neq \varnothing$ because otherwise the result is trivial, and we take a matrix $g \in H \setminus \mathcal{L}$ such that g has the smallest possible rank r. If g has at least $(q-1)/2$ nonzero rows, then we have $r \geqslant (q-1)/32$ in view of Observation 10.10, so it remains to deduce a contradiction starting from the assumption that

$$\text{all rows of } g \text{ are zero except a family of at most } \lfloor q/2 \rfloor - 1 \text{ rows.} \tag{10.6}$$

To this end, we construct the bipartite graph Γ with the edges passing between the two copies $\mathcal{Z} \times \{1\}$ and $\mathcal{Z} \times \{2\}$ so that $(i, 1)$ is adjacent to $(j, 2)$ if and only if $g(i|j) \neq 0$. Further, we get it immediately that the lack of the connectedness of the edge set of Γ would mean that there are partitions

$$\mathcal{Z} = Z_1 \cup Z_2 = Z_3 \cup Z_4 \;\text{ with }\; Z_1 \cap Z_2 = Z_3 \cap Z_4 = \varnothing$$

such that both the $Z_2 \times Z_3$ and $Z_1 \times Z_4$ blocks of g are zero matrices, and the remaining $Z_1 \times Z_3$ and $Z_2 \times Z_4$ blocks of g are both nonzero. In this case, the definition of H implies that the $\mathcal{Z} \times \mathcal{Z}$ paddings of the $Z_1 \times Z_3$ and $Z_2 \times Z_4$ blocks of g would still be in H, and, also, both paddings would have the ranks smaller than

r. Due to the minimality of r, this shows that the corresponding paddings of the $Z_1 \times Z_3$ and $Z_2 \times Z_4$ blocks of g are in $\mathcal{L}$, but this is impossible because their sum is $g \notin \mathcal{L}$. Therefore, indeed, we conclude that the edge set of Γ is connected. Further, according to the definition of H, the vertices $(i,1)$ and $(j,2)$ can be adjacent only if $i-j \in \{0,\, 1,\, q,\, q+1\}$, which implies that, in view of the connectedness of Γ and the bound (10.6), for any two indexes i and $\hat{\imath}$ of the nonzero rows of g, we have

$$i \in N(\hat{\imath},\, \lfloor q/2\rfloor - 1,\, \lfloor q/2\rfloor - 1),$$

where, for any $\tilde{\imath} \in \mathcal{Z}$ and nonnegative integers a, b, we define $N(\tilde{\imath}, a, b)$ as the set of all $i \in \mathcal{Z}$ such that $i - \tilde{\imath} = \alpha\, q + \beta$ with some integers α, β such that

$$-a \leqslant \alpha \leqslant b \ \text{ and } \ -a \leqslant \beta \leqslant b.$$

Also, we write $\mathcal{S}$ to denote the space of all $\mathcal{Z} \times \mathcal{Z}$ matrices over $\mathbb{K}$ whose all nonzero entries are collected in the rows with the indexes in $N(\hat{\imath},\, \lfloor q/2\rfloor - 1,\, \lfloor q/2\rfloor - 1)$, and we further define $H' = H \cap \mathcal{S}$ and $\mathcal{L}' = \mathcal{L} \cap \mathcal{S}$. In particular, we get

$$g \in H' \setminus \mathcal{L}' \tag{10.7}$$

immediately, and, since we have $\mathcal{L} \subseteq H$ due to Observation 10.7, we also obtain

$$\mathcal{L}' = (\mathcal{L} \cap \mathcal{S}) \subseteq (H \cap \mathcal{S}) = H'. \tag{10.8}$$

Further, we compute the dimension of H' as in Observation 10.9. Indeed, we have

$$|N(\hat{\imath},\, \lfloor q/2\rfloor - 1,\, \lfloor q/2\rfloor - 1)| = (2\lfloor q/2\rfloor - 1)^2$$

corresponding nonzero rows, and there are exactly four positions for the nonzero elements in every such row. Also, we observe that the columns covering all the nonzero entries of the matrices in H' have their indexes in the set

$$N(\hat{\imath},\, \lfloor q/2\rfloor,\, \lfloor q/2\rfloor - 1),$$

so there are exactly $(2\lfloor q/2\rfloor)^2$ such columns. In order to apply Observation 10.9, we remark that the corresponding graph on the vertex set

$$(N(\hat{\imath},\, \lfloor q/2\rfloor - 1,\, \lfloor q/2\rfloor - 1) \times \{1\}) \cup (N(\hat{\imath},\, \lfloor q/2\rfloor,\, \lfloor q/2\rfloor - 1) \times \{2\})$$

is connected, where $(i,1)$ and $(j,2)$ are adjacent if and only if $i-j \in \{0,\, 1,\, q,\, q+1\}$. Finally, we are ready to apply Observation 10.9, and we get

$$\dim_{\mathbb{K}} H' = 4(2\lfloor q/2\rfloor - 1)^2 - (2\lfloor q/2\rfloor - 1)^2 - (2\lfloor q/2\rfloor)^2 + 1$$

and hence

$$\dim_{\mathbb{K}} H' = 8(\lfloor q/2\rfloor)^2 - 12\lfloor q/2\rfloor + 4. \tag{10.9}$$

Now we switch to the space $\mathcal{L}'$ and note that it contains the matrices

$$\mathcal{A}(\alpha\, q + \beta) \ \text{ and } \ \mathcal{B}(\beta\, q + \alpha),$$

for all integers α, β with

$$1 - \lfloor q/2\rfloor \leqslant \alpha \leqslant \lfloor q/2\rfloor - 1 \ \text{ and } \ 1 - \lfloor q/2\rfloor \leqslant \beta \leqslant \lfloor q/2\rfloor - 2.$$

The family of all such matrices is linearly independent by Corollary 10.8, so we get

$$\dim_{\mathbb{K}} \mathcal{L}' \geqslant 2 \cdot (2\lfloor q/2\rfloor - 1) \cdot (2\lfloor q/2\rfloor - 2) = 8(\lfloor q/2\rfloor)^2 - 12\lfloor q/2\rfloor + 4. \tag{10.10}$$

A comparison of the conditions (10.8), (10.9) and (10.10) shows that $\mathcal{L}'$ and H' are equal, which leads immediately to the desired contradiction with (10.7). $\square$

Theorem 10.12. *Let $\mathbb{K}$ be a field, let $\mathcal{L}$ be the linear space as in* (10.5)*, let*

$$\gamma = \sum_{s=0}^{q-1} \left((\overline{e}_{-s} - \overline{e}_{1-s}) \otimes (\overline{\varepsilon}_{-s} + \overline{\varepsilon}_{-s-q})\right) + (\overline{e}_1 - \overline{e}_{1-q}) \otimes (\overline{\varepsilon}_{-q} + \overline{\varepsilon}_{1-q})$$

be a matrix, where $\overline{e}_i = \overline{\varepsilon}_i$ is the vector that has a one at the position $i \in \mathcal{Z}$ and a zero at every position in $\mathcal{Z} \setminus \{i\}$. If $\operatorname{char}\mathbb{K}$ does not divide $2(q+1)$, then

$$\min_{\ell \in \mathcal{L}} \operatorname{rk}(\gamma + \ell) \geqslant (q-1)/32. \tag{10.11}$$

Proof. We define the linear functional Υ on the $\mathcal{Z} \times \mathcal{Z}$ matrices by the formula

$$\Upsilon\left(\sum_{i \in \mathcal{Z}} \sum_{j \in \mathcal{Z}} \alpha_{ij} \, (\overline{e}_i \otimes \overline{\varepsilon}_j)\right) = \sum_{k \in \mathcal{Z}} (\alpha_{k+q\,k} - \alpha_{k+1\,k})$$

and check the equalities $\Upsilon(\mathcal{L}) = 0$ and $\Upsilon(\gamma) = 2(q+1)$. Therefore, it remains to note that γ lies in the space H as in Lemma 10.11 and apply the lemma. $\square$

The following observations deal with the diagonal blocks in $\Phi(q)$.

Observation 10.13. *For any $\chi \in \{1,2\}$, the $(\mathcal{Z} \times \{\chi\}) \times (\mathcal{Z} \times \{\chi\})$ block of any matrix in $\Phi(q)$ has a zero at the entry $((i,\chi),(j,\chi))$ unless $i - j \in \{0, 1, -1, q, -q\}$.*

Proof. Immediate from Definition 10.3. $\square$

Observation 10.14. *For any $k \in \mathcal{Z}$ and the family*

$$\Phi'' = \bigcup_{i \in \mathcal{Z}} \{A_1(i), A_2(i), B_1(i), B_2(i)\},$$

the only nonzero $((k,1),(k+1,1))$ entry in Φ'' is in $A_1(k)$, the only nonzero $((k,1),(k+q,1))$ entry in Φ'' is in $B_1(k)$, the only nonzero $((k,2),(k-q,2))$ entry in Φ'' is in $A_2(k)$, and the only nonzero $((k,2),(k-1,2))$ entry in Φ'' is in $B_2(k)$.

Proof. Immediate from Definition 10.3. $\square$

Observation 10.15. *The $\mathcal{J} \times \mathcal{J}$ blocks of the $4(q^2+1)$ matrices*

$$A_1(k), A_2(k), B_1(k), B_2(k) \text{ with } k \in \mathcal{Z}$$

are linearly independent with respect to any field $\mathbb{F}$.

Proof. Immediate from Observation 10.14. $\square$

We are ready to confirm the point (i) in Definition 8.3 for $\Phi(q)$.

Lemma 10.16. *The matrix*

$$\sum_{k \in \mathcal{Z}} (A_3(k) - A_2(k) - A_1(k) + B_3(k) - B_2(k) - B_1(k)) \tag{10.12}$$

is the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ padding of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

seen as the $\mathcal{I} \times \mathcal{I}$ matrix after the assignment of the index $(♘, 1)$ to its first row and first column and the index $(♘, 2)$ to the second row and second column.

Proof. Immediately from Definition 10.3, we have

$$A_3(k) - A_2(k) - A_1(k) = (O \oplus u''(k)) \otimes (u'(k) \oplus O) + (u'(k) \oplus O) \otimes (O \oplus u''(k))$$

and

$$B_3(k) - B_2(k) - B_1(k) = (O \oplus v''(k)) \otimes (v'(k) \oplus O) + (v'(k) \oplus O) \otimes (O \oplus v''(k)),$$

so we need to check that the matrix

$$\mathcal{C} = \sum_{k \in \mathcal{Z}} (u'(k) \otimes u''(k) + v'(k) \otimes v''(k)) \tag{10.13}$$

has zeros everywhere except the intersection of the row with the index $(♘, 1)$ and the column with the index $(♘, 2)$, where $\mathcal{C}$ has a one. In fact, the $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{2\})$ block of $\mathcal{C}$ is zero due to Observation 10.7, so it remains to examine the entries $((\hat{\imath}, 1), (\hat{\jmath}, 2))$ with either $\hat{\imath} = ♘$ or $\hat{\jmath} = ♘$. To this end, we check that the sum

$$\sum_{k \in \mathcal{Z}} (u'(k) \otimes u''(k)) \tag{10.14}$$

has ones at the positions $((♘, 1), (♘, 2))$, $((1-q, 1), (♘, 2))$, $((♘, 1), (1, 2))$, negative ones at the positions $((1, 1), (♘, 2))$, $((♘, 1), (0, 2))$, and zeros at the remaining places $((\hat{\imath}, 1), (\hat{\jmath}, 2))$ with either $\hat{\imath} = ♘$ or $\hat{\jmath} = ♘$. Similarly, the sum

$$\sum_{k \in \mathcal{Z}} (v'(k) \otimes v''(k)) \tag{10.15}$$

has ones at $((♘, 1), (0, 2))$, $((1, 1), (♘, 2))$, negative ones at $((1 - q, 1), (♘, 2))$, $((♘, 1), (1, 2))$, and zeros at all other entries $((\hat{\imath}, 1), (\hat{\jmath}, 2))$ with either $\hat{\imath} = ♘$ or $\hat{\jmath} = ♘$. Finally, we obtain the expression (10.13) as the sum of (10.14) and (10.15), and we see that the computed entries of $\mathcal{C}$ match the values required above. □

We proceed with the item (ii) of Definition 8.3.

Remark 10.17. Of course, the condition 'only if' can be replaced by 'if and only if' in the following lemma, since the 'if' part of the statement follows from Lemma 10.16.

Lemma 10.18. *Let $(x_{\alpha t})$ and $(y_{\alpha t})$ be families of elements of a field $\mathbb{F}$ indexed with $\alpha \in \{1, 2, 3\}$ and $t \in \mathcal{Z}$. Then the $\mathcal{J} \times \mathcal{J}$ block of the linear combination*

$$\sum_{k \in \mathcal{Z}} (x_{3k} A_3(k) - x_{2k} A_2(k) - x_{1k} A_1(k) + y_{3k} B_3(k) - y_{2k} B_2(k) - y_{1k} B_1(k)) \tag{10.16}$$

can be zero only if all the elements in $(x_{\alpha t})$ and $(y_{\alpha t})$ assume the same value.

Proof. If the $\mathcal{J} \times \mathcal{J}$ block of the matrix (10.16) is indeed zero, then, according to Corollary 10.8, the elements (x_{3t}) and (y_{3t}) should all be equal to the same scalar c. Further, Observation 10.15 shows that the values of (x_{1t}), (y_{1t}), (x_{2t}), (y_{2t}) are defined uniquely for any given c, and hence the space of all possible choices of $(x_{\alpha t})$, $(y_{\alpha t})$ is one dimensional. Therefore, any appropriate linear combination (10.16) should indeed use the family of the coefficients proportional to those in (10.12). □

We are ready to deal with the condition (iii) in Definition 8.3.

Lemma 10.19. *We consider an arbitrary field $\mathbb{F}$, a subset $\Phi' \subseteq \Phi(q)$ and the space*

$$V' = \sum_{\varphi \in \Phi'} (\operatorname{rows} \varphi)\, \mathbb{F}.$$

If $\dim_{\mathbb{F}} V' \leqslant q$*, then any nonzero vector* $v \in V'$ *has a nonzero element in at least one of its* $\mathcal{J}$ *coordinates.*

Proof. Since the indexing set of v is $\mathcal{I} \cup \mathcal{J}$, we can assume without loss of generality that at least one of the $\mathcal{I}$ coordinates of v is nonzero. This means that either

(1♘) the $(♘, 1)$ coordinate of v is nonzero, or
(2♘) the $(♘, 2)$ coordinate of v is nonzero.

In the case (1♘), we restrict every vector in V' to the $(\mathcal{Z} \cup \{♘\}) \times \{1\}$ block of the coordinates, which allows us to assume without loss of generality that

$$\Phi' \subseteq \bigcup_{k \in \mathcal{Z}} \{-A_1(k), -B_1(k)\}, \tag{10.17}$$

and, similarly, the case (2♘) allows us to restrict the consideration to

$$\Phi' \subseteq \bigcup_{k \in \mathcal{Z}} \{-A_2(k), -B_2(k)\}. \tag{10.18}$$

So, the case (1♘) requires us to show that no family of at most q vectors in

$$\big(u'(1), v'(1), \dots, u'(1+q^2), v'(1+q^2)\big) \tag{10.19}$$

contains $e_{♘}$ in its $\mathbb{F}$-linear span, and, similarly, the case (2♘) requires us to show that no family of at most q vectors in

$$\big(u''(1), v''(1), \dots, u''(1+q^2), v''(1+q^2)\big) \tag{10.20}$$

contains $\varepsilon_{♘}$ in its $\mathbb{F}$-linear span. In order to deal with the case (2♘), we define

$$q^2 - q + 1 \preccurlyeq q^2 - q \preccurlyeq q^2 - q - 1 \preccurlyeq \dots \preccurlyeq 2 - q \preccurlyeq 1 - q \preccurlyeq ♘$$

to be the total ordering of the indexing set $\mathcal{Z} \cup \{♘\}$, and we set up the $\mathbb{F}$-linear transformation defined by the images of the basis vectors

$$\varepsilon_i \to \sum_{j \preccurlyeq i} \varepsilon_j$$

for all $i \in \mathcal{Z} \cup \{♘\}$. In this case, the image of $\varepsilon_{♘}$ is the vector of all ones, and each of the corresponding images of the vectors (10.20) has at most q nonzero entries at the coordinates in $\mathcal{Z} \times \{2\}$, that is, it has at most q nonzero coordinates if we do not count the entry $(♘, 2)$. Therefore, the vector $\varepsilon_{♘}$ cannot be represented as a linear combination of less than $q+1$ vectors in (10.20), which completes the analysis of the case (2♘). In order to deal with the case (1♘), we refer to Observation 10.4. □

Remark 10.20. Alternatively, the case (1♘) follows by the transformation

$$e_i \to \sum_{j \preccurlyeq i} e_j$$

corresponding to the ordering

$$1 + q \preccurlyeq 1 + 2q \preccurlyeq 1 + 3q \preccurlyeq \dots \preccurlyeq 1 + (q^2 - 1)q \prec 1 + q^3 \preccurlyeq 1 \preccurlyeq ♘,$$

and we could complete the proof in a way similar to the case (2♘).

Our approach to the point (iv) in Definition 8.3 is a bit less straightforward, so we need to develop one further combinatorial technique.

Definition 10.21. We think of the *square lattice* $\mathbb{Z}^2$ as the graph in which a pair of vertices (u_1, u_2), (v_1, v_2) is adjacent if and only if $|u_1 - v_1| + |u_2 - v_2| = 1$.

The following observation is probably well known, but we did not find a reference.

Observation 10.22. *Let $F \subset \mathbb{Z}^2$ be a subset with $|F| = a$, and assume that there are exactly b adjacent pairs $(u, v) \subset \mathbb{Z}^2 \times \mathbb{Z}^2$ with $u \in F$ and $v \notin F$. Then $b \geqslant 4\sqrt{a}$.*

Proof. If F contains three vertices of some induced cycle of the length four, then the fourth vertex can be added to F without an increase of b. In the case when the subgraph induced by F is connected, this allows us to assume without loss of generality that F is the set of all integer points that lie inside some rectangle with the sides parallel to the corresponding coordinate axes, and then the result follows by a simple computation. In the case that it is not connected, we observe that the cases $a = 0$ and $a = 1$ are trivial and complete the proof by the induction on a. □

A similar result can be shown for the $q \times q$ square grid instead of $\mathbb{Z} \times \mathbb{Z}$.

Lemma 10.23. *For the subgraph of the square lattice $\mathbb{Z} \times \mathbb{Z}$ induced by the set of the vertices $Q = \{0, \ldots, q-1\} \times \{0, \ldots, q-1\}$, and, for any subset $F \subseteq Q$ with*

$$|F| = \alpha \leqslant q^2/2, \tag{10.21}$$

there exist at least $\sqrt{\alpha}$ adjacent pairs (i, j) with $i \in F$ and $j \in Q \setminus F$.

Proof. Let β be the total number of the adjacent pairs $(i, j) \in F \times (Q \setminus F)$. What we need to show is $\beta \geqslant \sqrt{\alpha}$, and, to this end, we subsequently apply Observation 10.22 to the sets F and $Q \setminus F$ seen as vertex families of the full square lattice $\mathbb{Z} \times \mathbb{Z}$.

Indeed, the number of the adjacent pairs $(u, v) \subset \mathbb{Z}^2 \times \mathbb{Z}^2$ with $u \in F$ and $v \notin F$ equals $\beta + \gamma(F)$, where $\gamma(F)$ is the number of the vertices of F which have the form (i, j) with either $i \in \{0, q-1\}$ or $j \in \{0, q-1\}$, assuming that each of the corner vertices $(0, 0)$, $(0, q-1)$, $(q-1, 0)$, $(q-1, q-1)$ is counted twice. Similarly, there are exactly $\beta + \gamma(Q \setminus F)$ adjacent pairs $(u, v) \subset \mathbb{Z}^2 \times \mathbb{Z}^2$ with $u \in Q \setminus F$ and $v \notin Q \setminus F$, and it is easy to note that $\gamma(F) + \gamma(Q \setminus F) = 4q$. We employ Observation 10.22:

$$\beta + \gamma(F) \geqslant 4\sqrt{\alpha} \quad \text{and} \quad \beta + \gamma(Q \setminus F) \geqslant 4\sqrt{q^2 - \alpha},$$

and we sum these inequalities to arrive at $2\beta + 4q \geqslant 4\sqrt{\alpha} + 4\sqrt{q^2 - \alpha}$ and

$$(\beta/2 + q)^2 \geqslant \alpha + 2\sqrt{\alpha(q^2 - \alpha)} + (q^2 - \alpha).$$

The further elementary transformations imply $\beta(\beta + 4q) \geqslant 8\sqrt{\alpha(q^2 - \alpha)}$ and

$$\beta^2(\beta + 4q)^2 \geqslant 64\alpha(q^2 - \alpha),$$

which becomes $\beta^2(\beta + 4q)^2 \geqslant 32\alpha q^2$ after an application of the inequality (10.21). Furthermore, since the desired condition is $\beta \geqslant \sqrt{\alpha}$, the inequality (10.21) allows us to assume without loss of generality that $\beta \leqslant q/\sqrt{2}$, so we get

$$\beta^2 \geqslant \frac{32\alpha}{(4 + 1/\sqrt{2})^2}$$

and deduce the condition $\beta \geqslant \sqrt{\alpha}$ by a simple computation. □

We return to $\mathcal{Z} = \mathbb{Z}/(1 + q^2)\mathbb{Z}$ and get the following immediate corollary.

Corollary 10.24. *For any subset $F' \subseteq \mathcal{Z}$ with $|F'| = \alpha \leqslant (q^2 + 1)/2$, there exist at least $\sqrt{\alpha}$ pairs $(i, j) \in F' \times (\mathcal{Z} \setminus F')$ such that $i - j \in \{-q, -1, 1, q\}$.*

Proof. The result follows from Lemma 10.23 as we have $x_1 - x_2 \in \{-q, -1, 1, q\}$ with $x_1 = r_1 q + s_1 \in \mathcal{Z}$, $x_2 = r_2 q + s_2 \in \mathcal{Z}$, $r_1, r_2, s_1, s_2 \in \{0, \ldots, q-1\}$ whenever the pairs (r_1, s_1) and (r_2, s_2) are adjacent on the square lattice. □

We give an easy combinatorial observation before we return to $\Phi(q)$.

Observation 10.25. *Let $d \geqslant 1$ be an integer, let $p, r, s_1, \dots, s_d$ be positive integers satisfying $s_1 + \dots + s_d = p > 3r$. If any subset $T \subseteq \{1, \dots, d\}$ satisfies either*

$$\sum_{t \in T} s_t \leqslant r \quad \text{or} \quad \sum_{t \in T} s_t \geqslant p - r,$$

then there exists $j \in \{1, \dots, d\}$ such that $s_j \geqslant p - r$.

Proof. If $s_1 + s_2 \leqslant r$, then we replace s_1 and s_2 by $s_1 + s_2$ and complete the proof by the induction on d. Otherwise, we take $T = \{1, 2\}$ and deduce $s_1 + s_2 \geqslant p - r > 2p/3$, so we have either $s_1 > p/3 > r$ or $s_2 > p/3 > r$. In the former case, we get $s_1 \geqslant p - r$ by the choice $T = \{1\}$, and the latter case gives $s_2 \geqslant p - r$ due to $T = \{2\}$. □

We proceed with the proof of the item (iv) in Definition 8.3.

Theorem 10.26. *Let $x = (x_t)$ and $y = (y_t)$ be two families of elements of a field $\mathbb{F}$ indexed with $t \in \mathcal{Z}$ each. If the linear combination*

$$m(x, y) = \sum_{k \in \mathcal{Z}} x_k \, \mathcal{A}(k) + \sum_{k \in \mathcal{Z}} y_k \, \mathcal{B}(k)$$

has the rank $\rho \leqslant q/28$, then there exists an element $c \in \mathbb{F}$ such that the string

$$\begin{pmatrix} x_1 \ y_1 \ \dots \ x_{1+q^2} \ y_{1+q^2} \end{pmatrix}$$

contains at most $512\rho^2$ entries different from c.

Proof. Let $C \subset \mathbb{F}$ be the set of all values of the elements x_k over all $k \in \mathcal{Z}$, and, for a fixed subset $D \subseteq C$, let F be the set of all $i \in \mathcal{Z}$ such that $x_i \in D$.

A special case. We write $\alpha = |F|$ and assume that $\alpha \leqslant (q^2 + 1)/2$. In this case, Corollary 10.24 shows that there are at least $\sqrt{\alpha}$ pairs $(i, j) \in F \times (\mathcal{Z} \setminus F)$ such that $i - j \in \{-1, 1, -q, q\}$, and, in particular, by the definition of F, this means that, for at least $\sqrt{\alpha}$ pairs $(i, j) \in \mathcal{Z} \times \mathcal{Z}$, both conditions $x_i \neq x_j$ and $i - j \in \{-1, 1, -q, q\}$ are satisfied. By Observation 10.7, the condition $x_{i-1} \neq x_i$ implies that either

$$[m(x, y)]_{ii} \neq 0 \quad \text{or} \quad [m(x, y)]_{i\,i-1} \neq 0, \tag{10.22}$$

and, in addition, either

$$[m(x, y)]_{i\,i-q} \neq 0 \quad \text{or} \quad [m(x, y)]_{i\,i-q-1} \neq 0. \tag{10.23}$$

Similarly, the condition $x_i \neq x_{i-q}$ implies that at least one of the conditions

$$[m(x, y)]_{i-q\,i-q} \neq 0 \quad \text{or} \quad [m(x, y)]_{i\,i-q} \neq 0, \tag{10.24}$$

is true, and, also, in addition, we have either

$$[m(x, y)]_{i-q+1\ i-q} \neq 0 \quad \text{or} \quad [m(x, y)]_{i+1\ i-q} \neq 0. \tag{10.25}$$

We remark that all the $4|\mathcal{Z}|$ conditions appearing in (10.22) and (10.23) address the different entries of $m(x, y)$, and, also, all the $4|\mathcal{Z}|$ conditions listed in (10.24) and (10.25) address the different entries of $m(x, y)$, over all $i \in \mathcal{Z}$. Further, according to the above consideration, the conditions $x_{i-1} \neq x_i$ and $x_{i-q} \neq x_i$ appear at least $\sqrt{\alpha}$ times in the total, which means that at least one of these conditions appears at least $\sqrt{\alpha}/2$ times, and hence, according to the corresponding conditions (10.22), (10.23) or (10.24), (10.25), we see that at least $\sqrt{\alpha}$ entries of $m(x, y)$ are nonzero. Using Observation 10.7 again, we see that every row and every column of $m(x, y)$ contain at least four nonzero elements each, so we get $\operatorname{rk} m(x, y) \geqslant \sqrt{\alpha}/16$

due to Observation 10.10. Since the rank of $m(x,y)$ is denoted by ρ in the formulation of the current theorem, we get $\rho \geqslant 16\sqrt{\alpha}$ and hence $256\,\rho^2 \geqslant \alpha$.

The consideration of the special case is now complete. Namely, we get that, for any subset F of the values involved in (x_k), it holds that either $|F| \leqslant 256\rho^2$ or $|F| > (q^2+1)/2$, but, in the latter situation, we can apply the considerations of the special case to the set $\mathcal{Z} \setminus F$ instead of F, and, therefore, we get that either $|F| \leqslant 256\rho^2$ or $|F| \geqslant q^2 - 256\rho^2 + 1$. Since $256\rho^2 < (q^2+1)/3$ by the inequality in the formulation of the theorem, we can apply Observation 10.25 and conclude that, for some $c \in C$, there are at least $q^2 - 256\rho^2 + 1$ values $i \in \mathcal{Z}$ such that $x_i = c$.

The argument similar to the one above but applied to (y_t) instead of (x_t) shows that, for some $c' \in C$, there are at least $q^2 - 256\rho^2 + 1$ values $j \in \mathcal{Z}$ such that $x_j = c'$. Finally, if we had $c' \neq c$, then, according to Observation 10.7, the matrix $m(x,y)$ would have at least $q^2 - 512\rho^2$ nonzero diagonal entries, and hence Observation 10.10 would imply $\operatorname{rk} m(x,y) \geqslant (q^2 - 512\rho^2)/16$, which implies $\operatorname{rk} m(x,y) > \rho$ again by the inequality in the formulation and contradicts to $\operatorname{rk} m(x,y) = \rho$. Thus, we have $c' = c$, and hence at most $512\rho^2$ values in (x_t) and (y_t) are different from c. □

Theorem 10.27. *Let $x = (x_{\alpha t})$ and $y = (y_{\alpha t})$ be families of elements of a field $\mathbb{F}$ indexed with $\alpha \in \{1,2,3\}$ and $t \in \mathcal{Z}$, and let M be the $\mathcal{J} \times \mathcal{J}$ block of the matrix*

$$\sum_{k \in \mathcal{Z}} \left(x_{3k} A_3(k) - x_{2k} A_2(k) - x_{1k} A_1(k) + y_{3k} B_3(k) - y_{2k} B_2(k) - y_{1k} B_1(k) \right).$$

If $\operatorname{rk} M = \rho \leqslant q/28$, then there exists an element $c \in \mathbb{F}$ such that the string

$$\Delta = \begin{pmatrix} x_{11} \ x_{21} \ x_{31} \ y_{11} \ y_{21} \ y_{31} \ \ldots \ x_{1\,1+q^2} \ x_{2\,1+q^2} \ x_{3\,1+q^2} \ y_{1\,1+q^2} \ y_{2\,1+q^2} \ y_{3\,1+q^2} \end{pmatrix}$$

contains at most $1561\,\rho^2$ entries different from c.

Proof. By Theorem 10.26, there exists an element $c \in \mathbb{F}$ such that the string

$$\begin{pmatrix} x_{31} \ y_{31} \ \ldots \ x_{3\,1+q^2} \ y_{3\,1+q^2} \end{pmatrix}$$

contains at most $512\rho^2$ entries different from c. Furthermore, if the string

$$\begin{pmatrix} x_{11} \ y_{11} \ \ldots \ x_{1\,1+q^2} \ y_{1\,1+q^2} \end{pmatrix} \tag{10.26}$$

contained more than $512\rho^2 + 12.5\rho$ entries different from c, then, by Observation 10.14, the $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{1\})$ block of M would have more than 25ρ nonzero entries. However, Observation 10.13 implies that the $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{1\})$ block of M has at most five nonzero entries in every row and every column, so we use Observation 10.10 and get that the $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{1\})$ block of M has the rank greater than ρ. This contradicts to the assumption $\operatorname{rk} M = \rho$ in the formulation, and hence, indeed, the string (10.26) contains at most $512\rho^2 + 12.5\rho$ entries different from c. A similar consideration but applied to the $(\mathcal{Z} \times \{2\}) \times (\mathcal{Z} \times \{2\})$ block of M instead of the corresponding $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{1\})$ block shows that the string

$$\begin{pmatrix} x_{21} \ y_{21} \ \ldots \ x_{2\,1+q^2} \ y_{2\,1+q^2} \end{pmatrix}$$

has at most $512\rho^2 + 12.5\rho$ entries different from c. Summing up, we get the desired bound $512\rho^2 + (512\rho^2 + 12.5\rho) + (512\rho^2 + 12.5\rho) \leqslant 1561\,\rho^2$ for Δ. □

We finalize the section with a further application of Theorem 10.27.

Corollary 10.28. *For any subset $U \subseteq \mathcal{J}$, we take the linear space $\mathcal{S}(U)$ consisting of all $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ matrices over a field $\mathbb{F}$ whose entries in $(\mathcal{J} \times \mathcal{J}) \setminus (U \times U)$ are all zero. If $q \geqslant 28|U|$, then there exists a subset $L(U) \subseteq \Phi(q)$ with*

$$|L(U)| \leqslant 1561\,|U|^2 \tag{10.27}$$

such that the $\mathcal{J} \times \mathcal{J}$ restriction of any matrix in

$$\mathcal{S}(U) \cap (\Phi(q)\,\mathbb{F}) \tag{10.28}$$

is the $\mathcal{J} \times \mathcal{J}$ restriction of some matrix in $L(U)\,\mathbb{F}$.

Proof. We take an arbitrary basis $(\beta_1, \ldots, \beta_r)$ of the $\mathbb{F}$-linear space (10.28), and we define $\mathbb{F}' = \mathbb{F}(\xi_1, \xi_2, \ldots)$ as the purely transcendental extension of the countably infinite degree over $\mathbb{F}$. By the definition of $\mathcal{S}(U)$, the rank of the $\mathcal{J} \times \mathcal{J}$ block of the generic element $\gamma = \xi_1\,\beta_1 + \ldots + \xi_r\,\beta_r$ is at most $|U|$, and hence, according to Theorem 10.27, there exists a subset $L(U) \subseteq \Phi(q)$ that matches the bound (10.27) and satisfies $\gamma \in L(U)\,\mathbb{F}' + m\,\mathbb{F}'$ with m being the padding of the $\mathcal{I} \times \mathcal{I}$ matrix as in Lemma 10.16. Since γ is generic, the matrices $\beta_1, \ldots, \beta_r$ belong to $L(U)\,\mathbb{F}' + m\mathbb{F}'$ as well, and, since these matrices have their entries in $\mathbb{F}$, they are in $L(U)\,\mathbb{F} + m\,\mathbb{F}$. □

11. Eliminating families. The proof

In this section, we deal with the condition (v) of Definition 8.3 in regard to the family $\Phi(q)$, which will allow us to build a sequence of eliminating families and proceed with the proof of the main result using the material of Section 9 above.

Remark 11.1. We still use the notations q and $\mathcal{Z}$ in the current section, where $q \geqslant 5$ is an integer, and $\mathcal{Z}$ is the residue group $\mathbb{Z}/(1+q^2)\mathbb{Z}$.

The argument presented this section makes an extensive use of the combinatorial structure of the matrices in $\Phi(q)$, and we need to consider the following concept, which is well known in combinatorial matrix theory [16, page 3].

Definition 11.2. Let I and J be indexing sets, and let M be an $I \times J$ matrix. The *pattern* of M is the set of all pairs $(i, j) \in I \times J$ for which $[M]_{ij} \neq 0$.

We proceed with an easy but useful lower bound on the rank of a tensor.

Definition 11.3. Let (I_1, I_2, I_3) be indexing sets, and let U be an $I_1 \times I_2 \times I_3$ tensor with entries in a field $\mathbb{F}$. For any $\chi \in \{1, 2, 3\}$, we write $\operatorname{grk}_\chi(U)$ to denote the rank of a generic linear combination of the χ-slices of U, that is,

$$\operatorname{grk}_\chi(U) = \operatorname{rk}\left(\sum_{i \in I_\chi} x_i\, u_i\right)$$

with u_i being the i-th χ-slice of U, and the elements (x_i) are taken in some field extension of $\mathbb{F}$ so that the family (x_i) is algebraically independent over $\mathbb{F}$.

Remark 11.4. In the notation of Definition 11.3, we have $\operatorname{grk}_\chi(U) \leqslant \operatorname{rk}_{\mathbb{F}} U$ because taking an extension of the ground field does not increase the rank of a tensor.

Observation 11.5. *Let I, J, K be indexing sets, and let T be an $I \times J \times K$ tensor with entries in a field $\mathbb{F}$. Let $(s_0, s_1, \ldots, s_m)$ be the family of all 3-slices of T with*

$$s_i = \left(\begin{array}{c|c|c} A_i & B_i & C_i \\ \hline D_i & E_i & F_i \\ \hline G_i & H_i & L_i \end{array}\right)$$

with $i \in \{0, 1, \ldots, m\}$, where the sizes of the blocks in the partitions do not depend on i. If the matrices F_0, H_0, L_0 are all zero, then $\operatorname{grk}_3(T) \geqslant \operatorname{rk} E_0 + \operatorname{grk}_3(T')$, where T' is the tensor for which $(L_1, \ldots, L_m)$ is the family of all its 3-slices.

Proof. We have

$$\operatorname{rk}\left(\begin{array}{c|c} y\,E_0 + E & F \\ \hline H & L \end{array}\right) \geqslant \operatorname{rk} E_0 + \operatorname{rk} L$$

if E, F, H, L, E_0 are matrices with entries in some field $\mathbb{K}$, and y is an element in some extension of $\mathbb{F}$ such that y is transcendent with respect to $\mathbb{K}$. □

Before we can discuss the condition (v) of Definition 8.3, we need to prove two general lemmas regarding the combinatorial structure of sparse tensors.

Lemma 11.6. *Let I, J, K be indexing sets, let M_1 be a family of $J \times K$ matrices over a field $\mathbb{F}$, let M_2 be a family of $I \times K$ matrices over $\mathbb{F}$, and let M_3 be a family of $I \times J$ matrices over $\mathbb{F}$. For any $\chi \in \{1, 2, 3\}$, we define $P(\chi)$ as the union of the patterns of the matrices M_χ and assume that $P(\chi)$ has at most δ entries in every row and at most δ entries in every column, with some $\delta > 0$. Further, let a tensor*

$$T \in O \operatorname{mod}_{\mathbb{F}}(M_1, M_2, M_3)$$

admit exactly c positions $(\hat\imath, \hat\jmath) \in I \times J$ for which there exists $k \in K$ such that

$$(\hat\jmath, k) \notin P(1) \text{ and } (\hat\imath, k) \notin P(2)$$

but still $T(\hat\imath|\hat\jmath|k) \neq 0$. Then $\operatorname{grk}_3(T) \geqslant c/(4\delta^4)$.

Proof. For any $k \in K$, we define I_{1k} as the set of all $i \in I$ such that $(i, k) \in P(2)$, and we consider the set J_{1k} consisting of all $j \in J$ for which $(j, k) \in P(1)$. We have

$$|I_{1k}| \leqslant \delta \text{ and } |J_{1k}| \leqslant \delta \tag{11.1}$$

due to the assumption on $P(1)$ and $P(2)$ in the formulation. Further, we define H_k as the $(I \setminus I_{1k}) \times (J \setminus J_{1k})$ block of the k-th 3-slice of T, and we write $I_{2k} \subseteq (I \setminus I_{1k})$ and $J_{2k} \subseteq (J \setminus J_{1k})$ to be the smallest indexing families for which

$$H_k \text{ is the padding of an } I_{2k} \times J_{2k} \text{ matrix.} \tag{11.2}$$

If we have $H_k = O$ for any $k \in K$, then $c = 0$ in the notation of the formulation, and then it suffices to check that $\operatorname{grk}_3(T) \geqslant 0$, which is trivial. So we assume that, for some $\kappa \in K$, the matrix $H_\varkappa$ has exactly $h \neq 0$ nonzero entries, which leads to

$$|I_{2k}| \leqslant h \text{ and } |J_{2k}| \leqslant h \tag{11.3}$$

immediately. We also get

$$\operatorname{rk} H_\varkappa \geqslant \frac{h}{\delta^2} \tag{11.4}$$

after an application of Observation 10.10, and, finally, we can proceed the proof by the induction on c. In particular, the $\varkappa$-th 3-slice of T has the form

$$\left(\begin{array}{c|c|c} * & * & * \\ \hline * & \mathcal{H} & O \\ \hline * & O & O \end{array}\right)$$

in which the partition of the rows corresponds to

$$(I_{1k},\, I_{2k},\, I \setminus (I_{1k} \cup I_{2k}))\,,$$

the partition of the columns corresponds to

$$(J_{1k},\ J_{2k},\ J \setminus (J_{1k} \cup J_{2k}))\,,$$

and the $*$ entries do not need to be specified. We also have

$$\operatorname{rk} \mathcal{H} \geqslant \frac{h}{\delta^2}$$

because of the conditions (11.2) and (11.4), and we get

$$\operatorname{grk}_3(T) \geqslant \operatorname{grk}_3(T') + \frac{h}{\delta^2} \tag{11.5}$$

in view of Observation 11.5, where T' is the

$$(I \setminus (I_{1k} \cup I_{2k})) \times (J \setminus (J_{1k} \cup J_{2k})) \times K$$

restriction of T. According to the conditions (11.1) and (11.3), the passing from T to T' requires the deletion of a family of 1-slices and a family of 2-slices of the cardinalities not exceeding $h+\delta$ each, so there are at most $(h+\delta)\delta$ entries of $P(3)$ which appear in the deleted 1-slices, and, similarly, there are at most $(h+\delta)\delta$ entries of $P(3)$ which appear in the deleted 2-slices. Therefore, we have $c' \geqslant c - 2(h+\delta)\delta$ for the quantity c' defined for T' in the same way as the quantity c was defined for T in the formulation of the theorem. By the inductive assumption, we get

$$\operatorname{grk}_3(T') \geqslant \frac{c'}{4\delta^4} \geqslant \frac{c - 2(h+\delta)\delta}{4\delta^4} \tag{11.6}$$

and complete the proof by a comparison to the inequality (11.5). □

Lemma 11.7. *Let T be an $I \times J \times K$ tensor with entries in a field $\mathbb{F}$ such that $\operatorname{rk}_{\mathbb{F}} T \leqslant r$. If, for any $(i,j,k) \in I \times J \times K$, there are at most δ choices of $\hat{\imath} \in I$ such that $T(\hat{\imath}|j|k) \neq 0$, and there are at most δ choices of $\hat{\jmath} \in J$ such that $T(i|\hat{\jmath}|k) \neq 0$, then there exist families $I' \subseteq I$ and $J' \subseteq J$ of the cardinalities not exceeding $r\delta^2$ each so that the $(I \setminus I') \times (J \setminus J') \times K$ restriction of T is the zero tensor.*

Proof. If all the 3-slices of T are zero, then there is nothing to prove, so we can assume that, for some $\varkappa \in K$, the $\varkappa$-th 3-slice of T equals a matrix H that contains exactly $h \neq 0$ nonzero entries. According to Observation 11.5, we get

$$\operatorname{grk}_3(T) \geqslant \operatorname{grk}_3(T') + \operatorname{rk} H,$$

where T' is the tensor obtained from T by the removal of all those i-th 1-slices for which there exists some $\hat{\jmath} \in J$ such that $T(i|\hat{\jmath}|\varkappa) \neq 0$ and the subsequent removal of all j-th 2-slices for which there exists an $\hat{\imath} \in I$ with $T(\hat{\imath}|j|\varkappa) \neq 0$. We also get

$$\operatorname{grk}_3(T) \geqslant \operatorname{grk}_3(T') + \frac{h}{\delta^2}$$

after an application of Observation 10.10, and we complete the proof by the induction because the tensor T' differs from T by the deletion of a family of the 1-slices and a family of the 2-slices of the cardinalities not exceeding h each. □

We are ready to switch to a discussion of the family $\Phi(q)$ as in Section 10.

Remark 11.8. In the rest of this section, we write $\mathcal{P}$ to denote the union of the patterns of the $\mathcal{J} \times \mathcal{J}$ restrictions of the matrices in $\Phi(q)$.

Observation 11.9. *For any $i, j \in \mathcal{Z}$ and $a \in \{1,2\}$, we have*

$$((i,a),(j,a)) \in \mathcal{P} \text{ if and only if } i-j \in \{-q,-1,0,1,q\},$$

and also

$$((i,1),(j,2)) \in \mathcal{P} \text{ if and only if } i-j \in \{0,1,q,q+1\}.$$

Proof. Immediate from Definition 10.3. □

We can apply the above technique to the tensor in the point (v) of Definition 8.3.

Lemma 11.10. *Let $\mathbb{K}$ be a field, let T be an $(\mathcal{I}\cup\mathcal{J})\times(\mathcal{I}\cup\mathcal{J})\times(\mathcal{I}\cup\mathcal{J})$ tensor in*

$$O \operatorname{mod}_{\mathbb{K}}(\Phi(q),\Phi(q),\Phi(q)),$$

and let $\mathcal{T}$ be the $\mathcal{J}\times\mathcal{J}\times\mathcal{J}$ block of T. If $\operatorname{rk}_{\mathbb{K}}\mathcal{T}\leqslant\rho$, then there is a subset $U\subseteq\mathcal{J}$ of the cardinality not exceeding $159\,408\,\rho$ so that every nonzero entry of $\mathcal{T}$ belongs to the union of the blocks $\mathcal{J}\times U\times U$, $U\times\mathcal{J}\times U$, $U\times U\times\mathcal{J}$.

Proof. Let D_1 be the set of all pairs $(j,k)\in\mathcal{J}\times\mathcal{J}$ such that there exists $\hat{\imath}\in\mathcal{J}$ with $(\hat{\imath},j)\notin\mathcal{P}$, $(\hat{\imath},k)\notin\mathcal{P}$ and $T(\hat{\imath}|j|k)\neq 0$, and, similarly, let D_2 be the set of all pairs $(i,k)\in\mathcal{J}\times\mathcal{J}$ such that there exists $\hat{\jmath}\in\mathcal{J}$ with $(i,\hat{\jmath})\notin\mathcal{P}$, $(\hat{\jmath},k)\notin\mathcal{P}$ and $T(i|\hat{\jmath}|k)\neq 0$. By Observation 11.9, every row and every column contains at most nine entries of $\mathcal{P}$, so we can apply Lemma 11.6 with $\delta=9$ to the tensor $\mathcal{T}$ to get $|D_1|\leqslant(4\cdot 9^4)\rho=26\,244\,\rho$ and $|D_2|\leqslant 26\,244\,\rho$. If I' is the set of all $i\in\mathcal{J}$ such that $(i,k')\in D_2$ is true for some $k'\in\mathcal{J}$, then we immediately get $|I'|\leqslant 26\,244\,\rho$, and, similarly, we define J' as the set of all $j\in\mathcal{J}$ such that $(j,k'')\in D_1$ is true for some $k''\in\mathcal{J}$, and we get $|J'|\leqslant 26\,244\,\rho$. We get that, for any triple

$$(\tilde{\imath},\tilde{\jmath},k)\in(\mathcal{J}\setminus I')\times(\mathcal{J}\setminus J')\times\mathcal{J},$$

neither $(\tilde{\jmath},k)\in D_1$ nor $(\tilde{\imath},k)\in D_2$ is true, so we can have $T(\tilde{\imath}|\tilde{\jmath}|k)\neq 0$ only if either

$$(\tilde{\imath},\tilde{\jmath})\in\mathcal{P} \tag{11.7}$$

or both the conditions

$$(\tilde{\imath},k)\in\mathcal{P} \text{ and } (\tilde{\jmath},k)\in\mathcal{P} \tag{11.8}$$

hold simultaneously. As said above, the pattern $\mathcal{P}$ has at most nine entries in every row and every column, which implies that, for any fixed $\tilde{\imath}$ and k, each of the conditions (11.7) and (11.8) can occur with at most nine different choices of $\tilde{\jmath}$, and, similarly, for any fixed $\tilde{\jmath}$ and k, each of the conditions (11.7) and (11.8) can occur for at most nine different $\tilde{\imath}$. Therefore, for any $k\in K$, the k-th 3-slice of the

$$(\mathcal{J}\setminus I')\times(\mathcal{J}\setminus J')\times\mathcal{J}$$

restriction of $\mathcal{T}$ contains at most 18 nonzero entries in every row and at most 18 nonzero entries in every column. This allows us to apply Lemma 11.7 with $\delta=18$, and we end up with the two families $I''\subseteq\mathcal{J}$ and $J''\subseteq\mathcal{J}$ such that

$$|I''|\leqslant 18^2\rho=324\,\rho,\ |J''|\leqslant 324\,\rho,$$

and the

$$(\mathcal{J}\setminus(I'\cup I''))\times(\mathcal{J}\setminus(J'\cup J''))\times\mathcal{J}$$

restriction of $\mathcal{T}$ is zero. In other words, the families $I_3=I'\cup I''$ and $J_3=J'\cup J''$ satisfy $|I_3|\leqslant(324+26244)\rho=26\,568\,\rho$, $|J_3|\leqslant 26\,568\,\rho$, and the

$$(\mathcal{J}\setminus I_3)\times(\mathcal{J}\setminus J_3)\times\mathcal{J}$$

restriction of $\mathcal{T}$ is zero. By the symmetry, there are J_1, K_1, I_2, $K_2 \subseteq \mathcal{J}$ with

$$|J_1| \leqslant 26\,568\,\rho,\ |K_1| \leqslant 26\,568\,\rho,\ |I_2| \leqslant 26\,568\,\rho,\ |K_2| \leqslant 26\,568\,\rho,$$

and, in addition, the

$$\mathcal{J} \times (\mathcal{J} \setminus J_1) \times (\mathcal{J} \setminus K_1) \text{ and } (\mathcal{J} \setminus I_2) \times \mathcal{J} \times (\mathcal{J} \setminus K_2)$$

restrictions of $\mathcal{T}$ are zero, so it remains to take $U = (I_1 \cup J_1) \cup (I_2 \cup J_2) \cup (I_3 \cup J_3)$. □

In other words, Lemma 11.10 reduces the number of the positions of potential nonzero entries of the tensor $T = T_1 + T_2 + T_3$ in the point (v) of Definition 8.3 corresponding to the family $\Phi(q)$ and a small rank bound ρ. The following lemma allows one to gather all the nonzero entries of T in some of its bounded size blocks for the price of the addition of three tensors whose slices of the corresponding directions are linear combinations of a bounded size subfamily of $\Phi(q)$.

Lemma 11.11. *Let $\mathbb{K}$ be a field, let T be an $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor in*

$$O \operatorname{mod}_{\mathbb{K}} (\Phi(q), \Phi(q), \Phi(q)),$$

and let $\mathcal{T}$ be the $\mathcal{J} \times \mathcal{J} \times \mathcal{J}$ block of T. If $\operatorname{rk}_{\mathbb{K}} \mathcal{T} \leqslant \rho \leqslant q/4\,500\,000$, then there exist

$$\Phi' \subseteq \Phi(q),\ \ W \subseteq \mathcal{Z},\ \ T' \in T \operatorname{mod}_{\mathbb{K}} (\Phi', \Phi', \Phi')$$

with $|\Phi'| \leqslant 4 \cdot 10^{13}\, \rho^2$ and $|W| \leqslant 4 \cdot 10^{12}\, (\rho + 18)^5$ such that all the nonzero entries of the $\mathcal{J} \times \mathcal{J} \times \mathcal{J}$ block of T' are in $(W \times \{1, 2\}) \times (W \times \{1, 2\}) \times (W \times \{1, 2\})$.

Proof. We apply Lemma 11.10 and find a subset $U \subset \mathcal{J}$ with $|U| \leqslant 159\,408\,\rho$ such that every nonzero entry of the $\mathcal{J} \times \mathcal{J} \times \mathcal{J}$ block of T belongs to the union of the blocks $\mathcal{J} \times U \times U$, $U \times \mathcal{J} \times U$, $U \times U \times \mathcal{J}$. Also, we define U' as the set of all $i \in \mathcal{Z}$ such that either $(i, 1) \in U$ or $(i, 2) \in U$, and we write $T = T_1 + T_2 + T_3$ with

$$T_1 \in O \operatorname{mod}_{\mathbb{K}} (\Phi(q), \varnothing, \varnothing),\ T_2 \in O \operatorname{mod}_{\mathbb{K}} (\varnothing, \Phi(q), \varnothing),\ T_3 \in O \operatorname{mod}_{\mathbb{K}} (\varnothing, \varnothing, \Phi(q)).$$

Now we define W as the set of all indexes $w \in \mathcal{Z}$ that can be written in the form $w = u + \alpha q + \beta$ with $\alpha, \beta \in \mathbb{Z}$ such that

$$|\alpha| \leqslant 1561(\rho + 18)^2 + 2, \quad |\beta| \leqslant 1561(\rho + 18)^2 + 2$$

and $u \in U'$. By Observation 11.9, for $e \in \{1, 2\}$, $\chi \in \{1, 2, 3\}$ and $k \in \mathcal{Z} \setminus W$, the

$$(U' \times \{1, 2\}) \times (U' \times \{1, 2\})$$

restriction of the (k, e)-th χ-slice of T_χ equals the corresponding restriction of the (k, e)-th χ-slice of T. We proceed with a special case to deal with such a restriction.

Special case. Assume that, for some $e \in \{1, 2\}$, $\chi \in \{1, 2, 3\}$ and $k \in \mathcal{Z} \setminus W$, the $\mathcal{J} \times \mathcal{J}$ padding of the $(U' \times \{1, 2\}) \times (U' \times \{1, 2\})$ restriction of the (k, e)-th χ-slice of T_χ cannot be obtained as the $\mathcal{J} \times \mathcal{J}$ restriction of some matrix in $\Phi(q)\, \mathbb{K}$.

Subcase A. If the $\mathcal{J} \times \mathcal{J}$ block of the (k, e)-th χ-slice of T_χ is not a $\mathbb{K}$-linear combination of the $\mathcal{J} \times \mathcal{J}$ blocks of at most $1561(\rho + 18)^2$ elements of $\Phi(q)$, then, according to Lemma 10.16 and Theorem 10.27, the rank of the $\mathcal{J} \times \mathcal{J}$ block of this slice is greater than $\rho + 18$. Using Observation 11.9 again, we see that the difference of the $\mathcal{J} \times \mathcal{J}$ blocks for the (k, e)-th χ-slices of T and T_χ has all its nonzero entries in a family of nine rows and nine columns, so we get $\operatorname{rk}_{\mathbb{K}} T > \rho$, which contradicts to the assumptions of the lemma and shows that the current Subcase A is void.

Subcase B. If there exists a subset $H \subseteq \Phi(q)$ with

$$|H| \leqslant 1561(\rho + 18)^2$$

such that the $\mathcal{J}\times\mathcal{J}$ block of the (k,e)-th χ-slice of T_χ belongs to the corresponding $\mathcal{J}\times\mathcal{J}$ restriction of $H\,\mathbb{K}$, then we consider the hypergraph G with the vertex set $\mathcal{Z}$ and the edge set labeled by H so that the edge corresponding to a given $h\in H$ contains all those indexes $i\in\mathcal{Z}$ for which the matrix h has a nonzero entry in either the row or column with the index (i,a), for some $a\in\{1,2\}$. Further, we define U'' as the union of all those connected components of G that have non-empty intersections with U' and note that the $\mathcal{J}\times\mathcal{J}$ padding of the

$$(U''\times\{1,2\})\times(U''\times\{1,2\})$$

restriction of the (k,e)-th χ-slice of T_χ is indeed the $\mathcal{J}\times\mathcal{J}$ restriction of some matrix in $H\,\mathbb{K}\subseteq\Phi(q)\,\mathbb{K}$. According to the initial assumption of the special case above, this means that the (k,e)-th χ-slice of T_χ has a nonzero entry

$$((\hat\imath,c),(\hat\jmath,d)) \text{ in } (U''\times\{1,2\})\times(U''\times\{1,2\}) \text{ outside } (U'\times\{1,2\})\times(U'\times\{1,2\}).$$

However, according to Definition 10.3, the elements $i,j\in\mathcal{Z}$ cannot be connected with an edge of G unless $i-j\in\{0,\,\pm1,\,\pm q,\,\pm1\pm q\}$, which implies that any index $u''\in U''$ can be written as $u''=u+\alpha q+\beta$ with $\alpha,\beta\in\mathbb{Z}$ such that

$$|\alpha|\leqslant 1561(\rho+18)^2,\quad |\beta|\leqslant 1561(\rho+18)^2$$

and $u\in U'$. A comparison with the definition of W and Observation 11.9 implies

$$((\hat\imath,c),(k,e))\notin\mathcal{P} \text{ and } ((\hat\jmath,d),(k,e))\notin\mathcal{P},$$

and hence the $((\hat\imath,c),(\hat\jmath,d))$ entry of the (k,e)-th χ-slice of T is nonzero, which contradicts the first sentence of the proof that forced every nonzero entry of the $\mathcal{J}\times\mathcal{J}\times\mathcal{J}$ block of T belong to the union of $\mathcal{J}\times U\times U$, $U\times\mathcal{J}\times U$, $U\times U\times\mathcal{J}$.

Therefore, neither Subcase A nor Subcase B is possible, and, since these subcases cover all possibilities, the assumption of the special case above is invalid. Therefore, we can proceed with the assumption that, for all

$$e\in\{1,2\},\ \chi\in\{1,2,3\},\ k\in\mathcal{Z}\setminus W,$$

the $\mathcal{J}\times\mathcal{J}$ padding of the

$$(U'\times\{1,2\})\times(U'\times\{1,2\})$$

restriction of the (k,e)-th χ-slice of T_χ is the $\mathcal{J}\times\mathcal{J}$ restriction of some matrix in $\Phi(q)\,\mathbb{K}$, and it remains to take Φ' equal to the set $L(U)$ as in Corollary 10.28. $\square$

We proceed with a deeper analysis of the combinatorial structure for each of the corresponding tensors (T_1,T_2,T_3) as in the point (v) of Definition 10.3. In particular, the following lemma deals with the corresponding off-diagonal blocks

$$(\mathcal{Z}\times\{1\})\times(\mathcal{Z}\times\{2\})\times(\mathcal{Z}\times\{2\}).$$

Lemma 11.12. *Let $\mathbb{K}$ be a field, let $W\subseteq\mathcal{Z}$, and let*

$$T_1\in O\,\mathrm{mod}_{\mathbb{K}}(\Phi(q),\varnothing,\varnothing),\ T_2\in O\,\mathrm{mod}_{\mathbb{K}}(\varnothing,\Phi(q),\varnothing),\ T_3\in O\,\mathrm{mod}_{\mathbb{K}}(\varnothing,\varnothing,\Phi(q))$$

be $(\mathcal{I}\cup\mathcal{J})\times(\mathcal{I}\cup\mathcal{J})\times(\mathcal{I}\cup\mathcal{J})$ tensors such that, for $i,j,p\in\mathcal{Z}$ and $\omega_1,\omega_2,\omega_3\in\{1,2\}$,

$$[T_1+T_2+T_3]_{(i,\omega_1),\,(j,\omega_2),\,(p,\omega_3)}=0 \text{ holds whenever } (i,j,p)\notin W\times W\times W. \tag{11.9}$$

If $\hat\imath,\ \hat\jmath\in\mathcal{Z}$, $k\in\mathcal{Z}\setminus W$ and $\chi\in\{1,2,3\}$, then the $(k,2)$-th χ-slice of T_χ can have a nonzero at the position $((\hat\imath,1),(\hat\jmath,2))$ only if the conditions

$$((k,2),(\hat\imath,1))\in\mathcal{P},\ ((k,2),(\hat\jmath,2))\in\mathcal{P},\ ((\hat\imath,1),(\hat\jmath,2))\in\mathcal{P} \tag{11.10}$$

hold simultaneously.

Proof. We note that the possible cases $\chi \in \{1, 2, 3\}$ correspond to each other up to the transpositions of the slices of the tensors involved in the consideration, so we can assume without loss of generality that $\chi = 3$. Also, we immediately note that the third of the conditions (11.10) holds automatically, or, in other words, we have

$$((\hat{\imath}, 1), (\hat{\jmath}, 2)) \in \mathcal{P} \tag{11.11}$$

because otherwise the $((\hat{\imath}, 1), (\hat{\jmath}, 2))$ position of any 3-slice of T_3 is zero. Also,

$$\text{either } ((k, 2), (\hat{\imath}, 1)) \in \mathcal{P} \text{ or } ((k, 2), (\hat{\jmath}, 2)) \in \mathcal{P} \tag{11.12}$$

is true because otherwise the $((\hat{\imath}, 1), (\hat{\jmath}, 2), (k, 2))$ entry of both T_1 and T_2 is zero, which contradicts to (11.9). Further, Observation 11.9 confirms the conditions

$$((i, a), (j, a)) \in \mathcal{P} \text{ if and only if } i - j \in \{-q, -1, 0, 1, q\} \tag{11.13}$$

and

$$((i, 1), (j, 2)) \in \mathcal{P} \text{ if and only if } i - j \in \{0, 1, q, q + 1\}, \tag{11.14}$$

for all $i, j \in \mathcal{J}$ and $a \in \{1, 2\}$. Also, for all $\alpha, \beta, \gamma \in \mathcal{J}$ and $\omega \in \{1, 2\}$, we have

$$\sum_{g \in \mathcal{Z}} T_1(\alpha|g, \omega|\gamma) = \sum_{g \in \mathcal{Z}} T_1(\alpha|\beta|g, \omega) = 0, \tag{11.15}$$

$$\sum_{g \in \mathcal{Z}} T_2(g, \omega|\beta|\gamma) = \sum_{g \in \mathcal{Z}} T_2(\alpha|\beta|g, \omega) = 0, \tag{11.16}$$

$$\sum_{g \in \mathcal{Z}} T_3(g, \omega|\beta|\gamma) = \sum_{g \in \mathcal{Z}} T_3(\alpha|g, \omega|\gamma) = 0$$

because the sums of the $\mathcal{Z} \times \{\omega\}$ coordinates of each of the vectors $u'(k)$, $v'(k)$, $u''(k)$, $v''(k)$ in Definition 10.3 are zero. We are ready to proceed with the case by case analysis of all possible $(\hat{\imath}, \hat{\jmath}, k)$, and, in view of the conditions (11.11) and (11.14), the cases that we need to consider are $\hat{\imath} - \hat{\jmath} \in \{0, 1, q, q + 1\}$.

Case 1. Assume that $\hat{\imath} = \hat{\jmath}$. According to the condition (11.13), the second option in (11.12) comes into play if and only if $k \in \{\hat{\imath} - q, \hat{\imath} - 1, \hat{\imath}, \hat{\imath} + 1, \hat{\imath} + q\}$, and, similarly, the condition (11.14) shows that the first option in (11.12) is true if and only if $k \in \{\hat{\imath} - q - 1, \hat{\imath} - q, \hat{\imath} - 1, \hat{\imath}\}$. Therefore, we can focus on the cases $k \in \{\hat{\imath} - q - 1, \hat{\imath} + 1, \hat{\imath} + q\}$ because whenever both of the options (11.12) are true, we take into account the condition (11.11) and get the desired condition (11.10).

The arguments in Cases 1.1–1.3 below are similar to each other, but, since they are not trivial and none of them seems to follow immediately from the other ones by the symmetry, we need to treat each of these cases separately.

Case 1.1. Assume $k = \hat{\imath} - q - 1$, which implies $(\hat{\imath}, \hat{\jmath}, k) = (k + q + 1, k + q + 1, k)$. In view of the condition (11.13), we get $((k + q + 1, 2), (k, 2)) \notin \mathcal{P}$, so the

$$((k + q + 1, 1), (k + q + 1, 2), (k, 2)) \tag{11.17}$$

entry of T_1 is zero, and hence the corresponding entry at the position (11.17) of T_2 is nonzero by the equality (11.9). According to the equalities (11.16), we get that the $(k, 2)$-th column of the $(k + q + 1, 2)$-th 2-slice of T_2 should contain some other nonzero entry besides the one at the position (11.17), and, in view of the condition (11.14), this nonzero entry should be located at one of the positions

$$((k + \delta, 1), (k + q + 1, 2), (k, 2)) \text{ with } \delta \in \{0, 1, q\}. \tag{11.18}$$

However, according to the conditions (11.13) and (11.14), both tensors T_1 and T_3 have zeros at every position in (11.18), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 1.1 is invalid.

Case 1.2. Assume $k = \hat{\imath} + 1$, which means that $(\hat{\imath}, \hat{\jmath}, k) = (k-1, k-1, k)$. In view of the condition (11.14), we get $((k-1, 1), (k, 2)) \notin \mathcal{P}$, so the

$$((k-1, 1), (k-1, 2), (k, 2)) \tag{11.19}$$

entry of T_2 is zero, and hence the corresponding entry at the position (11.19) of T_1 is nonzero by the equality (11.9). According to the equalities (11.15), we get that the $(k, 2)$-th column of the $(k-1, 1)$-th 1-slice of T_1 should contain some other nonzero entry besides the one at the position (11.19), and, in view of the condition (11.13), this nonzero entry should be located at one of the positions

$$((k-1, 1), (k+\delta, 2), (k, 2)) \text{ with } \delta \in \{-q, 0, 1, q\}. \tag{11.20}$$

However, according to the condition (11.14), both tensors T_2 and T_3 have zeros at every position in (11.20), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 1.2 is also invalid.

Case 1.3. Assume $k = \hat{\imath} + q$, which means that $(\hat{\imath}, \hat{\jmath}, k) = (k-q, k-q, k)$. In view of the condition (11.14), we get $((k-q, 1), (k, 2)) \notin \mathcal{P}$, so the

$$((k-q, 1), (k-q, 2), (k, 2)) \tag{11.21}$$

entry of T_2 is zero, and hence the corresponding entry at the position (11.21) of T_1 is nonzero by the equality (11.9). According to the equalities (11.15), we get that the $(k, 2)$-th column of the $(k-q, 1)$-th 1-slice of T_1 should contain some other nonzero entry besides the one at the position (11.21), and, in view of the condition (11.13), this nonzero entry should be located at one of the positions

$$((k-q, 1), (k+\delta, 2), (k, 2)) \text{ with } \delta \in \{-1, 0, 1, q\}. \tag{11.22}$$

However, according to the condition (11.14), both tensors T_2 and T_3 have zeros at every position in (11.22), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 1.3 is invalid as well.

We see that none of Cases 1.1–1.3 can be in effect, but, since these cases cover all the possibilities that remained in Case 1, we reach the desired condition as explained in the first paragraph of Case 1 and conclude the consideration of Case 1.

Case 2. Assume that $\hat{\imath} = \hat{\jmath} + 1$. The argument required in this situation looks similar to that in Case 1, but, again, since these cases do not seem to be equivalent by the symmetry, we need to proceed with a detailed consideration. Indeed, by the condition (11.14), the first option in (11.12) comes into play if and only if $k \in \{\hat{\jmath} - q, \hat{\jmath} - q + 1, \hat{\jmath}, \hat{\jmath} + 1\}$, and, similarly, the condition (11.13) shows that the second option in (11.12) is true if and only if $k \in \{\hat{\jmath} - q, \hat{\jmath} - 1, \hat{\jmath}, \hat{\jmath} + 1, \hat{\jmath} + q\}$. So we switch to the cases $k \in \{\hat{\jmath} - q + 1, \hat{\jmath} - 1, \hat{\jmath} + q\}$ because if the conditions (11.12) are both true, we use the condition (11.11) and get the desired condition (11.10).

Case 2.1. Assume $k = \hat{\jmath} - q + 1$, which implies $(\hat{\imath}, \hat{\jmath}, k) = (k+q, k+q-1, k)$. In view of the condition (11.13), we get $((k+q-1, 2), (k, 2)) \notin \mathcal{P}$, so the

$$((k+q, 1), (k+q-1, 2), (k, 2)) \tag{11.23}$$

entry of T_1 is zero, and hence the corresponding entry at the position (11.23) of T_2 is nonzero by the equality (11.9). According to the equalities (11.16), we get that the $(k, 2)$-th column of the $(k+q-1, 2)$-th 2-slice of T_2 should contain some other nonzero entry besides the one at the position (11.23), and, in view of the condition (11.14), this nonzero entry should be located at one of the positions

$$((k+\delta,1),(k+q-1,2),(k,2)) \text{ with } \delta \in \{0,1,q+1\}. \tag{11.24}$$

However, according to the conditions (11.13) and (11.14), both tensors T_1 and T_3 have zeros at every position in (11.24), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 2.1 is invalid.

Case 2.2. Assume $k = \hat{\jmath}-1$, which implies $(\hat{\imath},\hat{\jmath},k) = (k+2,k+1,k)$. In view of the condition (11.14), we get $((k+2,1),(k,2)) \notin \mathcal{P}$, so the

$$((k+2,1),(k+1,2),(k,2)) \tag{11.25}$$

entry of T_2 is zero, and hence the corresponding entry at the position (11.25) of T_1 is nonzero by the equality (11.9). According to the equalities (11.15), we get that the $(k,2)$-th column of the $(k+2,1)$-th 1-slice of T_1 should contain some other nonzero entry besides the one at the position (11.25), and, in view of the condition (11.13), this nonzero entry should be located at one of the positions

$$((k+2,1),(k+\delta,2),(k,2)) \text{ with } \delta \in \{-q,-1,0,q\}. \tag{11.26}$$

However, according to the condition (11.14), both tensors T_2 and T_3 have zeros at every position in (11.26), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 2.2 is invalid.

Case 2.3. Assume $k = \hat{\jmath}+q$, which implies $(\hat{\imath},\hat{\jmath},k) = (k-q+1,k-q,k)$. In view of the condition (11.14), we get $((k-q+1,1),(k,2)) \notin \mathcal{P}$, so the

$$((k-q+1,1),(k-q,2),(k,2)) \tag{11.27}$$

entry of T_2 is zero, and hence the corresponding entry at the position (11.27) of T_1 is nonzero by the equality (11.9). According to the equalities (11.15), we get that the $(k,2)$-th column of the $(k-q+1,1)$-th 1-slice of T_1 should contain some other nonzero entry besides the one at the position (11.27), and, in view of the condition (11.13), this nonzero entry should be located at one of the positions

$$((k-q+1,1),(k+\delta,2),(k,2)) \text{ with } \delta \in \{-1,0,1,q\}. \tag{11.28}$$

However, according to the condition (11.14), both tensors T_2 and T_3 have zeros at every position in (11.28), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 2.3 is invalid.

We see that none of Cases 2.1–2.3 can be in effect, but, since these cases cover all the possibilities that remained in Case 2, we reach the desired condition as explained in the first paragraph of Case 2 and conclude the consideration of Case 2.

Case 3. Assume that $\hat{\imath}-\hat{\jmath} = q$. In a way similar to the previous cases, we get that, according to the condition (11.14), the first option in (11.12) comes into play if and only if $k \in \{\hat{\jmath}-1,\hat{\jmath},\hat{\jmath}-1+q,\hat{\jmath}+q\}$, and, similarly, the condition (11.13) shows that the second option in (11.12) is true if and only if $k \in \{\hat{\jmath}-q,\hat{\jmath}-1,\hat{\jmath},\hat{\jmath}+1,\hat{\jmath}+q\}$. So we switch to the cases $k \in \{\hat{\jmath}-q,\hat{\jmath}+1,\hat{\jmath}+q-1\}$ because if the conditions (11.12) are both true, we use the condition (11.11) and get the desired condition (11.10).

Case 3.1. Assume $k = \hat{\jmath}-q$, which implies $(\hat{\imath},\hat{\jmath},k) = (k+2q,k+q,k)$. In view of the condition (11.14), we get $((k+2q,1),(k,2)) \notin \mathcal{P}$, so the

$$((k+2q,1),(k+q,2),(k,2)) \tag{11.29}$$

entry of T_2 is zero, and hence the corresponding entry at the position (11.29) of T_1 is nonzero by the equality (11.9). According to the equalities (11.15), we get that the $(k, 2)$-th column of the $(k + 2q, 1)$-th 1-slice of T_1 should contain some other nonzero entry besides the one at the position (11.29), and, in view of the condition (11.13), this nonzero entry should be located at one of the positions

$$((k + 2q, 1), (k + \delta, 2), (k, 2)) \text{ with } \delta \in \{-q, -1, 0, 1\}. \tag{11.30}$$

However, according to the condition (11.14), both tensors T_2 and T_3 have zeros at every position in (11.30), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 3.1 is invalid.

Case 3.2. Assume $k = \hat{\jmath} + 1$, which implies $(\hat{\imath}, \hat{\jmath}, k) = (k + q - 1, k - 1, k)$. In view of the condition (11.14), we get $((k + q - 1, 1), (k, 2)) \notin \mathcal{P}$, so the

$$((k + q - 1, 1), (k - 1, 2), (k, 2)) \tag{11.31}$$

entry of T_2 is zero, and hence the corresponding entry at the position (11.31) of T_1 is nonzero by the equality (11.9). According to the equalities (11.15), we get that the $(k, 2)$-th column of the $(k + q - 1, 1)$-th 1-slice of T_1 should contain some other nonzero entry besides the one at the position (11.31), and, in view of the condition (11.13), this nonzero entry should be located at one of the positions

$$((k + q - 1, 1), (k + \delta, 2), (k, 2)) \text{ with } \delta \in \{-q, 0, 1, q\}. \tag{11.32}$$

However, according to the condition (11.14), both tensors T_2 and T_3 have zeros at every position in (11.32), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 3.2 is invalid.

Case 3.3. Assume $k = \hat{\jmath} + q - 1$, which implies $(\hat{\imath}, \hat{\jmath}, k) = (k + 1, k + 1 - q, k)$. In view of the condition (11.13), we get $((k + 1 - q, 2), (k, 2)) \notin \mathcal{P}$, so the

$$((k + 1, 1), (k + 1 - q, 2), (k, 2)) \tag{11.33}$$

entry of T_1 is zero, and hence the corresponding entry at the position (11.33) of T_2 is nonzero by the equality (11.9). According to the equalities (11.16), we get that the $(k, 2)$-th column of the $(k + 1 - q, 2)$-th 2-slice of T_2 should contain some other nonzero entry besides the one at the position (11.33), and, in view of the condition (11.14), this nonzero entry should be located at one of the positions

$$((k + \delta, 1), (k + 1 - q, 2), (k, 2)) \text{ with } \delta \in \{0, q, q + 1\}. \tag{11.34}$$

However, according to the conditions (11.13) and (11.14), both tensors T_1 and T_3 have zeros at every position in (11.34), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 3.3 is invalid.

We see that none of Cases 3.1–3.3 can be in effect, but, since these cases cover all the possibilities that remained in Case 3, we reach the desired condition as explained in the first paragraph of Case 3 and conclude the consideration of Case 3.

Case 4. Assume that $\hat{\imath} - \hat{\jmath} = q + 1$. According to the condition (11.14), the first option in (11.12) comes into play if and only if $k \in \{\hat{\jmath}, \hat{\jmath} + 1, \hat{\jmath} + q, \hat{\jmath} + q + 1\}$, and, similarly, the condition (11.13) shows that the second option in (11.12) is true if and only if $k \in \{\hat{\jmath} - q, \hat{\jmath} - 1, \hat{\jmath}, \hat{\jmath} + 1, \hat{\jmath} + q\}$. So we switch to the cases $k \in \{\hat{\jmath} - q, \hat{\jmath} - 1, \hat{\jmath} + q + 1\}$ because if the conditions (11.12) are both true, we use the condition (11.11) and get the desired condition (11.10).

Case 4.1. Assume $k = \hat{\jmath} - q$, which implies $(\hat{\imath}, \hat{\jmath}, k) = (k + 2q + 1, k + q, k)$. In view of the condition (11.14), we get $((k + 2q + 1, 1), (k, 2)) \notin \mathcal{P}$, so the

$$((k + 2q + 1, 1), (k + q, 2), (k, 2)) \tag{11.35}$$

entry of T_2 is zero, and hence the corresponding entry at the position (11.35) of T_1 is nonzero by the equality (11.9). According to the equalities (11.15), we get that the $(k, 2)$-th column of the $(k+2q+1, 1)$-th 1-slice of T_1 should contain some other nonzero entry besides the one at the position (11.35), and, in view of the condition (11.13), this nonzero entry should be located at one of the positions

$$((k + 2q + 1, 1), (k + \delta, 2), (k, 2)) \text{ with } \delta \in \{-q, -1, 0, 1\}. \tag{11.36}$$

However, according to the condition (11.14), both tensors T_2 and T_3 have zeros at every position in (11.36), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 4.1 is invalid.

Case 4.2. Assume $k = \hat{\jmath} - 1$, which implies $(\hat{\imath}, \hat{\jmath}, k) = (k + q + 2, k + 1, k)$. In view of the condition (11.14), we get $((k + q + 2, 1), (k, 2)) \notin \mathcal{P}$, so the

$$((k + q + 2, 1), (k + 1, 2), (k, 2)) \tag{11.37}$$

entry of T_2 is zero, and hence the corresponding entry at the position (11.37) of T_1 is nonzero by the equality (11.9). According to the equalities (11.15), we get that the $(k, 2)$-th column of the $(k + q + 2, 1)$-th 1-slice of T_1 should contain some other nonzero entry besides the one at the position (11.37), and, in view of the condition (11.13), this nonzero entry should be located at one of the positions

$$((k + q + 2, 1), (k + \delta, 2), (k, 2)) \text{ with } \delta \in \{-q, -1, 0, q\}. \tag{11.38}$$

However, according to the condition (11.14), both tensors T_2 and T_3 have zeros at every position in (11.38), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 4.2 is invalid.

Case 4.3. Assume $k = \hat{\jmath} + q + 1$, which implies $(\hat{\imath}, \hat{\jmath}, k) = (k, k - q - 1, k)$. In view of the condition (11.13), we get $((k, 2), (k - q - 1, 2)) \notin \mathcal{P}$, so the

$$((k, 1), (k - q - 1, 2), (k, 2)) \tag{11.39}$$

entry of T_1 is zero, and hence the corresponding entry at the position (11.39) of T_2 is nonzero by the equality (11.9). According to the equalities (11.16), we get that the $(k, 2)$-th column of the $(k - q - 1, 2)$-th 2-slice of T_2 should contain some other nonzero entry besides the one at the position (11.39), and, in view of the condition (11.14), this nonzero entry should be located at one of the positions

$$((k + \delta, 1), (k - q - 1, 2), (k, 2)) \text{ with } \delta \in \{1, q, q + 1\}. \tag{11.40}$$

However, according to the conditions (11.13) and (11.14), both tensors T_1 and T_3 have zeros at every position in (11.40), and hence we get a contradiction to the equality (11.9). This means that the assumption of Case 4.3 is invalid.

The consideration of Cases 1–4 is now complete, and, since these cases cover all possibilities, the proof of the current lemma is now complete as well. $\square$

We need some further notational conventions before we proceed with a further analysis of the off-diagonal blocks of the tensor in the point (v) of Definition 8.3.

Notation 11.13. Using the conventions of Definition 10.3, we write

$$A_0(k) = (O \oplus u''(k)) \otimes (u'(k) \oplus O) + (u'(k) \oplus O) \otimes (O \oplus u''(k)),$$
$$B_0(k) = (O \oplus v''(k)) \otimes (v'(k) \oplus O) + (v'(k) \oplus O) \otimes (O \oplus v''(k)),$$

and, in particular, we note that

$$A_0(k) = A_3(k) - A_2(k) - A_1(k),\ B_0(k) = B_3(k) - B_2(k) - B_1(k).$$

Also, we write

$$\mathcal{M} = \sum_{k \in \mathcal{Z}} (A_0(k) + B_0(k))$$

to denote the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ padding of the $\mathcal{I} \times \mathcal{I}$ matrix as in Lemma 10.16.

Remark 11.14. It is clear that the linear spans

$$A_1(k)\,\mathbb{F} + A_2(k)\,\mathbb{F} + A_3(k)\,\mathbb{F} \ \text{ and } \ B_1(k)\,\mathbb{F} + B_2(k)\,\mathbb{F} + B_3(k)\,\mathbb{F}$$

do not change if the matrices $(A_3(k), B_3(k))$ are replaced with $(A_0(k), B_0(k))$. In particular, the linear span $\Phi(q)\,\mathbb{F}$ does not change under such a replacement as well.

Remark 11.15. We also adopt the set sum notation, so that, for any $A, B \subseteq \mathcal{Z}$, the sum $A + B$ is the set of all $c \in \mathcal{Z}$ such that $c = a + b$ with some $(a, b) \in A \times B$.

We are ready to clean up the off-diagonal blocks of the tensor $T_1 + T_2 + T_3$ as in the point (v) in Definition 8.3 with a further addition of three tensors whose slices of the corresponding directions are spanned by a bounded size subfamily of $\Phi(q)$.

Lemma 11.16. *Let $\rho \geqslant 1$ be an integer and $W \subseteq \mathcal{Z}$. Assume that*

$$\mathbb{K} \text{ is a field whose characteristic does not divide } 2(q+1), \tag{11.41}$$

$$q > 2 \cdot 10^7 \cdot |W|^2 + 4 \cdot 10^6 \cdot \rho^2 \cdot |W| + 32\,\rho + 1\,153. \tag{11.42}$$

Further, let three $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensors

$$T_1 \in O \operatorname{mod}_{\mathbb{K}} (\Phi(q), \varnothing, \varnothing),\ T_2 \in O \operatorname{mod}_{\mathbb{K}} (\varnothing, \Phi(q), \varnothing),\ T_3 \in O \operatorname{mod}_{\mathbb{K}} (\varnothing, \varnothing, \Phi(q))$$

be such that, for all $\omega_1, \omega_2, \omega_3 \in \{1, 2\}$ and $i, j, k \in \mathcal{Z}$, the conditions

$$[T_1 + T_2 + T_3]_{(i,\omega_1),\,(j,\omega_2),\,(k,\omega_3)} = 0 \ \textit{ whenever } \ (i, j, k) \notin W \times W \times W \tag{11.43}$$

are in effect, and, in addition, for some $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor Δ which has all its nonzero entries collected in the union of the blocks $(\mathcal{I} \cup \mathcal{J}) \times \mathcal{I} \times \mathcal{I}$, $\mathcal{I} \times (\mathcal{I} \cup \mathcal{J}) \times \mathcal{I}$ and $\mathcal{I} \times \mathcal{I} \times (\mathcal{I} \cup \mathcal{J})$, one has

$$\operatorname{rk}_{\mathbb{K}} (T_1 + T_2 + T_3 + \Delta) \leqslant \rho. \tag{11.44}$$

Then there exists a subset $\Phi'' \subseteq \Phi(q)$ with

$$|\Phi''| \leqslant 4 \cdot 10^{15} \cdot |W|^4 + 2 \cdot 10^{15} \cdot \left(\rho^2\,|W|^3 + \rho^4\,|W|^2\right) + 10^6 \cdot \rho^2 + 13 \cdot 10^6$$

such that the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor obtained by taking the

$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}),$$
$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}),$$
$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\})$$

blocks of $T_1 + T_2 + T_3$ and replacing all the remaining entries with zeros belongs to

$$O \operatorname{mod}_{\mathbb{K}} (\Phi''\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \Phi''\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \Phi''\,\mathbb{K} + \mathcal{M}\,\mathbb{K}). \tag{11.45}$$

Proof. We have that, for any $\chi \in \{1,2,3\}$ and $g \in \mathcal{I} \cup \mathcal{J}$, the g-th χ-slice of T_χ is

$$\sum_{c=0}^{2} \sum_{k \in \mathcal{Z}} \left(\alpha_c(\chi|g|k)\, A_c(k) + \beta_c(\chi|g|k)\, B_c(k)\right) \tag{11.46}$$

with some scalars $\alpha_c(\chi|g|k)$ and $\beta_c(\chi|g|k)$ in $\mathbb{K}$. To begin with, we take a deeper investigation of these scalars subdivided into several steps below.

The considerations of these steps follow analogously for different $\chi \in \{1,2,3\}$, so, in order to avoid unnecessary repetitions of similar arguments,

$$\text{we fix an arbitrary permutation } (\chi, \chi', \chi'') \text{ of the string } (1,2,3), \tag{11.47}$$

and we use the notation (11.47) in the rest of the current proof. Also, we write σ to denote the permutation of the coordinates in a string of the length three via

$$1 \to \chi,\ 2 \to \chi',\ 3 \to \chi''$$

so that the string $\sigma(\varkappa_1, \varkappa_2, \varkappa_3)$ has $\varkappa_1$ at the χ-th position, it has $\varkappa_2$ at the χ'-th position, and $\varkappa_3$ is at the remaining χ''-th position.

Step 1A. Suppose that, for some $\hat{\imath}$ and k in $\mathcal{Z}$, we have

$$\alpha_2(\chi|\hat{\imath},1|k) \neq 0 \ \text{ with } \ \{k, k-q\} \cap W = \varnothing \ \text{ and } \ \hat{\imath} \notin \{k, k+1\}. \tag{11.48}$$

Then, by Observation 10.14, the $((k,2),(k-q,2))$ entry of the $(\hat{\imath},1)$-th χ-slice of T_χ is nonzero, and hence, due to (11.43) and the assumption $k \notin W$ in (11.48), either

$$\text{the } ((\hat{\imath},1),(k-q,2)) \text{ entry of the } (k,2)\text{-th } \chi'\text{-slice of } T_{\chi'} \text{ is nonzero, or} \tag{11.49}$$

$$\text{the } ((\hat{\imath},1),(k,2)) \text{ entry of the } (k-q,2)\text{-th } \chi''\text{-slice of } T_{\chi''} \text{ is nonzero.} \tag{11.50}$$

However, in view of Observation 11.9 and the condition $\hat{\imath} \notin \{k, k+1\}$ in (11.48), we have either $((\hat{\imath},1),(k-q,2)) \notin \mathcal{P}$ or $((\hat{\imath},1),(k,2)) \notin \mathcal{P}$, and we see that, in view of the assumption $\{k, k-q\} \cap W = \varnothing$ in (11.48), the validity of any of the corresponding conditions (11.49) and (11.50) contradicts to Lemma 11.12.

Step 1B. Suppose that, for some $\hat{\imath}$ and k in $\mathcal{Z}$, we have

$$\beta_2(\chi|\hat{\imath},1|k) \neq 0 \ \text{ with } \ \{k, k-1\} \cap W = \varnothing \ \text{ and } \ \hat{\imath} \notin \{k, k+q\}. \tag{11.51}$$

Then, by Observation 10.14, the $((k,2),(k-1,2))$ entry of the $(\hat{\imath},1)$-th χ-slice of T_χ is nonzero, and hence, due to (11.43) and the assumption $k \notin W$ in (11.51), either

$$\text{the } ((\hat{\imath},1),(k-1,2)) \text{ entry of the } (k,2)\text{-th } \chi'\text{-slice of } T_{\chi'} \text{ is nonzero, or} \tag{11.52}$$

$$\text{the } ((\hat{\imath},1),(k,2)) \text{ entry of the } (k-1,2)\text{-th } \chi''\text{-slice of } T_{\chi''} \text{ is nonzero.} \tag{11.53}$$

However, in view of Observation 11.9 and the condition $\hat{\imath} \notin \{k, k+q\}$ in (11.51), we have either $((\hat{\imath},1),(k-1,2)) \notin \mathcal{P}$ or $((\hat{\imath},1),(k,2)) \notin \mathcal{P}$, and we see that, in view of the assumption $\{k, k-1\} \cap W = \varnothing$ in (11.51), the validity of any of the corresponding conditions (11.52) and (11.53) contradicts to Lemma 11.12.

The contradictions of Steps 1A and 1B show that neither (11.48) nor (11.51) can occur, indeed. Furthermore, we can define

$$W' = W + \{0,\ \pm 1,\ \pm q\} \tag{11.54}$$

to get the conditions

$$\alpha_2(\chi|\hat{\imath},1|k) = 0 \ \text{ whenever } \ k \in \mathcal{Z} \setminus W' \ \text{ and } \ \hat{\imath} \notin \{k, k+1\}, \tag{11.55}$$

$$\beta_2(\chi|\hat{\imath},1|k) = 0 \ \text{ whenever } \ k \in \mathcal{Z} \setminus W' \ \text{ and } \ \hat{\imath} \notin \{k, k+q\}. \tag{11.56}$$

Step 2. We also need to estimate the total count of the nonzero coefficients of the forms $\alpha_0(\chi|j,2|k)$ and $\beta_0(\chi|j,2|k)$ in the formula (11.46). To this end, we use Observation 11.9 and conclude that, for any $\hat{\jmath} \in \mathcal{Z}$, the sum of the

$$(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{2\})$$

blocks of the $(\hat{\jmath}, 2)$-th χ-slices of $T_{\chi'}$ and $T_{\chi''}$ has all its nonzero entries in the union of four rows and five columns, so the difference of the corresponding ranks of the

$$\sigma\left((\{\hat{\jmath}\} \times \{2\}) \times (\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{2\})\right) \tag{11.57}$$

restrictions of $T_1 + T_2 + T_3$ and T_χ is at most $4 + 5 = 9$. In view of (11.44), we have

$$\operatorname{rk}_{\mathbb{K}} \overline{T}(\chi, \hat{\jmath}) \leqslant \rho + 9, \tag{11.58}$$

where $\overline{T}(\chi, \hat{\jmath})$ is the matrix that arises as the restriction of T_χ to (11.57). Now we apply Theorem 10.26 to the inequality (11.58) and get a subset $S(\chi, \hat{\jmath}) \subset \mathcal{Z}$ with

$$|S(\chi, \hat{\jmath})| \leqslant 512(\rho + 9)^2 \tag{11.59}$$

and a scalar $\varphi(\chi, \hat{\jmath}) \in \mathbb{K}$ such that

$$\alpha_0(\chi|\hat{\jmath}, 2|k) = \beta_0(\chi|\hat{\jmath}, 2|k) = \varphi(\chi, \hat{\jmath}) \text{ whenever } k \notin S(\chi, \hat{\jmath}). \tag{11.60}$$

Further, as stated in Notation 11.13, we have

$$\sum_{k \in \mathcal{Z}} (A_0(k) + B_0(k)) = \mathcal{M},$$

and then we consider the tensor τ_χ obtained from T_χ by subtracting $\varphi(\chi, \hat{\jmath})\, \mathcal{M}$ from its $(\hat{\jmath}, 2)$-th χ-slice, for all $\hat{\jmath} \in \mathcal{Z}$ in sequence. Since the replacement

$$(T_1, T_2, T_3) \to (\tau_1, \tau_2, \tau_3)$$

has no effect on the desired condition as in (11.45), we can further assume that

$$\varphi(\chi, \hat{\jmath}) = 0 \text{ holds for all } \hat{\jmath} \in \mathcal{Z}. \tag{11.61}$$

Step 3. Now we assume $j \notin W$ and apply Observation 11.9 and Lemma 11.12 to see that the $(j, 2)$-th χ-slice of T_χ has the potential nonzero entries at the positions

$$\left(\begin{array}{c|ccccc} & (j,2) & (j-1,2) & (j+1,2) & (j-q,2) & (j+q,2) \\ \hline (j,1) & * & * & 0 & * & 0 \\ (j+1,1) & * & 0 & * & * & 0 \\ (j+q,1) & * & * & 0 & 0 & * \\ (j+q+1,1) & * & 0 & * & 0 & * \end{array}\right) \tag{11.62}$$

and zeros in all other places of the form $((i,1),(k,2))$ with $i,\, k \in \mathcal{Z}$, where the first row and column of (11.62) indicate the column and row indexes, respectively. In view of the formula (11.46), the $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{2\})$ block of the $(j, 2)$-th χ-slice of T_χ comes as a linear combination of the corresponding blocks of the matrices $(A_0(k), B_0(k))$, which implies that, according to Definition 10.3, the row and column sums in (11.62) should be zero, which allows us to rewrite the block (11.62) as

$$\left(\begin{array}{c|ccccc} & (j,2) & (j-1,2) & (j+1,2) & (j-q,2) & (j+q,2) \\ \hline (j,1) & p_{\chi j1} + p_{\chi j2} & -p_{\chi j1} & 0 & -p_{\chi j2} & 0 \\ (j+1,1) & -p_{\chi j2} + p_{\chi j3} & 0 & -p_{\chi j3} & p_{\chi j2} & 0 \\ (j+q,1) & -p_{\chi j1} + p_{\chi j4} & p_{\chi j1} & 0 & 0 & -p_{\chi j4} \\ (j+q+1,1) & -p_{\chi j3} - p_{\chi j4} & 0 & p_{\chi j3} & 0 & p_{\chi j4} \end{array}\right)$$

with the use of the parameters $(p_{\chi j\omega})$ in $\mathbb{K}$. Therefore, the $(\mathcal{Z}\times\{1\})\times(\mathcal{Z}\times\{2\})$ block of the $(j,2)$-th χ-slice of T_χ equals the corresponding block of some matrix in

$$A_0(j)\,\mathbb{K}+A_0(j+q)\,\mathbb{K}+B_0(j)\,\mathbb{K}+B_0(j+1)\,\mathbb{K},$$

and hence, in view of Lemma 10.18, for all $\chi\in\{1,2,3\}$ and $j\in\mathcal{Z}\setminus W$, we have

$$\alpha_0(\chi|j,2|i)=\beta_0(\chi|j,2|k)=\zeta_{\chi j}\ \text{ if }\ i\notin\{j,j+q\}\ \text{ and }\ k\notin\{j,j+1\},$$

for some scalars $(\zeta_{\chi j})$ in $\mathbb{K}$. The results of Step 2 imply $\zeta_{\chi j}=0$, and hence

$$\alpha_0(\chi|j,2|i)=0 \text{ for all } j\in\mathcal{Z}\setminus W \text{ and } i\in\mathcal{Z}\setminus\{j,j+q\}, \tag{11.63}$$

$$\beta_0(\chi|j,2|k)=0 \text{ for all } j\in\mathcal{Z}\setminus W \text{ and } k\in\mathcal{Z}\setminus\{j,j+1\}, \tag{11.64}$$

where the condition $j\in\mathcal{Z}\setminus W$ is due to the initial assumption of Step 3.

Step 4. Our further idea is to force the vanishing of the coefficients $\alpha_2(\chi|\widehat{\imath},1|k)$ and $\beta_2(\chi|\widehat{\imath},1|k)$ for all $k\notin W'$ and $\widehat{\imath}\in\mathcal{Z}$, or, in other words, we want to get rid of the assumption $\widehat{\imath}\notin\{k,k+1\}$ as in the condition (11.55), and, similarly, we want to remove the assumption $\widehat{\imath}\notin\{k,k+q\}$ as in the formula (11.56). To this end, we take $s\in\mathcal{Z}\setminus W'$ and write the equations that appear by taking, respectively,

- $\sigma((s+1,1),(s,2),(s-q,2))$,
- $\sigma((s,1),(s,2),(s-q,2))$,
- $\sigma((s+q,1),(s,2),(s-1,2))$,
- $\sigma((s,1),(s,2),(s-1,2))$

in the role of $((i,\omega_1),(j,\omega_2),(k,\omega_3))$ in the condition (11.43). As we can see after the application of the conditions (11.46), (11.63) and (11.64), these are

$$\begin{gathered}
-\alpha_2(\chi|s+1,1|s)-\alpha_0(\chi'|s,2|s)+\alpha_0(\chi''|s-q,2|s)=0,\\
-\alpha_2(\chi|s,1|s)+\alpha_0(\chi'|s,2|s)-\alpha_0(\chi''|s-q,2|s)=0,\\
-\beta_2(\chi|s+q,1|s)+\beta_0(\chi'|s,2|s)-\beta_0(\chi''|s-1,2|s)=0,\\
-\beta_2(\chi|s,1|s)-\beta_0(\chi'|s,2|s)+\beta_0(\chi''|s-1,2|s)=0,
\end{gathered}$$

and, here, the sum of the first two equations gives

$$\alpha_2(\chi|s,1|s)+\alpha_2(\chi|s+1,1|s)=0 \tag{11.65}$$

as the sum of the remaining two equations leads to

$$\beta_2(\chi|s,1|s)+\beta_2(\chi|s+q,1|s)=0. \tag{11.66}$$

Using the conditions (11.65) and (11.66), we are going to transform the coefficients in (11.46) in order to get rid of the nonzero coefficients $\alpha_2(\chi|j,1|s)$ and $\beta_2(\chi|j,1|s)$ as explained in the preamble of the current step. To this end, we write

$$\gamma(\chi,s)=\frac{\alpha_2(1|s,1|s)+\alpha_2(2|s,1|s)+\alpha_2(3|s,1|s)}{2}-\alpha_2(\chi|s,1|s) \tag{11.67}$$

and

$$\delta(\chi,s)=\frac{\beta_2(1|s,1|s)+\beta_2(2|s,1|s)+\beta_2(3|s,1|s)}{2}-\beta_2(\chi|s,1|s), \tag{11.68}$$

which is possible because the assumptions of the lemma imply $\operatorname{char}\mathbb{K}\neq 2$. Further, for any $s\in\mathcal{Z}\setminus W'$ and $\chi\in\{1,2,3\}$, we take

$$\widehat{\alpha}_2(\chi|s,1|s)=\widehat{\alpha}_2(\chi|s+1,1|s)=0 \tag{11.69}$$

and

$$\widehat{\beta}_2(\chi|s,1|s)=\widehat{\beta}_2(\chi|s+q,1|s)=0, \tag{11.70}$$

and, in particular, this setting confirms the vanishing of the desired coefficients as explained above. Also, we amend several coefficients in the (α_0) and (β_0) parts:

$$\widehat{\alpha}_0(\chi|s,2|s) = \alpha_0(\chi|s,2|s) - \gamma(\chi,s), \tag{11.71}$$

$$\widehat{\alpha}_0(\chi|s-q,2|s) = \alpha_0(\chi|s-q,2|s) + \gamma(\chi,s), \tag{11.72}$$

$$\widehat{\beta}_0(\chi|s,2|s) = \beta_0(\chi|s,2|s) + \delta(\chi,s), \tag{11.73}$$

$$\widehat{\beta}_0(\chi|s-1,2|s) = \beta_0(\chi|s-1,2|s) - \delta(\chi,s). \tag{11.74}$$

The $\mathcal{I}$ parts of (α_0) and (α_2) should also be changed:

$$\widehat{\alpha}_0(\chi|\text{♘},2|s') = \alpha_0(\chi|\text{♘},2|s') + \gamma(\chi,s') \tag{11.75}$$

with any $s' \in \{0,-1,\ldots,-q+1\} \setminus W'$ and

$$\widehat{\alpha}_2(\chi|\text{♘},1|s'') = \alpha_2(\chi|\text{♘},1|s'') - \alpha_2(\chi|s'',1|s'') \tag{11.76}$$

whenever $s'' \in \{0,-q,-2q,\ldots,-q(q-1)\} \setminus W'$.

Finally, concerning the $\mathcal{I}$ parts of (β_0) and (β_2), we take

$$\widehat{\beta}_0(\chi|\text{♘},2|1-q) = \beta_0(\chi|\text{♘},2|1-q) - \delta(\chi,1-q) \text{ if } 1-q \notin W' \tag{11.77}$$

and

$$\widehat{\beta}_2(\chi|\text{♘},1|1) = \beta_2(\chi|\text{♘},1|1) + \beta_2(\chi|1,1|1) \text{ if } 1 \notin W'. \tag{11.78}$$

Also, if either $\widehat{\alpha}_c(\chi|x,y|z)$ or $\widehat{\beta}_c(\chi|x,y|z)$ does not appear in (11.69)–(11.78), then it is defined to be equal to the corresponding value $\alpha_c(\chi|x,y|z)$ or $\beta_c(\chi|x,y|z)$. In the way similar to the conditions (11.46), we define the $(\mathcal{I}\cup\mathcal{J})\times(\mathcal{I}\cup\mathcal{J})\times(\mathcal{I}\cup\mathcal{J})$ tensors $\widehat{\tau}_\chi$ by declaring that, for any $g \in \mathcal{I}\cup\mathcal{J}$, the g-th χ-slice of $\widehat{\tau}_\chi$ is

$$\sum_{c=0}^{2}\sum_{k\in\mathcal{Z}}\Big(\widehat{\alpha}_c(\chi|g|k)\,A_c(k) + \widehat{\beta}_c(\chi|g|k)\,B_c(k)\Big). \tag{11.79}$$

Using Definition 10.3 and the conditions (11.65)–(11.78), we can check the equality $\widehat{\tau}_1+\widehat{\tau}_2+\widehat{\tau}_3 = T_1+T_2+T_3$ and hence conclude that

$$[\widehat{\tau}_1+\widehat{\tau}_2+\widehat{\tau}_3]_{(i,\,\omega_1),\,(j,\,\omega_2),\,(k,\,\omega_3)} = 0 \text{ whenever } (i,j,k)\notin W\times W\times W. \tag{11.80}$$

The goal of Step 4 is now accomplished as the conditions (11.55) and (11.69) imply

$$\widehat{\alpha}_2(\chi|\hat{\imath},1|k) = 0 \text{ for all } k\in\mathcal{Z}\setminus W' \text{ and } \hat{\imath}\in\mathcal{Z}, \tag{11.81}$$

and, similarly, the conditions (11.56) and (11.70) confirm that

$$\widehat{\beta}_2(\chi|\hat{\imath},1|k) = 0 \text{ for all } k\in\mathcal{Z}\setminus W' \text{ and } \hat{\imath}\in\mathcal{Z}. \tag{11.82}$$

Finally, we rewrite the conditions (11.63) and (11.64) in the new notation to get

$$\widehat{\alpha}_0(\chi|\hat{\jmath},2|k) = 0 \text{ for all } \hat{\jmath}\in\mathcal{Z}\setminus W \text{ and } k\in\mathcal{Z}\setminus\{\hat{\jmath},\,\hat{\jmath}+q\}, \tag{11.83}$$

$$\widehat{\beta}_0(\chi|\hat{\jmath},2|k) = 0 \text{ for all } \hat{\jmath}\in\mathcal{Z}\setminus W \text{ and } k\in\mathcal{Z}\setminus\{\hat{\jmath},\,\hat{\jmath}+1\}. \tag{11.84}$$

Finally, the situation $j\in W$ will require the results of Step 2, so we recall that

$$\widehat{\alpha}_0(\chi|j,2|k) = \widehat{\beta}_0(\chi|j,2|k) = 0 \text{ whenever } j\in W \text{ and } k\notin S(\chi,j), \tag{11.85}$$

as stated in the formulas (11.60) and (11.61).

Step 5. We proceed with a further information on the coefficients

$$\widehat{\alpha}_0(\chi|j,2|k) \text{ and } \widehat{\beta}_0(\chi|j,2|k)$$

as in (11.79) with the indices $j \in \mathcal{Z} \setminus W$. In fact, Steps 2 and 3 have already been devoted to these coefficients, but our further arguments require a characterization deeper than the conditions (11.63) and (11.64). In order to give such a description, we take an arbitrary index $\widehat{s} \in \mathcal{Z} \setminus W'$ and write the equations corresponding to

$$\sigma((\widehat{s}+1,1),(\widehat{s},2),(\widehat{s}-q,2)) \text{ and } \sigma((\widehat{s}+q,1),(\widehat{s},2),(\widehat{s}-1,2))$$

which we use in the role of $((i,\omega_1),(j,\omega_2),(k,\omega_3))$ in the condition (11.80). Namely, in view of the conditions (11.81), (11.82), (11.83) and (11.84), these equations are

$$-\widehat{\alpha}_0\left(\chi'|\widehat{s},2|\widehat{s}\right)+\widehat{\alpha}_0\left(\chi''|\widehat{s}-q,2|\widehat{s}\right)=\widehat{\beta}_0\left(\chi'|\widehat{s},2|\widehat{s}\right)-\widehat{\beta}_0\left(\chi''|\widehat{s}-1,2|\widehat{s}\right)=0,$$

and, since the permutation σ is arbitrary, they imply

$$\widehat{\alpha}_0\left(\chi|\widehat{s},2|\widehat{s}\right)=\widehat{\alpha}_0\left(\chi|\widehat{s}-q,2|\widehat{s}\right)=a(\widehat{s}) \text{ and } \widehat{\beta}_0\left(\chi|\widehat{s},2|\widehat{s}\right)=\widehat{\beta}_0\left(\chi|\widehat{s}-1,2|\widehat{s}\right)=b(\widehat{s}),$$

for some $a(\widehat{s})$ and $b(\widehat{s})$ in $\mathbb{K}$. Further, we take

$$\mathcal{W}=(W'+\{0,\pm1\}+\{0,\pm q\})\cup(W'+\{2q\}) \tag{11.86}$$

and, for any $\xi \in \mathcal{Z}\setminus\mathcal{W}$, we use the condition (11.80) with $((\xi,1),(\xi-q,2),(\xi-q,2))$ in the role of $((i,\omega_1),(j,\omega_2),(k,\omega_3))$, which implies

$$a(\xi)=b(\xi-q), \text{ for all } \xi \in \mathcal{Z}\setminus\mathcal{W},$$

and a similar application of (11.80) with $((\xi+1,1),(\xi,2),(\xi,2))$ gives

$$a(\xi)=b(\xi+1), \text{ for all } \xi \in \mathcal{Z}\setminus\mathcal{W},$$

and, finally, the use of $((\xi,1),(\xi,2),(\xi,2))$ leads us to

$$a(\xi)=b(\xi), \text{ for all } \xi \in \mathcal{Z}\setminus\mathcal{W}. \tag{11.87}$$

Summing up, we see that

$$a(\xi)=b(\xi)=b(\xi-q)=b(\xi+1), \text{ for all } \xi \in \mathcal{Z}\setminus\mathcal{W}, \tag{11.88}$$

and, using the conditions (11.54) and (11.86), we get that

$$|\mathcal{W}| \leqslant 25\,|W|. \tag{11.89}$$

In particular, the condition (11.88) shows that

$$\text{there can be at most } 50\,|W| \text{ problematic pairs } \{\xi,\zeta\}\subset\mathcal{Z}, \tag{11.90}$$

where a pair $\{\xi,\zeta\}\subset\mathcal{Z}$ is called *problematic* whenever $\xi-\zeta\in\{\pm1,\,\pm q\}$ and, in addition, at least one of the following conditions holds:

- $\xi \in W'$ (which means that $b(\xi)$ is not defined),
- $\zeta \in W'$ (which means that $b(\zeta)$ is not defined),
- neither ξ nor ζ belongs to W' but still $b(\xi)\neq b(\zeta)$.

In order to proceed with an application of Corollary 10.24 and Observation 10.25, we define the partition $S=\{S_1,\dots,S_\omega\}$ of $\mathcal{Z}$ as follows (as usual, by saying that S is a partition, we mean that $\mathcal{Z}=S_1\cup\ldots\cup S_\omega$ and that the sets in S are disjoint):

(S1) for any $\zeta\in W'$, the singleton set $\{\zeta\}$ belongs to S,
(S2) ξ_1 and ξ_2 in $\mathcal{Z}\setminus W'$ are in the same set in S if and only if $b(\xi_1)=b(\xi_2)$.

We proceed with an application of Corollary 10.24, and, in view of the further condition (11.90), we get that, for all $S' \subset S$, either

$$\left|\bigcup S'\right| \leqslant 2\,500\,|W|^2 + 1 \ \text{ or } \ \left|\bigcup S'\right| \geqslant |\mathcal{Z}| - 2\,500\,|W|^2 - 1$$

is true. Also, the assumption (11.42) justifies the application of Observation 10.25, and hence we get the lower bound

$$|S_t| \geqslant |\mathcal{Z}| - 2\,500\,|W|^2 - 1 \tag{11.91}$$

for some $S_t \in S$. Due to the definition of S, this means that $b(\xi) = \mathcal{C}$ for all $\xi \in S_t$ with some universal constant $\mathcal{C} \in \mathbb{K}$. These considerations allow us to define

$$\mathcal{W}' = \mathcal{W} \cup (\mathcal{Z} \setminus S_t) \tag{11.92}$$

and conclude that

$$a(\xi) = b(\xi) = \mathcal{C} \ \text{ for all } \ \xi \in \mathcal{Z} \setminus \mathcal{W}'$$

after the application of the equalities (11.87) and the statement of the item (S2) above. By the definition of a and b above in the current step, this means that

$$\widehat{\alpha}_0(\chi|j,2|j) = \widehat{\alpha}_0(\chi|j,2|j+q) = \mathcal{C} \text{ for all } j \notin \mathcal{V}, \tag{11.93}$$

$$\widehat{\beta}_0(\chi|j,2|j) = \widehat{\beta}_0(\chi|j,2|j+1) = \mathcal{C} \text{ for all } j \notin \mathcal{V}, \tag{11.94}$$

where $\mathcal{V}$ is the set $\mathcal{W}' + \{0,-1,-q\}$. Finally, we obtain the bound

$$|\mathcal{V}| \leqslant 3\,|\mathcal{W}'| \leqslant 3 \cdot \big(2\,525\,|W|^2 + 1\big) \tag{11.95}$$

after a straightforward comparison of the conditions (11.89), (11.91) and (11.92).

Step 6. Now we are going to put the arguments of the previous steps together in order to be able to proceed with the proof on the next step. Namely, we define the

$$(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$$

tensor $\widetilde{\tau}_\chi$ so that, for any $\chi \in \{1,2,3\}$, its g-th χ-slice is

$$\sum_{c=0}^{2} \sum_{k \in \mathcal{Z}} \Big(\widetilde{\alpha}_c(\chi|g|k)\, A_c(k) + \widetilde{\beta}_c(\chi|g|k)\, B_c(k)\Big) \tag{11.96}$$

with

$$\widetilde{\alpha}_c(\chi|j,\omega|k) = \begin{cases} \widehat{\alpha}_c(\chi|j,\omega|k) - \mathcal{C} & \text{if } c=0,\ \omega=2,\ k-j \in \{0,q\}, \\ \widehat{\alpha}_c(\chi|j,\omega|k) & \text{otherwise,} \end{cases}$$

$$\widetilde{\beta}_c(\chi|j,\omega|k) = \begin{cases} \widehat{\beta}_c(\chi|j,\omega|k) - \mathcal{C} & \text{if } c=0,\ \omega=2,\ k-j \in \{0,1\}, \\ \widehat{\beta}_c(\chi|j,\omega|k) & \text{otherwise.} \end{cases}$$

In other words, we consider the tensor

$$\widetilde{\tau}_1 + \widetilde{\tau}_2 + \widetilde{\tau}_3 = T_1 + T_2 + T_3 - \mathcal{C}\,(\mathcal{T}_a + \mathcal{T}_b) \tag{11.97}$$

in which the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensors $\mathcal{T}_a$ and $\mathcal{T}_b$ are defined as

$$\mathcal{T}_a = \sum_{\pi \in \mathcal{S}_3} \sum_{s \in \mathcal{Z}} \pi\left((O \oplus u''(s)) \otimes (u'(s) \oplus O) \otimes (O \oplus (\varepsilon_s + \varepsilon_{s-q}))\right),$$

$$\mathcal{T}_b = \sum_{\pi \in \mathcal{S}_3} \sum_{s \in \mathcal{Z}} \pi\left((O \oplus v''(s)) \otimes (v'(s) \oplus O) \otimes (O \oplus (\varepsilon_s + \varepsilon_{s-1}))\right).$$

Now, according to the conditions (11.83) and (11.84), we have

$$\widetilde{\alpha}_0(\chi|j,2|k) = \widetilde{\beta}_0(\chi|j,2|k) = 0 \text{ if } j \in \mathcal{Z} \setminus W \text{ and } k \notin \{j, j+1, j+q\}, \tag{11.98}$$

and, due to the definition of $\widetilde{\alpha}_c(\chi|j,\omega|k)$ and $\widetilde{\beta}_c(\chi|j,\omega|k)$ above, the corresponding conditions (11.93) and (11.94) lead to a stronger statement in the case $j \in \mathcal{V}$ as

$$\widetilde{\alpha}_0(\chi|j,2|k) = \widetilde{\beta}_0(\chi|j,2|k) = 0 \text{ holds whenever } j \notin \mathcal{V}. \tag{11.99}$$

Concerning the case $j \in W$, we apply the definition of $\widetilde{\alpha}_c(\chi|j,\omega|k)$ and $\widetilde{\beta}_c(\chi|j,\omega|k)$ together with the condition (11.85), and we get

$$\widetilde{\alpha}_0(\chi|j,2|k) = \widetilde{\beta}_0(\chi|j,2|k) = 0 \text{ if } j \in W \text{ and } k \notin S(\chi,j) \cup \{j, j+1, j+q\},$$

which can be rewritten as

$$\widetilde{\alpha}_0(\chi|j,2|k) = \widetilde{\beta}_0(\chi|j,2|k) = 0 \text{ if } j \in W \text{ and } k \notin S(\chi,j) \cup W' \tag{11.100}$$

due to (11.54). Taking the conditions (11.98), (11.99) and (11.100), we get that

$$\widetilde{\alpha}_0(\chi|j,2|k) = \widetilde{\beta}_0(\chi|j,2|k) = 0 \text{ if } k \in \mathcal{Z} \setminus \Omega \tag{11.101}$$

holds for all $j \in \mathcal{Z}$, where Ω is defined to be the set $(\mathcal{V} + \{0,1,q\}) \cup W' \cup W''$ with

$$W'' = \bigcup_{j \in W} \left(S(1,j) \cup S(2,j) \cup S(3,j) \right),$$

and the conditions (11.54), (11.59) and (11.95) lead to the estimate

$$|\Omega| \leqslant 9 \cdot \big(2\,525\,|W|^2 + 1\big) + 3 \cdot 512 \cdot |W| \cdot (\rho + 9)^2 + 5\,|W|. \tag{11.102}$$

Finally, the conditions (11.81) and (11.82) show that

$$\widetilde{\alpha}_2(\chi|j,1|k) = \widetilde{\beta}_2(\chi|j,1|k) = 0 \text{ if } k \in \mathcal{Z} \setminus \Omega \tag{11.103}$$

again for all $j \in \mathcal{Z}$.

Step 7. We proceed with the core of the argument and define

$$\ell_\chi \text{ as the } (\text{♘}, 2) \times (\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{2\}) \text{ block of } \widetilde{\tau}_\chi \tag{11.104}$$

for all $\chi \in \{1,2,3\}$, and, respectively, the conditions (11.101) and (11.103) imply

$$\operatorname{rk} \ell_2 \leqslant 2 \cdot |\Omega| \text{ and } \operatorname{rk} \ell_3 \leqslant 2 \cdot |\Omega|.$$

A further application of the rank bound (11.44) and equality (11.97) gives

$$\operatorname{rk}(\mathcal{C}\,\gamma + \ell_1) \leqslant 4 \cdot |\Omega| + \rho, \tag{11.105}$$

where γ is the $(\text{♘}, 2) \times (\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{2\})$ block of $(\mathcal{T}_a + \mathcal{T}_b)$. If we write $\mathcal{L}$ to denote the linear space of the $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{2\})$ restrictions of the matrices in

$$\sum_{s \in \mathcal{Z}} (u'(s) \otimes u''(s))\,\mathbb{K} + \sum_{s \in \mathcal{Z}} (v'(s) \otimes v''(s))\,\mathbb{K},$$

then we clearly have $\ell_1 \in \mathcal{L}$ because the 1-slices of $\widetilde{\tau}_1$ are linear combinations of the matrices in Definition 10.3. Further, a direct computation shows that γ is the matrix as in Theorem 10.12, and, also, the initial assumption (11.41) on the characteristic of $\mathbb{K}$ matches the requirement of this theorem. Therefore, a comparison of the conditions (10.11), (11.42), (11.102) and (11.105) shows that $\mathcal{C} = 0$, and hence

$$T_1 + T_2 + T_3 = \widetilde{\tau}_1 + \widetilde{\tau}_2 + \widetilde{\tau}_3$$

follows due to the equality (11.97). Also, the formula (11.105) now reads

$$\operatorname{rk}(\ell_1) \leqslant 4\,|\Omega| + \rho, \tag{11.106}$$

which is also at most $q/28$ due to (11.42). Therefore, we can apply Theorem 10.26 and conclude that there are $\pi(1) \in \mathbb{F}$ and $\Delta(1) \subseteq \mathcal{Z}$ with

$$|\Delta(1)| \leqslant 512 \cdot (4\,|\Omega| + \rho)^2 \tag{11.107}$$

such that the equalities

$$\widetilde{\alpha}_0(1|\text{♘}, 2|k) = \widetilde{\beta}_0(1|\text{♘}, 2|k) = \pi(1) \text{ are true whenever } k \in \mathcal{Z} \setminus \Delta(1). \tag{11.108}$$

Further, in a way similar to (11.104), we can define

$$\widetilde{\ell}_\chi \text{ as the } (\text{♘}, 1) \times (\mathcal{Z} \times \{2\}) \times (\mathcal{Z} \times \{2\}) \text{ block of } \widetilde{\tau}_\chi$$

for all $\chi \in \{1, 2, 3\}$, and, in view of the conditions (11.101), we get

$$\operatorname{rk} \widetilde{\ell}_2 \leqslant 2\,|\Omega|, \ \operatorname{rk} \widetilde{\ell}_3 \leqslant 2\,|\Omega|,$$

and hence the rank bound (11.44) implies

$$\operatorname{rk} \widetilde{\ell}_1 \leqslant 4\,|\Omega| + \rho.$$

We recall that, by Definition 10.3, the union of the patterns of the matrices $A_2(k)$ and $B_2(k)$ over all $k \in \mathcal{Z}$ has at most five entries in each row and each column, which implies, in view of the latter inequality and Observation 10.10, the fact that

$$\widetilde{\ell}_1 \text{ has at most } 25 \cdot (4\,|\Omega| + \rho) \text{ nonzero entries.} \tag{11.109}$$

A further application of Observation 10.14 to the condition (11.109) implies that, for some subset $\Delta'(1) \subseteq \mathcal{Z}$ with

$$|\Delta'(1)| \leqslant 25 \cdot (4\,|\Omega| + \rho)\,, \tag{11.110}$$

the equalities

$$\widetilde{\alpha}_2(1|\text{♘}, 1|k) = \widetilde{\beta}_2(1|\text{♘}, 1|k) = 0 \text{ are true whenever } k \in \mathcal{Z} \setminus \Delta'(1). \tag{11.111}$$

By the symmetry, the conditions (11.107), (11.108), (11.110) and (11.111) can be lifted from $\chi = 1$ to any $\chi \in \{1, 2, 3\}$, so we can find $\pi(\chi) \in \mathbb{F}$ and $\Delta(\chi) \subseteq \mathcal{Z}$ with

$$|\Delta(\chi)| \leqslant 512 \cdot (4\,|\Omega| + \rho)^2$$

such that

$$\widetilde{\alpha}_0(\chi|\text{♘}, 2|k) = \widetilde{\beta}_0(\chi|\text{♘}, 2|k) = \pi(\chi) \text{ holds for all } k \in \mathcal{Z} \setminus \Delta(\chi), \tag{11.112}$$

and, also, for some $\Delta'(\chi) \subseteq \mathcal{Z}$ with

$$|\Delta'(\chi)| \leqslant 25 \cdot (4\,|\Omega| + \rho)\,,$$

we have

$$\widetilde{\alpha}_2(\chi|\text{♘}, 1|k) = \widetilde{\beta}_2(\chi|\text{♘}, 1|k) = 0 \text{ for all } k \in \mathcal{Z} \setminus \Delta'(\chi). \tag{11.113}$$

Finally, we remark that, for any fixed $\chi \in \{1, 2, 3\}$, the transformation

$$\widetilde{\alpha}_0(\chi|\text{♘}, 2|k) \to \widetilde{\alpha}_0(\chi|\text{♘}, 2|k) - \pi(\chi), \ \widetilde{\beta}_0(\chi|\text{♘}, 2|k) = \widetilde{\beta}_0(\chi|\text{♘}, 2|k) - \pi(\chi)$$

corresponds to the subtraction of $\pi(\chi)\,\mathcal{M}$ from the χ-slice of $\widetilde{\tau}_\chi$ with the index $(\text{♘}, 2)$, where $\mathcal{M}$ is the matrix defined in Notation 11.13, and we note that the subset (11.45) does not change upon the addition of a scalar multiple of $\mathcal{M}$ to any slice. Therefore, we can define the desired set Φ'' as the union of all the families

$$\{A_1(k), A_2(k), A_3(k), B_1(k), B_2(k), B_3(k)\}$$

with $k \in \Omega \cup \Delta(1) \cup \Delta(2) \cup \Delta(3) \cup \Delta'(1) \cup \Delta'(2) \cup \Delta'(3)$ and complete the proof. $\square$

At this point, we need to clean up the diagonal blocks

$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\})\,,$$
$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\})$$

of the tensor $T_1 + T_2 + T_3$ in a way similar to how Lemma 11.16 allows us to get rid of the nonzero entries in the corresponding off-diagonal blocks. This requires two observations that can be proved by a straightforward use of Definition 10.3.

Remark 11.17. A permutation $\pi \in S_3$ determines the *braiding isomorphism* of the linear space of the $I \times I \times I$ tensors over a field, which is defined as the linear mapping satisfying $\pi(u_1 \otimes u_2 \otimes u_3) = u_{\pi(1)} \otimes u_{\pi(2)} \otimes u_{\pi(3)}$ for all (u_1, u_2, u_3).

Remark 11.18. In Observations 11.19 and 11.20 below, we follow the convention of Definition 10.3 and write e_s to denote the vector labeled with $(\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}$ so that it has a one at the coordinate $(s, 1)$ and zeros at all other places.

Observation 11.19. *Let $\mathbb{F}$ be a field with* char $\mathbb{F} \neq 2, 3$*, let $s \in \mathcal{Z}$ and*

$$\omega_s = \begin{cases} 1, & \textit{if } s \in \{-q+1, -2q+1, \ldots, -(q-1)q+1\}, \\ 0, & \textit{if } s \notin \{-q+1, -2q+1, \ldots, -(q-1)q+1\}, \end{cases}$$

$$\psi_{1s} = u'(s-1) \otimes u'(s-1) \otimes (e_{s-1} + 2e_s - 3e_{s+q} + \omega_s e_{\text{♘}})\,,$$
$$\psi_{2s} = u'(s+q-1) \otimes u'(s+q-1) \otimes (2e_{s+q-1} + e_{s+q} - 3e_{s-1} - \omega_s e_{\text{♘}})\,,$$
$$\psi_{3s} = v'(s-1) \otimes v'(s-1) \otimes (3e_{q+s} - e_{s-1} - 2e_{s+q-1} - 3\omega_s e_{\text{♘}})\,,$$
$$\psi_{4s} = v'(s) \otimes v'(s) \otimes (3e_{s-1} - 2e_s - e_{s+q} + 3\omega_s e_{\text{♘}})\,.$$

Then, for every $s \notin \{1, 2\}$, the tensor $\sum_{\pi \in S_3} \pi(\psi_{1s} + \psi_{2s} + \psi_{3s} + \psi_{4s})$ is zero.

Proof. Assume that the variables (x, y, z, u, w) represent the coordinate vectors

$$(e_{s-1},\, e_s,\, e_{s+q-1},\, e_{s+q},\, e_{\text{♘}}),$$

respectively. By Definition 10.3, the vector $u'(s-1)$ corresponds to $y - x - \omega_s w$, the vector $u'(s+q-1)$ is represented as $u - z - \omega_s w$, the vector $v'(s-1)$ comes from $x - z$, and the vector $v'(s)$ corresponds to $y - u$. Therefore, the equality

$$(y - x - \omega_s w)^2\,(x + 2y - 3u + \omega_s w) + (u - z - \omega_s w)^2\,(2z + u - 3x - \omega_s w) +$$
$$+ (x - z)^2\,(3u - x - 2z - 3\omega_s w) + (y - u)^2\,(3x - 2y - u + 3\omega_s w) = 0$$

states that the polynomials corresponding to $(\psi_{1s}, \psi_{2s}, \psi_{3s}, \psi_{4s})$ sum to zero. □

Observation 11.20. *Let $\mathbb{F}$ be a field with* char $\mathbb{F} \neq 2, 3$*, let $s \in \mathcal{Z}$ and*

$$\widetilde{\omega}_s = \begin{cases} 1, & \textit{if } s \in \{0, -q, -2q, \ldots, -(q-2)q\}, \\ 0, & \textit{if } s \notin \{0, -q, -2q, \ldots, -(q-2)q\}, \end{cases}$$

$$\varphi_{1s} = u'(s) \otimes u'(s) \otimes (3e_{s+1-q} - 2e_{s+1} - e_s - \widetilde{\omega}_s e_{\text{♘}})\,,$$
$$\varphi_{2s} = u'(s-q) \otimes u'(s-q) \otimes (3e_s - 2e_{s-q} - e_{s-q+1} + \widetilde{\omega}_s e_{\text{♘}})\,,$$
$$\varphi_{3s} = v'(s-q+1) \otimes v'(s-q+1) \otimes (-3e_s + e_{s-q+1} + 2e_{s+1} - 3\widetilde{\omega}_s e_{\text{♘}})\,,$$
$$\varphi_{4s} = v'(s-q) \otimes v'(s-q) \otimes (-3e_{s-q+1} + e_s + 2e_{s-q} + 3\widetilde{\omega}_s e_{\text{♘}})\,.$$

Then, for every $s \notin \{q, q+1\}$, the tensor $\sum_{\pi \in S_3} \pi(\varphi_{1s} + \varphi_{2s} + \varphi_{3s} + \varphi_{4s})$ is zero.

Proof. Assume that the variables (x, y, z, u, w) represent the coordinate vectors

$$(e_{s-q},\, e_{s-q+1},\, e_s,\, e_{s+1},\, e_{♘}),$$

respectively. By Definition 10.3, the vector $u'(s)$ corresponds to $u - z - \widetilde{\omega}_s w$, the vector $u'(s-q)$ is represented as $y - x - \widetilde{\omega}_s w$, the vector $v'(s-q+1)$ comes from $y - u$, and the vector $v'(s-q)$ corresponds to $x - z$. Therefore, the equality

$$(u - z - \widetilde{\omega}_s w)^2\,(3y - 2u - z - \widetilde{\omega}_s w) + (y - x - \widetilde{\omega}_s w)^2\,(3z - 2x - y + \widetilde{\omega}_s w) +$$
$$+ (y-u)^2\,(-3z + y + 2u - 3\widetilde{\omega}_s w) + (x-z)^2\,(-3y + z + 2x + 3\widetilde{\omega}_s w) = 0$$

states that the polynomials corresponding to $(\varphi_{1s}, \varphi_{2s}, \varphi_{3s}, \varphi_{4s})$ sum to zero. $\square$

We proceed with the diagonal block analogue of Lemma 11.16.

Lemma 11.21. *Let $\rho \geqslant 1$ be an integer and $W \subseteq \mathcal{Z}$, and assume that $\mathbb{K}$ is a field with* char $\mathbb{K} \neq 2, 3$. *Further, let three $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensors*

$$T_1 \in O\operatorname{mod}_{\mathbb{K}}(\Phi(q), \varnothing, \varnothing),\ T_2 \in O\operatorname{mod}_{\mathbb{K}}(\varnothing, \Phi(q), \varnothing),\ T_3 \in O\operatorname{mod}_{\mathbb{K}}(\varnothing, \varnothing, \Phi(q))$$

be such that, for all $\nu_1, \nu_2, \nu_3 \in \{1, 2\}$ and $i, j, k \in \mathcal{Z}$, the conditions

$$[T_1 + T_2 + T_3]_{(i,\,\nu_1),\,(j,\,\nu_2),\,(k,\,\nu_3)} = 0 \ \text{ whenever } \ (i, j, k) \notin W \times W \times W \tag{11.114}$$

are in effect, and, in addition, for some $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor Δ which has all its nonzero entries collected in the union of the blocks $(\mathcal{I} \cup \mathcal{J}) \times \mathcal{I} \times \mathcal{I}$, $\mathcal{I} \times (\mathcal{I} \cup \mathcal{J}) \times \mathcal{I}$ and $\mathcal{I} \times \mathcal{I} \times (\mathcal{I} \cup \mathcal{J})$, one has

$$\operatorname{rk}_{\mathbb{K}}(T_1 + T_2 + T_3 + \Delta) \leqslant \rho. \tag{11.115}$$

Then there exists a subset

$$\Phi' \subseteq \Phi(q) \ \text{ with } \ |\Phi'| \leqslant 151 \cdot (\rho + 1\,600\,|W| + 400)$$

such that the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor obtained by taking the

$$((\mathcal{Z} \cup \{♘\}) \times \{1\}) \times ((\mathcal{Z} \cup \{♘\}) \times \{1\}) \times ((\mathcal{Z} \cup \{♘\}) \times \{1\}) \tag{11.116}$$

block of $T_1 + T_2 + T_3$ and replacing all the remaining entries with zeros belongs to

$$O\operatorname{mod}_{\mathbb{K}}(\Phi'\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \Phi'\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \Phi'\,\mathbb{K} + \mathcal{M}\,\mathbb{K}). \tag{11.117}$$

Proof. We begin the argument in a way similar to the one in Lemma 11.16. Indeed, for any $\chi \in \{1, 2, 3\}$ and $g \in \mathcal{I} \cup \mathcal{J}$, the g-th χ-slice of T_χ is

$$\sum_{c=0}^{2} \sum_{k \in \mathcal{Z}} \left(\alpha_c(\chi|g|k)\, A_c(k) + \beta_c(\chi|g|k)\, B_c(k)\right) \tag{11.118}$$

with $\alpha_c(\chi|g|k)$ and $\beta_c(\chi|g|k)$ in $\mathbb{K}$. In fact, we are mostly focused on these values for $c = 1$ and $g \in (\mathcal{Z} \cup \{♘\}) \times \{1\}$ because all the other corresponding coefficients in (11.118) do not affect the block (11.116). In order to simplify the notation,

(11.119) we fix an arbitrary permutation (χ, χ', χ'') of the string $(1, 2, 3)$,

and we use the notation (11.119) in the rest of the current proof.

Further, we look back to Observation 10.14 and recall that, for any i and k in $\mathcal{Z}$, the $((k, 1), (k+1, 1))$ entry of the $(i, 1)$-th χ-slice of T_χ is nonzero if $\alpha_1(\chi|i, 1|k) \neq 0$, and, similarly, the $((k, 1), (k+q, 1))$ entry of the $(i, 1)$-th χ-slice of T_χ is nonzero if $\beta_1(\chi|i, 1|k) \neq 0$. A further application of Observation 11.9 shows that

(AA) the $((k, 1), (k+1, 1))$ entry of the $(i, 1)$-th χ-slice of $T_1 + T_2 + T_3$ is nonzero if $\alpha_1(\chi|i, 1|k) \neq 0$ and $i \notin \{k-q, k-q+1, k-1, k, k+1, k+2, k+q, k+q+1\}$, (BB) the $((k,1),(k+q,1))$ entry of the $(i,1)$-th χ-slice of $T_1+T_2+T_3$ is nonzero if $\beta_1(\chi|i,1|k) \neq 0$ and $i \notin \{k-q, k-1, k, k+1, k+q-1, k+q, k+q+1, k+2q\}$.

In order to simplify the notation in (AA) and (BB), we write

$$\delta = \{-q, -q+1, -1, 0, 1, 2, q, q+1\} \text{ and } \varepsilon = \{-q, -1, 0, 1, q-1, q, q+1, 2q\}.$$

In particular, in view of the assumption (11.114), the condition (AA) implies

$$\alpha_1(\chi|i,1|k) = 0 \text{ if } k \in \mathcal{Z} \setminus W \text{ and } i \notin \{k\} + \delta, \tag{11.120}$$

and the corresponding condition (BB) gives

$$\beta_1(\chi|i,1|k) = 0 \text{ if } k \in \mathcal{Z} \setminus W \text{ and } i \notin \{k\} + \varepsilon. \tag{11.121}$$

Further, in a way similar to Lemma 11.16, we write σ to denote the permutation of the coordinates in a string of the length three corresponding to

$$1 \to \chi,\ 2 \to \chi',\ 3 \to \chi''$$

so that the string $\sigma(\varkappa_1, \varkappa_2, \varkappa_3)$ has $\varkappa_1$ at the χ-th position, it has $\varkappa_2$ at the χ'-th position, and $\varkappa_3$ is at the remaining χ''-th position. Also, we define

$$W' = W + \{-1, 0, 1, 2\} + \{-q, 0, q, 2q\} \tag{11.122}$$

and proceed the argument with four separate steps, 1A, 2A, 1B and 2B.

Step 1A. Assume that, for some $h \in \mathcal{Z} \setminus W$, we have

$$\alpha_1(\chi|h+2,1|h) \neq 0. \tag{11.123}$$

In this case, the assumption (11.114) implies that the $\sigma((h+2,1),(h,1),(h,1))$ entry of $T_1+T_2+T_3$ is zero, and Observation 11.9 shows that the corresponding entries in $T_{\chi'}$ and $T_{\chi''}$ are also zero. Therefore, the $\sigma((h+2,1),(h,1),(h,1))$ entry of T_χ is zero as well, and hence, by the conditions (11.118) and (11.123), the values

$$\alpha_1(\chi|h+2,1|h-1),\ \beta_1(\chi|h+2,1|h),\ \beta_1(\chi|h+2,1|h-q)$$

are not simultaneously zero. In particular, in view of (11.120) and (11.121), at least one of the conditions $h-1 \in W$, $h \in W$, $h-q \in W$ is true, and hence $h \in W'$.

Step 2A. Assume that, for some $h \in \mathcal{Z}$ with $h-1 \notin W$, we have

$$\alpha_1(\chi|h-1,1|h) \neq 0. \tag{11.124}$$

In this case, the assumption (11.114) implies that the $\sigma((h-1,1),(h+1,1),(h+1,1))$ entry of $T_1+T_2+T_3$ is zero, and Observation 11.9 shows that the corresponding entries in $T_{\chi'}$ and $T_{\chi''}$ are also zero. Therefore, the $\sigma((h-1,1),(h+1,1),(h+1,1))$ entry of T_χ is zero, too, and, by the conditions (11.118) and (11.124), the values

$$\alpha_1(\chi|h-1,1|h+1),\ \beta_1(\chi|h-1,1|h+1),\ \beta_1(\chi|h-1,1|h-q+1)$$

are not simultaneously zero. In view of the conditions (11.120) and (11.121), this requires either $h+1 \in W$ or $h-q+1 \in W$, and hence we get $h \in W'$.

Step 1B. Assume that, for some $h \in \mathcal{Z} \setminus W$, we have

$$\beta_1(\chi|h+2q,1|h) \neq 0. \tag{11.125}$$

In this case, the formula (11.114) shows that the $\sigma((h+2q,1),(h,1),(h,1))$ entry of $T_1+T_2+T_3$ is zero, and Observation 11.9 shows that the corresponding entries in $T_{\chi'}$ and $T_{\chi''}$ are also zero. Therefore, the $\sigma((h+2q,1),(h,1),(h,1))$ entry of T_χ is zero as well, and hence, by the conditions (11.118) and (11.125), the values

$$\alpha_1(\chi|h+2q,1|h-1),\ \alpha_1(\chi|h+2q,1|h),\ \beta_1(\chi|h+2q,1|h-q)$$

are not simultaneously zero. In particular, in view of (11.120) and (11.121), at least one of the conditions $h-1 \in W$, $h \in W$, $h-q \in W$ is true, and hence $h \in W'$.

Step 2B. Assume that, for some $h \in \mathcal{Z}$ with $h-q \notin W$, we have

$$\beta_1(\chi|h-q,1|h) \neq 0. \tag{11.126}$$

In this case, the assumption (11.114) implies that the $\sigma((h-q,1),(h+q,1),(h+q,1))$ entry of $T_1+T_2+T_3$ is zero, and Observation 11.9 shows that the corresponding entries in $T_{\chi'}$ and $T_{\chi''}$ are also zero. Therefore, the $\sigma((h-q,1),(h+q,1),(h+q,1))$ entry of T_χ is zero, too, and, by the conditions (11.118) and (11.126), the values

$$\alpha_1(\chi|h-q,1|h+q-1),\ \alpha_1(\chi|h-q,1|h+q),\ \beta_1(\chi|h-q,1|h+q)$$

are not simultaneously zero. In view of the conditions (11.120) and (11.121), this requires either $h+q-1 \in W$ or $h+q \in W$, and hence we get $h \in W'$.

The consideration of Steps 1A, 2A, 1B, 2B is now complete. After an application of the conditions (11.120) and (11.121) to the conclusions of these steps, we get

$$\alpha_1(\chi|i,1|k) = 0 \text{ if } k \in \mathcal{Z} \setminus W' \text{ and } i \notin \{k\} + \widehat{\delta} \tag{11.127}$$

and

$$\beta_1(\chi|i,1|k) = 0 \text{ if } k \in \mathcal{Z} \setminus W' \text{ and } i \notin \{k\} + \widehat{\varepsilon} \tag{11.128}$$

with $\widehat{\delta} = \{-q,-q+1,0,1,q,q+1\}$ and $\widehat{\varepsilon} = \{-1,0,1,q-1,q,q+1\}$. Now we take an arbitrary $s \in \mathcal{Z} \setminus W'$ and proceed with two further separate steps, 3 and 4.

Step 3. We apply the condition (11.114) to the entries

$$\sigma((s-1,1),(s,1),(s+q,1)) \text{ and } \sigma((s-1,1),(s+q,1),(s,1)),$$

which gives

$$\beta_1(\chi|s-1,1|s) = -\alpha_1(\chi'|s+q,1|s-1) = -\alpha_1(\chi''|s+q,1|s-1)$$

and hence

$$\alpha_1(1|s+q,1|s-1) = \alpha_1(2|s+q,1|s-1) = \alpha_1(3|s+q,1|s-1) = g(s), \tag{11.129}$$

$$\beta_1(1|s-1,1|s) = \beta_1(2|s-1,1|s) = \beta_1(3|s-1,1|s) = -g(s), \tag{11.130}$$

for some $g(s) \in \mathbb{K}$. A further application of (11.114) to the entry

$$\sigma((s-1,1),(s+q,1),(s+q,1))$$

gives $\beta_1(\chi|s-1,1|s+q)+\alpha_1(\chi|s-1,1|s+q)+\alpha_1(\chi|s-1,1|s+q-1) = g(s)$, in which the first two summands of the left hand side are zero by (11.120) and (11.121), so

$$\alpha_1(\chi|s-1,1|s+q-1) = g(s) \tag{11.131}$$

is true for all $\chi \in \{1,2,3\}$. A similar application of (11.114) to the entry

$$\sigma((s+q,1),(s-1,1),(s-1,1))$$

gives $\alpha_1(\chi|s+q,1|s-2) + \beta_1(\chi|s+q,1|s-1-q) + \beta_1(\chi|s+q,1|s-1) = -g(s)$, and again the first two summands of the left hand side are zero, so we get

$$\beta_1(\chi|s+q,1|s-1) = -g(s). \tag{11.132}$$

A comparison of the conditions (11.129), (11.130), (11.131) and (11.132) gives

$$\alpha_1(\chi|s+q,1|s-1) = \alpha_1(\chi|s-1,1|s+q-1) = g(s), \tag{11.133}$$

$$\beta_1(\chi|s-1,1|s) = \beta_1(\chi|s+q,1|s-1) = -g(s), \tag{11.134}$$

for all $s \in \mathcal{Z} \setminus W'$ and $\chi \in \{1, 2, 3\}$.

Step 4. Now we apply the condition (11.114) with

$$\sigma((s, 1), (s + 1 - q, 1), (s + 1, 1)) \text{ and } \sigma((s, 1), (s + 1, 1), (s + 1 - q, 1)),$$

which gives

$$\beta_1(\chi|s, 1|s + 1 - q) = -\alpha_1(\chi'|s + 1 - q, 1|s) = -\alpha_1(\chi''|s + 1 - q, 1|s)$$

and hence

$$\alpha_1(1|s + 1 - q, 1|s) = \alpha_1(2|s + 1 - q, 1|s) = \alpha_1(3|s + 1 - q, 1|s) = \gamma(s), \tag{11.135}$$

$$\beta_1(1|s, 1|s + 1 - q) = \beta_1(2|s, 1|s + 1 - q) = \beta_1(3|s, 1|s + 1 - q) = -\gamma(s), \tag{11.136}$$

for some $\gamma(s) \in \mathbb{K}$. A further application of (11.114) to the entry

$$\sigma((s, 1), (s + 1 - q, 1), (s + 1 - q, 1))$$

gives $\beta_1(\chi|s, 1|s + 1 - 2q) + \alpha_1(\chi|s, 1|s + 1 - q) + \alpha_1(\chi|s, 1|s - q) = \gamma(s)$, and again the first two summands of the left hand side are zero, so we get

$$\alpha_1(\chi|s, 1|s - q) = \gamma(s) \tag{11.137}$$

for all $\chi \in \{1, 2, 3\}$. Another application of (11.114) to the entry

$$\sigma((s + 1 - q, 1), (s, 1), (s, 1))$$

gives $\alpha_1(\chi|s + 1 - q, 1|s - 1) + \beta_1(\chi|s + 1 - q, 1|s) + \beta_1(\chi|s + 1 - q, 1|s - q) = -\gamma(s)$, and, since the first two summands of the left hand side are zeros, we get

$$\beta_1(\chi|s + 1 - q, 1|s - q) = -\gamma(s). \tag{11.138}$$

Finally, the conditions (11.135), (11.136), (11.137) and (11.138) imply

$$\alpha_1(\chi|s + 1 - q, 1|s) = \alpha_1(\chi|s, 1|s - q) = \gamma(s), \tag{11.139}$$

$$\beta_1(\chi|s, 1|s + 1 - q) = \beta_1(\chi|s + 1 - q, 1|s - q) = -\gamma(s), \tag{11.140}$$

for all $s \in \mathcal{Z} \setminus W'$ and $\chi \in \{1, 2, 3\}$.

Step 5. Now we want to reduce the situation to the case $g(s) = 0$ and $\gamma(s) = 0$, where g and γ are the functions identified in Steps 3 and 4. The considerations of the current step are based on Observations 11.19 and 11.20, and, for any fixed

$$\chi \in \{1, 2, 3\} \text{ and } \hat\imath \in \mathcal{Z} \setminus (W' \cup \{1, 2, q, q + 1\}), \tag{11.141}$$

we define the transformations suggested by Observation 11.19:

$$\alpha_1(\chi|\hat\imath + q, 1|\hat\imath - 1) \longrightarrow \alpha_1(\chi|\hat\imath + q, 1|\hat\imath - 1) - g(\hat\imath), \tag{11.142}$$

$$\alpha_1(\chi|\hat\imath - 1, 1|\hat\imath - 1) \longrightarrow \alpha_1(\chi|\hat\imath - 1, 1|\hat\imath - 1) + g(\hat\imath)/3, \tag{11.143}$$

$$\alpha_1(\chi|\hat\imath, 1|\hat\imath - 1) \longrightarrow \alpha_1(\chi|\hat\imath, 1|\hat\imath - 1) + 2g(\hat\imath)/3, \tag{11.144}$$

$$\alpha_1(\chi|\text{♘}, 1|\hat\imath - 1) \longrightarrow \alpha_1(\chi|\text{♘}, 1|\hat\imath - 1) + \omega_{\hat\imath}\, g(\hat\imath)/3, \tag{11.145}$$

$$\alpha_1(\chi|\hat\imath - 1, 1|\hat\imath + q - 1) \longrightarrow \alpha_1(\chi|\hat\imath - 1, 1|\hat\imath + q - 1) - g(\hat\imath), \tag{11.146}$$

$$\alpha_1(\chi|\hat\imath + q - 1, 1|\hat\imath + q - 1) \longrightarrow \alpha_1(\chi|\hat\imath + q - 1, 1|\hat\imath + q - 1) + 2g(\hat\imath)/3, \tag{11.147}$$

$$\alpha_1(\chi|\hat\imath + q, 1|\hat\imath + q - 1) \longrightarrow \alpha_1(\chi|\hat\imath + q, 1|\hat\imath + q - 1) + g(\hat\imath)/3, \tag{11.148}$$

$$\alpha_1(\chi|\text{♘}, 1|\hat\imath + q - 1) \longrightarrow \alpha_1(\chi|\text{♘}, 1|\hat\imath + q - 1) - \omega_{\hat\imath}\, g(\hat\imath)/3, \tag{11.149}$$

$$\beta_1(\chi|\hat{\imath}+q,1|\hat{\imath}-1) \longrightarrow \beta_1(\chi|\hat{\imath}+q,1|\hat{\imath}-1)+g(\hat{\imath}), \tag{11.150}$$

$$\beta_1(\chi|\hat{\imath}-1,1|\hat{\imath}-1) \longrightarrow \beta_1(\chi|\hat{\imath}-1,1|\hat{\imath}-1)-g(\hat{\imath})/3, \tag{11.151}$$

$$\beta_1(\chi|\hat{\imath}+q-1,1|\hat{\imath}-1) \longrightarrow \beta_1(\chi|\hat{\imath}+q-1,1|\hat{\imath}-1)-2g(\hat{\imath})/3, \tag{11.152}$$

$$\beta_1(\chi|\text{♘},1|\hat{\imath}-1) \longrightarrow \beta_1(\chi|\text{♘},1|\hat{\imath}-1)-\omega_{\hat{\imath}}\, g(\hat{\imath}), \tag{11.153}$$

$$\beta_1(\chi|\hat{\imath}-1,1|\hat{\imath}) \longrightarrow \beta_1(\chi|\hat{\imath}-1,1|\hat{\imath})+g(\hat{\imath}), \tag{11.154}$$

$$\beta_1(\chi|\hat{\imath},1|\hat{\imath}) \longrightarrow \beta_1(\chi|\hat{\imath},1|\hat{\imath})-2g(\hat{\imath})/3, \tag{11.155}$$

$$\beta_1(\chi|\hat{\imath}+q,1|\hat{\imath}) \longrightarrow \beta_1(\chi|\hat{\imath}+q,1|\hat{\imath})-g(\hat{\imath})/3, \tag{11.156}$$

$$\beta_1(\chi|\text{♘},1|\hat{\imath}) \longrightarrow \beta_1(\chi|\text{♘},1|\hat{\imath})+\omega_{\hat{\imath}}\, g(\hat{\imath}) \tag{11.157}$$

with $\omega_{\hat{\imath}}$ as in Observation 11.19. In a similar way, we are ready to proceed with the second part of the transformation as governed by Observation 11.20:

$$\alpha_1(\chi|\hat{\imath}+1-q,1|\hat{\imath}) \longrightarrow \alpha_1(\chi|\hat{\imath}+1-q,1|\hat{\imath})-\gamma(\hat{\imath}), \tag{11.158}$$

$$\alpha_1(\chi|\hat{\imath}+1,1|\hat{\imath}) \longrightarrow \alpha_1(\chi|\hat{\imath}+1,1|\hat{\imath})+2\gamma(\hat{\imath})/3, \tag{11.159}$$

$$\alpha_1(\chi|\hat{\imath},1|\hat{\imath}) \longrightarrow \alpha_1(\chi|\hat{\imath},1|\hat{\imath})+\gamma(\hat{\imath})/3, \tag{11.160}$$

$$\alpha_1(\chi|\text{♘},1|\hat{\imath}) \longrightarrow \alpha_1(\chi|\text{♘},1|\hat{\imath})+\widetilde{\omega}_{\hat{\imath}}\,\gamma(\hat{\imath})/3, \tag{11.161}$$

$$\alpha_1(\chi|\hat{\imath},1|\hat{\imath}-q) \longrightarrow \alpha_1(\chi|\hat{\imath},1|\hat{\imath}-q)-\gamma(\hat{\imath}), \tag{11.162}$$

$$\alpha_1(\chi|\hat{\imath}-q,1|\hat{\imath}-q) \longrightarrow \alpha_1(\chi|\hat{\imath}-q,1|\hat{\imath}-q)+2\gamma(\hat{\imath})/3, \tag{11.163}$$

$$\alpha_1(\chi|\hat{\imath}-q+1,1|\hat{\imath}-q) \longrightarrow \alpha_1(\chi|\hat{\imath}-q+1,1|\hat{\imath}-q)+\gamma(\hat{\imath})/3, \tag{11.164}$$

$$\alpha_1(\chi|\text{♘},1|\hat{\imath}-q) \longrightarrow \alpha_1(\chi|\text{♘},1|\hat{\imath}-q)-\widetilde{\omega}_{\hat{\imath}}\,\gamma(\hat{\imath})/3, \tag{11.165}$$

$$\beta_1(\chi|\hat{\imath},1|\hat{\imath}+1-q) \longrightarrow \beta_1(\chi|\hat{\imath},1|\hat{\imath}+1-q)+\gamma(\hat{\imath}), \tag{11.166}$$

$$\beta_1(\chi|\hat{\imath}-q+1,1|\hat{\imath}+1-q) \longrightarrow \beta_1(\chi|\hat{\imath}-q+1,1|\hat{\imath}+1-q)-\gamma(\hat{\imath})/3, \tag{11.167}$$

$$\beta_1(\chi|\hat{\imath}+1,1|\hat{\imath}+1-q) \longrightarrow \beta_1(\chi|\hat{\imath}+1,1|\hat{\imath}+1-q)-2\gamma(\hat{\imath})/3, \tag{11.168}$$

$$\beta_1(\chi|\text{♘},1|\hat{\imath}+1-q) \longrightarrow \beta_1(\chi|\text{♘},1|\hat{\imath}+1-q)+\widetilde{\omega}_{\hat{\imath}}\,\gamma(\hat{\imath}), \tag{11.169}$$

$$\beta_1(\chi|\hat{\imath}+1-q,1|\hat{\imath}-q) \longrightarrow \beta_1(\chi|\hat{\imath}+1-q,1|\hat{\imath}-q)+\gamma(\hat{\imath}), \tag{11.170}$$

$$\beta_1(\chi|\hat{\imath},1|\hat{\imath}-q) \longrightarrow \beta_1(\chi|\hat{\imath},1|\hat{\imath}-q)-\gamma(\hat{\imath})/3, \tag{11.171}$$

$$\beta_1(\chi|\hat{\imath}-q,1|\hat{\imath}-q) \longrightarrow \beta_1(\chi|\hat{\imath}-q,1|\hat{\imath}-q)-2\gamma(\hat{\imath})/3, \tag{11.172}$$

$$\beta_1(\chi|\text{♘},1|\hat{\imath}-q) \longrightarrow \beta_1(\chi|\text{♘},1|\hat{\imath}-q)-\widetilde{\omega}_{\hat{\imath}}\,\gamma(\hat{\imath}) \tag{11.173}$$

with $\widetilde{\omega}_{\hat{\imath}}$ as in Observation 11.20. Finally, we take

$$\alpha_c(\varkappa|g|k) \longrightarrow \alpha_c(\varkappa|g|k), \quad \beta_{c'}(\varkappa'|g'|k') \longrightarrow \beta_{c'}(\varkappa'|g'|k') \tag{11.174}$$

for all

$$\varkappa,\ \varkappa' \in \{1,2,3\},\ \ c,\ c' \in \{0,1,2\},\ \ g,\ g' \in \mathcal{I}\cup\mathcal{J},\ \ k,\ k' \in \mathcal{Z}$$

such that the quantities appearing in (11.174) are not involved in (11.142)–(11.173). We write $\Psi_{\chi\hat{\imath}}$ to denote the transformation (11.142)–(11.174) and $\Psi_{\chi\hat{\imath}}(T_\chi)$ for the tensor obtained as in (11.118) but with all the coefficients $\alpha_c(\chi|g|k)$ and $\beta_c(\chi|g|k)$ replaced by their images under the mapping (11.142)–(11.174). In other words, for $\chi \in \{1,2,3\}$ and $g \in \mathcal{I} \cup \mathcal{J}$, we declare that the g-th χ-slice of $\Psi_{\chi\hat{\imath}}(T_\chi)$ is

$$\sum_{c=0}^{2} \sum_{k \in \mathcal{Z}} \left(\Psi_{\chi\hat{\imath}}\left(\alpha_c(\chi|g|k)\right) A_c(k) + \Psi_{\chi\hat{\imath}}\left(\beta_c(\chi|g|k)\right) B_c(k)\right).$$

We remark that all the coefficients $\alpha_c(\varkappa|g|k)$ and $\beta_c(\varkappa|g|k)$ that are involved in the formulas (11.142)–(11.173) have $\varkappa = \chi$, and we immediately have

$$\Psi_{\chi\hat{\imath}}(T_{\chi'}) = T_{\chi'} \quad \text{and} \quad \Psi_{\chi\hat{\imath}}(T_{\chi''}) = T_{\chi''} \tag{11.175}$$

for any $\hat{\imath}$ satisfying (11.141). Also, we immediately have

$$\Psi_{\chi\hat{\imath}}(T_\chi) - T_\chi = g(\hat{\imath})\pi_\chi(\psi_{1\hat{\imath}} + \psi_{2\hat{\imath}} + \psi_{3\hat{\imath}} + \psi_{4\hat{\imath}}) - \gamma(\hat{\imath})\pi_\chi(\varphi_{1\hat{\imath}} + \varphi_{2\hat{\imath}} + \varphi_{3\hat{\imath}} + \varphi_{4\hat{\imath}}) \tag{11.176}$$

in the notation of Observations 11.19 and 11.20, where π_χ denotes the braiding isomorphism of the permutation $(1,2,3) \to (\chi', \chi'', \chi)$. We proceed by taking the mappings $\Psi_{\chi\hat{\imath}}$ for all $(\hat{\imath}, \chi)$ in sequence, where, again, we assume

$$\hat{\imath} \in \mathcal{Z} \setminus (W' \cup \{1, 2, q, q+1\}) \quad \text{and} \quad \chi \in \{1,2,3\},$$

and we use $\widetilde{\alpha}_c(\chi|g|k)$ and $\widetilde{\beta}_c(\chi|g|k)$ to denote the images of $\alpha_c(\chi|g|k)$ and $\beta_c(\chi|g|k)$ under the composition of all such mappings, respectively. So we get the

$$(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$$

tensors (τ_1, τ_2, τ_3) in a way similar to the formula (11.118), that is, we declare that, for any $\chi \in \{1,2,3\}$ and $g \in \mathcal{I} \cup \mathcal{J}$, the g-th χ-slice of τ_χ is

$$\sum_{c=0}^{2} \sum_{k \in \mathcal{Z}} \left(\widetilde{\alpha}_c(\chi|g|k)\, A_c(k) + \widetilde{\beta}_c(\chi|g|k)\, B_c(k)\right). \tag{11.177}$$

In view of Observations 11.19, 11.20 and formulas (11.175), (11.176), we obtain

$$\tau_1 + \tau_2 + \tau_3 = T_1 + T_2 + T_3 \tag{11.178}$$

and proceed with the use of (11.133), (11.142) and (11.146) to get the condition

$$\widetilde{\alpha}_1(\chi|\hat{\imath}+q, 1|\hat{\imath}-1) = \widetilde{\alpha}_1(\chi|\hat{\imath}-1, 1|\hat{\imath}+q-1) = 0 \tag{11.179}$$

for $\hat{\imath} \in \mathcal{Z} \setminus (W' \cup \{1,2,q,q+1\})$, and we use (11.134), (11.150), (11.154) to get

$$\widetilde{\beta}_1(\chi|\hat{\imath}+q, 1|\hat{\imath}-1) = \widetilde{\beta}_1(\chi|\hat{\imath}-1, 1|\hat{\imath}) = 0. \tag{11.180}$$

Again, for $\hat{\imath} \in \mathcal{Z} \setminus (W' \cup \{1,2,q,q+1\})$, we use (11.139), (11.158), (11.162) to get

$$\widetilde{\alpha}_1(\chi|\hat{\imath}+1-q, 1|\hat{\imath}) = \widetilde{\alpha}_1(\chi|\hat{\imath}, 1|\hat{\imath}-q) = 0, \tag{11.181}$$

and the formulas (11.140), (11.166) and (11.170) lead us to

$$\widetilde{\beta}_1(\chi|\hat{\imath}+1-q, 1|\hat{\imath}-q) = \widetilde{\beta}_1(\chi|\hat{\imath}, 1|\hat{\imath}+1-q) = 0. \tag{11.182}$$

Finally, the quantities involved in (11.127) and (11.128) do not appear in any of the formulas (11.142)–(11.173), so the condition (11.174) is in effect, and hence

$$\widetilde{\alpha}_1(\chi|i, 1|k) = 0 \ \text{ if } \ k \in \mathcal{Z} \setminus W' \text{ and } i \in \mathcal{Z} \setminus (\{k\} + \widehat{\delta}) \tag{11.183}$$

$$\widetilde{\beta}_1(\chi|i, 1|k) = 0 \ \text{ if } \ k \in \mathcal{Z} \setminus W' \text{ and } i \in \mathcal{Z} \setminus (\{k\} + \widehat{\varepsilon}). \tag{11.184}$$

again with $\widehat{\delta} = \{-q, -q+1, 0, 1, q, q+1\}$ and $\widehat{\varepsilon} = \{-1, 0, 1, q-1, q, q+1\}$. We write

$$W'' = (W' \cup \{1, 2, q, q+1\}) + \{0, \pm 1\} + \{0, \pm q\},$$

$$\mathcal{W} = (W' \cup \{1, 2, q, q+1\}) + \{0, \pm 1, \pm 2\} + \{0, \pm q, \pm 2q\} \tag{11.185}$$

and use the formulas (11.179)–(11.184) to conclude that

$$\widetilde{\alpha}_1(\chi|i, 1|k) = 0 \text{ if } k \in \mathcal{Z} \setminus W'' \text{ and } i \in \mathcal{Z} \setminus \{k, k+1\}, \tag{11.186}$$

$$\widetilde{\beta}_1(\chi|i, 1|k) = 0 \text{ if } k \in \mathcal{Z} \setminus W'' \text{ and } i \in \mathcal{Z} \setminus \{k, k+q\}. \tag{11.187}$$

Step 6. The goals of the preceding step are reached, so we proceed to work with the coefficients as in (11.177), and we apply the condition (11.114) to the entries

$$((\ell, 1), (\ell, 1), (\ell+1, 1)) \text{ and } ((\ell, 1), (\ell+1, 1), (\ell+1, 1)) \text{ with } \ell \in \mathcal{Z} \setminus \mathcal{W}.$$

Indeed, since we have $T_1 + T_2 + T_3 = \tau_1 + \tau_2 + \tau_3$ due to (11.178), we get

$$-\widetilde{\alpha}_1(1|\ell, 1|\ell) - \widetilde{\alpha}_1(2|\ell, 1|\ell) + \widetilde{\alpha}_1(3|\ell+1, 1|\ell) = 0, \tag{11.188}$$

$$\widetilde{\alpha}_1(1|\ell, 1|\ell) - \widetilde{\alpha}_1(2|\ell+1, 1|\ell) - \widetilde{\alpha}_1(3|\ell+1, 1|\ell) = 0 \tag{11.189}$$

in view of (11.186) and (11.187), and, furthermore, the sum of the equalities (11.188) and (11.189) gives $\widetilde{\alpha}_1(2|\ell, 1|\ell) + \widetilde{\alpha}_1(2|\ell+1, 1|\ell) = 0$. In fact, we obtain

$$\widetilde{\alpha}_1(\chi|\ell, 1|\ell) + \widetilde{\alpha}_1(\chi|\ell+1, 1|\ell) = 0, \tag{11.190}$$

for all $\chi \in \{1, 2, 3\}$, in view of the symmetry of our construction. We proceed with a similar application of the condition (11.114) to the entries

$$((\ell, 1), (\ell, 1), (\ell+q, 1)) \text{ and } ((\ell, 1), (\ell+q, 1), (\ell+q, 1)),$$

which gives, after a further application of (11.186) and (11.187),

$$-\widetilde{\beta}_1(1|\ell, 1|\ell) - \widetilde{\beta}_1(2|\ell, 1|\ell) + \widetilde{\beta}_1(3|\ell+q, 1|\ell) = 0,$$

$$\widetilde{\beta}_1(1|\ell, 1|\ell) - \widetilde{\beta}_1(2|\ell+q, 1|\ell) - \widetilde{\beta}_1(3|\ell+q, 1|\ell) = 0$$

and hence $\widetilde{\beta}_1(2|\ell, 1|\ell) + \widetilde{\beta}_1(2|\ell+q, 1|\ell) = 0$, and we obtain

$$\widetilde{\beta}_1(\chi|\ell, 1|\ell) + \widetilde{\beta}_1(\chi|\ell+q, 1|\ell) = 0, \tag{11.191}$$

for all $\chi \in \{1, 2, 3\}$, by the symmetry. The contribution of the coefficients involved in (11.190) to the sum $\tau_1 + \tau_2 + \tau_3$ is represented by the formula

$$\sum_{\chi=1}^{3} \tau_{\alpha\chi\ell} \tag{11.192}$$

in which, for any $\chi \in \{1, 2, 3\}$,

$$\text{the } (\ell, 1)\text{-th } \chi\text{-slice of } \tau_{\alpha\chi\ell} \text{ is } \widetilde{\alpha}_1(\chi|\ell, 1|\ell)\, A_1(\ell), \tag{11.193}$$

$$\text{the } (\ell+1, 1)\text{-th } \chi\text{-slice of } \tau_{\alpha\chi\ell} \text{ is } \widetilde{\alpha}_1(\chi|\ell+1, 1|\ell)\, A_1(\ell), \tag{11.194}$$

and all the other entries of $\tau_{\alpha\chi\ell}$ are zero. We directly compute the sum (11.192) with the use of Definition 10.3 and conditions (11.193), (11.194), which gives

$$\sum_{\chi=1}^{3} \tau_{\alpha\chi\ell} = \pi_{1\ell} \otimes u'(\ell) \otimes u'(\ell) + u'(\ell) \otimes \pi_{2\ell} \otimes u'(\ell) + u'(\ell) \otimes u'(\ell) \otimes \pi_{3\ell} \tag{11.195}$$

provided that

$$\pi_{\chi\ell} \in (e_{\ell+1} - e_\ell)\, \mathbb{K},$$

and, with a further application of Definition 10.3, we note that

$$e_{\ell+1} - e_\ell \in \{u'(\ell),\, u'(\ell) + e_{\text{♘}}\}$$

for all ℓ. Similarly, we compute the contribution to the sum $\tau_1 + \tau_2 + \tau_3$ made by the coefficients in (11.191), and this contribution is expressed by the formula

$$\sum_{\chi=1}^{3} \tau_{\beta\chi\ell} \tag{11.196}$$

in which, for any $\chi \in \{1, 2, 3\}$,

$$\text{the } (\ell, 1)\text{-th } \chi\text{-slice of } \tau_{\beta\chi\ell} \text{ is } \widetilde{\beta}_1(\chi|\ell, 1|\ell)\, B_1(\ell), \tag{11.197}$$

$$\text{the } (\ell + q, 1)\text{-th } \chi\text{-slice of } \tau_{\beta\chi\ell} \text{ is } \widetilde{\beta}_1(\chi|\ell + q, 1|\ell)\, B_1(\ell), \tag{11.198}$$

and all the other entries of $\tau_{\beta\chi\ell}$ are zero. Again, we can compute the sum (11.196) with the use of Definition 10.3 and conditions (11.197), (11.198), which gives

$$\sum_{\chi=1}^{3} \tau_{\beta\chi\ell} = \widetilde{\pi}_{1\ell} \otimes v'(\ell) \otimes v'(\ell) + v'(\ell) \otimes \widetilde{\pi}_{2\ell} \otimes v'(\ell) + v'(\ell) \otimes v'(\ell) \otimes \widetilde{\pi}_{3\ell} \tag{11.199}$$

in which we assume

$$\widetilde{\pi}_{\chi\ell} \in (e_\ell - e_{\ell+q})\, \mathbb{K},$$

and, again, with an application of Definition 10.3, we note that

$$e_\ell - e_{\ell+q} \in \{v'(\ell),\, v'(\ell) + e_{\text{♘}}\},$$

again for all ℓ. Therefore, the formulas (11.195) and (11.199) allow us to assume, without loss of generality, that the summands in (11.190) and (11.191) are zero, and then a further application of (11.186) and (11.187) implies

$$\widetilde{\alpha}_1(\chi|i, 1|k) = \widetilde{\beta}_1(\chi|i, 1|k) = 0 \text{ if } k \in \mathcal{Z} \setminus \mathcal{W},\ i \in \mathcal{Z},\ \chi \in \{1, 2, 3\}. \tag{11.200}$$

Therefore, we can now focus on the coefficients $\widetilde{\alpha}_1(\chi|\text{♘}, 1|k)$ and $\widetilde{\beta}_1(\chi|\text{♘}, 1|k)$, and, for any $\chi \in \{1, 2, 3\}$, we define Θ_χ as the set of all $k \in \mathcal{Z}$ for which we have either

$$\widetilde{\alpha}_1(\chi|\text{♘}, 1|k) \neq 0 \text{ or } \widetilde{\beta}_1(\chi|\text{♘}, 1|k) \neq 0.$$

If, for some $\chi \in \{1, 2, 3\}$, the condition

$$|\Theta_\chi| \leqslant 25 \cdot (\rho + 4\,|\mathcal{W}|) \tag{11.201}$$

was false, then, due to Observations 10.10 and 10.14, the

$$(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{1\})$$

block of the $(\text{♘}, 1)$-th χ-slice of τ_χ would have the rank greater than $\rho + 4\,|\mathcal{W}|$. In view of (11.200), we conclude that, in this case, the $(\mathcal{Z} \times \{1\}) \times (\mathcal{Z} \times \{1\})$ block of the $(\text{♘}, 1)$-th χ-slice of $\tau_1 + \tau_2 + \tau_3$ has the rank greater than ρ, which contradicts to the rank bound (11.115) and shows the validity of (11.201) for all $\chi \in \{1, 2, 3\}$.

Step 7. We are now ready to finalize the proof. To this end, we define the set Φ' as required in the formulation of the lemma by declaring that Φ' is the collection of all matrices $A_1(k)$ and $B_1(k)$ with $k \in \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \mathcal{W}$. The fact that the

$$(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$$

padding of the desired block (11.116) of $\tau_1+\tau_2+\tau_3 = T_1+T_2+T_3$ belongs to (11.117) is now immediate from the definition of Θ_χ and condition (11.200). In fact, together with the inequality (11.201), the condition (11.200) confirms that

$$|\Phi'| \leqslant 2\,|\mathcal{W}| + 2\cdot 3\cdot 25\cdot(\rho + 4\,|\mathcal{W}|) \leqslant 151\cdot(\rho + 4\,|\mathcal{W}|),$$

and, since the conditions (11.122) and (11.185) imply the inequalities

$$|\mathcal{W}| \leqslant 25\cdot(4 + |W'|) \leqslant 25\cdot(4 + 16\,|W|),$$

we get the desired bound $|\Phi'| \leqslant 151\cdot(\rho + 1\,600\,|W| + 400)$. □

12. Waring rank is not additive

In this section, we put together the results discussed earlier and finalize the proof of Claim 5.6, which is the main technical contribution of the paper. In particular, this allows us to obtain Theorem 3.9, but, also, as an outcome of Sections 10 and 11, we get a sequence of eliminating families consisting of rank one matrices.

Theorem 12.1. *Assume that q, ρ, r, π, σ, δ are nonnegative integers, let $\mathbb{F}$ be a field such that* char $\mathbb{F}$ *does not divide* $6(q+1)$. *We consider the matrix*

$$m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and assume $\sigma \geqslant 1\,561\,r^2$ and $q \geqslant \max\{5,\, \delta,\, 28r\}$. Also, we assume that either

$$\big(q \geqslant 3\cdot 10^{47}\cdot(\rho+18)^{10}\big) \wedge \big(\pi \geqslant 15\cdot 10^{96}\cdot(\rho+18)^{20}\big) \tag{12.1}$$

or

$$\pi \geqslant |\Phi(q)| + 1. \tag{12.2}$$

Then $\Phi(q)$ is a candidate family of the type $(\mathbb{F}, m, \rho, r, \pi, \sigma, \delta)$.

Proof. We need to confirm the conditions (i)–(v) in Definition 8.3. As said above, the points (i) and (ii) follow from Lemmas 10.16 and 10.18, respectively, and the only assumption that this requires is $q \geqslant 5$ as in Remark 10.1. Further, the item (iii) comes from Lemma 10.19 and employs the further assumption $q \geqslant \delta$.

Concerning the point (iv), we proceed with Theorem 10.27, which is applicable due to the assumption $q \geqslant 28r$ above. So we take an arbitrary linear combination

$$M = \sum_{k\in\mathcal{Z}} \big(x_{3k}A_3(k) - x_{2k}A_2(k) - x_{1k}A_1(k) + y_{3k}B_3(k) - y_{2k}B_2(k) - y_{1k}B_1(k)\big)$$

of $\Phi(q)$, and we see that, whenever the $\mathcal{J}\times\mathcal{J}$ block of M has the rank at most r, there exists an element c such that the string

$$\big(x_{11}\ x_{21}\ x_{31}\ y_{11}\ y_{21}\ y_{31}\ \ldots\ x_{1\,1+q^2}\ x_{2\,1+q^2}\ x_{3\,1+q^2}\ y_{1\,1+q^2}\ y_{2\,1+q^2}\ y_{3\,1+q^2}\big)$$

contains at most $1561\,r^2$ entries different from c. Now we apply the item (i) as above and see that the $\mathcal{J}\times\mathcal{J}$ block of the sum of all the matrices in $\Phi(q)$ is zero. Therefore, the $\mathcal{J}\times\mathcal{J}$ block of M comes from a linear combination of at most $1561\,r^2$ matrices in $\Phi(q)$, and, since these matrices are rank one, the sum of their row spaces has the dimension at most $1561\,r^2 \leqslant \sigma$, which is needed in the point (iv).

Since the condition (12.2) voids the remaining point (v), we assume the validity of the inequalities (12.1) in the rest of this proof. Indeed, we consider an arbitrary field extension $\mathbb{K} \supseteq \mathbb{F}$ and a pair of $(\mathcal{I}\cup\mathcal{J})\times(\mathcal{I}\cup\mathcal{J})\times(\mathcal{I}\cup\mathcal{J})$ tensors

$$T \in O\,\mathrm{mod}_{\mathbb{K}}\,(\Phi(q), \Phi(q), \Phi(q))$$

and Δ such that all entries outside the

$$\mathcal{I} \times \mathcal{I} \times \mathcal{I},\ \mathcal{I} \times \mathcal{I} \times \mathcal{J},\ \mathcal{I} \times \mathcal{J} \times \mathcal{I},\ \mathcal{J} \times \mathcal{I} \times \mathcal{I}$$

blocks of Δ are zero, and we assume that the further condition $\mathrm{rk}_{\mathbb{K}}\,(T + \Delta) \leqslant \rho$ is true. We apply Lemma 11.11 and proceed with two subsets

$$\Phi' \subseteq \Phi(q),\ \ W \subseteq \mathcal{Z}$$

and a tensor $T' \in T \bmod_{\mathbb{K}} (\Phi', \Phi', \Phi')$ satisfying

$$|\Phi'| \leqslant 4 \cdot 10^{13} \cdot \rho^2 \ \text{ and }\ |W| \leqslant 4 \cdot 10^{12} \cdot (\rho + 18)^5$$

such that all the nonzero entries of the $\mathcal{J} \times \mathcal{J} \times \mathcal{J}$ block of T' are in

$$(W \times \{1, 2\}) \times (W \times \{1, 2\}) \times (W \times \{1, 2\}).$$

Further, we write $\rho' = \mathrm{rk}_{\mathbb{K}}\,(T' + \Delta)$, and, by the arguments above, we have

$$\rho' \leqslant \rho + 3|\Phi'| \leqslant 12 \cdot 10^{13} \cdot \rho^2 + \rho,$$

which allows an application of Lemmas 11.16 and 11.21 to the tensor T' in view of the bound (12.1) in the formulation of the current theorem. So, Lemma 11.21 gives

$$\widetilde{\Phi}' \subseteq \Phi(q) \ \text{ with }\ |\widetilde{\Phi}'| \leqslant 151 \cdot (\rho' + 1\,600\,|W| + 400)$$

so that the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor obtained by taking the

$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\})$$

block of T' and replacing all the remaining entries with zeros belongs to

$$O \bmod_{\mathbb{K}} (\widetilde{\Phi}'\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \widetilde{\Phi}'\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \widetilde{\Phi}'\,\mathbb{K} + \mathcal{M}\,\mathbb{K}),$$

where $\mathcal{M}$ is the padding $m((\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}))$. In view of the symmetry shown in Observation 10.4, another application of Lemma 11.21 gives a family

$$\widetilde{\Phi}'' \subseteq \Phi(q) \ \text{ with }\ |\widetilde{\Phi}''| \leqslant 151 \cdot (\rho' + 1\,600\,|W| + 400)$$

so that the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor obtained by taking the

$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\})$$

block of T' and replacing all the remaining entries with zeros belongs to

$$O \bmod_{\mathbb{K}} (\widetilde{\Phi}''\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \widetilde{\Phi}''\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \widetilde{\Phi}''\,\mathbb{K} + \mathcal{M}\,\mathbb{K}).$$

We proceed with a similar application of Lemma 11.16, which gives two families

$$\widehat{\Phi}' \subseteq \Phi(q),\ \ \widehat{\Phi}'' \subseteq \Phi(q)$$

with the cardinalities not exceeding

$$7 \cdot 10^{96} \cdot (\rho + 18)^{20}$$

so that the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor obtained by taking the union of the

$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}),$$
$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}),$$
$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}),$$

blocks of T' and replacing all the remaining entries with zeros belongs to

$$O \bmod_{\mathbb{K}} (\widehat{\Phi}'\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \widehat{\Phi}'\,\mathbb{K} + \mathcal{M}\,\mathbb{K},\ \widehat{\Phi}'\,\mathbb{K} + \mathcal{M}\,\mathbb{K}),$$

and the $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ tensor obtained by taking the union of the

$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}),$$

$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}),$$
$$((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{2\}) \times ((\mathcal{Z} \cup \{\text{♘}\}) \times \{1\}),$$

blocks of T' and replacing all the remaining entries with zeros belongs to

$$O \operatorname{mod}_{\mathbb{K}} (\widehat{\Phi}'' \, \mathbb{K} + \mathcal{M} \, \mathbb{K}, \, \widehat{\Phi}'' \, \mathbb{K} + \mathcal{M} \, \mathbb{K}, \, \widehat{\Phi}'' \, \mathbb{K} + \mathcal{M} \, \mathbb{K}).$$

Therefore, we can define the families required in the point (v) as

$$\Phi_1 = \Phi_2 = \Phi_3 = \Phi' \cup \widetilde{\Phi}' \cup \widetilde{\Phi}'' \cup \widehat{\Phi}' \cup \widehat{\Phi}''$$

and note that, indeed, we have

$$T \in O \operatorname{mod}_{\mathbb{K}} (\Phi_1 \, \mathbb{K} + \mathcal{M} \, \mathbb{K}, \Phi_2 \, \mathbb{K} + \mathcal{M} \, \mathbb{K}, \Phi_3 \, \mathbb{K} + \mathcal{M} \, \mathbb{K}).$$

Also, the immediate bound on the cardinality

$$|\Phi_1| \leqslant |\Phi'| + |\widehat{\Phi}'| + |\widehat{\Phi}''| + |\widetilde{\Phi}'| + |\widetilde{\Phi}''|$$

is sufficient to see that $|\Phi_1| < \pi$, and hence the sum of the row spaces of the matrices in Φ_1 does indeed have the dimension at most π as a linear space over $\mathbb{K}$. □

We are ready to proceed with the explicit bounds of the forms $c_t(\rho)$ and $s_t(\rho)$ as in Theorem 9.2. Indeed, the following corollary deals with the case $t = 1$ and uses Theorem 12.1 with the condition (12.1), whilst, in Corollary 12.3 below, we cover the situation $t > 1$ with the corresponding case (12.2) of Theorem 12.1.

Corollary 12.2. *For any integer $\rho \geqslant 1$ and any field with* $\operatorname{char} \mathbb{F} \neq 2, 3$*, the matrix*

$$m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has an eliminating family Φ with respect to $(\mathbb{F}, \rho)$ so that Φ consists of at most

$$s_1(\rho) \leqslant 15 \cdot 10^{194} \cdot (\rho + 18)^{40}$$

symmetric matrices of the order not exceeding

$$c_1(\rho) \leqslant 5 \cdot 10^{194} \cdot (\rho + 18)^{40}$$

which are all rank one.

Proof. In view of Definition 8.7, we need to find a candidate family of the type $(\mathbb{F}, m, \rho, \rho, \pi, \sigma, \pi + \sigma)$ with some appropriate π and σ. According to Theorem 12.1, we can pick $\Phi = \Phi(q)$ whenever the following conditions are satisfied:

$$\operatorname{char} \mathbb{F} \text{ does not divide } 6(q+1) \tag{12.3}$$

and $3 \cdot 10^{47} \cdot (\rho + 18)^{10} \leqslant q$, and, in addition, there exist integers δ, π, σ such that

$$1561 \, \rho^2 \leqslant \sigma, \quad 15 \cdot 10^{96} \cdot (\rho + 18)^{20} \leqslant \pi, \quad \sigma + \pi \leqslant \delta \leqslant q.$$

Here, the inequalities can be solved easily, and, concerning the condition (12.3), if it does not hold for some q, then it can be satisfied by replacing $q \to q + 1$. It remains to recall that, according to Definition 10.3, the family $\Phi(q)$ consists of $6(q^2 + 1)$ symmetric rank one matrices of the size $2(q^2 + 2) \times 2(q^2 + 2)$ each. □

We proceed with eliminating families that reduce $\operatorname{skp}(\mathbb{F}, t, I)$ to $\operatorname{skp}(\mathbb{F}, t - 1, I')$ with $t \geqslant 2$, and these families turn out to be smaller than those in Corollary 12.2.

Corollary 12.3. *If $\rho \geqslant 1$ and $t \geqslant 2$ are integers, and $\mathbb{F}$ is a field with* char $\mathbb{F} \neq 2, 3$, *then there exists a full rank $2^t \times 2^t$ skew projector μ, an indexing set I' with*

$$|I'| \leqslant c_t(\rho) \leqslant 2^t \cdot 22\,000\,000 \cdot \left(\left\lceil \frac{\rho}{2^{t-2}} \right\rceil \right)^4$$

and a subset $\Phi_t \subseteq \operatorname{skp}(\mathbb{F}, t-1, I')$ of the cardinality at most

$$s_t(\rho) \leqslant 132\,000\,000 \cdot \left(\left\lceil \frac{\rho}{2^{t-2}} \right\rceil \right)^4$$

so that Φ_t is an eliminating family for the matrix μ with respect to $\mathbb{F}$ and ρ.

Proof. We take the matrix m as in Theorem 12.1 and write

$$\mu = \mathrm{LSP}_{t-1}(m)$$

with the use of the notation of Corollary 9.7. Indeed, according to Observation 9.3, the matrix μ is a full rank $2^t \times 2^t$ skew projector, and we intend to pick

$$\Phi_t = \mathrm{LSP}_{t-1}\left(\Phi(q)\right) \tag{12.4}$$

with some appropriate q. By Definition 8.7, Corollary 9.7 and Theorem 12.1, the choice (12.4) is sufficient whenever

$$\text{char } \mathbb{F} \text{ does not divide } 6(q+1) \tag{12.5}$$

and, in addition, there exist positive integers r, σ, δ such that

$$2^{t-2}\, r \geqslant \rho, \quad 2^t\, \sigma + 2^{t-1}\, \sigma \leqslant 2^{t-1}\, \delta, \quad q \geqslant \max\{28r, \delta\}, \quad 1561\, r^2 \leqslant \sigma. \tag{12.6}$$

Indeed, the choice

$$r = \left\lceil \frac{\rho}{2^{t-2}} \right\rceil$$

leads to a solution with $\sigma = 1561 r^2$ and $\delta = 3\sigma$, so it suffices to take

$$q = 3 \cdot 1561 \cdot \left(\left\lceil \frac{\rho}{2^{t-2}} \right\rceil \right)^2$$

in order to be able to fulfill the inequalities (12.6). Further, the condition (12.5) can be reached with the replacement $q \to q+1$ if it is not yet satisfied, and, according to Definitions 9.4 and 10.3, the family (12.4) consists of the $6(q^2+1)$ matrices of the size $2^t(q^2+2)$, so we can finalize the proof with a straightforward computation. □

Finally, according to Theorem 9.2, the result of Claim 5.6 is valid with

$$\mu\left(k, \rho, |W|\right) \leqslant |W| \cdot \left(\sum_{t=1}^{k} c_t(\rho) \right) \cdot s_2(\rho) \cdot s_3(\rho) \cdot \ldots \cdot s_k(\rho)$$

and

$$\sigma\left(k, \rho, |W|\right) \leqslant |W| \cdot s_1(\rho) \cdot s_2(\rho) \cdot s_3(\rho) \cdot \ldots \cdot s_k(\rho),$$

and the application of Corollaries 12.2 and 12.3 returns the bounds matching with those in Notation 5.5. This completes the proof of Claim 5.6, and, in particular, due to Corollary 6.12 and Remark 7.6, we arrive at Theorem 3.9.

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