Every convex pentagon has some vertex such that the sum of distances to the other four vertices is greater than its perimeter

Juan Moreno Borrallo

June 28, 2023

email: juan.morenoborrallo@gmail.com

Abstract

In this paper it is solved the case $n = 5$ of the problem 1.345 of the Crux Mathematicorum journal, proposed by Paul Erdös and Esther Szekeres in 1988. The problem was solved for $n \geq 6$ by János Pach and the solution published by the Crux Mathematicorum journal, leaving the case $n = 5$ open to the reader. In september 2021, user23571113 posed this problem at the post https://math.stackexchange.com/questions/4243661/prove-that-for-one-vertex-of-a-convex-pentagon-the-sum-of-distances-to-the-othe/4519514#4519514, and it has finally been solved.

2010MSC: 51M04

Keywords. *Erdös, Szekeres, Convex pentagon, Sum of distances, Perimeter.*

We want to show that, in every convex pentagon, there exists one vertex such that the sum of distances from this vertex to the other four is greater than the perimeter of the pentagon.

If we denote as S_d the sum of the five diagonals, the statement is true for every convex pentagon satisfying

$$p < \frac{2}{3}S_d$$

This can be easily showed noting that the perimeter and the sum of distances from some vertex to the other four have two edges in common. Thus, the sum of the three non-common edges must be equal or greater than the two diagonals with the common vertex, and this must hold for each vertex of the pentagon, generating five inequalities to hold simultaneously. Noting that, putting together the inequalities, each edge is counted three times, and each diagonal twice, proves the necessary condition for the statement not to hold:

$$3p \geq 2S_d$$

Thus, the statement is true for every convex pentagon satisfying

$$p < \frac{2}{3}S_d$$

Another result we will use is noting that, for any pair of vertices U and V, if we denote as S_U and S_V the sum of distances from U and V respectively, then we have that

$$S_U + S_V > 3 \mid UV \mid +p$$

Therefore, if some distance between two vertices is equal or greater than $\frac{p}{3}$, then the statement is true.

The result can be proved using the triangle inequality as follows:

Label the vertices $U_1, \ldots, U_5$ such that consecutive vertices have consecutive indices.

There are two cases, non-consecutive and consecutive vertices.

- Case 1:

The vertices are non-consecutive; without loss of generality, we consider vertices U_1 and U_3. Let P be the point of intersection of the segments U_1U_4 and U_3U_5. Using the triangle inequality we get

$$\begin{aligned}
&|U_1P| + |PU_3| > |U_1U_3|, \\
&|U_4P| + |PU_5| > |U_4U_5| \\
\implies &|U_1U_4| + |U_3U_5| > |U_1U_3| + |U_4U_5| \\
&\implies s_{U_1} + s_{U_3} = |U_1U_2| + |U_1U_3| + |U_1U_4| + |U_1U_5| \\
&\qquad + |U_3U_1| + |U_3U_2| + |U_3U_4| + |U_3U_5| \\
&\qquad > 3|U_1U_3| + |U_1U_2| + |U_2U_3| + |U_3U_4| + |U_4U_5| + |U_5U_1| \\
&\qquad = 3|U_1U_3| + p.
\end{aligned}$$

- Case 2:

The vertices are consecutive; without loss of generality, we consider vertices U_1 and U_2. Just like before we get

$$
\begin{aligned}
&|U_1U_3| + |U_2U_4| > |U_1U_2| + |U_3U_4|,\\
&|U_1U_4| + |U_2U_5| > |U_1U_2| + |U_4U_5|\\
&\implies s_{U_1} + s_{U_2} = |U_1U_2| + |U_1U_3| + |U_1U_4| + |U_1U_5|\\
&\qquad\qquad\qquad + |U_2U_1| + |U_2U_3| + |U_2U_4| + |U_2U_5|\\
&\qquad\qquad\qquad > 3|U_1U_2| + |U_1U_2| + |U_2U_3| + |U_3U_4| + |U_4U_5| + |U_5U_1|\\
&\qquad\qquad\qquad = 3|U_1U_2| + p.
\end{aligned}
$$

Without loss of generality, suppose that $p = 1$. Assume that there is no vertex such that the sum of distances to the other four is greater than the perimeter. This implies that $S_d < 1.5$, and thus the average of the lengths of the diagonals can be at most 0.3.

Using the Law of Cosines, every side s_i can be calculated as

$$s_i^2 = d_j^2 + d_k^2 - 2d_jd_k\cos(\theta_i)$$

Where θ_i is the interior angle of the star pentagon at the joining vertex of the diagonals d_j and d_k.

The sum of the interior angles of any star pentagon is equal to 180º; therefore, the average of the interior angles is equal to 36º. If we calculate the length of some s_i using the average of the interior angles of the star pentagon, and the maximum average length of the diagonals, we get that

$$s_i^2 = (0.3)^2 + (0.3)^2 - 2(0.3)(0.3)\cos(36)$$

$$s_i^2 = 0.18(1 - \cos(36))$$

$$s_i \approx 0.18541$$

However, the average length of the sides of the convex pentagon, as the perimeter is equal to 1, equals 0.2. Therefore, there would be needed sides s_j greater than s_i to achieve the perimeter's length, and that could only be achieved (i) with diagonals of greater length than the average, or/and (ii) with interior angles of the star pentagon greater than the average.

Note that, as showed before, every diagonal can be at most equal to $\frac{p}{3}$. If we plug this in the formula for s_j with the average interior angle of the star pentagon, we get that

$$s_j^2 = \frac{2}{9}(1 - \cos(36))$$

$$s_j \approx 0.206$$

Furthermore, $s_j \geq 0.2$ with both diagonals equal to $\frac{p}{3}$ only if $\theta_i > 34.9152°$; and if we set $\theta_i = 36°$, both diagonals need to be equal or greater than $\frac{1+\sqrt{5}}{10} \approx 0.3236$ to have some $s_j \geq 0.2$. Finally, $s_j \geq 0.2$ with both diagonals equal to 0.3 only if $\theta_i > 38.9424°$.

This shows that no matter which greater-than-the-average diagonals we use, or which interior angles of the star pentagon we plug in, we can obtain at most sides such that the sum of the sides s_i obtained with less-or-equal-to-the-average diagonals and interior angles of the star pentagon, and the sides s_j obtained with greater-than-the-average diagonals and/or interior angles of the star pentagon, is less than the perimeter of the convex pentagon; otherwise, the sum of lengths of the diagonals would be greater than the maximum possible of 1.5, and/or the sum of the interior angles of the star pentagon would be greater than 180°.

As we reach a contradiction, the initial assumption that there is no vertex such that the sum of distances to the other four is greater than the perimeter can not be true; therefore, in any convex pentagon there exists at least one vertex such that the sum of distances to the other four is greater than the perimeter.