SOME 3D-DETERMINANT PROPERTIES FOR CALCULATING OF CUBIC-MATRIX OF ORDER 2 AND ORDER 3

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Abstract. In this paper we have studied some properties for determinant-calculating for cubic-matrix of order 2 and order 3. These properties are analogous to some properties for determinants of square matrix we have proved and noted that these properties also are applicable (or not in some details) on this concept for cubic-matrix of orders 2 and 3. All results in this paper, are presented in detail during the theorem proofs.

1. Introduction

In this paper we prove some properties for determinant of cubic-matrix of order 2 and order 3. In the paper [2], we have defined the concept of determinant for cubic-matrix of order 2 and order 3, and we have prove some basic properties for calculating this determinants. This idea for developing this concept, it came simply from the determinant of 2D square matrices [19, 20, 21, 24, 25, 26, 29], as well as determinant of rectangular matrices [3, 15, 16, 17, 18, 30, 31, 32, 33]. In paper [1] we have prove that the Laplace expansion method is valid for calculating the determinant of cubic-matrix for orders 2 and 3. Encouraged by geometric intuition, in this paper we are trying to give an idea and visualize the meaning of the determinants for the cubic-matrix. Our early research mainly lies between geometry, algebra, matrix theory, etc., (see [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]).

This paper is continuation of the ideas that arise based on previous researches of 3D matrix ring with element from any whatever field F see [22], but here we study the case when the field F is the field of real numbers $\mathbb{R}$ also is continuation of our research [2] and [1] related to the study of the properties of determinants for cubic-matrix of order 2 and 3. In this paper we follow a different method from method which is studied in [23].

2. Results for more properties of determinants of cubic-matrix of order 2 and order 3

In this section, all proofs of our theorems are based on the definition of determinant for cubic-matrix of orders 2 and 3, presented in the papers [2] and [1], and results obtained in these papers.

The proofs of the following Theorems are too loaded with indices to calculate, and we are trying to make them a little simpler by separating them case by case, to avoid the difficulty of calculations!

Theorem 1. *Let's be A and B 3D-cubic matrix with same order (second and third order matrices), then we have that:*

$$\det(A+B) = \det(A) + \det(B)$$

Proof. **Case 1**. The cubic-matrix A of order 2, (and B has order 2), we will proof the case 1 for each "horizontal layer", "vertical page" and "vertical layer", as following:

1. For plan $i = 1$: Let A and B be cubic-matrix of order 2, where all elements on the plan $i = 1$ are identical in both matrices, then based on definition of determinant of cubic-matrix presented in [2] and [1] we have:

$$\det[A_{2\times2\times2}] + \det[B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) + \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ b_{211} & b_{221} & b_{212} & b_{222} \end{array}\right)$$

$$= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211} + a_{111}\cdot b_{222} - a_{112}\cdot b_{221} - a_{121}\cdot b_{212} + a_{122}\cdot b_{211}$$

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while,

$$\det[A_{2\times2\times2} + B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211}+b_{211} & a_{221}+b_{221} & a_{212}+b_{212} & a_{222}+b_{222} \end{array}\right)$$

$$= a_{111}\cdot(a_{222}+b_{222}) - a_{112}\cdot(a_{221}+b_{221}) - a_{121}\cdot(a_{212}+b_{212}) + a_{122}\cdot(a_{211}+b_{211})$$

$$= a_{111}\cdot a_{222} + a_{111}\cdot b_{222} - a_{112}\cdot a_{221} - a_{112}\cdot b_{221} - a_{121}\cdot a_{212} - a_{121}\cdot b_{212} + a_{122}\cdot a_{211} + a_{122}\cdot b_{211}$$

If we compare results of above equations, we can see that we have the same result in both cases. Similarly we will proof for all other cases.

2. For plan $i = 2$: Let A and B be cubic-matrices of order 2, where all elements on the plan $i = 2$ are identical in both matrices, then we have:

$$\det[A_{2\times2\times2}] + \det[B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) + \det\left(\begin{array}{cc|cc} b_{111} & b_{121} & b_{112} & b_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right)$$

$$= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211} + b_{111}\cdot a_{222} - b_{112}\cdot a_{221} - b_{121}\cdot a_{212} + b_{122}\cdot a_{211}$$

while,

$$\det[A_{2\times2\times2} + B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111}+b_{111} & a_{121}+b_{121} & a_{112}+b_{112} & a_{122}+b_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right)$$

$$= (a_{111}+b_{111})\cdot a_{222} - (a_{112}+b_{112})\cdot a_{221} - (a_{121}+b_{121})\cdot a_{212} + (a_{122}+b_{122})\cdot a_{211}$$

$$= a_{111}\cdot a_{222} + b_{111}\cdot a_{222} - a_{112}\cdot a_{221} - b_{112}\cdot a_{221} - a_{121}\cdot a_{212} - b_{121}\cdot a_{212} + a_{122}\cdot a_{211} + b_{122}\cdot a_{211}$$

If we compare results of above equations, we can see that we have the same result in both cases.

3. For plan $j = 1$: Let A and B be cubic-matrices of order 2, where all elements on the plan $j = 1$ are identical in both matrices, then we have:

$$\det[A_{2\times2\times2}] + \det[B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) + \det\left(\begin{array}{cc|cc} a_{111} & b_{121} & a_{112} & b_{122} \\ a_{211} & b_{221} & a_{212} & b_{222} \end{array}\right)$$

$$= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211} + a_{111}\cdot b_{222} - a_{112}\cdot b_{221} - b_{121}\cdot a_{212} + b_{122}\cdot a_{211}$$

while,

$$\det[A_{2\times2\times2} + B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121}+b_{121} & a_{112} & a_{122}+b_{122} \\ a_{211} & a_{221}+b_{221} & a_{212} & a_{222}+b_{222} \end{array}\right)$$

$$= a_{111}\cdot(a_{222}+b_{222}) - a_{112}\cdot(a_{221}+b_{221}) - (a_{121}+b_{121})\cdot a_{212} + (a_{122}+b_{122})\cdot a_{211}$$

$$= a_{111}\cdot a_{222} + a_{111}\cdot b_{222} - a_{112}\cdot a_{221} - a_{112}\cdot b_{221} - a_{121}\cdot a_{212} - b_{121}\cdot a_{212} + a_{122}\cdot a_{211} + b_{122}\cdot a_{211}.$$

If we compare results of above equations, we can see that we have the same result in both cases.

4. For plan $j = 2$: Let A and B be cubic-matrices of order 2, where all elements on the plan $j = 2$ are identical in both matrices, then we have:

$$\det[A_{2\times2\times2}] + \det[B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) + \det\left(\begin{array}{cc|cc} b_{111} & a_{121} & b_{112} & a_{122} \\ b_{211} & a_{221} & b_{212} & a_{222} \end{array}\right)$$

$$= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211} + b_{111}\cdot a_{222} - b_{112}\cdot a_{221} - a_{121}\cdot b_{212} + a_{122}\cdot b_{211}$$

while,

$$\det[A_{2\times2\times2} + B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111}+b_{111} & a_{121} & a_{112}+b_{112} & a_{122} \\ a_{211}+b_{211} & a_{221} & a_{212}+b_{212} & a_{222} \end{array}\right)$$

$$= (a_{111}+b_{111})\cdot a_{222} - (a_{112}+b_{112})\cdot a_{221} - a_{121}\cdot(a_{212}+b_{212}) + a_{122}\cdot(a_{211}+b_{211})$$

$$= a_{111}\cdot a_{222} + b_{111}\cdot a_{222} - a_{112}\cdot a_{221} - b_{112}\cdot a_{221} - a_{121}\cdot a_{212} - a_{121}\cdot b_{212} + a_{122}\cdot a_{211} + a_{122}\cdot b_{211}$$

If we compare results of above equations, we can see that we have the same result in both cases.

5. For plan $k = 1$: Let A and B be cubic-matrices of order 2, where all elements on the plan $k = 1$ are identical in both matrices, then we have:

$$\det[A_{2\times2\times2}] + \det[B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) + \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & b_{112} & b_{122} \\ a_{211} & a_{221} & b_{212} & b_{222} \end{array}\right)$$

$$= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211} + a_{111}\cdot b_{222} - b_{112}\cdot a_{221} - a_{121}\cdot b_{212} + b_{122}\cdot a_{211}$$

while,

$$\det[A_{2\times2\times2} + B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112}+b_{112} & a_{122}+b_{122} \\ a_{211} & a_{221} & a_{212}+b_{212} & a_{222}+b_{222} \end{array}\right)$$

$$= a_{111}\cdot(a_{222}+b_{222}) - (a_{112}+b_{112})\cdot a_{221} - a_{121}\cdot(a_{212}+b_{212}) + (a_{122}+b_{122})\cdot a_{211}$$

$$= a_{111}\cdot a_{222} + a_{111}\cdot b_{222} - a_{112}\cdot a_{221} - b_{112}\cdot a_{221} - a_{121}\cdot a_{212} - a_{121}\cdot b_{212} + a_{122}\cdot a_{211} + b_{122}\cdot a_{211}$$

If we compare results of above equations, we can see that we have the same result in both cases.

6. For plan $k = 2$: Let A and B be cubic-matrices of order 2, where all elements on the plan $k = 2$ are identical in both matrices, then we have:

$$\det[A_{2\times2\times2}] + \det[B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) + \det\left(\begin{array}{cc|cc} b_{111} & b_{121} & a_{112} & a_{122} \\ b_{211} & b_{221} & a_{212} & a_{222} \end{array}\right)$$

$$= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211} + b_{111}\cdot a_{222} - a_{112}\cdot b_{221} - b_{121}\cdot a_{212} + a_{122}\cdot b_{211}$$

while,

$$\det[A_{2\times2\times2} + B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111}+b_{111} & a_{121}+b_{121} & a_{112} & a_{122} \\ a_{211}+b_{211} & a_{221}+b_{221} & a_{212} & a_{222} \end{array}\right)$$

$$= (a_{111}+b_{111})\cdot a_{222} - a_{112}\cdot(a_{221}+b_{221}) - (a_{121}+b_{121})\cdot a_{212} + a_{122}\cdot(a_{211}+b_{211})$$

$$= a_{111}\cdot a_{222} + b_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{112}\cdot b_{221} - a_{121}\cdot a_{212} - b_{121}\cdot a_{212} + a_{122}\cdot a_{211} + a_{122}\cdot b_{211}$$

If we compare results of above equations, we can see that we have the same result in both cases.

Case 2. The cubic-matrix A of order 3, (and B has order 3), we will proof the case 1 for each "horizontal layer", "vertical page" and "vertical layer", as following:

1. For plan $i = 1$: Let A and B be cubic-matrices of order 3, where all elements on the plan $i = 1$ and $i = 2$ are identical in both matrices, then we have:

$$\det[A_{3\times3\times3}] + \det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$+ \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ b_{311} & b_{321} & b_{331} & b_{312} & b_{322} & b_{332} & b_{313} & b_{323} & b_{333} \end{array}\right)$$

$$\begin{aligned} &= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} \\ &+ a_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} \\ &- a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} - a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} \\ &- a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} \\ &- a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} + a_{131}\cdot a_{212}\cdot a_{323} \\ &- a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\ &+ a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{311} - a_{133}\cdot a_{221}\cdot a_{312} \\ &+ a_{133}\cdot a_{222}\cdot a_{311}\} + \{a_{111}\cdot a_{222}\cdot b_{333} - a_{111}\cdot a_{232}\cdot b_{323} - a_{111}\cdot a_{223}\cdot b_{332} + a_{111}\cdot a_{233}\cdot b_{322} \\ &- a_{112}\cdot a_{221}\cdot b_{333} + a_{112}\cdot a_{223}\cdot b_{331} + a_{112}\cdot a_{231}\cdot b_{323} - a_{112}\cdot a_{233}\cdot b_{321} + a_{113}\cdot a_{221}\cdot b_{332} \\ &- a_{113}\cdot a_{222}\cdot b_{331} - a_{113}\cdot a_{231}\cdot b_{322} + a_{113}\cdot a_{232}\cdot b_{321} - a_{121}\cdot a_{212}\cdot b_{333} + a_{121}\cdot a_{213}\cdot b_{332} \\ &+ a_{121}\cdot a_{232}\cdot b_{313} - a_{121}\cdot a_{233}\cdot b_{312} + a_{122}\cdot a_{211}\cdot b_{333} - a_{122}\cdot a_{213}\cdot b_{331} - a_{122}\cdot a_{231}\cdot b_{313} \\ &+ a_{122}\cdot a_{233}\cdot b_{311} - a_{123}\cdot a_{211}\cdot b_{332} + a_{123}\cdot a_{212}\cdot b_{331} + a_{123}\cdot a_{231}\cdot b_{312} - a_{123}\cdot a_{232}\cdot b_{311} \\ &+ a_{131}\cdot a_{212}\cdot b_{323} - a_{131}\cdot a_{213}\cdot b_{322} - a_{131}\cdot a_{222}\cdot b_{313} + a_{131}\cdot a_{223}\cdot b_{312} - a_{132}\cdot a_{211}\cdot b_{323} \\ &+ a_{132}\cdot a_{213}\cdot b_{321} + a_{132}\cdot a_{221}\cdot b_{313} - a_{132}\cdot a_{223}\cdot b_{311} + a_{133}\cdot a_{211}\cdot b_{322} - a_{133}\cdot a_{212}\cdot b_{311} \\ &- a_{133}\cdot a_{221}\cdot b_{312} + a_{133}\cdot a_{222}\cdot b_{311}\} \end{aligned}$$

while,

$$\det[A_{3\times3\times3} + B_{3\times3\times3}] = \det\left(\begin{array}{ccc} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311}+b_{311} & a_{321}+b_{321} & a_{331}+b_{331} \end{array}\right|$$

$$\left(\begin{array}{ccc|ccc} a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{312}+b_{312} & a_{322}+b_{322} & a_{332}+b_{332} & a_{313}+b_{313} & a_{323}+b_{323} & a_{333}+b_{333} \end{array}\right)$$

$$\begin{aligned}
&= a_{111}\cdot a_{222}\cdot(a_{333}+b_{333}) - a_{111}\cdot a_{232}\cdot(a_{323}+b_{323}) - a_{111}\cdot a_{223}\cdot(a_{332}+b_{332}) + a_{111}\cdot a_{233}\cdot(a_{322}+b_{322}) \\
&-a_{112}\cdot a_{221}\cdot(a_{333}+b_{333}) + a_{112}\cdot a_{223}\cdot(a_{331}+b_{331}) + a_{112}\cdot a_{231}\cdot(a_{323}+b_{323}) - a_{112}\cdot a_{233}\cdot(a_{321}+b_{321}) \\
&+a_{113}\cdot a_{221}\cdot(a_{332}+b_{332}) - a_{113}\cdot a_{222}\cdot(a_{331}+b_{331}) - a_{113}\cdot a_{231}\cdot(a_{322}+b_{322}) + a_{113}\cdot a_{232}\cdot(a_{321}+b_{321}) \\
&-a_{121}\cdot a_{212}\cdot(a_{333}+b_{333}) + a_{121}\cdot a_{213}\cdot(a_{332}+b_{332}) + a_{121}\cdot a_{232}\cdot(a_{313}+b_{313}) - a_{121}\cdot a_{233}\cdot(a_{312}+b_{312}) \\
&+a_{122}\cdot a_{211}\cdot(a_{333}+b_{333}) - a_{122}\cdot a_{213}\cdot(a_{331}+b_{331}) - a_{122}\cdot a_{231}\cdot(a_{313}+b_{313}) + a_{122}\cdot a_{233}\cdot(a_{311}+b_{311}) \\
&-a_{123}\cdot a_{211}\cdot(a_{332}+b_{332}) + a_{123}\cdot a_{212}\cdot(a_{331}+b_{331}) + a_{123}\cdot a_{231}\cdot(a_{312}+b_{312}) - a_{123}\cdot a_{232}\cdot(a_{311}+b_{311}) \\
&+a_{131}\cdot a_{212}\cdot(a_{323}+b_{323}) - a_{131}\cdot a_{213}\cdot(a_{322}+b_{322}) - a_{131}\cdot a_{222}\cdot(a_{313}+b_{313}) + a_{131}\cdot a_{223}\cdot(a_{312}+b_{312}) \\
&-a_{132}\cdot a_{211}\cdot(a_{323}+b_{323}) + a_{132}\cdot a_{213}\cdot(a_{321}+b_{321}) + a_{132}\cdot a_{221}\cdot(a_{313}+b_{313}) - a_{132}\cdot a_{223}\cdot(a_{311}+b_{311}) \\
&+a_{133}\cdot a_{211}\cdot(a_{322}+b_{322}) - a_{133}\cdot a_{212}\cdot(a_{311}+b_{311}) - a_{133}\cdot a_{221}\cdot(a_{312}+b_{312}) + a_{133}\cdot a_{222}\cdot(a_{311}+b_{311}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\det[A_{3\times3\times3}+B_{3\times3\times3}] &= a_{111}\cdot a_{222}\cdot a_{333} + a_{111}\cdot a_{222}\cdot b_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{232}\cdot b_{323} \\
&-a_{111}\cdot a_{223}\cdot a_{332} - a_{111}\cdot a_{223}\cdot b_{332} + a_{111}\cdot a_{233}\cdot a_{322} + a_{111}\cdot a_{233}\cdot b_{322} - a_{112}\cdot a_{221}\cdot a_{333} \\
&-a_{112}\cdot a_{221}\cdot b_{333} + a_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot a_{223}\cdot b_{331} + a_{112}\cdot a_{231}\cdot a_{323} + a_{112}\cdot a_{231}\cdot b_{323} \\
&-a_{112}\cdot a_{233}\cdot a_{321} - a_{112}\cdot a_{233}\cdot b_{321} + a_{113}\cdot a_{221}\cdot a_{332} + a_{113}\cdot a_{221}\cdot b_{332} - a_{113}\cdot a_{222}\cdot a_{331} \\
&-a_{113}\cdot a_{222}\cdot b_{331} - a_{113}\cdot a_{231}\cdot a_{322} - a_{113}\cdot a_{231}\cdot b_{322} + a_{113}\cdot a_{232}\cdot a_{321} + a_{113}\cdot a_{232}\cdot b_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} - a_{121}\cdot a_{212}\cdot b_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{213}\cdot b_{332} + a_{121}\cdot a_{232}\cdot a_{313} \\
&+a_{121}\cdot a_{232}\cdot b_{313} - a_{121}\cdot a_{233}\cdot a_{312} - a_{121}\cdot a_{233}\cdot b_{312} + a_{122}\cdot a_{211}\cdot a_{333} + a_{122}\cdot a_{211}\cdot b_{333} \\
&-a_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot a_{213}\cdot b_{331} - a_{122}\cdot a_{231}\cdot a_{313} - a_{122}\cdot a_{231}\cdot b_{313} + a_{122}\cdot a_{233}\cdot a_{311} \\
&+a_{122}\cdot a_{233}\cdot b_{311} - a_{123}\cdot a_{211}\cdot a_{332} - a_{123}\cdot a_{211}\cdot b_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{212}\cdot b_{331} \\
&+a_{123}\cdot a_{231}\cdot a_{312} + a_{123}\cdot a_{231}\cdot b_{312} - a_{123}\cdot a_{232}\cdot a_{311} - a_{123}\cdot a_{232}\cdot b_{311} + a_{131}\cdot a_{212}\cdot a_{323} \\
&+a_{131}\cdot a_{212}\cdot b_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{213}\cdot b_{322} - a_{131}\cdot a_{222}\cdot a_{313} - a_{131}\cdot a_{222}\cdot b_{313} \\
&+a_{131}\cdot a_{223}\cdot a_{312} + a_{131}\cdot a_{223}\cdot b_{312} - a_{132}\cdot a_{211}\cdot a_{323} - a_{132}\cdot a_{211}\cdot b_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{213}\cdot b_{321} + a_{132}\cdot a_{221}\cdot a_{313} + a_{132}\cdot a_{221}\cdot b_{313} - a_{132}\cdot a_{223}\cdot a_{311} - a_{132}\cdot a_{223}\cdot b_{311} \\
&+a_{133}\cdot a_{211}\cdot a_{322} + a_{133}\cdot a_{211}\cdot b_{322} - a_{133}\cdot a_{212}\cdot a_{311} - a_{133}\cdot a_{212}\cdot b_{311} - a_{133}\cdot a_{221}\cdot a_{312} \\
&-a_{133}\cdot a_{221}\cdot b_{312} + a_{133}\cdot a_{222}\cdot a_{311} + a_{133}\cdot a_{222}\cdot b_{311}.
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

2. For plan $i = 2$: Let A and B be cubic-matrices of order 3, where all elements on the plan $i = 1$ and $i = 3$ are identical in both matrices, then we have:

$$\det[A_{3\times3\times3}] + \det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$+\det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ b_{211} & b_{221} & b_{231} & b_{212} & b_{222} & b_{232} & b_{213} & b_{223} & b_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} \\
&+a_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} \\
&-a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} - a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} \\
&-a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} \\
&-a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} + a_{131}\cdot a_{212}\cdot a_{323} \\
&-a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{311} - a_{133}\cdot a_{221}\cdot a_{312}
\end{aligned}$$

$$+a_{133}\cdot a_{222}\cdot a_{311}\}+\{a_{111}\cdot b_{222}\cdot a_{333}-a_{111}\cdot b_{232}\cdot a_{323}-a_{111}\cdot b_{223}\cdot a_{332}+a_{111}\cdot b_{233}\cdot a_{322}$$
$$-a_{112}\cdot b_{221}\cdot a_{333}+a_{112}\cdot b_{223}\cdot a_{331}+a_{112}\cdot b_{231}\cdot a_{323}-a_{112}\cdot b_{233}\cdot a_{321}+a_{113}\cdot b_{221}\cdot a_{332}$$
$$-a_{113}\cdot b_{222}\cdot a_{331}-a_{113}\cdot b_{231}\cdot a_{322}+a_{113}\cdot b_{232}\cdot a_{321}-a_{121}\cdot b_{212}\cdot a_{333}+a_{121}\cdot b_{213}\cdot a_{332}$$
$$+a_{121}\cdot b_{232}\cdot a_{313}-a_{121}\cdot b_{233}\cdot a_{312}+a_{122}\cdot b_{211}\cdot a_{333}-a_{122}\cdot b_{213}\cdot a_{331}-a_{122}\cdot b_{231}\cdot a_{313}$$
$$+a_{122}\cdot b_{233}\cdot a_{311}-a_{123}\cdot b_{211}\cdot a_{332}+a_{123}\cdot b_{212}\cdot a_{331}+a_{123}\cdot b_{231}\cdot a_{312}-a_{123}\cdot b_{232}\cdot a_{311}$$
$$+a_{131}\cdot b_{212}\cdot a_{323}-a_{131}\cdot b_{213}\cdot a_{322}-a_{131}\cdot b_{222}\cdot a_{313}+a_{131}\cdot b_{223}\cdot a_{312}-a_{132}\cdot b_{211}\cdot a_{323}$$
$$+a_{132}\cdot b_{213}\cdot a_{321}+a_{132}\cdot b_{221}\cdot a_{313}-a_{132}\cdot b_{223}\cdot a_{311}+a_{133}\cdot b_{211}\cdot a_{322}-a_{133}\cdot b_{212}\cdot a_{311}$$
$$-a_{133}\cdot b_{221}\cdot a_{312}+a_{133}\cdot b_{222}\cdot a_{311}\}$$

while,

$$\det[A_{3\times3\times3}+B_{3\times3\times3}]=\det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133}\\ a_{211}+b_{211} & a_{221}+b_{221} & a_{231}+b_{231} & a_{212}+b_{212} & a_{222}+b_{222} & a_{232}+b_{232} & a_{213}+b_{213} & a_{223}+b_{223} & a_{233}+b_{233}\\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333}\end{array}\right)$$

$$=a_{111}\cdot(a_{222}+b_{222})\cdot a_{333}-a_{111}\cdot(a_{232}+b_{232})\cdot a_{323}-a_{111}\cdot(a_{223}+b_{223})\cdot a_{332}+a_{111}\cdot(a_{233}+b_{233})\cdot a_{322}$$
$$-a_{112}\cdot(a_{221}+b_{221})\cdot a_{333}+a_{112}\cdot(a_{223}+b_{223})\cdot a_{331}+a_{112}\cdot(a_{231}+b_{231})\cdot a_{323}-a_{112}\cdot(a_{233}+b_{233})\cdot a_{321}$$
$$+a_{113}\cdot(a_{221}+b_{221})\cdot a_{332}-a_{113}\cdot(a_{222}+b_{222})\cdot a_{331}-a_{113}\cdot(a_{231}+b_{231})\cdot a_{322}+a_{113}\cdot(a_{232}+b_{232})\cdot a_{321}$$
$$-a_{121}\cdot(a_{212}+b_{212})\cdot a_{333}+a_{121}\cdot(a_{213}+b_{213})\cdot a_{332}+a_{121}\cdot(a_{232}+b_{232})\cdot a_{313}-a_{121}\cdot(a_{233}+b_{233})\cdot a_{312}$$
$$+a_{122}\cdot(a_{211}+b_{211})\cdot a_{333}-a_{122}\cdot(a_{213}+b_{213})\cdot a_{331}-a_{122}\cdot(a_{231}+b_{231})\cdot a_{313}+a_{122}\cdot(a_{233}+b_{233})\cdot a_{311}$$
$$-a_{123}\cdot(a_{211}+b_{211})\cdot a_{332}+a_{123}\cdot(a_{212}+b_{212})\cdot a_{331}+a_{123}\cdot(a_{231}+b_{231})\cdot a_{312}-a_{123}\cdot(a_{232}+b_{232})\cdot a_{311}$$
$$+a_{131}\cdot(a_{212}+b_{212})\cdot a_{323}-a_{131}\cdot(a_{213}+b_{213})\cdot a_{322}-a_{131}\cdot(a_{222}+b_{222})\cdot a_{313}+a_{131}\cdot(a_{223}+b_{223})\cdot a_{312}$$
$$-a_{132}\cdot(a_{211}+b_{211})\cdot a_{323}+a_{132}\cdot(a_{213}+b_{213})\cdot a_{321}+a_{132}\cdot(a_{221}+b_{221})\cdot a_{313}-a_{132}\cdot(a_{223}+b_{223})\cdot a_{311}$$
$$+a_{133}\cdot(a_{211}+b_{211})\cdot a_{322}-a_{133}\cdot(a_{212}+b_{212})\cdot a_{311}-a_{133}\cdot(a_{221}+b_{221})\cdot a_{312}+a_{133}\cdot(a_{222}+b_{222})\cdot a_{311}.$$

Hence,

$$\det[A_{3\times3\times3}+B_{3\times3\times3}]=a_{111}\cdot a_{222}\cdot a_{333}+a_{111}\cdot b_{222}\cdot a_{333}-a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot b_{232}\cdot a_{323}$$
$$-a_{111}\cdot a_{223}\cdot a_{332}-a_{111}\cdot b_{223}\cdot a_{332}+a_{111}\cdot a_{233}\cdot a_{322}+a_{111}\cdot b_{233}\cdot a_{322}-a_{112}\cdot a_{221}\cdot a_{333}-a_{112}\cdot b_{221}\cdot a_{333}$$
$$+a_{112}\cdot a_{223}\cdot a_{331}+a_{112}\cdot b_{223}\cdot a_{331}+a_{112}\cdot a_{231}\cdot a_{323}+a_{112}\cdot b_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}-a_{112}\cdot b_{233}\cdot a_{321}$$
$$+a_{113}\cdot a_{221}\cdot a_{332}+a_{113}\cdot b_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot b_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}-a_{113}\cdot b_{231}\cdot a_{322}$$
$$+a_{113}\cdot a_{232}\cdot a_{321}+a_{113}\cdot b_{232}\cdot a_{321}-a_{121}\cdot a_{212}\cdot a_{333}-a_{121}\cdot b_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot b_{213}\cdot a_{332}$$
$$+a_{121}\cdot a_{232}\cdot a_{313}+a_{121}\cdot b_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot a_{312}-a_{121}\cdot b_{233}\cdot a_{312}+a_{122}\cdot a_{211}\cdot a_{333}+a_{122}\cdot b_{211}\cdot a_{333}$$
$$-a_{122}\cdot a_{213}\cdot a_{331}-a_{122}\cdot b_{213}\cdot a_{331}-a_{122}\cdot a_{231}\cdot a_{313}-a_{122}\cdot b_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}+a_{122}\cdot b_{233}\cdot a_{311}$$
$$-a_{123}\cdot a_{211}\cdot a_{332}-a_{123}\cdot b_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot b_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}+a_{123}\cdot b_{231}\cdot a_{312}$$
$$-a_{123}\cdot a_{232}\cdot a_{311}-a_{123}\cdot b_{232}\cdot a_{311}+a_{131}\cdot a_{212}\cdot a_{323}+a_{131}\cdot b_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot b_{213}\cdot a_{322}$$
$$-a_{131}\cdot a_{222}\cdot a_{313}-a_{131}\cdot b_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}+a_{131}\cdot b_{223}\cdot a_{312}-a_{132}\cdot a_{211}\cdot a_{323}-a_{132}\cdot b_{211}\cdot a_{323}$$
$$+a_{132}\cdot a_{213}\cdot a_{321}+a_{132}\cdot b_{213}\cdot a_{321}+a_{132}\cdot a_{221}\cdot a_{313}+a_{132}\cdot b_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}-a_{132}\cdot b_{223}\cdot a_{311}$$
$$+a_{133}\cdot a_{211}\cdot a_{322}+a_{133}\cdot b_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{311}-a_{133}\cdot b_{212}\cdot a_{311}-a_{133}\cdot a_{221}\cdot a_{312}-a_{133}\cdot b_{221}\cdot a_{312}$$
$$+a_{133}\cdot a_{222}\cdot a_{311}+a_{133}\cdot b_{222}\cdot a_{311}$$

If we compare results of above equations, we can see that we have the same result in both cases.

3. For plan $i=3$: Let A and B be cubic-matrices of order 3, where all elements on the plan $i=2$ and $i=3$ are identical in both matrices, then we have:

$$\det[A_{3\times3\times3}]+\det[B_{3\times3\times3}]=\det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133}\\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233}\\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333}\end{array}\right)$$

$$+\det\left(\begin{array}{ccc|ccc|ccc} b_{111} & b_{121} & b_{131} & b_{112} & b_{122} & b_{132} & b_{113} & b_{123} & b_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{311} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}\} \\
&+\{b_{111}\cdot a_{222}\cdot a_{333} - b_{111}\cdot a_{232}\cdot a_{323} - b_{111}\cdot a_{223}\cdot a_{332} + b_{111}\cdot a_{233}\cdot a_{322} - b_{112}\cdot a_{221}\cdot a_{333} + b_{112}\cdot a_{223}\cdot a_{331} \\
&+b_{112}\cdot a_{231}\cdot a_{323} - b_{112}\cdot a_{233}\cdot a_{321} + b_{113}\cdot a_{221}\cdot a_{332} - b_{113}\cdot a_{222}\cdot a_{331} - b_{113}\cdot a_{231}\cdot a_{322} + b_{113}\cdot a_{232}\cdot a_{321} \\
&-b_{121}\cdot a_{212}\cdot a_{333} + b_{121}\cdot a_{213}\cdot a_{332} + b_{121}\cdot a_{232}\cdot a_{313} - b_{121}\cdot a_{233}\cdot a_{312} + b_{122}\cdot a_{211}\cdot a_{333} - b_{122}\cdot a_{213}\cdot a_{331} \\
&-b_{122}\cdot a_{231}\cdot a_{313} + b_{122}\cdot a_{233}\cdot a_{311} - b_{123}\cdot a_{211}\cdot a_{332} + b_{123}\cdot a_{212}\cdot a_{331} + b_{123}\cdot a_{231}\cdot a_{312} - b_{123}\cdot a_{232}\cdot a_{311} \\
&+b_{131}\cdot a_{212}\cdot a_{323} - b_{131}\cdot a_{213}\cdot a_{322} - b_{131}\cdot a_{222}\cdot a_{313} + b_{131}\cdot a_{223}\cdot a_{312} - b_{132}\cdot a_{211}\cdot a_{323} + b_{132}\cdot a_{213}\cdot a_{321} \\
&+b_{132}\cdot a_{221}\cdot a_{313} - b_{132}\cdot a_{223}\cdot a_{311} + b_{133}\cdot a_{211}\cdot a_{322} - b_{133}\cdot a_{212}\cdot a_{311} - b_{133}\cdot a_{221}\cdot a_{312} + b_{133}\cdot a_{222}\cdot a_{311}\}
\end{aligned}$$

while,

$$\det[A_{3\times3\times3} + B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111}+b_{111} & a_{121}+b_{121} & a_{131}+b_{131} & a_{112}+b_{112} & a_{122}+b_{122} & a_{132}+b_{132} & a_{113}+b_{113} & a_{123}+b_{123} & a_{133}+b_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= (a_{111}+b_{111})\cdot a_{222}\cdot a_{333} - (a_{111}+b_{111})\cdot a_{232}\cdot a_{323} - (a_{111}+b_{111})\cdot a_{223}\cdot a_{332} + (a_{111}+b_{111})\cdot a_{233}\cdot a_{322} \\
&-(a_{112}+b_{112})\cdot a_{221}\cdot a_{333} + (a_{112}+b_{112})\cdot a_{223}\cdot a_{331} + (a_{112}+b_{112})\cdot a_{231}\cdot a_{323} - (a_{112}+b_{112})\cdot a_{233}\cdot a_{321} \\
&+(a_{113}+b_{113})\cdot a_{221}\cdot a_{332} - (a_{113}+b_{113})\cdot a_{222}\cdot a_{331} - (a_{113}+b_{113})\cdot a_{231}\cdot a_{322} + (a_{113}+b_{113})\cdot a_{232}\cdot a_{321} \\
&-(a_{121}+b_{121})\cdot a_{212}\cdot a_{333} + (a_{121}+b_{121})\cdot a_{213}\cdot a_{332} + (a_{121}+b_{121})\cdot a_{232}\cdot a_{313} - (a_{121}+b_{121})\cdot a_{233}\cdot a_{312} \\
&+(a_{122}+b_{122})\cdot a_{211}\cdot a_{333} - (a_{122}+b_{122})\cdot a_{213}\cdot a_{331} - (a_{122}+b_{122})\cdot a_{231}\cdot a_{313} + (a_{122}+b_{122})\cdot a_{233}\cdot a_{311} \\
&-(a_{123}+b_{123})\cdot a_{211}\cdot a_{332} + (a_{123}+b_{123})\cdot a_{212}\cdot a_{331} + (a_{123}+b_{123})\cdot a_{231}\cdot a_{312} - (a_{123}+b_{123})\cdot a_{232}\cdot a_{311} \\
&+(a_{131}+b_{131})\cdot a_{212}\cdot a_{323} - (a_{131}+b_{131})\cdot a_{213}\cdot a_{322} - (a_{131}+b_{131})\cdot a_{222}\cdot a_{313} + (a_{131}+b_{131})\cdot a_{223}\cdot a_{312} \\
&-(a_{132}+b_{132})\cdot a_{211}\cdot a_{323} + (a_{132}+b_{132})\cdot a_{213}\cdot a_{321} + (a_{132}+b_{132})\cdot a_{221}\cdot a_{313} - (a_{132}+b_{132})\cdot a_{223}\cdot a_{311} \\
&+(a_{133}+b_{133})\cdot a_{211}\cdot a_{322} - (a_{133}+b_{133})\cdot a_{212}\cdot a_{311} - (a_{133}+b_{133})\cdot a_{221}\cdot a_{312} + (a_{133}+b_{133})\cdot a_{222}\cdot a_{311}
\end{aligned}$$

Hence,

$$\begin{aligned}
\det[A_{3\times3\times3}] + B_{3\times3\times3}] &= a_{111}\cdot a_{222}\cdot a_{333} + b_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - b_{111}\cdot a_{232}\cdot a_{323} \\
&-a_{111}\cdot a_{223}\cdot a_{332} - b_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} + b_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} - b_{112}\cdot a_{221}\cdot a_{333} \\
&+a_{112}\cdot a_{223}\cdot a_{331} + b_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot a_{231}\cdot a_{323} + b_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} - b_{112}\cdot a_{233}\cdot a_{321} \\
&+a_{113}\cdot a_{221}\cdot a_{332} + b_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - b_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} - b_{113}\cdot a_{231}\cdot a_{322} \\
&+a_{113}\cdot a_{232}\cdot a_{321} + b_{113}\cdot a_{232}\cdot a_{321} - a_{121}\cdot a_{212}\cdot a_{333} - b_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + b_{121}\cdot a_{213}\cdot a_{332} \\
&+a_{121}\cdot a_{232}\cdot a_{313} + b_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} - b_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} + b_{122}\cdot a_{211}\cdot a_{333} \\
&-a_{122}\cdot a_{213}\cdot a_{331} - b_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot a_{231}\cdot a_{313} - b_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} + b_{122}\cdot a_{233}\cdot a_{311} \\
&-a_{123}\cdot a_{211}\cdot a_{332} - b_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + b_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} + b_{123}\cdot a_{231}\cdot a_{312} \\
&-a_{123}\cdot a_{232}\cdot a_{311} - b_{123}\cdot a_{232}\cdot a_{311} + a_{131}\cdot a_{212}\cdot a_{323} + b_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - b_{131}\cdot a_{213}\cdot a_{322} \\
&-a_{131}\cdot a_{222}\cdot a_{313} - b_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} + b_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} - b_{132}\cdot a_{211}\cdot a_{323} \\
&+a_{132}\cdot a_{213}\cdot a_{321} + b_{132}\cdot a_{213}\cdot a_{321} + a_{132}\cdot a_{221}\cdot a_{313} + b_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} - b_{132}\cdot a_{223}\cdot a_{311} \\
&+a_{133}\cdot a_{211}\cdot a_{322} + b_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{311} - b_{133}\cdot a_{212}\cdot a_{311} - a_{133}\cdot a_{221}\cdot a_{312} - b_{133}\cdot a_{221}\cdot a_{312} \\
&+a_{133}\cdot a_{222}\cdot a_{311} + b_{133}\cdot a_{222}\cdot a_{311}.
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

4. For plan $j = 1$: Let A and B be cubic-matrices of order 3, where all elements on the plan $j = 1$ and $j = 2$ are identical in both matrices, then have:

$$\det[A_{3\times3\times3}] + \det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$+ \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & b_{131} & a_{112} & a_{122} & b_{132} & a_{113} & a_{123} & b_{133} \\ a_{211} & a_{221} & b_{231} & a_{212} & a_{222} & b_{232} & a_{213} & a_{223} & b_{233} \\ a_{311} & a_{321} & b_{331} & a_{312} & a_{322} & b_{332} & a_{313} & a_{323} & b_{333} \end{array}\right)$$

$$\begin{aligned}
&= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{311} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}\} \\
&+\{a_{111}\cdot a_{222}\cdot b_{333} - a_{111}\cdot b_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot b_{332} + a_{111}\cdot b_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot b_{333} + a_{112}\cdot a_{223}\cdot b_{331} \\
&+a_{112}\cdot b_{231}\cdot a_{323} - a_{112}\cdot b_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot b_{332} - a_{113}\cdot a_{222}\cdot b_{331} - a_{113}\cdot b_{231}\cdot a_{322} + a_{113}\cdot b_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot b_{333} + a_{121}\cdot a_{213}\cdot b_{332} + a_{121}\cdot b_{232}\cdot a_{313} - a_{121}\cdot b_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot b_{333} - a_{122}\cdot a_{213}\cdot b_{331} \\
&-a_{122}\cdot b_{231}\cdot a_{313} + a_{122}\cdot b_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot b_{332} + a_{123}\cdot a_{212}\cdot b_{331} + a_{123}\cdot b_{231}\cdot a_{312} - a_{123}\cdot b_{232}\cdot a_{311} \\
&+b_{131}\cdot a_{212}\cdot a_{323} - b_{131}\cdot a_{213}\cdot a_{322} - b_{131}\cdot a_{222}\cdot a_{313} + b_{131}\cdot a_{223}\cdot a_{312} - b_{132}\cdot a_{211}\cdot a_{323} + b_{132}\cdot a_{213}\cdot a_{321} \\
&+b_{132}\cdot a_{221}\cdot a_{313} - b_{132}\cdot a_{223}\cdot a_{311} + b_{133}\cdot a_{211}\cdot a_{322} - b_{133}\cdot a_{212}\cdot a_{311} - b_{133}\cdot a_{221}\cdot a_{312} + b_{133}\cdot a_{222}\cdot a_{311}\}
\end{aligned}$$

while,

$$\det[A_{3\times3\times3} + B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131}+b_{131} & a_{112} & a_{122} & a_{132}+b_{132} & a_{113} & a_{123} & a_{133}+b_{133} \\ a_{211} & a_{221} & a_{231}+b_{231} & a_{212} & a_{222} & a_{232}+b_{232} & a_{213} & a_{223} & a_{233}+b_{233} \\ a_{311} & a_{321} & a_{331}+b_{331} & a_{312} & a_{322} & a_{332}+b_{332} & a_{313} & a_{323} & a_{333}+b_{333} \end{array}\right)$$

$$\begin{aligned}
&= a_{111}\cdot a_{222}\cdot(a_{333}+b_{333}) - a_{111}\cdot(a_{232}+b_{232})\cdot a_{323} - a_{111}\cdot a_{223}\cdot(a_{332}+b_{332}) + a_{111}\cdot(a_{233}+b_{233})\cdot a_{322} \\
&-a_{112}\cdot a_{221}\cdot(a_{333}+b_{333}) + a_{112}\cdot a_{223}\cdot(a_{331}+b_{331}) + a_{112}\cdot(a_{231}+b_{231})\cdot a_{323} - a_{112}\cdot(a_{233}+b_{233})\cdot a_{321} \\
&+a_{113}\cdot a_{221}\cdot(a_{332}+b_{332}) - a_{113}\cdot a_{222}\cdot(a_{331}+b_{331}) - a_{113}\cdot(a_{231}+b_{231})\cdot a_{322} + a_{113}\cdot(a_{232}+b_{232})\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot(a_{333}+b_{333}) + a_{121}\cdot a_{213}\cdot(a_{332}+b_{332}) + a_{121}\cdot(a_{232}+b_{232})\cdot a_{313} - a_{121}\cdot(a_{233}+b_{233})\cdot a_{312} \\
&+a_{122}\cdot a_{211}\cdot(a_{333}+b_{333}) - a_{122}\cdot a_{213}\cdot(a_{331}+b_{331}) - a_{122}\cdot(a_{231}+b_{231})\cdot a_{313} + a_{122}\cdot(a_{233}+b_{233})\cdot a_{311} \\
&-a_{123}\cdot a_{211}\cdot(a_{332}+b_{332}) + a_{123}\cdot a_{212}\cdot(a_{331}+b_{331}) + a_{123}\cdot(a_{231}+b_{231})\cdot a_{312} - a_{123}\cdot(a_{232}+b_{232})\cdot a_{311} \\
&+(a_{131}+b_{131})\cdot a_{212}\cdot a_{323} - (a_{131}+b_{131})\cdot a_{213}\cdot a_{322} - (a_{131}+b_{131})\cdot a_{222}\cdot a_{313} + (a_{131}+b_{131})\cdot a_{223}\cdot a_{312} \\
&-(a_{132}+b_{132})\cdot a_{211}\cdot a_{323} + (a_{132}+b_{132})\cdot a_{213}\cdot a_{321} + (a_{132}+b_{132})\cdot a_{221}\cdot a_{313} - (a_{132}+b_{132})\cdot a_{223}\cdot a_{311} \\
&+(a_{133}+b_{133})\cdot a_{211}\cdot a_{322} - (a_{133}+b_{133})\cdot a_{212}\cdot a_{311} - (a_{133}+b_{133})\cdot a_{221}\cdot a_{312} + (a_{133}+b_{133})\cdot a_{222}\cdot a_{311} \\
&= a_{111}\cdot a_{222}\cdot a_{333} + a_{111}\cdot a_{222}\cdot b_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot b_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} - a_{111}\cdot a_{223}\cdot b_{332} \\
&+a_{111}\cdot a_{233}\cdot a_{322} + a_{111}\cdot b_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} - a_{112}\cdot a_{221}\cdot b_{333} + a_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot b_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} + a_{112}\cdot b_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} - a_{112}\cdot b_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} + a_{113}\cdot a_{221}\cdot b_{332} \\
&-a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{222}\cdot b_{331} - a_{113}\cdot a_{231}\cdot a_{322} - a_{113}\cdot b_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} + a_{113}\cdot b_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} - a_{121}\cdot a_{212}\cdot b_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{213}\cdot b_{332} + a_{121}\cdot a_{232}\cdot a_{313} + a_{121}\cdot b_{232}\cdot a_{313} \\
&-a_{121}\cdot a_{233}\cdot a_{312} - a_{121}\cdot b_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} + a_{122}\cdot a_{211}\cdot b_{333} - a_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot a_{213}\cdot b_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} - a_{122}\cdot b_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} + a_{122}\cdot b_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} - a_{123}\cdot a_{211}\cdot b_{332} \\
&+a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{212}\cdot b_{331} + a_{123}\cdot a_{231}\cdot a_{312} + a_{123}\cdot b_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} - a_{123}\cdot b_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} + b_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - b_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} - b_{131}\cdot a_{222}\cdot a_{313} \\
&+a_{131}\cdot a_{223}\cdot a_{312} + b_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} - b_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} + b_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} + b_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} - b_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} + b_{133}\cdot a_{211}\cdot a_{322} \\
&-a_{133}\cdot a_{212}\cdot a_{311} - b_{133}\cdot a_{212}\cdot a_{311} - a_{133}\cdot a_{221}\cdot a_{312} - b_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311} + b_{133}\cdot a_{222}\cdot a_{311}
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

5. For plan $j = 2$: Let A and B be cubic-matrices of order 3, where all elements on the plan $j = 1$ and $j = 3$ are identical in both matrices, then have:

$$\det[A_{3\times3\times3}] + \det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$+ \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & b_{121} & a_{131} & a_{112} & b_{122} & a_{132} & a_{113} & b_{123} & a_{133} \\ a_{211} & b_{221} & a_{231} & a_{212} & b_{222} & a_{232} & a_{213} & b_{223} & a_{233} \\ a_{311} & b_{321} & a_{331} & a_{312} & b_{322} & a_{332} & a_{313} & b_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}\} \\
&+\{a_{111}\cdot b_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot b_{323} - a_{111}\cdot b_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot b_{322} - a_{112}\cdot b_{221}\cdot a_{333} + a_{112}\cdot b_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot b_{323} - a_{112}\cdot a_{233}\cdot b_{321} + a_{113}\cdot b_{221}\cdot a_{332} - a_{113}\cdot b_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot b_{322} + a_{113}\cdot a_{232}\cdot b_{321} \\
&-b_{121}\cdot a_{212}\cdot a_{333} + b_{121}\cdot a_{213}\cdot a_{332} + b_{121}\cdot a_{232}\cdot a_{313} - b_{121}\cdot a_{233}\cdot a_{312} + b_{122}\cdot a_{211}\cdot a_{333} - b_{122}\cdot a_{213}\cdot a_{331} \\
&-b_{122}\cdot a_{231}\cdot a_{313} + b_{122}\cdot a_{233}\cdot a_{311} - b_{123}\cdot a_{211}\cdot a_{332} + b_{123}\cdot a_{212}\cdot a_{331} + b_{123}\cdot a_{231}\cdot a_{312} - b_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot b_{323} - a_{131}\cdot a_{213}\cdot b_{322} - a_{131}\cdot b_{222}\cdot a_{313} + a_{131}\cdot b_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot b_{323} + a_{132}\cdot a_{213}\cdot b_{321} \\
&+a_{132}\cdot b_{221}\cdot a_{313} - a_{132}\cdot b_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot b_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot b_{221}\cdot a_{312} + a_{133}\cdot b_{222}\cdot a_{311}\}
\end{aligned}$$

while,

$$\det[A_{3\times3\times3} + B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121}+b_{121} & a_{131} & a_{112} & a_{122}+b_{122} & a_{132} & a_{113} & a_{123}+b_{123} & a_{133} \\ a_{211} & a_{221}+b_{221} & a_{231} & a_{212} & a_{222}+b_{222} & a_{232} & a_{213} & a_{223}+b_{223} & a_{233} \\ a_{311} & a_{321}+b_{321} & a_{331} & a_{312} & a_{322}+b_{322} & a_{332} & a_{313} & a_{323}+b_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= a_{111}\cdot(a_{222}+b_{222})\cdot a_{333} - a_{111}\cdot a_{232}\cdot(a_{323}+b_{323}) - a_{111}\cdot(a_{223}+b_{223})\cdot a_{332} + a_{111}\cdot a_{233}\cdot(a_{322}+b_{322}) \\
&-a_{112}\cdot(a_{221}+b_{221})\cdot a_{333} + a_{112}\cdot(a_{223}+b_{223})\cdot a_{331} + a_{112}\cdot a_{231}\cdot(a_{323}+b_{323}) - a_{112}\cdot a_{233}\cdot(a_{321}+b_{321}) \\
&+a_{113}\cdot(a_{221}+b_{221})\cdot a_{332} - a_{113}\cdot(a_{222}+b_{222})\cdot a_{331} - a_{113}\cdot a_{231}\cdot(a_{322}+b_{322}) + a_{113}\cdot a_{232}\cdot(a_{321}+b_{321}) \\
&-(a_{121}+b_{121})\cdot a_{212}\cdot a_{333} + (a_{121}+b_{121})\cdot a_{213}\cdot a_{332} + (a_{121}+b_{121})\cdot a_{232}\cdot a_{313} - (a_{121}+b_{121})\cdot a_{233}\cdot a_{312} \\
&+(a_{122}+b_{122})\cdot a_{211}\cdot a_{333} - (a_{122}+b_{122})\cdot a_{213}\cdot a_{331} - (a_{122}+b_{122})\cdot a_{231}\cdot a_{313} + (a_{122}+b_{122})\cdot a_{233}\cdot a_{311} \\
&-(a_{123}+b_{123})\cdot a_{211}\cdot a_{332} + (a_{123}+b_{123})\cdot a_{212}\cdot a_{331} + (a_{123}+b_{123})\cdot a_{231}\cdot a_{312} - (a_{123}+b_{123})\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot(a_{323}+b_{323}) - a_{131}\cdot a_{213}\cdot(a_{322}+b_{322}) - a_{131}\cdot(a_{222}+b_{222})\cdot a_{313} + a_{131}\cdot(a_{223}+b_{223})\cdot a_{312} \\
&-a_{132}\cdot a_{211}\cdot(a_{323}+b_{323}) + a_{132}\cdot a_{213}\cdot(a_{321}+b_{321}) + a_{132}\cdot(a_{221}+b_{221})\cdot a_{313} - a_{132}\cdot(a_{223}+b_{223})\cdot a_{311} \\
&+a_{133}\cdot a_{211}\cdot(a_{322}+b_{322}) - a_{133}\cdot a_{212}\cdot(a_{321}+b_{321}) - a_{133}\cdot(a_{221}+b_{221})\cdot a_{312} + a_{133}\cdot(a_{222}+b_{222})\cdot a_{311} \\
&= a_{111}\cdot a_{222}\cdot a_{333} + a_{111}\cdot b_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{232}\cdot b_{323} - a_{111}\cdot a_{223}\cdot a_{332} - a_{111}\cdot b_{223}\cdot a_{332} \\
&+a_{111}\cdot a_{233}\cdot a_{322} + a_{111}\cdot a_{233}\cdot b_{322} - a_{112}\cdot a_{221}\cdot a_{333} - a_{112}\cdot b_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot b_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} + a_{112}\cdot a_{231}\cdot b_{323} - a_{112}\cdot a_{233}\cdot a_{321} - a_{112}\cdot a_{233}\cdot b_{321} + a_{113}\cdot a_{221}\cdot a_{332} + a_{113}\cdot b_{221}\cdot a_{332} \\
&-a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot b_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} - a_{113}\cdot a_{231}\cdot b_{322} + a_{113}\cdot a_{232}\cdot a_{321} + a_{113}\cdot a_{232}\cdot b_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} - b_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + b_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} + b_{121}\cdot a_{232}\cdot a_{313} \\
&-a_{121}\cdot a_{233}\cdot a_{312} - b_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} + b_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} - b_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} - b_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} + b_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} - b_{123}\cdot a_{211}\cdot a_{332} \\
&+a_{123}\cdot a_{212}\cdot a_{331} + b_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} + b_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} - b_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} + a_{131}\cdot a_{212}\cdot b_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{213}\cdot b_{322} - a_{131}\cdot a_{222}\cdot a_{313} - a_{131}\cdot b_{222}\cdot a_{313} \\
&+a_{131}\cdot a_{223}\cdot a_{312} + a_{131}\cdot b_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} - a_{132}\cdot a_{211}\cdot b_{323} + a_{132}\cdot a_{213}\cdot a_{321} + a_{132}\cdot a_{213}\cdot b_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} + a_{132}\cdot b_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} - a_{132}\cdot b_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} + a_{133}\cdot a_{211}\cdot b_{322} \\
&-a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{212}\cdot b_{321} - a_{133}\cdot a_{221}\cdot a_{312} - a_{133}\cdot b_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311} + a_{133}\cdot b_{222}\cdot a_{311}
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

6. For plan $j = 3$: Let A and B be cubic-matrices of order 3, where all elements on the plan $j = 3$ and $j = 3$ are identical in both matrices, then have:

$$\det[A_{3\times3\times3}] + \det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$+ \det\left(\begin{array}{ccc|ccc|ccc} b_{111} & a_{121} & a_{131} & b_{112} & a_{122} & a_{132} & b_{113} & a_{123} & a_{133} \\ b_{211} & a_{221} & a_{231} & b_{212} & a_{222} & a_{232} & b_{213} & a_{223} & a_{233} \\ b_{311} & a_{321} & a_{331} & b_{312} & a_{322} & a_{332} & b_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}\} \\
&+\{b_{111}\cdot a_{222}\cdot a_{333} - b_{111}\cdot a_{232}\cdot a_{323} - b_{111}\cdot a_{223}\cdot a_{332} + b_{111}\cdot a_{233}\cdot a_{322} - b_{112}\cdot a_{221}\cdot a_{333} + b_{112}\cdot a_{223}\cdot a_{331} \\
&+b_{112}\cdot a_{231}\cdot a_{323} - b_{112}\cdot a_{233}\cdot a_{321} + b_{113}\cdot a_{221}\cdot a_{332} - b_{113}\cdot a_{222}\cdot a_{331} - b_{113}\cdot a_{231}\cdot a_{322} + b_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot b_{212}\cdot a_{333} + a_{121}\cdot b_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot b_{313} - a_{121}\cdot a_{233}\cdot b_{312} + a_{122}\cdot b_{211}\cdot a_{333} - a_{122}\cdot b_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot b_{313} + a_{122}\cdot a_{233}\cdot b_{311} - a_{123}\cdot b_{211}\cdot a_{332} + a_{123}\cdot b_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot b_{312} - a_{123}\cdot a_{232}\cdot b_{311} \\
&+a_{131}\cdot b_{212}\cdot a_{323} - a_{131}\cdot b_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot b_{313} + a_{131}\cdot a_{223}\cdot b_{312} - a_{132}\cdot b_{211}\cdot a_{323} + a_{132}\cdot b_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot b_{313} - a_{132}\cdot a_{223}\cdot b_{311} + a_{133}\cdot b_{211}\cdot a_{322} - a_{133}\cdot b_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot b_{312} + a_{133}\cdot a_{222}\cdot b_{311}\}
\end{aligned}$$

while,

$$\det[A_{3\times3\times3} + B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111}+b_{111} & a_{121} & a_{131} & a_{112}+b_{112} & a_{122} & a_{132} & a_{113}+b_{113} & a_{123} & a_{133} \\ a_{211}+b_{211} & a_{221} & a_{231} & a_{212}+b_{212} & a_{222} & a_{232} & a_{213}+b_{213} & a_{223} & a_{233} \\ a_{311}+b_{311} & a_{321} & a_{331} & a_{312}+b_{312} & a_{322} & a_{332} & a_{131}+b_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= (a_{111}+b_{111})\cdot a_{222}\cdot a_{333} - (a_{111}+b_{111})\cdot a_{232}\cdot a_{323} - (a_{111}+b_{111})\cdot a_{223}\cdot a_{332} + (a_{111}+b_{111})\cdot a_{233}\cdot a_{322} \\
&-(a_{112}+b_{112})\cdot a_{221}\cdot a_{333} + (a_{112}+b_{112})\cdot a_{223}\cdot a_{331} + (a_{112}+b_{112})\cdot a_{231}\cdot a_{323} - (a_{112}+b_{112})\cdot a_{233}\cdot a_{321} \\
&+(a_{113}+b_{113})\cdot a_{221}\cdot a_{332} - (a_{113}+b_{113})\cdot a_{222}\cdot a_{331} - (a_{113}+b_{113})\cdot a_{231}\cdot a_{322} + (a_{113}+b_{113})\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot (a_{212}+b_{212})\cdot a_{333} + a_{121}\cdot (a_{213}+b_{213})\cdot a_{332} + a_{121}\cdot a_{232}\cdot (a_{313}+b_{313}) - a_{121}\cdot a_{233}\cdot (a_{312}+b_{312}) \\
&+a_{122}\cdot (a_{211}+b_{211})\cdot a_{333} - a_{122}\cdot (a_{213}+b_{213})\cdot a_{331} - a_{122}\cdot a_{231}\cdot (a_{313}+b_{313}) + a_{122}\cdot a_{233}\cdot (a_{311}+b_{311}) \\
&-a_{123}\cdot (a_{211}+b_{211})\cdot a_{332} + a_{123}\cdot (a_{212}+b_{212})\cdot a_{331} + a_{123}\cdot a_{231}\cdot (a_{312}+b_{312}) - a_{123}\cdot a_{232}\cdot (a_{311}+b_{311}) \\
&+a_{131}\cdot (a_{212}+b_{212})\cdot a_{323} - a_{131}\cdot (a_{213}+b_{213})\cdot a_{322} - a_{131}\cdot a_{222}\cdot (a_{313}+b_{313}) + a_{131}\cdot a_{223}\cdot (a_{312}+b_{312}) \\
&-a_{132}\cdot (a_{211}+b_{211})\cdot a_{323} + a_{132}\cdot (a_{213}+b_{213})\cdot a_{321} + a_{132}\cdot a_{221}\cdot (a_{313}+b_{313}) - a_{132}\cdot a_{223}\cdot (a_{311}+b_{311}) \\
&+a_{133}\cdot (a_{211}+b_{211})\cdot a_{322} - a_{133}\cdot (a_{212}+b_{212})\cdot a_{321} - a_{133}\cdot a_{221}\cdot (a_{312}+b_{312}) + a_{133}\cdot a_{222}\cdot (a_{311}+b_{311}) \\
&= a_{111}\cdot a_{222}\cdot a_{333} + b_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - b_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} - b_{111}\cdot a_{223}\cdot a_{332} \\
&+a_{111}\cdot a_{233}\cdot a_{322} + b_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} - b_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} + b_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} + b_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} - b_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} + b_{113}\cdot a_{221}\cdot a_{332} \\
&-a_{113}\cdot a_{222}\cdot a_{331} - b_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} - b_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} + b_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} - a_{121}\cdot b_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot b_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} + a_{121}\cdot a_{232}\cdot b_{313} \\
&-a_{121}\cdot a_{233}\cdot a_{312} - a_{121}\cdot a_{233}\cdot b_{312} + a_{122}\cdot a_{211}\cdot a_{333} + a_{122}\cdot b_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot b_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} - a_{122}\cdot a_{231}\cdot b_{313} + a_{122}\cdot a_{233}\cdot a_{311} + a_{122}\cdot a_{233}\cdot b_{311} - a_{123}\cdot a_{211}\cdot a_{332} - a_{123}\cdot b_{211}\cdot a_{332} \\
&+a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot b_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} + a_{123}\cdot a_{231}\cdot b_{312} - a_{123}\cdot a_{232}\cdot a_{311} - a_{123}\cdot a_{232}\cdot b_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} + a_{131}\cdot b_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot b_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} - a_{131}\cdot a_{222}\cdot b_{313} \\
&+a_{131}\cdot a_{223}\cdot a_{312} + a_{131}\cdot a_{223}\cdot b_{312} - a_{132}\cdot a_{211}\cdot a_{323} - a_{132}\cdot b_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} + a_{132}\cdot b_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} + a_{132}\cdot a_{221}\cdot b_{313} - a_{132}\cdot a_{223}\cdot a_{311} - a_{132}\cdot a_{223}\cdot b_{311} + a_{133}\cdot a_{211}\cdot a_{322} + a_{133}\cdot b_{211}\cdot a_{322} \\
&-a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot b_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} - a_{133}\cdot a_{221}\cdot b_{312} + a_{133}\cdot a_{222}\cdot a_{311} + a_{133}\cdot a_{222}\cdot b_{311}
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

7. For plan $k = 1$: Let A and B be cubic-matrices of order 3, where all elements on the plan $k = 1$ and $k = 2$ are identical in both matrices, then have:

$$\det[A_{3\times3\times3}] + \det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$+ \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & b_{113} & b_{123} & b_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & b_{213} & b_{223} & b_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & b_{313} & b_{323} & b_{333} \end{array}\right)$$

$$\begin{aligned}
&= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}\} \\
&+\{a_{111}\cdot a_{222}\cdot b_{333} - a_{111}\cdot a_{232}\cdot b_{323} - a_{111}\cdot b_{223}\cdot a_{332} + a_{111}\cdot b_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot b_{333} + a_{112}\cdot b_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot b_{323} - a_{112}\cdot b_{233}\cdot a_{321} + b_{113}\cdot a_{221}\cdot a_{332} - b_{113}\cdot a_{222}\cdot a_{331} - b_{113}\cdot a_{231}\cdot a_{322} + b_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot b_{333} + a_{121}\cdot b_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot b_{313} - a_{121}\cdot b_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot b_{333} - a_{122}\cdot b_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot b_{313} + a_{122}\cdot b_{233}\cdot a_{311} - b_{123}\cdot a_{211}\cdot a_{332} + b_{123}\cdot a_{212}\cdot a_{331} + b_{123}\cdot a_{231}\cdot a_{312} - b_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot b_{323} - a_{131}\cdot b_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot b_{313} + a_{131}\cdot b_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot b_{323} + a_{132}\cdot b_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot b_{313} - a_{132}\cdot b_{223}\cdot a_{311} + b_{133}\cdot a_{211}\cdot a_{322} - b_{133}\cdot a_{212}\cdot a_{321} - b_{133}\cdot a_{221}\cdot a_{312} + b_{133}\cdot a_{222}\cdot a_{311}\}
\end{aligned}$$

while,

$$\det[A_{3\times3\times3} + B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113}+b_{113} & a_{123}+b_{123} & a_{133}+b_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213}+b_{213} & a_{223}+b_{223} & a_{233}+b_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313}+b_{313} & a_{323}+b_{323} & a_{333}+b_{333} \end{array}\right)$$

$$\begin{aligned}
&= a_{111}\cdot a_{222}\cdot(a_{333}+b_{333}) - a_{111}\cdot a_{232}\cdot(a_{323}+b_{323}) - a_{111}\cdot(a_{223}+b_{223})\cdot a_{332} + a_{111}\cdot(a_{233}+b_{233})\cdot a_{322} \\
&-a_{112}\cdot a_{221}\cdot(a_{333}+b_{333}) + a_{112}\cdot(a_{223}+b_{223})\cdot a_{331} + a_{112}\cdot a_{231}\cdot(a_{323}+b_{323}) - a_{112}\cdot(a_{233}+b_{233})\cdot a_{321} \\
&+(a_{113}+b_{113})\cdot a_{221}\cdot a_{332} - (a_{113}+b_{113})\cdot a_{222}\cdot a_{331} - (a_{113}+b_{113})\cdot a_{231}\cdot a_{322} + (a_{113}+b_{113})\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot(a_{333}+b_{333}) + a_{121}\cdot(a_{213}+b_{213})\cdot a_{332} + a_{121}\cdot a_{232}\cdot(a_{313}+b_{313}) - a_{121}\cdot(a_{233}+b_{233})\cdot a_{312} \\
&+a_{122}\cdot a_{211}\cdot(a_{333}+b_{333}) - a_{122}\cdot(a_{213}+b_{213})\cdot a_{331} - a_{122}\cdot a_{231}\cdot(a_{313}+b_{313}) + a_{122}\cdot(a_{233}+b_{233})\cdot a_{311} \\
&-(a_{123}+b_{123})\cdot a_{211}\cdot a_{332} + (a_{123}+b_{123})\cdot a_{212}\cdot a_{331} + (a_{123}+b_{123})\cdot a_{231}\cdot a_{312} - (a_{123}+b_{123})\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot(a_{323}+b_{323}) - a_{131}\cdot(a_{213}+b_{213})\cdot a_{322} - a_{131}\cdot a_{222}\cdot(a_{313}+b_{313}) + a_{131}\cdot(a_{223}+b_{223})\cdot a_{312} \\
&-a_{132}\cdot a_{211}\cdot(a_{323}+b_{323}) + a_{132}\cdot(a_{213}+b_{213})\cdot a_{321} + a_{132}\cdot a_{221}\cdot(a_{313}+b_{313}) - a_{132}\cdot(a_{223}+b_{223})\cdot a_{311} \\
&+(a_{133}+b_{133})\cdot a_{211}\cdot a_{322} - (a_{133}+b_{133})\cdot a_{212}\cdot a_{321} - (a_{133}+b_{133})\cdot a_{221}\cdot a_{312} + (a_{133}+b_{133})\cdot a_{222}\cdot a_{311} \\
&= a_{111}\cdot a_{222}\cdot a_{333} + a_{111}\cdot a_{222}\cdot b_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{232}\cdot b_{323} - a_{111}\cdot a_{223}\cdot a_{332} - a_{111}\cdot b_{223}\cdot a_{332} \\
&+a_{111}\cdot a_{233}\cdot a_{322} + a_{111}\cdot b_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} - a_{112}\cdot a_{221}\cdot b_{333} + a_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot b_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} + a_{112}\cdot a_{231}\cdot b_{323} - a_{112}\cdot a_{233}\cdot a_{321} - a_{112}\cdot b_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} + b_{113}\cdot a_{221}\cdot a_{332} \\
&-a_{113}\cdot a_{222}\cdot a_{331} - b_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} - b_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} + b_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} - a_{121}\cdot a_{212}\cdot b_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot b_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} + a_{121}\cdot a_{232}\cdot b_{313} \\
&-a_{121}\cdot a_{233}\cdot a_{312} - a_{121}\cdot b_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} + a_{122}\cdot a_{211}\cdot b_{333} - a_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot b_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} - a_{122}\cdot a_{231}\cdot b_{313} + a_{122}\cdot a_{233}\cdot a_{311} + a_{122}\cdot b_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} - b_{123}\cdot a_{211}\cdot a_{332} \\
&+a_{123}\cdot a_{212}\cdot a_{331} + b_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} + b_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} - b_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} + a_{131}\cdot a_{212}\cdot b_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot b_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} - a_{131}\cdot a_{222}\cdot b_{313} \\
&+a_{131}\cdot a_{223}\cdot a_{312} + a_{131}\cdot b_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} - a_{132}\cdot a_{211}\cdot b_{323} + a_{132}\cdot a_{213}\cdot a_{321} + a_{132}\cdot b_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} + a_{132}\cdot a_{221}\cdot b_{313} - a_{132}\cdot a_{223}\cdot a_{311} - a_{132}\cdot b_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} + b_{133}\cdot a_{211}\cdot a_{322} \\
&-a_{133}\cdot a_{212}\cdot a_{321} - b_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} - b_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311} + b_{133}\cdot a_{222}\cdot a_{311}
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

8. For plan $k = 2$: Let A and B be cubic-matrices of order 3, where all elements on the plan $k = 1$ and $k = 3$ are identical in both matrices, then we have:

$$\det[A_{3\times3\times3}] + \det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$+ \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & b_{112} & b_{122} & b_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & b_{212} & b_{222} & b_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & b_{312} & b_{322} & b_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}\} \\
&+\{a_{111}\cdot b_{222}\cdot a_{333} - a_{111}\cdot b_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot b_{332} + a_{111}\cdot a_{233}\cdot b_{322} - b_{112}\cdot a_{221}\cdot a_{333} + b_{112}\cdot a_{223}\cdot a_{331} \\
&+b_{112}\cdot a_{231}\cdot a_{323} - b_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot b_{332} - a_{113}\cdot b_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot b_{322} + a_{113}\cdot b_{232}\cdot a_{321} \\
&-a_{121}\cdot b_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot b_{332} + a_{121}\cdot b_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot b_{312} + b_{122}\cdot a_{211}\cdot a_{333} - b_{122}\cdot a_{213}\cdot a_{331} \\
&-b_{122}\cdot a_{231}\cdot a_{313} + b_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot b_{332} + a_{123}\cdot b_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot b_{312} - a_{123}\cdot b_{232}\cdot a_{311} \\
&+a_{131}\cdot b_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot b_{322} - a_{131}\cdot b_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot b_{312} - b_{132}\cdot a_{211}\cdot a_{323} + b_{132}\cdot a_{213}\cdot a_{321} \\
&+b_{132}\cdot a_{221}\cdot a_{313} - b_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot b_{322} - a_{133}\cdot b_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot b_{312} + a_{133}\cdot b_{222}\cdot a_{311}\}.
\end{aligned}$$

Whereas,

$$\det[A_{3\times3\times3} + B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112}+b_{112} & a_{122}+b_{122} & a_{132}+b_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212}+b_{212} & a_{222}+b_{222} & a_{232}+b_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312}+b_{312} & a_{322}+b_{322} & a_{332}+b_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= a_{111}\cdot(a_{222}+b_{222})\cdot a_{333} - a_{111}\cdot(a_{232}+b_{232})\cdot a_{323} - a_{111}\cdot a_{223}\cdot(a_{332}+b_{332}) + a_{111}\cdot a_{233}\cdot(a_{322}+b_{322}) \\
&-(a_{112}+b_{112})\cdot a_{221}\cdot a_{333} + (a_{112}+b_{112})\cdot a_{223}\cdot a_{331} + (a_{112}+b_{112})\cdot a_{231}\cdot a_{323} - (a_{112}+b_{112})\cdot a_{233}\cdot a_{321} \\
&+a_{113}\cdot a_{221}\cdot(a_{332}+b_{332}) - a_{113}\cdot(a_{222}+b_{222})\cdot a_{331} - a_{113}\cdot a_{231}\cdot(a_{322}+b_{322}) + a_{113}\cdot(a_{232}+b_{232})\cdot a_{321} \\
&-a_{121}\cdot(a_{212}+b_{212})\cdot a_{333} + a_{121}\cdot a_{213}\cdot(a_{332}+b_{332}) + a_{121}\cdot(a_{232}+b_{232})\cdot a_{313} - a_{121}\cdot a_{233}\cdot(a_{312}+b_{312}) \\
&+(a_{122}+b_{122})\cdot a_{211}\cdot a_{333} - (a_{122}+b_{122})\cdot a_{213}\cdot a_{331} - (a_{122}+b_{122})\cdot a_{231}\cdot a_{313} + (a_{122}+b_{122})\cdot a_{233}\cdot a_{311} \\
&-a_{123}\cdot a_{211}\cdot(a_{332}+b_{332}) + a_{123}\cdot(a_{212}+b_{212})\cdot a_{331} + a_{123}\cdot a_{231}\cdot(a_{312}+b_{312}) - a_{123}\cdot(a_{232}+b_{232})\cdot a_{311} \\
&+a_{131}\cdot(a_{212}+b_{212})\cdot a_{323} - a_{131}\cdot a_{213}\cdot(a_{322}+b_{322}) - a_{131}\cdot(a_{222}+b_{222})\cdot a_{313} + a_{131}\cdot a_{223}\cdot(a_{312}+b_{312}) \\
&-(a_{132}+b_{132})\cdot a_{211}\cdot a_{323} + (a_{132}+b_{132})\cdot a_{213}\cdot a_{321} + (a_{132}+b_{132})\cdot a_{221}\cdot a_{313} - (a_{132}+b_{132})\cdot a_{223}\cdot a_{311} \\
&+a_{133}\cdot a_{211}\cdot(a_{322}+b_{322}) - a_{133}\cdot(a_{212}+b_{212})\cdot a_{321} - a_{133}\cdot a_{221}\cdot(a_{312}+b_{312}) + a_{133}\cdot(a_{222}+b_{222})\cdot a_{311} \\
&= a_{111}\cdot a_{222}\cdot a_{333} + a_{111}\cdot b_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot b_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} - a_{111}\cdot a_{223}\cdot b_{332} \\
&+a_{111}\cdot a_{233}\cdot a_{322} + a_{111}\cdot a_{233}\cdot b_{322} - a_{112}\cdot a_{221}\cdot a_{333} - b_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} + b_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} + b_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} - b_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} + a_{113}\cdot a_{221}\cdot b_{332} \\
&-a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot b_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} - a_{113}\cdot a_{231}\cdot b_{322} + a_{113}\cdot a_{232}\cdot a_{321} + a_{113}\cdot b_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} - a_{121}\cdot b_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{213}\cdot b_{332} + a_{121}\cdot a_{232}\cdot a_{313} + a_{121}\cdot b_{232}\cdot a_{313} \\
&-a_{121}\cdot a_{233}\cdot a_{312} - a_{121}\cdot a_{233}\cdot b_{312} + a_{122}\cdot a_{211}\cdot a_{333} + b_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} - b_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} - b_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} + b_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} - a_{123}\cdot a_{211}\cdot b_{332} \\
&+a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot b_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} + a_{123}\cdot a_{231}\cdot b_{312} - a_{123}\cdot a_{232}\cdot a_{311} - a_{123}\cdot b_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} + a_{131}\cdot b_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{213}\cdot b_{322} - a_{131}\cdot a_{222}\cdot a_{313} - a_{131}\cdot b_{222}\cdot a_{313} \\
&+a_{131}\cdot a_{223}\cdot a_{312} + a_{131}\cdot a_{223}\cdot b_{312} - a_{132}\cdot a_{211}\cdot a_{323} - b_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} + b_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} + b_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} - b_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} + a_{133}\cdot a_{211}\cdot b_{322} \\
&-a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot b_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} - a_{133}\cdot a_{221}\cdot b_{312} + a_{133}\cdot a_{222}\cdot a_{311} + a_{133}\cdot b_{222}\cdot a_{311}
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

9. For plan $k = 3$: Let A and B be cubic-matrices of order 3, where all elements on the plan $k = 2$ and $k = 3$ are identical in both matrices, then we have:

$$\det[A_{3\times3\times3}] + \det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$+ \det\left(\begin{array}{ccc|ccc|ccc} b_{111} & b_{121} & b_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ b_{211} & b_{221} & b_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ b_{311} & b_{321} & b_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= \{a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}\} \\
&+\{b_{111}\cdot a_{222}\cdot a_{333} - b_{111}\cdot a_{232}\cdot a_{323} - b_{111}\cdot a_{223}\cdot a_{332} + b_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot b_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot b_{331} \\
&+a_{112}\cdot b_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot b_{321} + a_{113}\cdot b_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot b_{331} - a_{113}\cdot b_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot b_{321} \\
&-b_{121}\cdot a_{212}\cdot a_{333} + b_{121}\cdot a_{213}\cdot a_{332} + b_{121}\cdot a_{232}\cdot a_{313} - b_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot b_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot b_{331} \\
&-a_{122}\cdot b_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot b_{311} - a_{123}\cdot b_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot b_{331} + a_{123}\cdot b_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot b_{311} \\
&+b_{131}\cdot a_{212}\cdot a_{323} - b_{131}\cdot a_{213}\cdot a_{322} - b_{131}\cdot a_{222}\cdot a_{313} + b_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot b_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot b_{321} \\
&+a_{132}\cdot b_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot b_{311} + a_{133}\cdot b_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot b_{321} - a_{133}\cdot b_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot b_{311}\}.
\end{aligned}$$

Whereas,

$$\det[A_{3\times3\times3} + B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111}+b_{111} & a_{121}+b_{121} & a_{131}+b_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211}+b_{211} & a_{221}+b_{221} & a_{231}+b_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311}+b_{311} & a_{321}+b_{321} & a_{331}+b_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= (a_{111}+b_{111})\cdot a_{222}\cdot a_{333} - (a_{111}+b_{111})\cdot a_{232}\cdot a_{323} - (a_{111}+b_{111})\cdot a_{223}\cdot a_{332} + (a_{111}+b_{111})\cdot a_{233}\cdot a_{322} \\
&-a_{112}\cdot(a_{221}+b_{221})\cdot a_{333} + a_{112}\cdot a_{223}\cdot(a_{331}+b_{331}) + a_{112}\cdot(a_{231}+b_{231})\cdot a_{323} - a_{112}\cdot a_{233}\cdot(a_{321}+b_{321}) \\
&+a_{113}\cdot(a_{221}+b_{221})\cdot a_{332} - a_{113}\cdot a_{222}\cdot(a_{331}+b_{331}) - a_{113}\cdot(a_{231}+b_{231})\cdot a_{322} + a_{113}\cdot a_{232}\cdot(a_{321}+b_{321}) \\
&-(a_{121}+b_{121})\cdot a_{212}\cdot a_{333} + (a_{121}+b_{121})\cdot a_{213}\cdot a_{332} + (a_{121}+b_{121})\cdot a_{232}\cdot a_{313} - (a_{121}+b_{121})\cdot a_{233}\cdot a_{312} \\
&+a_{122}\cdot(a_{211}+b_{211})\cdot a_{333} - a_{122}\cdot a_{213}\cdot(a_{331}+b_{331}) - a_{122}\cdot(a_{231}+b_{231})\cdot a_{313} + a_{122}\cdot a_{233}\cdot(a_{311}+b_{311}) \\
&-a_{123}\cdot(a_{211}+b_{211})\cdot a_{332} + a_{123}\cdot a_{212}\cdot(a_{331}+b_{331}) + a_{123}\cdot(a_{231}+b_{231})\cdot a_{312} - a_{123}\cdot a_{232}\cdot(a_{311}+b_{311}) \\
&+(a_{131}+b_{131})\cdot a_{212}\cdot a_{323} - (a_{131}+b_{131})\cdot a_{213}\cdot a_{322} - (a_{131}+b_{131})\cdot a_{222}\cdot a_{313} + (a_{131}+b_{131})\cdot a_{223}\cdot a_{312} \\
&-a_{132}\cdot(a_{211}+b_{211})\cdot a_{323} + a_{132}\cdot a_{213}\cdot(a_{321}+b_{321}) + a_{132}\cdot(a_{221}+b_{221})\cdot a_{313} - a_{132}\cdot a_{223}\cdot(a_{311}+b_{311}) \\
&+a_{133}\cdot(a_{211}+b_{211})\cdot a_{322} - a_{133}\cdot a_{212}\cdot(a_{321}+b_{321}) - a_{133}\cdot(a_{221}+b_{221})\cdot a_{312} + a_{133}\cdot a_{222}\cdot(a_{311}+b_{311}) \\
&= a_{111}\cdot a_{222}\cdot a_{333} + b_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - b_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} - b_{111}\cdot a_{223}\cdot a_{332} \\
&+a_{111}\cdot a_{233}\cdot a_{322} + b_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} - a_{112}\cdot b_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot a_{223}\cdot b_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} + a_{112}\cdot b_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} - a_{112}\cdot a_{233}\cdot b_{321} + a_{113}\cdot a_{221}\cdot a_{332} + a_{113}\cdot b_{221}\cdot a_{332} \\
&-a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{222}\cdot b_{331} - a_{113}\cdot a_{231}\cdot a_{322} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} + a_{113}\cdot a_{232}\cdot b_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} - b_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + b_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} + b_{121}\cdot a_{232}\cdot a_{313} \\
&-a_{121}\cdot a_{233}\cdot a_{312} - b_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} + a_{122}\cdot b_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot a_{213}\cdot b_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} - a_{122}\cdot b_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} + a_{122}\cdot a_{233}\cdot b_{311} - a_{123}\cdot a_{211}\cdot a_{332} - a_{123}\cdot b_{211}\cdot a_{332} \\
&+a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{212}\cdot b_{331} + a_{123}\cdot a_{231}\cdot a_{312} + a_{123}\cdot b_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} - a_{123}\cdot a_{232}\cdot b_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} + b_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - b_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} - b_{131}\cdot a_{222}\cdot a_{313} \\
&+a_{131}\cdot a_{223}\cdot a_{312} + b_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} - a_{132}\cdot b_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} + a_{132}\cdot a_{213}\cdot b_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} + a_{132}\cdot b_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} - a_{132}\cdot a_{223}\cdot b_{311} + a_{133}\cdot a_{211}\cdot a_{322} + a_{133}\cdot b_{211}\cdot a_{322} \\
&-a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{212}\cdot b_{321} - a_{133}\cdot a_{221}\cdot a_{312} - a_{133}\cdot b_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311} + a_{133}\cdot a_{222}\cdot b_{311}.
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

□

Theorem 2. *Let it be the cubic matrix B, which is formed by multiplying a plane (in the indices j and k, it does not work in the index i) of the matrix A by a scalar α, and addition another plane, then we have:*

$$\det(A) = \det(B)$$

Proof. **Case 1.** The cubic-matrix A of order 2, (and B has order 2), we will proof the case 1 for each "vertical page" and "vertical layer", as following:

1. For plan $j = 1$: Let us add first vertical page to second vertical page while multiplying by a scalar α.

$$\begin{aligned}\det[A_{2\times2\times2}] &= \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) \\ &= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}.\end{aligned}$$

Whereas,

$$\begin{aligned}\det[B_{2\times2\times2}] &= \det\left(\begin{array}{cc|cc} a_{111} & a_{121}+\alpha\cdot a_{111} & a_{112} & a_{122}+\alpha\cdot a_{112} \\ a_{211} & a_{221}+\alpha\cdot a_{211} & a_{212} & a_{222}+\alpha\cdot a_{212} \end{array}\right) \\ &= a_{111}\cdot(a_{222}+\alpha\cdot a_{212}) - (a_{122}+\alpha\cdot a_{112})\cdot a_{221} - (a_{121}+\alpha\cdot a_{111})\cdot a_{212} + a_{122}\cdot(a_{221}+\alpha\cdot a_{211}).\end{aligned}$$

After expanding further we get the following result

$$\det[B_{2\times2\times2}] = a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}$$

If we compare results of above equations, we can see that we have the same result in both cases.

2. For plan $j = 2$: Let us add second vertical page to first vertical page while multiplying by a scalar α.

$$\begin{aligned}\det[A_{2\times2\times2}] &= \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) \\ &= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}.\end{aligned}$$

Whereas,

$$\begin{aligned}\det[B_{2\times2\times2}] &= \det\left(\begin{array}{cc|cc} a_{111}+\alpha\cdot a_{121} & a_{121} & a_{112}+\alpha\cdot a_{122} & a_{122} \\ a_{211}+\alpha\cdot a_{221} & a_{221} & a_{212}+\alpha\cdot a_{222} & a_{222} \end{array}\right) \\ &= (a_{111}+\alpha\cdot a_{121})\cdot a_{222} - (a_{112}+\alpha\cdot a_{122})\cdot a_{221} - a_{121}\cdot(a_{212}+\alpha\cdot a_{222}) + a_{122}\cdot(a_{211}+\alpha\cdot a_{221}).\end{aligned}$$

After expanding further we get the following result

$$\det[B_{2\times2\times2}] = a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}$$

If we compare results of above equations, we can see that we have the same result in both cases.

3. For plan $k = 1$: Let us add first vertical page to second vertical layer while multiplying by a scalar α.

$$\begin{aligned}\det[A_{2\times2\times2}] &= \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) \\ &= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}.\end{aligned}$$

Whereas,

$$\begin{aligned}\det[B_{2\times2\times2}] &= \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112}+\alpha\cdot a_{111} & a_{122}+\alpha\cdot a_{121} \\ a_{211} & a_{221} & a_{212}+\alpha\cdot a_{211} & a_{222}+\alpha\cdot a_{221} \end{array}\right) \\ &= a_{111}\cdot(a_{222}+\alpha\cdot a_{221}) - (a_{112}+\alpha\cdot a_{111})\cdot a_{221} - a_{121}\cdot(a_{212}+\alpha\cdot a_{211}) + (a_{122}+\alpha\cdot a_{121})\cdot a_{211}.\end{aligned}$$

After expanding further we get the following result

$$\det[B_{2\times2\times2}] = a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}$$

If we compare results of above equations, we can see that we have the same result in both cases.

4. For plan $k = 2$: Let us add second vertical page to first vertical layer while multiplying by a scalar α.

$$\det[A_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right)$$
$$= a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}.$$

Whereas,

$$\det[B_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111}+\alpha\cdot a_{112} & a_{121}+\alpha\cdot a_{122} & a_{112} & a_{122} \\ a_{211}+\alpha\cdot a_{212} & a_{221}+\alpha\cdot a_{222} & a_{212} & a_{222} \end{array}\right)$$
$$= (a_{111}+\alpha\cdot a_{112})\cdot a_{222} - a_{112}\cdot(a_{221}+\alpha\cdot a_{222}) - (a_{121}+\alpha\cdot a_{122})\cdot a_{212} + a_{122}\cdot(a_{211}+\alpha\cdot a_{212}).$$

After expanding further we get the following result

$$\det[B_{2\times2\times2}] = a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}.$$

If we compare results of above equations, we can see that we have the same result in both cases.

Case 2. The cubic-matrix A of order 3, (and B has order 3), we will proof the case 2 for each "vertical page" and "vertical layer", as following:

1. For plan $j = 1$: Let us add first vertical page to second vertical page while multiplying by a scalar α.

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}.
\end{aligned}$$

Whereas,

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121}+\alpha\cdot a_{111} & a_{131} & a_{112} & a_{122}+\alpha\cdot a_{112} & a_{132} & a_{113} & a_{123}+\alpha\cdot a_{113} & a_{133} \\ a_{211} & a_{221}+\alpha\cdot a_{211} & a_{231} & a_{212} & a_{222}+\alpha\cdot a_{212} & a_{232} & a_{213} & a_{223}+\alpha\cdot a_{213} & a_{233} \\ a_{311} & a_{321}+\alpha\cdot a_{311} & a_{331} & a_{312} & a_{322}+\alpha\cdot a_{312} & a_{332} & a_{313} & a_{323}+\alpha\cdot a_{313} & a_{333} \end{array}\right)$$

$$\begin{aligned}
&= a_{111}\cdot(a_{222}+\alpha\cdot a_{212})\cdot a_{333} - a_{111}\cdot a_{232}\cdot(a_{323}+\alpha\cdot a_{313}) - a_{111}\cdot(a_{223}+\alpha\cdot a_{213})\cdot a_{332} + a_{111}\cdot a_{233}\cdot(a_{322}+\alpha\cdot a_{312}) \\
&-a_{112}\cdot(a_{221}+\alpha\cdot a_{211})\cdot a_{333} + a_{112}\cdot(a_{223}+\alpha\cdot a_{213})\cdot a_{331} + a_{112}\cdot a_{231}\cdot(a_{323}+\alpha\cdot a_{313}) - a_{112}\cdot a_{233}\cdot(a_{321}+\alpha\cdot a_{311}) \\
&+a_{113}\cdot(a_{221}+\alpha\cdot a_{211})\cdot a_{332} - a_{113}\cdot(a_{222}+\alpha\cdot a_{212})\cdot a_{331} - a_{113}\cdot a_{231}\cdot(a_{322}+\alpha\cdot a_{312}) + a_{113}\cdot a_{232}\cdot(a_{321}+\alpha\cdot a_{311}) \\
&-(a_{121}+\alpha\cdot a_{111})\cdot a_{212}\cdot a_{333} + (a_{121}+\alpha\cdot a_{111})\cdot a_{213}\cdot a_{332} + (a_{121}+\alpha\cdot a_{111})\cdot a_{232}\cdot a_{313} - (a_{121}+\alpha\cdot a_{111})\cdot a_{233}\cdot a_{312} \\
&+(a_{122}+\alpha\cdot a_{112})\cdot a_{211}\cdot a_{333} - (a_{122}+\alpha\cdot a_{112})\cdot a_{213}\cdot a_{331} - (a_{122}+\alpha\cdot a_{112})\cdot a_{231}\cdot a_{313} + (a_{122}+\alpha\cdot a_{112})\cdot a_{233}\cdot a_{311} \\
&-(a_{123}+\alpha\cdot a_{113})\cdot a_{211}\cdot a_{332} + (a_{123}+\alpha\cdot a_{113})\cdot a_{212}\cdot a_{331} + (a_{123}+\alpha\cdot a_{113})\cdot a_{231}\cdot a_{312} - (a_{123}+\alpha\cdot a_{113})\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot(a_{323}+\alpha\cdot a_{313}) - a_{131}\cdot a_{213}\cdot(a_{322}+\alpha\cdot a_{312}) - a_{131}\cdot(a_{222}+\alpha\cdot a_{212})\cdot a_{313} + a_{131}\cdot(a_{223}+\alpha\cdot a_{213})\cdot a_{312} \\
&-a_{132}\cdot a_{211}\cdot(a_{323}+\alpha\cdot a_{313}) + a_{132}\cdot a_{213}\cdot(a_{321}+\alpha\cdot a_{311}) + a_{132}\cdot(a_{221}+\alpha\cdot a_{211})\cdot a_{313} - a_{132}\cdot(a_{223}+\alpha\cdot a_{213})\cdot a_{311} \\
&+a_{133}\cdot a_{211}\cdot(a_{322}+\alpha\cdot a_{312}) - a_{133}\cdot a_{212}\cdot(a_{321}+\alpha\cdot a_{311}) - a_{133}\cdot(a_{221}+\alpha\cdot a_{211})\cdot a_{312} + a_{133}\cdot(a_{222}+\alpha\cdot a_{212})\cdot a_{311}.
\end{aligned}$$

After expanding further we get the following result

$$\begin{aligned}
&= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\
&+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\
&-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\
&-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\
&+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\
&+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}.
\end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

2. For plan $j = 2$: Let us add first vertical page to third vertical page while multiplying by a scalar α.

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned} &= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\ &+ a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\ &- a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\ &- a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\ &+ a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\ &+ a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}. \end{aligned}$$

Whereas,

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131}+\alpha\cdot a_{111} & a_{112} & a_{122} & a_{132}+\alpha\cdot a_{112} & a_{113} & a_{123} & a_{133}+\alpha\cdot a_{113} \\ a_{211} & a_{221} & a_{231}+\alpha\cdot a_{211} & a_{212} & a_{222} & a_{232}+\alpha\cdot a_{212} & a_{213} & a_{223} & a_{233}+\alpha\cdot a_{213} \\ a_{311} & a_{321} & a_{331}+\alpha\cdot a_{311} & a_{312} & a_{322} & a_{332}+\alpha\cdot a_{312} & a_{313} & a_{323} & a_{333}+\alpha\cdot a_{313} \end{array}\right)$$

$$\begin{aligned} &= a_{111}\cdot a_{222}\cdot(a_{333}+\alpha\cdot a_{313}) - a_{111}\cdot(a_{232}+\alpha\cdot a_{212})\cdot a_{323} - a_{111}\cdot a_{223}\cdot(a_{332}+\alpha\cdot a_{312}) + a_{111}\cdot(a_{233}+\alpha\cdot a_{213})\cdot a_{322} \\ &- a_{112}\cdot a_{221}\cdot(a_{333}+\alpha\cdot a_{313}) + a_{112}\cdot a_{223}\cdot(a_{331}+\alpha\cdot a_{311}) + a_{112}\cdot(a_{231}+\alpha\cdot a_{211})\cdot a_{323} - a_{112}\cdot(a_{233}+\alpha\cdot a_{213})\cdot a_{321} \\ &+ a_{113}\cdot a_{221}\cdot(a_{332}+\alpha\cdot a_{312}) - a_{113}\cdot a_{222}\cdot(a_{331}+\alpha\cdot a_{311}) - a_{113}\cdot(a_{231}+\alpha\cdot a_{211})\cdot a_{322} + a_{113}\cdot(a_{232}+\alpha\cdot a_{212})\cdot a_{321} \\ &- a_{121}\cdot a_{212}\cdot(a_{333}+\alpha\cdot a_{313}) + a_{121}\cdot a_{213}\cdot(a_{332}+\alpha\cdot a_{312}) + a_{121}\cdot(a_{232}+\alpha\cdot a_{212})\cdot a_{313} - a_{121}\cdot(a_{233}+\alpha\cdot a_{213})\cdot a_{312} \\ &+ a_{122}\cdot a_{211}\cdot(a_{333}+\alpha\cdot a_{313}) - a_{122}\cdot a_{213}\cdot(a_{331}+\alpha\cdot a_{311}) - a_{122}\cdot(a_{231}+\alpha\cdot a_{211})\cdot a_{313} + a_{122}\cdot(a_{233}+\alpha\cdot a_{213})\cdot a_{311} \\ &- a_{123}\cdot a_{211}\cdot(a_{332}+\alpha\cdot a_{312}) + a_{123}\cdot a_{212}\cdot(a_{331}+\alpha\cdot a_{311}) + a_{123}\cdot(a_{231}+\alpha\cdot a_{211})\cdot a_{312} - a_{123}\cdot(a_{232}+\alpha\cdot a_{212})\cdot a_{311} \\ &+ (a_{131}+\alpha\cdot a_{111})\cdot a_{212}\cdot a_{323} - (a_{131}+\alpha\cdot a_{111})\cdot a_{213}\cdot a_{322} - (a_{131}+\alpha\cdot a_{111})\cdot a_{222}\cdot a_{313} + (a_{131}+\alpha\cdot a_{111})\cdot a_{223}\cdot a_{312} \\ &- (a_{132}+\alpha\cdot a_{112})\cdot a_{211}\cdot a_{323} + (a_{132}+\alpha\cdot a_{112})\cdot a_{213}\cdot a_{321} + (a_{132}+\alpha\cdot a_{112})\cdot a_{221}\cdot a_{313} - (a_{132}+\alpha\cdot a_{112})\cdot a_{223}\cdot a_{311} \\ &+ (a_{133}+\alpha\cdot a_{113})\cdot a_{211}\cdot a_{322} - (a_{133}+\alpha\cdot a_{113})\cdot a_{212}\cdot a_{321} - (a_{133}+\alpha\cdot a_{113})\cdot a_{221}\cdot a_{312} + (a_{133}+\alpha\cdot a_{113})\cdot a_{222}\cdot a_{311}. \end{aligned}$$

After expanding further we get the following result

$$\begin{aligned} &= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\ &+ a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\ &- a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\ &- a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\ &+ a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\ &+ a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}. \end{aligned}$$

If we compare results of above equations, we can see that we have the same result in both cases.

3. For plan $j = 3$: Let us add second vertical page to third vertical page while multiplying by a scalar α.

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$\begin{aligned} &= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} \\ &+ a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321} \\ &- a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} \\ &- a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311} \\ &+ a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321} \\ &+ a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}. \end{aligned}$$

Whereas,

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131}+\alpha\cdot a_{121} & a_{112} & a_{122} & a_{132}+\alpha\cdot a_{122} & a_{113} & a_{123} & a_{133}+\alpha\cdot a_{123} \\ a_{211} & a_{221} & a_{231}+\alpha\cdot a_{221} & a_{212} & a_{222} & a_{232}+\alpha\cdot a_{222} & a_{213} & a_{223} & a_{233}+\alpha\cdot a_{223} \\ a_{311} & a_{321} & a_{331}+\alpha\cdot a_{321} & a_{312} & a_{322} & a_{332}+\alpha\cdot a_{322} & a_{313} & a_{323} & a_{333}+\alpha\cdot a_{323} \end{array}\right)$$

$$= a_{111}\cdot a_{222}\cdot(a_{333}+\alpha\cdot a_{323}) - a_{111}\cdot(a_{232}+\alpha\cdot a_{222})\cdot a_{323} - a_{111}\cdot a_{223}\cdot(a_{332}+\alpha\cdot a_{312}) + a_{111}\cdot(a_{233}+\alpha\cdot a_{223})\cdot a_{322}$$
$$-a_{112}\cdot a_{221}\cdot(a_{333}+\alpha\cdot a_{323}) + a_{112}\cdot a_{223}\cdot(a_{331}+\alpha\cdot a_{321}) + a_{112}\cdot(a_{231}+\alpha\cdot a_{211})\cdot a_{323} - a_{112}\cdot(a_{233}+\alpha\cdot a_{223})\cdot a_{321}$$
$$+a_{113}\cdot a_{221}\cdot(a_{332}+\alpha\cdot a_{322}) - a_{113}\cdot a_{222}\cdot(a_{331}+\alpha\cdot a_{321}) - a_{113}\cdot(a_{231}+\alpha\cdot a_{211})\cdot a_{322} + a_{113}\cdot(a_{232}+\alpha\cdot a_{222})\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot(a_{333}+\alpha\cdot a_{323}) + a_{121}\cdot a_{213}\cdot(a_{332}+\alpha\cdot a_{322}) + a_{121}\cdot(a_{232}+\alpha\cdot a_{212})\cdot a_{313} - a_{121}\cdot(a_{233}+\alpha\cdot a_{223})\cdot a_{312}$$
$$+a_{122}\cdot a_{211}\cdot(a_{333}+\alpha\cdot a_{323}) - a_{122}\cdot a_{213}\cdot(a_{331}+\alpha\cdot a_{321}) - a_{122}\cdot(a_{231}+\alpha\cdot a_{211})\cdot a_{313} + a_{122}\cdot(a_{233}+\alpha\cdot a_{223})\cdot a_{311}$$
$$-a_{123}\cdot a_{211}\cdot(a_{332}+\alpha\cdot a_{322}) + a_{123}\cdot a_{212}\cdot(a_{331}+\alpha\cdot a_{321}) + a_{123}\cdot(a_{231}+\alpha\cdot a_{211})\cdot a_{312} - a_{123}\cdot(a_{232}+\alpha\cdot a_{222})\cdot a_{311}$$
$$+(a_{131}+\alpha\cdot a_{121})\cdot a_{212}\cdot a_{323} - (a_{131}+\alpha\cdot a_{121})\cdot a_{213}\cdot a_{322} - (a_{131}+\alpha\cdot a_{121})\cdot a_{222}\cdot a_{313} + (a_{131}+\alpha\cdot a_{121})\cdot a_{223}\cdot a_{312}$$
$$-(a_{132}+\alpha\cdot a_{122})\cdot a_{211}\cdot a_{323} + (a_{132}+\alpha\cdot a_{122})\cdot a_{213}\cdot a_{321} + (a_{132}+\alpha\cdot a_{122})\cdot a_{221}\cdot a_{313} - (a_{132}+\alpha\cdot a_{122})\cdot a_{223}\cdot a_{311}$$
$$+(a_{133}+\alpha\cdot a_{123})\cdot a_{211}\cdot a_{322} - (a_{133}+\alpha\cdot a_{123})\cdot a_{212}\cdot a_{321} - (a_{133}+\alpha\cdot a_{123})\cdot a_{221}\cdot a_{312} + (a_{133}+\alpha\cdot a_{123})\cdot a_{222}\cdot a_{311}.$$

After expanding further we get the following result

$$= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331}$$
$$+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331}$$
$$-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321}$$
$$+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}.$$

If we compare results of above equations, we can see that we have the same result in both cases.

4. For plan $k = 1$: Let us add first vertical layer to second vertical layer while multiplying by a scalar α.

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331}$$
$$+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331}$$
$$-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321}$$
$$+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}.$$

Whereas,

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112}+\alpha\cdot a_{111} & a_{122}+\alpha\cdot a_{121} & a_{132}+\alpha\cdot a_{131} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212}+\alpha\cdot a_{211} & a_{222}+\alpha\cdot a_{221} & a_{232}+\alpha\cdot a_{231} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312}+\alpha\cdot a_{311} & a_{322}+\alpha\cdot a_{321} & a_{332}+\alpha\cdot a_{331} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{111}\cdot(a_{222}+\alpha\cdot a_{221})\cdot a_{333} - a_{111}\cdot(a_{232}+\alpha\cdot a_{231})\cdot a_{323} - a_{111}\cdot a_{223}\cdot(a_{332}+\alpha\cdot a_{331}) + a_{111}\cdot a_{233}\cdot(a_{322}+\alpha\cdot a_{321})$$
$$-(a_{112}+\alpha\cdot a_{111})\cdot a_{221}\cdot a_{333} + (a_{112}+\alpha\cdot a_{111})\cdot a_{223}\cdot a_{331} + (a_{112}+\alpha\cdot a_{111})\cdot a_{231}\cdot a_{323} - (a_{112}+\alpha\cdot a_{111})\cdot a_{233}\cdot a_{321}$$
$$+a_{113}\cdot a_{221}\cdot(a_{332}+\alpha\cdot a_{331}) - a_{113}\cdot(a_{222}+\alpha\cdot a_{221})\cdot a_{331} - a_{113}\cdot a_{231}\cdot(a_{322}+\alpha\cdot a_{321}) + a_{113}\cdot(a_{232}+\alpha\cdot a_{231})\cdot a_{321}$$
$$-a_{121}\cdot(a_{212}+\alpha\cdot a_{211})\cdot a_{333} + a_{121}\cdot a_{213}\cdot(a_{332}+\alpha\cdot a_{331}) + a_{121}\cdot(a_{232}+\alpha\cdot a_{231})\cdot a_{313} - a_{121}\cdot a_{233}\cdot(a_{312}+\alpha\cdot a_{311})$$
$$+(a_{122}+\alpha\cdot a_{121})\cdot a_{211}\cdot a_{333} - (a_{122}+\alpha\cdot a_{121})\cdot a_{213}\cdot a_{331} - (a_{122}+\alpha\cdot a_{121})\cdot a_{231}\cdot a_{313} + (a_{122}+\alpha\cdot a_{121})\cdot a_{233}\cdot a_{311}$$
$$-a_{123}\cdot a_{211}\cdot(a_{332}+\alpha\cdot a_{331}) + a_{123}\cdot(a_{212}+\alpha\cdot a_{211})\cdot a_{331} + a_{123}\cdot a_{231}\cdot(a_{312}+\alpha\cdot a_{311}) - a_{123}\cdot(a_{232}+\alpha\cdot a_{231})\cdot a_{311}$$
$$+a_{131}\cdot(a_{212}+\alpha\cdot a_{211})\cdot a_{323} - a_{131}\cdot a_{213}\cdot(a_{322}+\alpha\cdot a_{321}) - a_{131}\cdot(a_{222}+\alpha\cdot a_{221})\cdot a_{313} + a_{131}\cdot a_{223}\cdot(a_{312}+\alpha\cdot a_{311})$$
$$-(a_{132}+\alpha\cdot a_{131})\cdot a_{211}\cdot a_{323} + (a_{132}+\alpha\cdot a_{131})\cdot a_{213}\cdot a_{321} + (a_{132}+\alpha\cdot a_{131})\cdot a_{221}\cdot a_{313} - (a_{132}+\alpha\cdot a_{131})\cdot a_{223}\cdot a_{311}$$
$$+a_{133}\cdot a_{211}\cdot(a_{322}+\alpha\cdot a_{321}) - a_{133}\cdot(a_{212}+\alpha\cdot a_{211})\cdot a_{321} - a_{133}\cdot a_{221}\cdot(a_{312}+\alpha\cdot a_{311}) + a_{133}\cdot(a_{222}+\alpha\cdot a_{221})\cdot a_{311}.$$

After expanding further we get the following result

$$= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331}$$
$$+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331}$$
$$-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}$$
$$+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}+a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311}.$$

If we compare results of above equations, we can see that we have the same result in both cases.

5. For plan $k = 2$: Let us add first vertical layer to third vertical layer while multiplying by a scalar α.

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{111}\cdot a_{222}\cdot a_{333}-a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot a_{332}+a_{111}\cdot a_{233}\cdot a_{322}-a_{112}\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot a_{331}$$
$$+a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}+a_{113}\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot a_{312}+a_{122}\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot a_{331}$$
$$-a_{122}\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}-a_{123}\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}$$
$$+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}+a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311}.$$

Whereas,

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113}+\alpha\cdot a_{111} & a_{123}+\alpha\cdot a_{121} & a_{133}+\alpha\cdot a_{131} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213}+\alpha\cdot a_{211} & a_{223}+\alpha\cdot a_{221} & a_{233}+\alpha\cdot a_{231} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313}+\alpha\cdot a_{311} & a_{323}+\alpha\cdot a_{321} & a_{333}+\alpha\cdot a_{331} \end{array}\right)$$

$$= a_{111}\cdot a_{222}\cdot(a_{333}+\alpha\cdot a_{331})-a_{111}\cdot a_{232}\cdot(a_{323}+\alpha\cdot a_{321})-a_{111}\cdot(a_{223}+\alpha\cdot a_{221})\cdot a_{332}+a_{111}\cdot(a_{233}+\alpha\cdot a_{231})\cdot a_{322}$$
$$-a_{112}\cdot a_{221}\cdot(a_{333}+\alpha\cdot a_{331})+a_{112}\cdot(a_{223}+\alpha\cdot a_{221})\cdot a_{331}+a_{112}\cdot a_{231}\cdot(a_{323}+\alpha\cdot a_{321})-a_{112}\cdot(a_{233}+\alpha\cdot a_{231})\cdot a_{321}$$
$$+(a_{113}+\alpha\cdot a_{111})\cdot a_{221}\cdot a_{332}-(a_{113}+\alpha\cdot a_{111})\cdot a_{222}\cdot a_{331}-(a_{113}+\alpha\cdot a_{111})\cdot a_{231}\cdot a_{322}+(a_{113}+\alpha\cdot a_{111})\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot(a_{333}+\alpha\cdot a_{331})+a_{121}\cdot(a_{213}+\alpha\cdot a_{211})\cdot a_{332}+a_{121}\cdot a_{232}\cdot(a_{313}+\alpha\cdot a_{311})-a_{121}\cdot(a_{233}+\alpha\cdot a_{231})\cdot a_{312}$$
$$+a_{122}\cdot a_{211}\cdot(a_{333}+\alpha\cdot a_{331})-a_{122}\cdot(a_{213}+\alpha\cdot a_{211})\cdot a_{331}-a_{122}\cdot a_{231}\cdot(a_{313}+\alpha\cdot a_{311})+a_{122}\cdot(a_{233}+\alpha\cdot a_{231})\cdot a_{311}$$
$$-(a_{123}+\alpha\cdot a_{121})\cdot a_{211}\cdot a_{332}+(a_{123}+\alpha\cdot a_{121})\cdot a_{212}\cdot a_{331}+(a_{123}+\alpha\cdot a_{121})\cdot a_{231}\cdot a_{312}-(a_{123}+\alpha\cdot a_{121})\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot(a_{323}+\alpha\cdot a_{321})-a_{131}\cdot(a_{213}+\alpha\cdot a_{211})\cdot a_{322}-a_{131}\cdot a_{222}\cdot(a_{313}+\alpha\cdot a_{311})+a_{131}\cdot(a_{223}+\alpha\cdot a_{221})\cdot a_{312}$$
$$-a_{132}\cdot a_{211}\cdot(a_{323}+\alpha\cdot a_{321})+a_{132}\cdot(a_{213}+\alpha\cdot a_{211})\cdot a_{321}+a_{132}\cdot a_{221}\cdot(a_{313}+\alpha\cdot a_{311})-a_{132}\cdot(a_{223}+\alpha\cdot a_{221})\cdot a_{311}$$
$$+(a_{133}+\alpha\cdot a_{131})\cdot a_{211}\cdot a_{322}-(a_{133}+\alpha\cdot a_{131})\cdot a_{212}\cdot a_{321}-(a_{133}+\alpha\cdot a_{131})\cdot a_{221}\cdot a_{312}+(a_{133}+\alpha\cdot a_{131})\cdot a_{222}\cdot a_{311}$$
$$+a_{133}\cdot a_{211}\cdot(a_{322}+\alpha\cdot a_{221})-a_{133}\cdot(a_{212}+\alpha\cdot a_{211})\cdot a_{321}-a_{133}\cdot a_{221}\cdot(a_{312}+\alpha\cdot a_{211})+a_{133}\cdot(a_{222}+\alpha\cdot a_{221})\cdot a_{311}.$$

After expanding further we get the following result,

$$= a_{111}\cdot a_{222}\cdot a_{333}-a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot a_{332}+a_{111}\cdot a_{233}\cdot a_{322}-a_{112}\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot a_{331}$$
$$+a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}+a_{113}\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot a_{312}+a_{122}\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot a_{331}$$
$$-a_{122}\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}-a_{123}\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}$$
$$+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}+a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311}.$$

If we compare results of above equations, we can see that we have the same result in both cases.

6. For plan $k = 3$: Let us add first vertical layer to third vertical layer while multiplying by a scalar α.

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{111}\cdot a_{222}\cdot a_{333}-a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot a_{332}+a_{111}\cdot a_{233}\cdot a_{322}-a_{112}\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot a_{331}$$
$$+a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}+a_{113}\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot a_{312}+a_{122}\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot a_{331}$$
$$-a_{122}\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}-a_{123}\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}$$
$$+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}+a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311}.$$

Whereas,

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113}+\alpha\cdot a_{112} & a_{123}+\alpha\cdot a_{122} & a_{133}+\alpha\cdot a_{132} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213}+\alpha\cdot a_{212} & a_{223}+\alpha\cdot a_{222} & a_{233}+\alpha\cdot a_{232} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313}+\alpha\cdot a_{312} & a_{323}+\alpha\cdot a_{322} & a_{333}+\alpha\cdot a_{332} \end{array}\right)$$

$$= a_{111}\cdot a_{222}\cdot(a_{333}+\alpha\cdot a_{332}) - a_{111}\cdot a_{232}\cdot(a_{323}+\alpha\cdot a_{322}) - a_{111}\cdot(a_{223}+\alpha\cdot a_{222})\cdot a_{332} + a_{111}\cdot(a_{233}+\alpha\cdot a_{232})\cdot a_{322}$$
$$-a_{112}\cdot a_{221}\cdot(a_{333}+\alpha\cdot a_{332}) + a_{112}\cdot(a_{223}+\alpha\cdot a_{222})\cdot a_{331} + a_{112}\cdot a_{231}\cdot(a_{323}+\alpha\cdot a_{322}) - a_{112}\cdot(a_{233}+\alpha\cdot a_{232})\cdot a_{321}$$
$$+(a_{113}+\alpha\cdot a_{112})\cdot a_{221}\cdot a_{332} - (a_{113}+\alpha\cdot a_{112})\cdot a_{222}\cdot a_{331} - (a_{113}+\alpha\cdot a_{112})\cdot a_{231}\cdot a_{322} + (a_{113}+\alpha\cdot a_{112})\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot(a_{333}+\alpha\cdot a_{332}) + a_{121}\cdot(a_{213}+\alpha\cdot a_{212})\cdot a_{332} + a_{121}\cdot a_{232}\cdot(a_{313}+\alpha\cdot a_{312}) - a_{121}\cdot(a_{233}+\alpha\cdot a_{232})\cdot a_{312}$$
$$+a_{122}\cdot a_{211}\cdot(a_{333}+\alpha\cdot a_{332}) - a_{122}\cdot(a_{213}+\alpha\cdot a_{212})\cdot a_{331} - a_{122}\cdot a_{231}\cdot(a_{313}+\alpha\cdot a_{312}) + a_{122}\cdot(a_{233}+\alpha\cdot a_{232})\cdot a_{311}$$
$$-(a_{123}+\alpha\cdot a_{122})\cdot a_{211}\cdot a_{332} + (a_{123}+\alpha\cdot a_{122})\cdot a_{212}\cdot a_{331} + (a_{123}+\alpha\cdot a_{122})\cdot a_{231}\cdot a_{312} - (a_{123}+\alpha\cdot a_{122})\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot(a_{323}+\alpha\cdot a_{322}) - a_{131}\cdot(a_{213}+\alpha\cdot a_{212})\cdot a_{322} - a_{131}\cdot a_{222}\cdot(a_{313}+\alpha\cdot a_{312}) + a_{131}\cdot(a_{223}+\alpha\cdot a_{222})\cdot a_{312}$$
$$-a_{132}\cdot a_{211}\cdot(a_{323}+\alpha\cdot a_{322}) + a_{132}\cdot(a_{213}+\alpha\cdot a_{212})\cdot a_{321} + a_{132}\cdot a_{221}\cdot(a_{313}+\alpha\cdot a_{312}) - a_{132}\cdot(a_{223}+\alpha\cdot a_{222})\cdot a_{311}$$
$$+(a_{133}+\alpha\cdot a_{132})\cdot a_{211}\cdot a_{322} - (a_{133}+\alpha\cdot a_{132})\cdot a_{212}\cdot a_{321} - (a_{133}+\alpha\cdot a_{132})\cdot a_{221}\cdot a_{312} + (a_{133}+\alpha\cdot a_{132})\cdot a_{222}\cdot a_{311}$$
$$+a_{133}\cdot a_{211}\cdot(a_{322}+\alpha\cdot a_{222}) - a_{133}\cdot(a_{212}+\alpha\cdot a_{212})\cdot a_{321} - a_{133}\cdot a_{221}\cdot(a_{312}+\alpha\cdot a_{212}) + a_{133}\cdot(a_{222}+\alpha\cdot a_{222})\cdot a_{311}.$$

After expanding further we get the following result,

$$= a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331}$$
$$+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321}$$
$$-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331}$$
$$-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot a_{323} - a_{131}\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{313} + a_{131}\cdot a_{223}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{323} + a_{132}\cdot a_{213}\cdot a_{321}$$
$$+a_{132}\cdot a_{221}\cdot a_{313} - a_{132}\cdot a_{223}\cdot a_{311} + a_{133}\cdot a_{211}\cdot a_{322} - a_{133}\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot a_{312} + a_{133}\cdot a_{222}\cdot a_{311}.$$

If we compare results of above equations, we can see that we have the same result in both cases.

Remark 1. *This theorem does not hold for plan "Horizontal Layers".*

Theorem 3. *Suppose that A is 3D Determinant with two identical "Vertical Pages" or two identical "Vertical Layers".*

Then $|A| = 0$.

Proof. **Case 1.** The cubic-matrix A of order 2 with two identical "Vertical Pages" or two identical "Vertical Layers", we will proof the case 1, as following:

1. For two identical "Vertical Pages":

$$\begin{aligned}\det[A_{2\times2\times2}] &= \det\left(\begin{array}{cc|cc} a_{111} & a_{111} & a_{112} & a_{112} \\ a_{211} & a_{211} & a_{212} & a_{212} \end{array}\right) \\ &= a_{111}\cdot a_{212} - a_{112}\cdot a_{211} - a_{111}\cdot a_{212} + a_{112}\cdot a_{211} \\ &= 0.\end{aligned}$$

2. For two identical "Vertical Layers":

$$\begin{aligned}\det[A_{2\times2\times2}] &= \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{111} & a_{121} \\ a_{211} & a_{221} & a_{211} & a_{221} \end{array}\right) \\ &= a_{111}\cdot a_{221} - a_{111}\cdot a_{221} - a_{121}\cdot a_{211} + a_{121}\cdot a_{211} \\ &= 0.\end{aligned}$$

Case 2. The cubic-matrix A of order 3 with two identical "Vertical Pages" or two identical "Vertical Layers", we will proof the case 1, as following:

1. For two identical "Vertical Pages", first "Vertical Page" identical to second "Vertical Page":

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{111} & a_{131} & a_{112} & a_{112} & a_{132} & a_{113} & a_{113} & a_{133} \\ a_{211} & a_{211} & a_{231} & a_{212} & a_{212} & a_{232} & a_{213} & a_{213} & a_{233} \\ a_{311} & a_{311} & a_{331} & a_{312} & a_{312} & a_{332} & a_{313} & a_{313} & a_{333} \end{array}\right)$$

$$= a_{111} \cdot a_{212} \cdot a_{333} - a_{111} \cdot a_{232} \cdot a_{313} - a_{111} \cdot a_{213} \cdot a_{332} + a_{111} \cdot a_{233} \cdot a_{312} - a_{112} \cdot a_{211} \cdot a_{333} + a_{112} \cdot a_{213} \cdot a_{331}$$
$$+ a_{112} \cdot a_{231} \cdot a_{313} - a_{112} \cdot a_{233} \cdot a_{311} + a_{113} \cdot a_{211} \cdot a_{332} - a_{113} \cdot a_{212} \cdot a_{331} - a_{113} \cdot a_{231} \cdot a_{312} + a_{113} \cdot a_{232} \cdot a_{311}$$
$$- a_{111} \cdot a_{212} \cdot a_{333} + a_{111} \cdot a_{213} \cdot a_{332} + a_{111} \cdot a_{232} \cdot a_{313} - a_{111} \cdot a_{233} \cdot a_{312} + a_{112} \cdot a_{211} \cdot a_{333} - a_{112} \cdot a_{213} \cdot a_{331}$$
$$- a_{112} \cdot a_{231} \cdot a_{313} + a_{112} \cdot a_{233} \cdot a_{311} - a_{113} \cdot a_{211} \cdot a_{332} + a_{113} \cdot a_{212} \cdot a_{331} + a_{113} \cdot a_{231} \cdot a_{312} - a_{113} \cdot a_{232} \cdot a_{311}$$
$$+ a_{131} \cdot a_{212} \cdot a_{313} - a_{131} \cdot a_{213} \cdot a_{312} - a_{131} \cdot a_{212} \cdot a_{313} + a_{131} \cdot a_{213} \cdot a_{312} - a_{132} \cdot a_{211} \cdot a_{313} + a_{132} \cdot a_{213} \cdot a_{311}$$
$$+ a_{132} \cdot a_{211} \cdot a_{313} - a_{132} \cdot a_{213} \cdot a_{311} + a_{133} \cdot a_{211} \cdot a_{312} - a_{133} \cdot a_{212} \cdot a_{311} - a_{133} \cdot a_{211} \cdot a_{312} + a_{133} \cdot a_{212} \cdot a_{311} = 0.$$

2. For two identical ”Vertical Pages”, first ”Vertical Page” identical to third ”Vertical Page”:

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{111} & a_{112} & a_{122} & a_{112} & a_{113} & a_{123} & a_{113} \\ a_{211} & a_{221} & a_{211} & a_{212} & a_{222} & a_{212} & a_{213} & a_{223} & a_{213} \\ a_{311} & a_{321} & a_{311} & a_{312} & a_{322} & a_{312} & a_{313} & a_{323} & a_{313} \end{array}\right)$$

$$= a_{111} \cdot a_{222} \cdot a_{313} - a_{111} \cdot a_{212} \cdot a_{323} - a_{111} \cdot a_{223} \cdot a_{312} + a_{111} \cdot a_{213} \cdot a_{322} - a_{112} \cdot a_{221} \cdot a_{313} + a_{112} \cdot a_{223} \cdot a_{311}$$
$$+ a_{112} \cdot a_{211} \cdot a_{323} - a_{112} \cdot a_{213} \cdot a_{321} + a_{113} \cdot a_{221} \cdot a_{312} - a_{113} \cdot a_{222} \cdot a_{311} - a_{113} \cdot a_{211} \cdot a_{322} + a_{113} \cdot a_{212} \cdot a_{321}$$
$$- a_{121} \cdot a_{212} \cdot a_{313} + a_{121} \cdot a_{213} \cdot a_{312} + a_{121} \cdot a_{212} \cdot a_{313} - a_{121} \cdot a_{213} \cdot a_{312} + a_{122} \cdot a_{211} \cdot a_{313} - a_{122} \cdot a_{213} \cdot a_{311}$$
$$- a_{122} \cdot a_{211} \cdot a_{313} + a_{122} \cdot a_{213} \cdot a_{311} - a_{123} \cdot a_{211} \cdot a_{312} + a_{123} \cdot a_{212} \cdot a_{311} + a_{123} \cdot a_{211} \cdot a_{312} - a_{123} \cdot a_{212} \cdot a_{311}$$
$$+ a_{111} \cdot a_{212} \cdot a_{323} - a_{111} \cdot a_{213} \cdot a_{322} - a_{111} \cdot a_{222} \cdot a_{313} + a_{111} \cdot a_{223} \cdot a_{312} - a_{112} \cdot a_{211} \cdot a_{323} + a_{112} \cdot a_{213} \cdot a_{321}$$
$$+ a_{112} \cdot a_{221} \cdot a_{313} - a_{112} \cdot a_{223} \cdot a_{311} + a_{113} \cdot a_{211} \cdot a_{322} - a_{113} \cdot a_{212} \cdot a_{321} - a_{113} \cdot a_{221} \cdot a_{312} + a_{113} \cdot a_{222} \cdot a_{311} = 0.$$

3. For two identical ”Vertical Pages”, second ”Vertical Page” identical to third ”Vertical Page”:

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{121} & a_{112} & a_{122} & a_{122} & a_{113} & a_{123} & a_{123} \\ a_{211} & a_{221} & a_{221} & a_{212} & a_{222} & a_{222} & a_{213} & a_{223} & a_{223} \\ a_{311} & a_{321} & a_{321} & a_{312} & a_{322} & a_{322} & a_{313} & a_{323} & a_{323} \end{array}\right)$$

$$= a_{111} \cdot a_{222} \cdot a_{323} - a_{111} \cdot a_{222} \cdot a_{323} - a_{111} \cdot a_{223} \cdot a_{322} + a_{111} \cdot a_{223} \cdot a_{322} - a_{112} \cdot a_{221} \cdot a_{323} + a_{112} \cdot a_{223} \cdot a_{321}$$
$$+ a_{112} \cdot a_{221} \cdot a_{323} - a_{112} \cdot a_{223} \cdot a_{321} + a_{113} \cdot a_{221} \cdot a_{322} - a_{113} \cdot a_{222} \cdot a_{321} - a_{113} \cdot a_{221} \cdot a_{322} + a_{113} \cdot a_{222} \cdot a_{321}$$
$$- a_{121} \cdot a_{212} \cdot a_{323} + a_{121} \cdot a_{213} \cdot a_{322} + a_{121} \cdot a_{222} \cdot a_{313} - a_{121} \cdot a_{223} \cdot a_{312} + a_{122} \cdot a_{211} \cdot a_{323} - a_{122} \cdot a_{213} \cdot a_{321}$$
$$- a_{122} \cdot a_{221} \cdot a_{313} + a_{122} \cdot a_{223} \cdot a_{311} - a_{123} \cdot a_{211} \cdot a_{322} + a_{123} \cdot a_{212} \cdot a_{321} + a_{123} \cdot a_{221} \cdot a_{312} - a_{123} \cdot a_{222} \cdot a_{311}$$
$$+ a_{121} \cdot a_{212} \cdot a_{323} - a_{121} \cdot a_{213} \cdot a_{322} - a_{121} \cdot a_{222} \cdot a_{313} + a_{121} \cdot a_{223} \cdot a_{312} - a_{122} \cdot a_{211} \cdot a_{323} + a_{122} \cdot a_{213} \cdot a_{321}$$
$$+ a_{122} \cdot a_{221} \cdot a_{313} - a_{122} \cdot a_{223} \cdot a_{311} + a_{123} \cdot a_{211} \cdot a_{322} - a_{123} \cdot a_{212} \cdot a_{321} - a_{123} \cdot a_{221} \cdot a_{312} + a_{123} \cdot a_{222} \cdot a_{311} = 0.$$

4. For two identical ”Vertical Layers”, first ”Vertical Layer” identical to second ”Vertical Layer”:

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{111} & a_{121} & a_{131} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{211} & a_{221} & a_{231} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{311} & a_{321} & a_{331} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{111} \cdot a_{221} \cdot a_{333} - a_{111} \cdot a_{231} \cdot a_{323} - a_{111} \cdot a_{223} \cdot a_{331} + a_{111} \cdot a_{233} \cdot a_{321} - a_{111} \cdot a_{221} \cdot a_{333} + a_{111} \cdot a_{223} \cdot a_{331}$$
$$+ a_{111} \cdot a_{231} \cdot a_{323} - a_{111} \cdot a_{233} \cdot a_{321} + a_{113} \cdot a_{221} \cdot a_{331} - a_{113} \cdot a_{221} \cdot a_{331} - a_{113} \cdot a_{231} \cdot a_{321} + a_{113} \cdot a_{231} \cdot a_{321}$$
$$- a_{121} \cdot a_{211} \cdot a_{333} + a_{121} \cdot a_{213} \cdot a_{331} + a_{121} \cdot a_{231} \cdot a_{313} - a_{121} \cdot a_{233} \cdot a_{311} + a_{121} \cdot a_{211} \cdot a_{333} - a_{121} \cdot a_{213} \cdot a_{331}$$
$$- a_{121} \cdot a_{231} \cdot a_{313} + a_{121} \cdot a_{233} \cdot a_{311} - a_{123} \cdot a_{211} \cdot a_{331} + a_{123} \cdot a_{211} \cdot a_{331} + a_{123} \cdot a_{231} \cdot a_{311} - a_{123} \cdot a_{231} \cdot a_{311}$$
$$+ a_{131} \cdot a_{211} \cdot a_{323} - a_{131} \cdot a_{213} \cdot a_{321} - a_{131} \cdot a_{221} \cdot a_{313} + a_{131} \cdot a_{223} \cdot a_{311} - a_{131} \cdot a_{211} \cdot a_{323} + a_{131} \cdot a_{213} \cdot a_{321}$$
$$+ a_{131} \cdot a_{221} \cdot a_{313} - a_{131} \cdot a_{223} \cdot a_{311} + a_{133} \cdot a_{211} \cdot a_{321} - a_{133} \cdot a_{211} \cdot a_{321} - a_{133} \cdot a_{221} \cdot a_{311} + a_{133} \cdot a_{221} \cdot a_{311} = 0.$$

5. For two identical ”Vertical Layers”, first ”Vertical Layer” identical to third ”Vertical Layer”:

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{311} & a_{321} & a_{331} \end{array}\right)$$

$$= a_{111} \cdot a_{222} \cdot a_{331} - a_{111} \cdot a_{232} \cdot a_{321} - a_{111} \cdot a_{221} \cdot a_{332} + a_{111} \cdot a_{231} \cdot a_{322} - a_{112} \cdot a_{221} \cdot a_{331} + a_{112} \cdot a_{221} \cdot a_{331}$$
$$+ a_{112} \cdot a_{231} \cdot a_{321} - a_{112} \cdot a_{231} \cdot a_{321} + a_{111} \cdot a_{221} \cdot a_{332} - a_{111} \cdot a_{222} \cdot a_{331} - a_{111} \cdot a_{231} \cdot a_{322} + a_{111} \cdot a_{232} \cdot a_{321}$$
$$- a_{121} \cdot a_{212} \cdot a_{331} + a_{121} \cdot a_{211} \cdot a_{332} + a_{121} \cdot a_{232} \cdot a_{311} - a_{121} \cdot a_{231} \cdot a_{312} + a_{122} \cdot a_{211} \cdot a_{331} - a_{122} \cdot a_{211} \cdot a_{331}$$
$$- a_{122} \cdot a_{231} \cdot a_{311} + a_{122} \cdot a_{231} \cdot a_{311} - a_{121} \cdot a_{211} \cdot a_{332} + a_{121} \cdot a_{212} \cdot a_{331} + a_{121} \cdot a_{231} \cdot a_{312} - a_{121} \cdot a_{232} \cdot a_{311}$$
$$+ a_{131} \cdot a_{212} \cdot a_{321} - a_{131} \cdot a_{211} \cdot a_{322} - a_{131} \cdot a_{222} \cdot a_{311} + a_{131} \cdot a_{221} \cdot a_{312} - a_{132} \cdot a_{211} \cdot a_{321} + a_{132} \cdot a_{211} \cdot a_{321}$$
$$+ a_{132} \cdot a_{221} \cdot a_{311} - a_{132} \cdot a_{221} \cdot a_{311} + a_{131} \cdot a_{211} \cdot a_{322} - a_{131} \cdot a_{212} \cdot a_{321} - a_{131} \cdot a_{221} \cdot a_{312} + a_{131} \cdot a_{222} \cdot a_{311} = 0.$$

6. For two identical "Vertical Layers", second "Vertical Layer" identical to third "Vertical Layer":

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{112} & a_{122} & a_{132} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{212} & a_{222} & a_{232} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{312} & a_{322} & a_{332} \end{array}\right)$$

$$= a_{111}\cdot a_{222}\cdot a_{332} - a_{111}\cdot a_{232}\cdot a_{322} - a_{111}\cdot a_{222}\cdot a_{332} + a_{111}\cdot a_{232}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{332} + a_{112}\cdot a_{222}\cdot a_{331}$$

$$+a_{112}\cdot a_{231}\cdot a_{322} - a_{112}\cdot a_{232}\cdot a_{321} + a_{112}\cdot a_{221}\cdot a_{332} - a_{112}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{112}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot a_{332} + a_{121}\cdot a_{212}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{312} - a_{121}\cdot a_{232}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{332} - a_{122}\cdot a_{212}\cdot a_{331}$$

$$-a_{122}\cdot a_{231}\cdot a_{312} + a_{122}\cdot a_{232}\cdot a_{311} - a_{122}\cdot a_{211}\cdot a_{332} + a_{122}\cdot a_{212}\cdot a_{331} + a_{122}\cdot a_{231}\cdot a_{312} - a_{122}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot a_{322} - a_{131}\cdot a_{212}\cdot a_{322} - a_{131}\cdot a_{222}\cdot a_{312} + a_{131}\cdot a_{222}\cdot a_{312} - a_{132}\cdot a_{211}\cdot a_{322} + a_{132}\cdot a_{212}\cdot a_{321}$$

$$+a_{132}\cdot a_{221}\cdot a_{312} - a_{132}\cdot a_{222}\cdot a_{311} + a_{132}\cdot a_{211}\cdot a_{322} - a_{132}\cdot a_{212}\cdot a_{321} - a_{132}\cdot a_{221}\cdot a_{312} + a_{132}\cdot a_{222}\cdot a_{311} = 0.$$

□

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