On new type of sequences convergent to the Euler-Mascheroni constant

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ABSTRACT

In this paper, we present a new sequence that converges to the Euler constant. We use the Cramer's rule to determine the best possible constants of this sequence.

Keywords: Euler-Mascheroni constant, Rate of convergence, Cramer's rule

1. Introduction

In the theory of special function, an important thing is to consider the approximate formulas of mathematical constants or special functions, and to determine their best possible constants.

These approximate formulas are widely used in mathematics and engineering.

Many mathematicians have tried to find new types of approximate formulas and study their related properties.

The Euler–Mascheroni constant γ is given by

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right) = 0.57721566490115328\cdots. \tag{1.1}$$

Recently, many researchers are preoccupied to improve rate of convergence of remarkable sequences convergent towards γ; see, for example, [1-4]. We list some main results:

$$\sum_{k=1}^{n}\frac{1}{k}-\ln n = \gamma + O(n^{-1}), \tag{1.2}$$

$$\sum_{k=1}^{n-1}\frac{1}{k}+\frac{1}{2n}-\ln n = \gamma + O(n^{-2}), \tag{1.3}$$

$$\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right) = \gamma + O(n^{-2}), \tag{1.4}$$

$$\sum_{k=1}^{n-1}\frac{1}{k}+\frac{1}{(6\pm 2\sqrt{6})n}-\ln\left(n\mp\frac{1}{\sqrt{6}}\right) = \gamma + O(n^{-3}), \tag{1.5}$$

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$$\sum_{k=1}^{n}\frac{1}{k}-\ln\frac{n^3+\frac{3}{2}n^2+\frac{227}{240}n+\frac{107}{480}}{n^2+n+\frac{97}{240}}=\gamma+O(n^{-6}), \tag{1.6}$$

$$\sum_{k=1}^{n}\frac{1}{k}-\ln\left(1+\frac{1}{2n}+\frac{1}{24n^2}-\frac{1}{48n^3}+\frac{23}{5760n^4}\right)=\gamma+O(n^{-5}), \tag{1.7}$$

Dawei [5], using continued fraction approximation, provided faster sequence convergent to γ as follows,

$$L_{r,n}=1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{rn}-\ln n-\cfrac{a_1}{n+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\cfrac{a_4 n}{n+\ddots}}}}.. \tag{1.8}$$

You [6] provided new classes of convergent sequences for the Euler–Mascheroni constant as follows

$$r_m(n)=\sum_{k=1}^{n}\frac{1}{k}-\ln n-\sum_{k=1}^{m}\ln\left(1+\frac{a_k}{n^k}\right), \tag{1.9}$$

where

$$a_1=\frac{1}{2},a_2=\frac{1}{24},a_3=-\frac{1}{24},a_4=\frac{143}{5760},a_5=-\frac{1}{160},a_6=-\frac{151}{290304},a_7=-\frac{1}{896},\cdots.$$

In this paper, we provide a new class of sequence convergent to Euler-Mascheroni constant.

2. Approximations for the Euler-Mascheroni constant

Here we give new classes of convergent sequence for the Euler-Mascheroni constant.

Lemma([7, 8]). If $(x_n)_{n\geq 1}$ is convergent to zero and there exists the limit

$$\lim_{n\to\infty} n^s(x_n-x_{n+1})=L\in[-\infty,+\infty] \tag{2.1}$$

with $s>1$, then

$$\lim_{n\to\infty} n^{s-1}x_n=\frac{L}{s-1}. \tag{2.2}$$

Using **Lemma**, we can see that the rate of convergence of the sequence $(x_n)_{n\geq 1}$ increases together with the value s satisfying (2.1).

Theorem. For the Euler-Mascheroni constant, we have the following convergent sequence,

$$\gamma_n^k=1+\frac{1}{2}+\cdots+\frac{1}{n-k-1}+\frac{a_k}{n-k}+\frac{a_{k-1}}{n-k+1}+\cdots+\frac{a_0}{n}-\ln n. \tag{2.3}$$

For any fixed k, we can obtain the sequence with the coefficients of which rate of convergence is n^{-m} ($m\leq k+2$).

Proof. We need to find the value of the parameters $a_0,a_1,\cdots,a_k\in(-\infty,+\infty)$ which produces the best approximation of (2.3).

The method to measure the accuracy of the approximation is to say that the approximation (2.3) is better as $\gamma_n^k - \gamma$ quicker converges to zero. Using (2.3), we have

$$\gamma_{n-1}^k - \gamma_n^k = -\frac{1}{n-k-1} + \sum_{i=0}^{k} a_i \left(\frac{1}{n-1-i} - \frac{1}{n-i} \right) - \ln\left(1 - \frac{1}{n}\right) \tag{2.4}$$

Developing in power series in $1/n$, we have, from (2.4)

$$\gamma_{n-1}^k - \gamma_n^k = \sum_{m=2}^{\infty} \left(\frac{1}{m} - (k+1)^{m-1} + \sum_{i=0}^{k} a_i \left((i+1)^{m-1} - i^{m-1} \right) \right) x^m . \tag{2.5}$$

From **Lemma**, we know that the convergent rate of the sequence $(\gamma_n^k)_{n \ge 1}$ is even higher as the value s ($s \le k+3$) satisfying (2.1).

We find the coefficients in (2.5) to satisfy (2.1).

These coefficients are a solution of systems of linear equations in matrix form as

$$D \cdot A = B, \tag{2.6}$$

where D, A, B are matrices given by

$$D = \begin{pmatrix} 1^1 - 0^1 & 2^1 - 1^1 & \cdots & (k+1)^1 - k^1 \\ 1^2 - 0^2 & 2^2 - 1^2 & \cdots & (k+1)^2 - k^2 \\ \cdots & \cdots & & \cdots \\ 1^{k+1} - 0^{k+1} & 2^{k+1} - 1^{k+1} & \cdots & (k+1)^{k+1} - k^{k+1} \end{pmatrix},$$

$$A = \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_k \end{pmatrix},$$

$$B = \begin{pmatrix} (k+1) - \frac{1}{2} \\ (k+1)^2 - \frac{1}{3} \\ \cdots\cdots\cdots\cdots \\ (k+1)^{k+1} - \frac{1}{k+2} \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$\det D = \begin{vmatrix} 1^1 - 0^1 & 2^1 - 1^1 & \cdots & (k+1)^1 - k^1 \\ 1^2 - 0^2 & 2^2 - 1^2 & \cdots & (k+1)^2 - k^2 \\ \cdots & \cdots & & \cdots \\ 1^{k+1} - 0^{k+1} & 2^{k+1} - 1^{k+1} & \cdots & (k+1)^{k+1} - k^{k+1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2^1 & \cdots & (k+1)^1 \\ 1 & 2^2 & \cdots & (k+1)^2 \\ \cdots & \cdots & & \cdots \\ 1 & 2^{k+1} & \cdots & (k+1)^{k+1} \end{vmatrix}$$

$$= (k+1)! \begin{vmatrix} 1^1 & 2^1 & \cdots & k^1 \\ 1^2 & 2^2 & \cdots & k^2 \\ \cdots & \cdots & & \cdots \\ 1^k & 2^k & \cdots & k^k \end{vmatrix} = \prod_{i=1}^{k+1} i! \ .$$

Using the Cramer's rule, we solve the systems of linear equations (2.6).
Since the determinant is nonzero, we can use the Cramer's rule to find a solution.
We find the solution by substitution of the elements of the vector B for the j-th (j=1,2,…, k+1) column of D.

$$\det D_j = \begin{vmatrix} 1^1 - 0^1 & \cdots & (k+1) - \frac{1}{2} & \cdots & (k+1)^1 - k^1 \\ 1^2 - 0^2 & \cdots & (k+1)^2 - \frac{1}{3} & \cdots & (k+1)^2 - k^2 \\ \cdots & & \cdots & & \cdots \\ 1^{k+1} - 0^{k+1} & \cdots & (k+1)^{k+1} - \frac{1}{k+2} & \cdots & (k+1)^{k+1} - k^{k+1} \end{vmatrix} = D - M_j,$$

where M_j are matrices given by

$$M_j = \begin{vmatrix} 1^1 - 0^1 & \cdots & \frac{1}{2} & \cdots & (k+1)^1 - k^1 \\ 1^2 - 0^2 & \cdots & \frac{1}{3} & \cdots & (k+1)^2 - k^2 \\ \cdots & & \cdots & & \cdots \\ 1^{k+1} - 0^{k+1} & \cdots & \frac{1}{k+2} & \cdots & (k+1)^{k+1} - k^{k+1} \end{vmatrix}.$$

Thus, the Cramer's rule allows us to find a solution given by

$$a_{j-1} = \frac{D_j}{D} = 1 - \frac{M_j}{\prod_{i=1}^{k+1} i!} \quad , j = \overline{1, k+1}. \tag{2.7}$$

For any fixed k, we can obtain the sequence with the coefficients of which rate of convergence is n^{-m} ($m \le k+2$).
The first few best possible constants can be found as follows:

(i) if $k = 0$, $a_0 = \frac{1}{2}$, then, the rate of convergence of $(\gamma_n^0 - \gamma)_{n\ge1}$ is n^{-2}.

(ii) if $k = 0$, $a_0 \ne \frac{1}{2}$, then the rate of convergence of $(\gamma_n^0 - \gamma)_{n\ge1}$ is n^{-1}.

We repeat our approach to determine the coefficients $a_0, a_1, a_2, \cdots$.

In fact, we can easily compute by the *Mathematica*. For example, if $k=4$,

$$a_0=\frac{95}{288}, a_1=\frac{951}{720}, a_2=\frac{552}{720}, a_3=\frac{73}{720}, a_4=\frac{1413}{1440},$$

then the rate of convergence of $(\gamma_n^4-\gamma)_{n\geq 1}$ is n^{-6}.

The proof of Theorem is thus completed.

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