LAGRANGIAN APPROACH TO DERIVING THE GRAVITY EQUATIONS IN A 3D-BRANE UNIVERSE WITHOUT EQUIDISTANCE POSTULATE.

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Abstract. The 3D-brane universe model is an alternative non-Einsteinian theory of gravity. The initial version of this theory uses the so-called equidistance postulate. Recently a new version of theory was started which is free of this postulate. In this paper we continue building the new version of theory by applying the Lagrangian approach to it.

1. Introduction.

The 3D-brane universe model was first suggested in [1] (see also [2] and [3]). This theory is based on a criticism of the standard concept of four-dimensional spacetime and on the assertion that the spacetime is not a physical continuum. It is a mathematical continuum that subdivides into a dense foliation of 3D-branes each representing some instantaneous state of the physical universe. From the standard relativity this mathematical continuum inherits the metric and some other attributes. The initial version of theory (see [1,4–8] and [9–11]) was built using the so-called equidistance postulate.

Postulate 1.1. *Watches of any two comoving observers can be synchronized.*

Comoving observers in this context are those observers whose world lines (see [12] and [13]) are perpendicular to all 3D-branes in the spacetime foliation. Geometrically the postulate 1.1 means that the segments of world lines of comoving observers enclosed between any two given 3D-branes all are of the same length.

A new version of the 3D-brane universe model was started in [14]. It does not use the postulate 1.1 and deals with arbitrary foliations of 3D-branes in the spacetime. Spacial coordinates x^1, x^2, x^3 which are constant along world lines of comoving observers are called comoving coordinates. A time variable t is called a brane time if it is constant throughout each 3D-brane of the spacetime foliation:

$$t\,\Big|_{\,\text{3D-brane}}=\text{const}\,. \tag{1.1}$$

Using some brane time variable we define the associated brane time coordinate:

$$x^0=c_{\text{gr}}\,t. \tag{1.2}$$

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The constant c_{gr} in (1.2) is interpreted as the speed of gravitational waves. As it was declared in [7], in the 3D-brane universe model this constant is not necessarily equal to the speed of electromagnetic waves c_{el}, which is the speed of light.

Comoving coordinates and brane time variables obeying the condition (1.1) do exist. If we complement comoving coordinates x^1, x^2, x^3 with a brane time coordinate x^0 from (1.2), then we get four-dimensional coordinates x^0, x^1, x^2, x^3 associated with the foliation of 3D-branes in the spacetime. The coordinates x^0, x^1, x^2, x^3 are special ones. The four-dimensional metric tensor in these coordinates is given by a block-diagonal matrix:

$$G_{ij}=\begin{Vmatrix} g_{00} & 0 & 0 & 0\\ 0 & -g_{11} & -g_{12} & -g_{13}\\ 0 & -g_{21} & -g_{22} & -g_{23}\\ 0 & -g_{31} & -g_{32} & -g_{33}\end{Vmatrix}. \tag{1.3}$$

The quantities g_{ij} from the lower right diagonal block of the matrix (1.3) are interpreted as the components of a time-dependent three-dimensional metric:

$$g_{ij}=g_{ij}(t,x^1,x^2,x^3),\qquad 1\leqslant i,j\leqslant 3. \tag{1.4}$$

The quantity g_{00} from the upper left diagonal block in the matrix (1.3) is interpreted as a time dependent scalar function

$$g_{00}=g_{00}(t,x^1,x^2,x^3). \tag{1.5}$$

By substituting (1.3) into the standard Einstein's gravity equation

$$r_{ij}-\frac{r}{2}\,G_{ij}-\Lambda\,G_{ij}=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{ij} \tag{1.6}$$

in [14] three groups of differential equations for the function (1.5) and for the metric (1.4) were derived (see (4.35), (4.36), and (4.37) therein). The first group of equations is the most numerous:

$$\begin{gathered}
\frac{g_{00}^{-2}}{2\,c_{\text{gr}}}\biggl(g_{ij}\sum_{k=1}^{3}b^k_k-b_{ij}\biggr)\,\dot g_{00}+\frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(g^{kq}\,g_{ij}-\delta^k_i\,\delta^q_j\Bigr)\,\nabla_{kq}\,g_{00}\,-\\
-\,\frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(g^{kq}\,g_{ij}-\delta^k_i\,\delta^q_j\Bigr)\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}+g_{00}^{-1}\biggl(\frac{1}{c_{\text{gr}}}\,\dot b_{ij}\,-\\
-\sum_{k=1}^{3}\frac{1}{c_{\text{gr}}}\,\dot b^k_k\,g_{ij}-\sum_{k=1}^{3}(b_{ki}\,b^k_j+b_{kj}\,b^k_i)-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k\,-\\
-\,\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\sum_{k=1}^{3}b^k_k\,b_{ij}\biggr)+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{ij},
\end{gathered}\tag{1.7}$$

where $1\leqslant i,j\leqslant 3$. Here in (1.7) and in (1.6) γ is Newton's gravitational constant (see [6]), Λ is the cosmological constant (see [7]), R_{ij} are the components of the three-dimensional Ricci tensor of the metric (1.4), R is the three-dimensional scalar curvature, and b_{ij} are given by the formula

$$b_{ij}=\frac{\dot{g}_{ij}}{2\,c_{\text{gr}}}=\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial g_{ij}}{\partial t}=\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}. \tag{1.8}$$

The quantities b^i_j in (1.7) are produced by raising one of the two indices in (1.8).

The second group of equations derived in [14] is written as follows:

$$\sum_{k=1}^{3}\nabla_k\,b^k_i-\sum_{k=1}^{3}\nabla_i\,b^k_k+\frac{1}{2}\,g_{00}^{-1}\sum_{k=1}^{3}\Bigl(b^k_k\,\nabla_i\,g_{00}-b^k_i\,\nabla_k\,g_{00}\Bigr)=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{i\,0}, \tag{1.9}$$

where $1\leqslant i\leqslant 3$. The third group comprises exactly one equation. It is written as

$$-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\frac{R}{2}\,g_{00}-\Lambda\,g_{00}=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{00}. \tag{1.10}$$

The quantities T_{ij}, $T_{i\,0}$, and T_{00} in (1.7), (1.9), and (1.10) are the components of the energy-momentum tensor (see [17]) from (1.6). In the present paper we consider two of the three groups of equations (1.7), (1.9), (1.10) and apply the Lagrangian approach to deriving them. These are the equations (1.7) and (1.10). As for the equations (1.9), we exclude them from the theory thus making our theory non-equivalent to the standard relativity.

2. Action integral and its reduction.

In the standard theory of relativity the Einstein equation (1.6) is derived with the use of the following standard action integral (see § 2 in Chapter V of [18]):

$$S_{\text{gr}}=-\frac{c_{\text{gr}}^3}{16\,\pi\,\gamma}\int(r+2\,\Lambda)\,\sqrt{-\det G}\;d^4x. \tag{2.1}$$

Here r is the four-dimensional scalar curvature associated with the metric (1.3). Applying (1.2) and (1.3) to (2.1) we write (2.1) in the three-dimensional form

$$S_{\text{gr}}=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\iint(r+2\,\Lambda)\,\sqrt{\det g}\,\sqrt{g_{00}}\;d^3x\,dt. \tag{2.2}$$

In [14] the following formula for the scalar curvature r was derived:

$$\begin{aligned}r=g_{00}^{-2}\,\frac{\dot{g}_{00}}{c_{\text{gr}}}\sum_{k=1}^{3}b^k_k+g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_{kq}\,g_{00}\,-\\-\frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}-2\,g_{00}^{-1}\sum_{k=1}^{3}\frac{\dot{b}^k_k}{c_{\text{gr}}}\,-\\-R-g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q.\end{aligned} \tag{2.3}$$

Let's take the first and the fourth terms in the right hand side of the formula (2.3). When substituting them into (2.2) we get the following time integral:

$$\begin{aligned} I=\int\limits_v^u\left(g_{00}^{-2}\,\frac{\dot g_{00}}{c_{\text{gr}}}\sum_{k=1}^{3}b^k_k-2\,g_{00}^{-1}\sum_{k=1}^{3}\frac{\dot b^k_k}{c_{\text{gr}}}\right)\sqrt{\det g}\,\sqrt{g_{00}}\,dt=\\ =\int\limits_v^u\left(g_{00}^{-3/2}\,\frac{\dot g_{00}}{c_{\text{gr}}}\sum_{k=1}^{3}b^k_k-2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{\dot b^k_k}{c_{\text{gr}}}\right)\sqrt{\det g}\,dt.\end{aligned}\tag{2.4}$$

The further transformation of the integral (2.4) yields

$$\begin{aligned} I=\int\limits_v^u\frac{\partial}{\partial t}\left(-2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{b^k_k}{c_{\text{gr}}}\right)\sqrt{\det g}\,dt=\\ =-2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{b^k_k}{c_{\text{gr}}}\,\sqrt{\det g}\,\Biggr|_v^u+\int\limits_v^u 2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{b^k_k}{c_{\text{gr}}}\,\frac{\partial(\sqrt{\det g}\,)}{\partial t}\,dt.\end{aligned}\tag{2.5}$$

The non-integral term in (2.5) can be omitted since non-integral terms do not affect differential equations derived from action integrals. The integral term in (2.5) is transformed using Jacobi's formula for differentiating determinants (see [19]):

$$\frac{\partial(\sqrt{\det g}\,)}{\partial t}=\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\frac{\partial g_{kq}}{\partial t}\,\sqrt{\det g}\,.\tag{2.6}$$

Applying the formulas (2.6) and (1.8) to the integral (2.5), we derive

$$I=-2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{b^k_k}{c_{\text{gr}}}\,\sqrt{\det g}\,\Biggr|_v^u+\int\limits_v^u 2\,g_{00}^{-1/2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,\sqrt{\det g}\,dt.\tag{2.7}$$

Due to the formula (2.7) the action integral (2.2) is written as

$$S_{\text{gr}}=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\iint(\rho+2\,\Lambda)\,\sqrt{\det g}\,\sqrt{g_{00}}\,d^3x\,dt,\tag{2.8}$$

where the scalar function ρ is given by the following formula:

$$\begin{aligned}\rho=g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_{kq}\,g_{00}-\frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,-\\ -R-g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k+g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q.\end{aligned}\tag{2.9}$$

The formula (2.8) is analogous to the formula (2.2) in [5], while the formula (2.9) is an analog of the formula (2.6) therein. Unlike the initial action (2.2), the action integral (2.8) is of the first order with respect to the time derivatives of g_{ij} and g_{00}.

3. Lagrangians of the gravitational field and matter.

Action integrals are usually written as time integrals of Lagrangians, while Lagrangians are spacial integrals of Lagrangian densities. Therefore we write (2.8) as

$$S_{\text{gr}} = \int L_{\text{gr}}\,dt, \qquad\qquad L_{\text{gr}} = \int \mathcal{L}_{\text{gr}}\,\sqrt{\det g}\;d^3x. \tag{3.1}$$

Matter has its own action integral and its own Lagrangian:

$$S_{\text{mat}} = \int L_{\text{mat}}\,dt, \qquad\qquad L_{\text{mat}} = \int \mathcal{L}_{\text{mat}}\,\sqrt{\det g}\;d^3x. \tag{3.2}$$

The Lagrangian density in (3.1) for the gravitational field is given by the formula

$$\mathcal{L}_{\text{gr}} = -\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\sqrt{g_{00}}\,(\rho + 2\,\Lambda), \tag{3.3}$$

where ρ is taken from (2.9). The square root of g_{00} is inherited from the four-dimensional action. Therefore here in the three-dimensional approach we do not include it to (3.1) and (3.2) and attribute it to the Lagrangian density (3.3).

Due to (2.9) the Lagrangian density (3.3) depends on g_{00} from (1.5), on g_{ij} from (1.4) and on the time derivatives of these dynamic variables. The time derivatives of g_{ij} are replaced by b_{ij} from (1.8). Therefore we write

$$L_{\text{gr}} = L_{\text{gr}}(g, \dot{g}, \mathbf{g}, \mathbf{b}). \tag{3.4}$$

Here g and $\dot{g}$ represent g_{00} and $\dot{g}_{00}$, while $\mathbf{g}$ and $\mathbf{b}$ represent g_{ij} and b_{ij}. The Lagrangian density of matter can depend on some auxiliary dynamic variables responsible for the state of matter. Like in [5], we denote these auxiliary dynamic variables through $Q_1, \ldots, Q_n$ and their time derivatives through $\dot{Q}_1, \ldots, \dot{Q}_n$:

$$\dot{Q}_i = \frac{\partial Q_i}{\partial t}. \tag{3.5}$$

The relationships (3.5) are analogous to (1.8). Using them, we write

$$L_{\text{mat}} = L_{\text{mat}}(g, \dot{g}, \mathbf{g}, \mathbf{b}, \mathbf{Q}, \dot{\mathbf{Q}}). \tag{3.6}$$

Each argument in the argument lists of L_{gr} and L_{mat} in (3.4) and (3.6) represents not only the corresponding group of dynamic variables, but some finite number of partial derivatives of them with respect to the spacial variables x^1, x^2, x^3.

The total action integral of the gravitational field and matter is the sum of action integrals (3.1) and (3.2). We write it as

$$S = \int L\,dt, \qquad L = \int \mathcal{L}\,\sqrt{\det g}\;d^3x, \qquad \mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}. \tag{3.7}$$

The next step in developing the theory is to apply the stationary action principle (see [20]) to the action integral S in (3.7). Applying this principle formally, we get three groups of differential equations. The first group is written as

$$-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=0,\tag{3.8}$$

where $1\leqslant i,j\leqslant 3$. This group of equations is associated with the dynamic variables g_{ij} and b_{ij}. The second group of equations is associated with g_{00} and $\dot{g}_{00}$:

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta\dot{g}_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{L}}{\delta\dot{g}_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta g_{00}}\Bigr)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=0.\tag{3.9}$$

This group of equations comprises exactly one equation. And the third group of equations, which is associated with $Q_1,\,\ldots\,,\,Q_n$ and $\dot{Q}_1,\,\ldots\,,\,\dot{Q}_n$, is written as

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta\dot{Q}_i}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},\dot{Q}}}-c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{L}}{\delta\dot{Q}_i}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},\dot{\mathbf{Q}}}}=0,\tag{3.10}$$

where $1\leqslant i\leqslant 3$. The equations (3.8) and (3.9) describe the evolution of the gravitational field, while the equations (3.10) describe the evolution of matter.

Below we shall not transform the equations (3.10) since in the present paper the variables $Q_1,\,\ldots\,,\,Q_n$ are not specified and no explicit expression for the Lagrangian density of matter $\mathcal{L}_{\text{mat}}$ in (3.2) is given. As for the equations (3.8) and (3.9), we shall transform them. Let's denote

$$\begin{aligned}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}&=-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\\&\quad-\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}},\end{aligned}\tag{3.11}$$

$$\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}=-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{kq}}\,g_{ki}\,g_{qj}.\tag{3.12}$$

The formulas (3.11) and (3.12) are analogs of the formulas (3.20) and (3.21) from [5]. Apart from these two formulas we consider the following ones:

$$\begin{aligned}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}&=-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta\dot{g}_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\\&\quad-c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta\dot{g}_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}\Bigr)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}},\end{aligned}\tag{3.13}$$

$$\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{00}}=-\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}\,g_{00}^2.\tag{3.14}$$

Taking into account (3.7) and applying (3.11) to (3.8), we derive

$$-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=-\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}.\tag{3.15}$$

Similarly, taking into account (3.7) and applying (3.13) to (3.9), we derive

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta\dot{g}_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-c_{\text{gr}}\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta\dot{g}_{00}}\right)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^{q}_{q}+\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{00}}\right)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=-\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}. \tag{3.16}$$

The rest is to derive explicit expressions for the left hand sides of the equations (3.15) and (3.16) using (3.1), (3.3), and (2.9).

4. The equations for the three-dimensional metric.

In implicit form the required differential equations for the three-dimensional metric g_{ig} are written as the Euler-Lagrange equations (3.15). In order to make it explicit we need to calculate the partial variational derivatives in the left hand side of the equations (3.15). Let's introduce a small variation to b_{ij} as follows:

$$\hat{b}_{ij}=b_{ij}(t,x^1,x^2,x^3)+\varepsilon\,h_{ij}(t,x^1,x^2,x^3). \tag{4.1}$$

Here $\varepsilon\to 0$ is a small parameter and $h_{ij}(t,x^1,x^2,x^3)$ are arbitrary smooth functions with compact support (see [21]). Then the partial variational derivative of the Lagrangian density $\mathcal{L}_{\text{gr}}$ with respect to b_{ij} is defined through the formula

$$\hat{L}_{\text{gr}}=L_{\text{gr}}+\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}h_{ij}\,\sqrt{\det g}\,d^3x+\ldots, \tag{4.2}$$

where L_{gr} is taken from (3.1) and $\hat{L}_{\text{gr}}$ is its deflection upon substituting $\hat{b}_{ig}$ for b_{ij}. The Lagrangian density $\mathcal{L}_{\text{gr}}$ in (4.2) is given by the formula (3.3). It depends on b_{ij} only through the last two terms in the right hand side of the formula (2.9) for ρ. Similar terms are available in the formula (2.6) from [5]. Therefore we can apply the formula (6.3) from [5] having slightly modified it:

$$\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=\frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{8\,\pi\,\gamma}\biggl(b^{ij}-\sum_{k=1}^{3}b^k_k\,g^{ij}\biggr). \tag{4.3}$$

Now, according to (3.15), we should differentiate the partial variational derivative (4.3) with respect to the time variable t:

$$\begin{aligned}-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}&=\frac{c_{\text{gr}}^3\,g_{00}^{-3/2}}{16\,\pi\,\gamma}\biggl(\frac{b^{ij}}{2}-\sum_{k=1}^{3}b^k_k\,\frac{g^{ij}}{2}\biggr)\,\dot{g}_{00}-\\ &-\frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{16\,\pi\,\gamma}\biggl(\frac{1}{c_{\text{gr}}}\,\dot{b}^{ij}-\sum_{k=1}^{3}\frac{1}{c_{\text{gr}}}\,\dot{b}^k_k\,g^{ij}+\sum_{k=1}^{3}2\,b^k_k\,b^{ij}\biggr).\end{aligned} \tag{4.4}$$

In deriving (4.4) we used the formula for differentiating the inverse matrix:

$$\dot{g}^{ij}=-\sum_{k=1}^{3}\sum_{q=1}^{3}g^{ik}\,\dot{g}_{kq}\,g^{qj}. \tag{4.5}$$

Along with (4.5) in deriving (4.4) we applied the formula (1.8) for $\dot g_{kq}$.

The second term in the left hand side of (3.15) is transformed as follows:

$$-\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^q_q=-\frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{16\,\pi\,\gamma}\Bigl(\sum_{k=1}^{3}b^k_k\,b^{ij}-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,g^{ij}\Bigr).\tag{4.6}$$

The third term in the left hand side of (3.15) comprises the partial variational derivative of $\mathcal{L}_{\text{gr}}$ with respect to g_{ij}. In order to calculate this derivative we introduce a small variation of the metric:

$$\hat g_{ij}=g_{ij}(t,x^1,x^2,x^3)+\varepsilon\,h_{ij}(t,x^1,x^2,x^3).\tag{4.7}$$

Despite the relationship (1.8) the variations (4.1) and (4.7) are treated as independent. Here again $\varepsilon\to 0$ is a small parameter and $h_{ij}(t,x^1,x^2,x^3)$ are arbitrary smooth functions with compact support. The partial variational derivative of the Lagrangian density $\mathcal{L}_{\text{gr}}$ with respect to g_{ij} is defined through the formula

$$\hat L_{\text{gr}}=L_{\text{gr}}+\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\substack{g,\dot g,\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\tag{4.8}$$

The second integral L_{gr} in (3.1) upon substituting (3.3) into it and upon applying the formula (2.9) can be subdivided into six integrals:

$$L_{\text{gr}}=L_1+L_2+L_3+L_4+L_5+L_6.\tag{4.9}$$

Here is the first of these six integrals:

$$L_1=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,g_{00}^{-1/2}\,\nabla_{kq}\,g_{00}\,\sqrt{\det g}\;d^3x.\tag{4.10}$$

The second summand in the right hand side of (4.9) is similar to (4.10):

$$L_2=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\frac{g_{00}^{-3/2}}{2}\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,\sqrt{\det g}\;d^3x.\tag{4.11}$$

The third term in the right hand side of (4.9) comprises the scalar curvature R:

$$L_3=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int g_{00}^{1/2}\,R\,\sqrt{\det g}\;d^3x.\tag{4.12}$$

The fourth and fifth terms in the right hand side of (4.9) are similar to each other:

$$L_4=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,b^k_q\,b^q_k\,\sqrt{\det g}\;d^3x,\tag{4.13}$$

$$L_5=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,b^k_k\,b^q_q\,\sqrt{\det g}\;d^3x.\tag{4.14}$$

The sixth term in the right hand side of (4.9) comprises the cosmological constant:

$$L_6=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int g_{00}^{1/2}\,2\,\Lambda\,\sqrt{\det g}\,d^3x. \tag{4.15}$$

In order to derive the explicit expression for the partial variational derivative in (4.8) we need to substitute (4.7) for g_{ij} in each of the integrals (4.10), (4.11), (4.12), (4.13), (4.14), (4.15) and then expand each of them with respect to the small parameter ε up to the first order.

In the formula (4.10) we see the double covariant derivative $\nabla_{kq}\,g_{00}$. It is calculated using the components Γ^s_{kq} of the metric connection for the metric (1.4):

$$\nabla_{kq}\,g_{00}=\frac{g_{00}}{\partial x^k\,\partial x^q}-\sum_{s=1}^{3}\Gamma^s_{kq}\,\frac{g_{00}}{\partial x^s}. \tag{4.16}$$

The connection components Γ^s_{kq} are given by the Levi-Civita formula:

$$\Gamma^s_{kq}=\frac{1}{2}\sum_{r=1}^{3}g^{sr}\left(\frac{\partial g_{rq}}{\partial x^k}+\frac{\partial g_{kr}}{\partial x^q}-\frac{\partial g_{kq}}{\partial x^r}\right) \tag{4.17}$$

(see § 7 of Chapter III in [22]). Applying (4.7) to the term g^{sr} in (4.10), we get

$$\hat{g}^{sr}=g^{sr}-\varepsilon\sum_{k=1}^{3}\sum_{q=1}^{3}g^{sk}\,h_{kq}\,g^{qr}+\ldots\,. \tag{4.18}$$

Through dots in (4.2), (4.8), and (4.18) we denote higher order terms with respect to the small parameter ε. The formula (4.18) is analogous to (4.5). Applying (4.7) and (4.18) to (4.17), we derive the formula

$$\hat{\Gamma}^s_{kq}=\Gamma^s_{kq}+\frac{\varepsilon}{2}\sum_{r=1}^{3}g^{sr}\,\left(\nabla_k\,h_{rq}+\nabla_q\,h_{kr}-\nabla_r\,h_{kq}\right)+\ldots\,. \tag{4.19}$$

Then we apply (4.19) to (4.16). As a result we get

$$\begin{aligned}\hat{\nabla}_{kq}\,g_{00}=\nabla_{kq}\,g_{00}-\frac{\varepsilon}{2}\sum_{r=1}^{3}\sum_{s=1}^{3}g^{sr}\bigl(\nabla_k\,h_{rq}\,+\\+\,\nabla_q\,h_{kr}-\nabla_r\,h_{kq}\bigr)\,\nabla_s\,g_{00}+\ldots\,.\end{aligned} \tag{4.20}$$

Apart from the double covariant derivative (4.16), the integral L_1 in (4.10) comprises g^{kq} and the square root $\sqrt{\det g}$. The term g^{kq} is handled with the use of the formula (4.18). For the square root $\sqrt{\det g}$ we write

$$\sqrt{\det\hat{g}}=\sqrt{\det g}+\frac{\varepsilon}{2}\sum_{r=1}^{3}\sum_{s=1}^{3}g^{rs}\,h_{rs}\,\sqrt{\det g}+\ldots\,. \tag{4.21}$$

Like the time derivative in (2.6), the formula (4.21) is derived using Jacobi's formula for differentiating determinants (see [19]).

Now we apply (4.18), (4.20) and (4.21) to the integral (4.10). As a result we get

$$\begin{gathered}\hat{L}_1=L_1+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\left(g^{ik}\,g^{qj}-\frac{1}{2}\,g^{kq}\,g^{ij}\right)\cdot\\ \cdot\nabla_{kq}\,g_{00}\,h_{ij}\sqrt{\det g}\;d^3x+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}g_{00}^{-1/2}\,\frac{g^{kq}\,g^{sr}}{2}\cdot\\ \cdot\left(\nabla_k\,h_{rq}+\nabla_q\,h_{kr}-\nabla_r\,h_{kq}\right)\nabla_s\,g_{00}\,\sqrt{\det g}\;d^3x+\ldots.\end{gathered}\tag{4.22}$$

The second integral in (4.22) is transformed integrating by parts:

$$\begin{gathered}\hat{L}_1=L_1+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\left(g^{ik}\,g^{qj}-\frac{1}{2}\,g^{kq}\,g^{ij}\right)\cdot\\ \cdot\nabla_{kq}\,g_{00}\,h_{ij}\sqrt{\det g}\;d^3x-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}(2\,g^{ik}\,g^{qj}-\\ -\,g^{ij}\,g^{kq})\,\nabla_{kq}(g_{00}^{1/2})\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots.\end{gathered}\tag{4.23}$$

The integration by parts in spaces with metric is based on the formula

$$\int\limits_{\Omega}\sum_{k=1}^{3}\nabla_k z^k\sqrt{\det g}\,d^3x=\int\limits_{\partial\Omega}g(\mathbf{z},\mathbf{n})\,dS.\tag{4.24}$$

This formula (4.24) is a three-dimensional version of the formula (4.14) from Chapter IV of [18]. Note that $\nabla_{kq}(g_{00}^{1/2})$ can be written as

$$\nabla_{kq}(g_{00}^{1/2})=\frac{1}{2}\,g_{00}^{-1/2}\,\nabla_{kq}g_{00}-\frac{1}{4}\,g_{00}^{-3/2}\,\nabla_k g_{00}\,\nabla_q g_{00}.\tag{4.25}$$

Applying the relationship (4.25) to (4.23), we get

$$\begin{gathered}\hat{L}_1=L_1+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{g_{00}^{-3/2}}{2}\left(g^{ik}\,g^{qj}-\frac{1}{2}\,g^{kq}\,g^{ij}\right)\cdot\\ \cdot\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots.\end{gathered}\tag{4.26}$$

The second integral (4.11) is more simple than the first one since the covariant derivatives $\nabla_k\,g_{00}$ and $\nabla_q\,g_{00}$ do not use the connection components (4.17). Applying (4.18) and (4.21) to this integral, we derive

$$\begin{gathered}\hat{L}_2=L_2-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{g_{00}^{-3/2}}{2}\left(g^{ik}\,g^{qj}-\frac{1}{2}\,g^{kq}\,g^{ij}\right)\cdot\\ \cdot\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots.\end{gathered}\tag{4.27}$$

The third integral (4.12) is the most complicated. It comprises the three-dimensional scalar curvature R. The scalar curvature R is calculated in several steps. First of all the curvature tensor is calculated. Its components are

$$R^k_{qij} = \frac{\partial \Gamma^k_{jq}}{\partial r^i} - \frac{\partial \Gamma^k_{iq}}{\partial r^j} + \sum_{s=1}^{3} \Gamma^k_{is}\,\Gamma^s_{jq} - \sum_{s=1}^{3} \Gamma^k_{js}\,\Gamma^s_{iq}. \tag{4.28}$$

(see (1.1) in Chapter V of [18]). We apply (4.19) to (4.28). This yields

$$\hat{R}^k_{qij} = R^k_{qij} + \varepsilon\,\left(\nabla_i Y^k_{jq} - \nabla_j Y^k_{iq}\right) + \ldots\,, \tag{4.29}$$

where the following notations are introduced:

$$Y^s_{kq} = \frac{1}{2}\sum_{r=1}^{3} g^{sr}\,\left(\nabla_k\, h_{rq} + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\right) \tag{4.30}$$

The Ricci tensor is produced from the curvature tensor (4.28). Its components are

$$R_{qj} = \sum_{k=1}^{3} R^k_{qkj}. \tag{4.31}$$

Applying (4.29) to the formula (4.31), we derive

$$\hat{R}_{qj} = R_{qj} + \varepsilon \sum_{k=1}^{3}\left(\nabla_k Y^k_{jq} - \nabla_j Y^k_{kq}\right) + \ldots\,. \tag{4.32}$$

The scalar curvature is produced from the Ricci tensor

$$R = \sum_{q=1}^{3}\sum_{j=1}^{3} g^{qj}\,R_{qj}. \tag{4.33}$$

Applying (4.18) and (4.32) to the formula (4.33), we obtain

$$\hat{R} = R - \varepsilon \sum_{i=1}^{3}\sum_{j=1}^{3} R^{ij}\,h_{ij} + \varepsilon \sum_{k=1}^{3} \nabla_k Z^k + \ldots\,, \tag{4.34}$$

where the following notations are introduced:

$$Z^k = \sum_{q=1}^{3}\sum_{j=1}^{3}\bigl(g^{jq}\,Y^k_{jq} - g^{kq}\,Y^j_{jq}\bigr). \tag{4.35}$$

Now we are ready to apply (4.34) to the third integral L_3 in (4.12). Along with (4.34) we apply the formula (4.21). As a result we get

$$\begin{aligned}\hat{L}_3 = L_3 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3} g_{00}^{1/2}\left(R^{ij} - \frac{R}{2}\,g^{ij}\right)h_{ij}\,\sqrt{\det g}\;d^3x\,+\\ +\,\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3} g_{00}^{1/2}\,\nabla_k Z^k\,\sqrt{\det g}\;d^3x + \ldots\,.\end{aligned} \tag{4.36}$$

The second integral in (4.36) is transformed integrating by parts:

$$\begin{aligned}\hat{L}_3=L_3-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}g_{00}^{1/2}\left(R^{ij}-\frac{R}{2}\,g^{ij}\right)h_{ij}\,\sqrt{\det g}\;d^3x\,-\\ -\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}Z^k\,\nabla_{\!k}(g_{00}^{1/2})\,\sqrt{\det g}\;d^3x+\ldots\,.\end{aligned}\tag{4.37}$$

In order to make the second integral in (4.37) explicit we calculate Z^k explicitly by substituting (4.30) into (4.35). This yields

$$Z^k=\sum_{q=1}^{3}\nabla_{\!q}\,h^{kq}-\sum_{q=1}^{3}\sum_{r=1}^{3}g^{kq}\,\nabla_{\!q}\,h^r_r.\tag{4.38}$$

Before substituting (4.38) into (4.37) we transform it as follows:

$$Z^k=\sum_{q=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3}(g^{ki}\,g^{jq}\,\nabla_{\!q}\,h_{ij}-g^{kq}\,g^{ij}\,\nabla_{\!q}\,h_{ij}).\tag{4.39}$$

Now we substitute (4.39) into (4.37) and apply integration by parts:

$$\begin{aligned}\hat{L}_3=L_3-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}g_{00}^{1/2}\left(R^{ij}-\frac{R}{2}\,g^{ij}\right)h_{ij}\,\sqrt{\det g}\;d^3x\,+\\ +\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{ki}\,g^{jq}-g^{kq}\,g^{ij})\,\nabla_{\!kq}(g_{00}^{1/2})\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\end{aligned}\tag{4.40}$$

The integral L_4 in (4.13) is much easier to handle than the previous one. It is because it does not comprise any spacial derivative of the metric g_{ij}. Before applying (4.18) and (4.21) we write (4.13) as

$$L_4=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,g^{ki}\,b_{iq}\,g^{qj}\,b_{jk}\,\sqrt{\det g}\;d^3x.\tag{4.41}$$

Then, applying (4.18) and (4.21) to (4.41), we get

$$\begin{aligned}\hat{L}_4=L_4+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,b^k_q\,b^q_k\,\frac{g^{ij}}{2}\,h_{ij}\,\sqrt{\det g}\;d^3x\,-\\ -\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}g_{00}^{-1/2}(b^{ik}\,b^j_k+b^{jk}\,b^i_k)\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\end{aligned}\tag{4.42}$$

The integral L_5 in (4.14) is treated similarly. First of all it is rewritten as

$$L_5=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,g^{ik}\,b_{ik}\,g^{qj}\,b_{qj}\,\sqrt{\det g}\;d^3x.\tag{4.43}$$

Then, applying (4.18) and (4.21) to (4.43), we get

$$\begin{gathered}\hat{L}_5=L_5+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}2\,g_{00}^{-1/2}\,b^k_k\,b^{ij}\,h_{ij}\,\sqrt{\det g}\;d^3x-\\-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,b^k_k\,b^q_q\,\frac{g^{ij}}{2}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\end{gathered}\tag{4.44}$$

The formulas (4.42) and (4.44) are analogous to the formulas (6.10) and (6.11) in [5]. The difference is in the factor $g_{00}^{-1/2}$, which is taken to be equal to 1 in [5].

The integral L_6 in (4.15) is the most simple among the six integrals from (4.9). Applying (4.21) to this integral, we obtain

$$\hat{L}_6=L_6-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}g_{00}^{1/2}\,2\,\Lambda\,\frac{g^{ij}}{2}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\tag{4.45}$$

Now we can put the formulas (4.26), (4.27), (4.40), (4.42), (4.44), and (4.45) together and derive the formula for the required partial variational derivative

$$\begin{gathered}\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\right)_{\substack{g,\dot{g},\mathbf{b}\\\mathbf{Q},\dot{\mathbf{Q}}}}=\frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{16\,\pi\,\gamma}\left(\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k\,\frac{g^{ij}}{2}-\sum_{k=1}^{3}(b^{ik}\,b^j_k+b^{jk}\,b^i_k)+\right.\\\left.+\sum_{k=1}^{3}2\,b^k_k\,b^{ij}-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,\frac{g^{ij}}{2}\right)-\frac{c_{\text{gr}}^4\,g_{00}^{1/2}}{16\,\pi\,\gamma}\left(R^{ij}-\frac{R}{2}\,g^{ij}+\Lambda\,g^{ij}\right)+\\+\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{ki}\,g^{jq}-g^{kq}\,g^{ij})\,\nabla_{kq}(g_{00}^{1/2}).\end{gathered}\tag{4.46}$$

The next step is to put the formulas (4.4), (4.6), and (4.46) together and then apply the equation (3.15) to them. As a result we get the equation

$$\begin{gathered}\frac{g_{00}^{-2}}{2\,c_{\text{gr}}}\left(\sum_{k=1}^{3}b^k_k\,g^{ij}-b^{ij}\right)\dot{g}_{00}+\frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{kq}\,g^{ij}-g^{ki}\,g^{jq})\,\nabla_{kq}\,g_{00}-\\-\frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{kq}\,g^{ij}-g^{ki}\,g^{jq})\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}+g_{00}^{-1}\left(\frac{1}{c_{\text{gr}}}\,\dot{b}^{ij}-\right.\\-\sum_{k=1}^{3}\frac{1}{c_{\text{gr}}}\,\dot{b}^k_k\,g^{ij}+\sum_{k=1}^{3}(b^{ik}\,b^j_k+b^{jk}\,b^i_k)-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k\,\frac{g^{ij}}{2}-\\\left.-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,\frac{g^{ij}}{2}+\sum_{k=1}^{3}b^k_k\,b^{ij}\right)+R^{ij}-\frac{R}{2}\,g^{ij}+\Lambda\,g^{ij}=\frac{16\,\pi\,\gamma}{c_{\text{gr}}^4\,g_{00}^{1/2}}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}.\end{gathered}\tag{4.47}$$

In order to compare (4.47) with (1.7) we need to lower indices i and j in (4.47). Doing it, we take into account the following relationship:

$$\dot{b}^{ij}=\sum_{k=1}^{3}\sum_{q=1}^{3}g^{ik}\,\dot{b}_{kq}\,g^{qj}-\sum_{k=1}^{3}2\,c_{\text{gr}}\,(b^{ik}\,b^j_k+b^{jk}\,b^i_k).\tag{4.48}$$

We derive (4.48) using (1.8). Now, applying (4.48) and (3.12), we write (4.47) as

$$\begin{gathered}\frac{g_{00}^{-2}}{2\,c_{\text{gr}}}\biggl(\sum_{k=1}^{3} b^k_k\,g_{ij}-b_{ij}\biggr)\,\dot g_{00}+\frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{kq}\,g_{ij}-\delta^k_i\,\delta^q_j)\,\nabla_{kq}\,g_{00}-\\ -\frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{kq}\,g_{ij}-\delta^k_i\,\delta^q_j)\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}+g_{00}^{-1}\biggl(\frac{1}{c_{\text{gr}}}\,\dot b_{ij}-\\ -\sum_{k=1}^{3}\frac{1}{c_{\text{gr}}}\,\dot b^k_k\,g_{ij}-\sum_{k=1}^{3}(b_{ki}\,b^k_j+b_{kj}\,b^k_i)-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k\,\frac{g_{ij}}{2}-\\ -\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,\frac{g_{ij}}{2}+\sum_{k=1}^{3}b^k_k\,b_{ij}\biggr)+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=-\frac{16\,\pi\,\gamma}{c_{\text{gr}}^4\,g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}.\end{gathered}\tag{4.49}$$

Comparing (4.49) with (1.7) we derive the relationship

$$T_{ij}=-\frac{2}{g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}\quad\text{for}\quad 1\leqslant i,j\leqslant 3.\tag{4.50}$$

The relationship (4.50) is analogous to the relationship (3.25) in [7].

Theorem 4.1. *The gravity equations* (1.7) *are equivalent to the Euler-Lagrange equations* (3.15), *which are explicitly written in the form of* (4.47) *or* (4.49).

5. The equation for the time scale function.

The time-dependent scalar function (1.5) arises as the temporal component of the four-dimensional metric (1.3). It is assumed to be positive. In [14] this function was introduced through the formula

$$g_{00}(t,x^1,x^21,x^3)=\Bigl(\frac{\partial t'}{\partial t}\Bigr)^2,\tag{5.1}$$

where t' is the proper time of the comoving observer with the comoving coordinates x^1, x^2, x^3 (see (2.3) and (3.2) in [14]). Due to the formula (5.1) the function g_{00} is called the time scale function. It is described by the Euler-Lagrange equation (3.16). Our goal in this section is to write this equation in an explicit form.

Looking at the formula (3.3) and taking into account (2.9), we find that $\mathcal{L}_{\text{gr}}$ does not depend on the time derivative $\dot g_{00}$. Therefore

$$\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=0.\tag{5.2}$$

Due to (5.2) the Euler-Lagrange equation (3.16) reduces to

$$\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=-\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}.\tag{5.3}$$

In order to calculate the partial variational derivative in the left hand side of the equation (5.3) we introduce the small variation of the time scale function:

$$\hat g_{00}=g_{00}(t,x^1,x^2,x^3)+\varepsilon\,h(t,x^1,x^2,x^3).\tag{5.4}$$

Here $\varepsilon \to 0$ is a small parameter and $h(t, x^1, x^2, x^3)$ is an arbitrary smooth function with compact support. The small variation (5.4) is applied to the Lagrangian L_{gr} in (3.1). Then the partial variational derivative of the Lagrangian density $\mathcal{L}_{\text{gr}}$ with respect to g_{00} is defined through the formula

$$\hat{L}_{\text{gr}} = L_{\text{gr}} + \varepsilon \int \Bigl(\frac{\delta \mathcal{L}_{\text{gr}}}{\delta g_{00}}\Bigr)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} h\,\sqrt{\det g}\,d^3x + \ldots . \tag{5.5}$$

Like in the previous section, here we subdivide the Lagrangian L_{gr} into six parts using the formula (4.9) for this purpose. The integrals L_1, L_2, L_3, L_4, L_5, and L_6 in (4.9) are given by the formulas (4.10), (4.11), (4.12), (4.13), (4.14), and (4.15). Applying (5.4) to the first of these six integrals, we get

$$\begin{aligned} \hat{L}_1 = L_1 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,g_{00}^{-1/2}\,\nabla_{kq}\,h\,\sqrt{\det g}\,d^3x\,+\\ +\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\frac{g_{00}^{-3/2}}{2}\,\nabla_{kq}\,g_{00}\,h\,\sqrt{\det g}\,d^3x + \ldots . \end{aligned} \tag{5.6}$$

The first integral in (5.6) is transformed integrating by parts:

$$\begin{aligned} \hat{L}_1 = L_1 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\frac{3\,g_{00}^{-5/2}}{4}\,\nabla_{k}\,g_{00}\,\nabla_{q}\,g_{00}\,h\,\sqrt{\det g}\,d^3x\,+\\ +\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,g_{00}^{-3/2}\,\nabla_{kq}\,g_{00}\,h\,\sqrt{\det g}\,d^3x + \ldots . \end{aligned} \tag{5.7}$$

Then we apply (5.4) to the integral L_2 in (4.11):

$$\begin{aligned} \hat{L}_2 = L_2 + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,g_{00}^{-3/2}\,\nabla_{k}\,g_{00}\,\nabla_{q}\,h\,\sqrt{\det g}\,d^3x\,-\\ -\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\frac{3\,g_{00}^{-5/2}}{4}\,\nabla_{k}\,g_{00}\,\nabla_{q}\,g_{00}\,h\,\sqrt{\det g}\,d^3x + \ldots . \end{aligned} \tag{5.8}$$

The first integral in (5.8) is transformed integrating by parts:

$$\begin{aligned} \hat{L}_2 = L_2 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,g_{00}^{-3/2}\,\nabla_{kq}\,g_{00}\,h\,\sqrt{\det g}\,d^3x\,+\\ +\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\frac{3\,g_{00}^{-5/2}}{4}\,\nabla_{k}\,g_{00}\,\nabla_{q}\,g_{00}\,h\,\sqrt{\det g}\,d^3x + \ldots . \end{aligned} \tag{5.9}$$

The next in turn is the integral L_3 in (4.12). Applying (5.4) to it, we get

$$\hat{L}_3 = L_3 + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma} \int \frac{g_{00}^{-1/2}}{2}\,R\,h\,\sqrt{\det g}\,d^3x + \ldots . \tag{5.10}$$

Then we proceed to the integral L_4 in (4.13). Applying (5.4) to it, we obtain

$$\hat{L}_4 = L_4 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{g_{00}^{-3/2}}{2}\,b_q^k\,b_k^q\,h\,\sqrt{\det g}\,d^3x + \ldots. \tag{5.11}$$

The integral L_5 in (4.14) is treated similarly. Applying (5.4) to it, we derive

$$\hat{L}_5 = L_5 + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{g_{00}^{-3/2}}{2}\,b_k^k\,b_q^q\;h\sqrt{\det g}\,d^3x + \ldots. \tag{5.12}$$

And finally, we come to the integral L_6 in (4.15). Applying (5.4) to it, we get

$$\hat{L}_6 = L_6 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int g_{00}^{-1/2}\,\Lambda\,h\,\sqrt{\det g}\,d^3x. \tag{5.13}$$

Now we can put the formulas (5.7), (5.9), (5.10), (5.11), (5.12), and (5.13) together and apply all of them to (5.5). As a result we obtain

$$\begin{gathered}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{00}}\Bigr)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = \frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\frac{g_{00}^{-1/2}}{2}\,(R-2\Lambda)\,+\\ +\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\frac{g_{00}^{-3/2}}{2}\biggl(\sum_{k=1}^{3}\sum_{q=1}^{3}b_k^k\,b_q^q - \sum_{k=1}^{3}\sum_{q=1}^{3}b_q^k\,b_k^q\biggr).\end{gathered} \tag{5.14}$$

Then by substituting (5.14) into (5.3) we derive the equation

$$-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b_q^k\,b_k^q + \frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b_k^k\,b_q^q + \frac{R}{2}\,g_{00} - \Lambda\,g_{00} = -\frac{16\,\pi\,\gamma}{c_{\text{gr}}^4}\,g_{00}^{3/2}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}. \tag{5.15}$$

If we recall the relationship $g^{00} = g_{00}^{-1}$ and apply the formula (3.14) derived from it, then the equation (5.15) can be rewritten as

$$-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b_q^k\,b_k^q + \frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b_k^k\,b_q^q + \frac{R}{2}\,g_{00} - \Lambda\,g_{00} = \frac{16\,\pi\,\gamma}{c_{\text{gr}}^4\,g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{00}}. \tag{5.16}$$

Comparing the equation (5.16) with the equation (1.10) we derive

$$T_{00} = \frac{2}{g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{00}}. \tag{5.17}$$

The relationship (5.17) is similar to the relationship (4.50).

Theorem 5.1. *The gravity equation* (1.10) *is equivalent to the Euler-Lagrange equation* (3.16)*, which is explicitly written in the form of* (5.16) *or* (5.17)*.*

The Lagrangian density of matter $\mathcal{L}_{\text{mat}}$ in the right hand sides of the equations (4.47), (4.49), (5.16), and (5.17) is not specified explicitly. It will be specified when considering specific sorts of matter such as gases, liquids, and solids.

6. Conclusions.

Theorems 4.1 and 5.1 constitute the main result of the present paper. They show that within the new version of the 3D-brane universe model that does not use the equidistance postulate 1.1 the gravity equations can be derived in a purely three-dimensional Lagrangian approach. The 3D-brane universe model is an alternative non-Einsteinian theory, though it still inherits many features from the standard Einsteinian theory. The 3D-brane universe model is more flexible than four-dimensional theories. It can admit more differences from the standard theory in future provided these differences will be motivated by experimental data and astronomical observations.

7. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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