One Piece

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Abstract

At first we tile the plane by 8-gons. Then we present a way to tile the plane by k-gons for a every fixed k for all natural numbers k larger than two. We use an infinite number of equal tiles to cover the plane.

1 Introduction

It is a widespread opinion that one can tile the plane $\mathbb{R}^2$ only with triangles, squares and regular 6-gons. This is wrong. Here we show another possibility.

Proposition 1. There is a tiling of the plane by 8-gons.

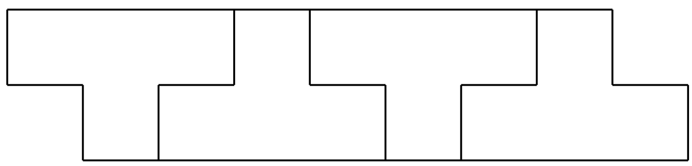
Proof. Instead of a written proof we prerfer to show a picture. See Figure 1. □

Figure 1:

We see four equal 8-gons.

With infinite many of these tiles

we can cover the plane.



We think that it is useful to repeat the definition of a *simple polygon*.
A simple polygon with k vertices consists of k points $(x_1,y_1),(x_2,y_2),\ldots(x_{k-1},y_{k-1}),(x_k,y_k)$ called *vertices*, and the straight lines between the vertices, where $k>2$. It is homeomorphic to a circle. We demand that there are no three consecutive collinear points

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$(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$ for $1 \leq i \leq k-2$. Also we demand that the three points $(x_k, y_k), (x_1, y_1), (x_2, y_2)$ and $(x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)$ are not collinear.
We call this just described simple polygon a *k-gon*.

Definition 1. We call a polygon a *piece* if and only if it is one half of a regular 6-gon and it has 5 vertices. Please see the picture Figure 2.
We use the word *doublepiece* as a synonym for a regular 6-gon. See Figure 2, too.

Definition 2. Let t be any natural number.
We call a simple polygon a t row piece, if and only if t pieces are put in a row. Two pieces are joined together at a common edge.
We call a simple polygon a t row doublepiece, if and only if t doublepieces are put in a row.
We call a simple polygon a t shift square, if and only if t squares are put upon the other, where the squares have sidelength 1 and each square is shifted by $\frac{1}{2}$. (Every number between 0 and 1 also works.)

See the example in Figure 3. There we show a 2 row piece and a 3 row doublepiece. See also the 3 shift square in Figure 4.
Note that a 1 row piece is just a piece, and a 1 row doublepiece is a doublepiece and also a regular 6-gon and a 1 shift square is a square.

Proposition 2. One can tile the plane with infinite copies of a t row piece for all natural numbers t; also we can tile the plane with infinite copies of a t row doublepiece for all t. Also we can tile the plane with infinite copies of a t shift square for all t.

Proof. Nearly trivial. □

Proposition 3. A t row piece has $5 + 2 \cdot (\mathsf{t} - 1)$ vertices. A t row doublepiece has $2 + 4 \cdot \mathsf{t}$ vertices. A t shift square has $4 \cdot \mathsf{t}$ vertices.

Proof. Easy. The proofs are by induction. □

2 Tiling

Theorem 1. There exists for all natural numbers k larger than 2 a tiling of $\mathbb{R}^2$ with k-gons, where infinite copies of a single tile are used.

Proof. For $k = 3$ and $k = 4$ and $k = 6$ the theorem is well-known. For $k = 5$ please see Figure 2. We use one piece and Proposition 2.
Now let k be a natural number larger than 6.

Lemma 1. It holds $k \equiv p \bmod 4$, where $p \in \{0, 1, 2, 3\}$.

Proof. Well-known. □

We discuss the four possibilities.

• Possibility 1: $p = 0$. In this case we get a suitable t from the equation $k = 4 \cdot \mathsf{t}$ and we take a t shift square as a k-gon. Please see Figure 3.
The sequence of the numbers of k is $8, 12, 16, \ldots$.

• Possibility 2: $p = 1$. These numbers are odd. Note that the set $\{5 + 2 \cdot (\mathsf{t} - 1) \mid \mathsf{t} \in \mathbb{N}\}$ contains all odd numbers larger than 3. See Proposition 3. We take a suitable t row piece as a k-gon. We get t from the equation $k = 4 \cdot \mathsf{t} + 1$.
The sequence of the numbers of k is $9, 13, 17, \ldots$.

• Possibility 3: $p = 2$. We get t from $k = 4 \cdot \mathsf{t} + 2$. We take a suitable t row doublepiece as a k-gon.
The sequence of the numbers of k is $10, 14, 18, \ldots$.

• Possibility 4: $p = 3$. These numbers also are odd.
The sequence of the numbers of k is $7, 11, 15, 19, \ldots$.

The theorem is proven. $\square$

It follows some figures.

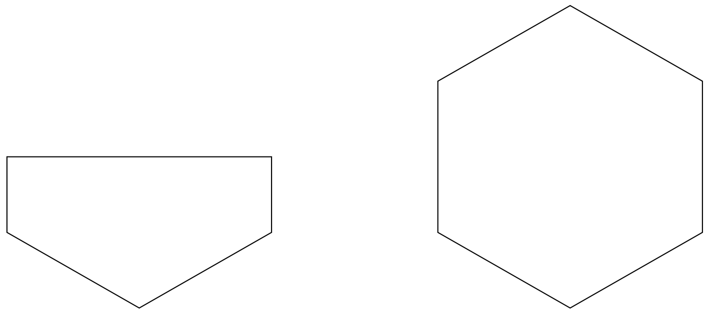

Figure 2:

On the right

we see a piece

and a doublepiece

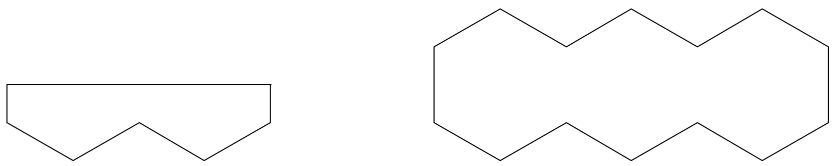

Figure 3:

On the right side

we see a 2 row piece

and a 3 row doublepiece

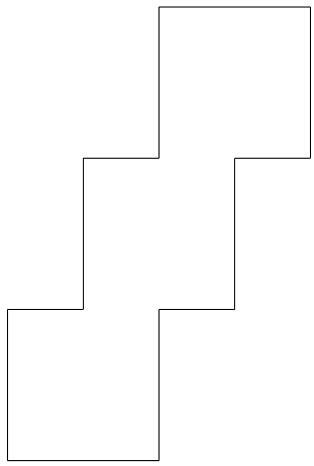

Figure 4:

On the right side

we see a

3 shift square