Traversable Wormholes in (2+1) dimensions and Gravity's Rainbow

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Abstract

We investigate the traversable wormholes in $(2+1)$ dimensions in the context of Gravity's Rainbow which may be one of the approaches to quantum gravity. The cases in the presence of cosmological constant and Casimir energy are studied.

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Contents

1 Introduction

Quantum gravity has been studied and investigated for a long time by lots of researchers. However we could not obtain even the twinkles of quantum gravity. Wormholes may give us some hints toward quantum gravity which may be different from black holes. Wormholes are the solutions to the Einstein field equations that have topological structures with a throat connecting two asymptotically regions of spacetime , called mouth.[1], [2], [3]. They can be interpreted, classically, as instantons describing a tunneling between two distant regions. The traversable wormhole solution was at first investigated by Morris and Thorne to verify the realistic possibilities of travelling through it [4]. A wormhole needs a negative energy in order to form and keep wormhole structure[4], though the standard energy conditions are violated. Without such a exotic matter which is defined to be violated energy conditions , a wormhole cannot keep the structure to collapses. Such a situation may suggest that wormholes belong to the realm of a possible quantum gravity[14] . Casimir energy is one of the candidates as such a negative energy. A Casimir wormhole in D dimensions [6] can be constructed if $D > 3$. But a three dimensional traversable Casimir wormhole cannot be constructed since it has an event horizon [5][6]. Namely, three dimensional case is very different from D(> 3) dimensional case. A traversable wormhole has two ways , which situation is very different from a black hole which has one way through horizon. In addition the cosmological constant is also the candidate of the exotic matters.The negative energy is realized if $\Lambda < 0$.

In this report we will study wormholes in (2+1) dimensions , considering Gravity Rainbow. Traversable wormholes in (2+1) dimensions were studied in the past [5][9][10].
Note that the (2+1)case is very different from the case of (3+1) in the Morris Thorne wormhole
It is generally believed that the geometry of spacetime should be fundamentally described by quantum gravity above and around the Planck energy[11]. Several candidates for that description are investigated with difficulties; string theory, non-commutative geometry, loop quantum gravity , Lorentzian dynamical triangulations , etc. A key points are to study how to transit between the fundamental quantum description and the effective low energy description in terms of classical general relativity. We explore the possibility that wormhole geometries are sustained by the Casimir energy and the Cosmological constant in three dimensional space-time in the context of Gravity's Rainbow. Gravitational Rainbow consists of a distortion of the space-time metric around the Planck energy. A wormhole is a solution to the Einstein Field Equation, which is structured by two mouthes and a throat. A traversable wormhole has no space-time singularity and no horizon . Semi-classical theory of quantum gravity may play a crucial role since we have no full quantum gravity as yet
The outline of the present paper is as follows: In Sec.2, we review the wormhole in (2+1) dimensional gravity with no Gravity Rainbow. In Sec.3 we obtain the metrics of the wormholes in (2+1) dimensions in the presence of Gravity Rainbow.

2 Traversable Wormholes in (2+1) dimensions with no gravitational Rainbow

The general metric of a traversable circularly symmetric wormhole in three dimensions is represented as

$$ds^2 = -e^{2\phi(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2 d\theta^2 \quad (1)$$

$$= g_{\mu\nu}dx^\mu dx^\nu \quad (2)$$

where $\phi(r)$ and $b(r)$ are the red-shift and shape functions, respectively.
A wormhole solution satisfies at the throat : $b(r_0) = r_0$ and $\acute{b}(r_0) < 1$. The latter condition is induced from the flaring-out condition of the throat : $\frac{b-\acute{b}(r)r}{2b^2} > 0$. Asymptotic flatness imposes $b(r)/r \to 0$ as $r \to +\infty$. In addition, in order for the wormhole to be traversable , one must demand that there exists no horizon (two ways travel is approved) , so that $\phi(r)$ must be finite everywhere. Here $g_{\mu\nu}$ is expressed as,

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\phi(r)} & 0 & 0 \\ 0 & \frac{1}{1-b(r)/r} & 0 \\ 0 & 0 & r^2 \end{pmatrix} \tag{3}$$

$g^{\mu\nu}$ as the inverse of $g_{\mu\nu}$ is

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\phi(r)} & 0 & 0 \\ 0 & 1-b(r)/r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix} \tag{4}$$

Next we calculate the Christoffel symbols:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \tag{5}$$

Non-vanishing Christoffel symbols are

$$\begin{aligned} \Gamma^0_{01} &= \Gamma^0_{10} = \phi\prime \\ \Gamma^1_{00} &= \left(1 - \frac{b}{r}\right) e^{2\phi} \phi\prime \\ \Gamma^1_{11} &= \frac{1}{2}\left(1 - \frac{b}{r}\right)^{-1} \frac{\acute{b}r - b}{r^2} = \frac{\acute{b}r - b}{2r(r-b)} \\ \Gamma^1_{22} &= -r\left(1 - \frac{b}{r}\right) \\ \Gamma^2_{21} &= \Gamma^2_{12} = \frac{1}{r} \end{aligned}$$

The Ricci tensor is calculated as

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\lambda}_{\lambda\alpha}\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\alpha}_{\lambda\mu}$$

We describe $0, 1, 2$ as t, r, θ respectively. The components are

$$\begin{aligned} R_{tt} &= \left(-\frac{1}{2}\frac{\acute{b}r - b}{r^2} + \frac{1}{r}\left(1 - \frac{b}{r}\right)\right) e^{2\phi}\acute{\phi} + \left(1 - \frac{b}{r}\right) e^{2\phi}\acute{\phi}^2 + \left(1 - \frac{b}{r}\right) e^{2\phi}\phi\prime\prime \\ R_{rr} &= -\phi\prime\prime - \phi\prime^2 + \frac{1}{2}\left(\phi\prime + \frac{1}{r}\right)\left(1 - \frac{b}{r}\right)^{-1} \frac{b\prime r - b}{r^2} \\ R_{\theta\theta} &= -\phi\prime r\left(1 - \frac{b}{r}\right) + \frac{b\prime}{2} - \frac{b}{2r} \end{aligned}$$

The Ricci scalar is calculated as

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22}$$

$$R = \left(\frac{b\prime}{r} + \frac{b}{r^2} - \frac{2}{r}\right)\phi\prime - 2\left(1 - \frac{b}{r}\right)\phi\prime^2 - 2\left(1 - \frac{b}{r}\right)\phi\prime\prime + \frac{b\prime}{r^2} - \frac{b}{r^3} \tag{6}$$

The Einstein tensor: $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ has no vanishing components

$$\begin{aligned} G_{tt} &= R_{tt} - \frac{1}{2} R g_{tt} = \frac{r b\prime - b}{2r^3} e^{2\phi} \\ G_{rr} &= \frac{\phi\prime}{r} \\ G_{\theta\theta} &= r\left(r - b\right)\left[\phi\prime^2 + \phi\prime\prime + \frac{b - b\prime r}{2r\left(r - b\right)}\phi\prime\right] \end{aligned}$$

Here it is better to treat the Einstein equation in an orthonormal basis of observers who remain always at rest in the coordinate system , since the stress-energy tensor is interpreted most eaaily. Basis transformation:

$$\vec{e_{\dot{t}}} = \frac{\partial x^t}{\partial x^{\dot{t}}} \vec{e_t} = e^{-\phi} \vec{e_t} \tag{7}$$

$$\vec{e_{\dot{r}}} = \frac{\partial x^r}{\partial x^{\dot{r}}} \vec{e_r} = (1 - \frac{b}{r})^{1/2} \vec{e_r} \tag{8}$$

$$\vec{e_{\dot{\theta}}} = \frac{\partial x^\theta}{\partial x^{\dot{\theta}}} \vec{e_\theta} = \frac{1}{r} \vec{e_\theta} \tag{9}$$

Metric representation:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= \vec{e_\mu} \cdot \vec{e_\nu} dx^\mu dx^\nu \\ &= \vec{e_\mu} \cdot \vec{e_\nu} \frac{\partial x^\mu}{\partial x^{\dot{\mu}}} dx^{\dot{\mu}} \frac{\partial x^\nu}{\partial x^{\dot{\nu}}} dx^{\dot{\nu}} \\ &= \vec{e_\mu} \frac{\partial x^\mu}{\partial x^{\dot{\mu}}} \cdot \vec{e_\nu} \frac{\partial x^\nu}{\partial x^{\dot{\nu}}} dx^{\dot{\mu}} dx^{\dot{\nu}} \\ &= \vec{e_{\dot{\mu}}} \cdot \vec{e_{\dot{\nu}}} dx^{\dot{\mu}} dx^{\dot{\nu}} \\ &= g_{\dot{\mu}\dot{\nu}} dx^{\dot{\mu}} dx^{\dot{\nu}} \end{aligned}$$

Components;

$$\begin{aligned}
g_{\dot{\mu}\dot{\nu}} &= \vec{e_{\dot{\mu}}} \cdot \vec{e_{\dot{\nu}}} \\
g_{\dot{t}\dot{t}} &= \vec{e_{\dot{t}}} \cdot \vec{e_{\dot{t}}} = e^{-\phi}\vec{e_t} \cdot e^{-\phi}\vec{e_t} \\
&= e^{-2\phi}\vec{e_t} \cdot \vec{e_t} = e^{-2\phi} g_{tt} = e^{-2\phi}\left(-e^{-2\phi}\right) = -1 \\
g_{\dot{r}\dot{r}} &= \vec{e_{\dot{r}}} \cdot \vec{e_{\dot{r}}} = \left(1 - \frac{b}{r}\right)^{1/2} \vec{e_r} \cdot \left(1 - \frac{b}{r}\right)^{1/2} \vec{e_r} \\
&= \left(1 - \frac{b}{r}\right) \vec{e_r} \cdot \vec{e_r} = \left(1 - \frac{b}{r}\right) g_{rr} = \left(1 - \frac{b}{r}\right)\left(1 - \frac{b}{r}\right)^{-1} = 1 \\
g_{\dot{\theta}\dot{\theta}} &= \vec{e_{\dot{\theta}}} \cdot \vec{e_{\dot{\theta}}} = \frac{1}{r}\vec{e_\theta} \cdot \frac{1}{r}\vec{e_\theta} = \frac{1}{r^2}\vec{e_\theta} \cdot \vec{e_\theta} = \frac{1}{r^2} g_{\theta\theta} = \frac{1}{r^2} r^2 = 1
\end{aligned}$$

So we obtain the simplified metric,

$$g_{\dot{\mu}\dot{\nu}} = \begin{pmatrix} g_{\dot{t}\dot{t}} & 0 & 0 \\ 0 & g_{\dot{r}\dot{r}} & 0 \\ 0 & 0 & g_{\dot{\theta}\dot{\theta}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \eta_{\dot{\mu}\dot{\nu}} \tag{10}$$

The Einstein tensors in this orthonormal frame;

$$G_{\dot{\mu}\dot{\nu}} = \frac{\partial x^\mu}{\partial x^{\dot{\mu}}} \frac{\partial x^\nu}{\partial x^{\dot{\nu}}} G_{\mu\nu} \tag{11}$$

$$\begin{aligned}
G_{\dot{t}\dot{t}} &= \frac{\partial x^t}{\partial x^{\dot{t}}} \frac{\partial x^t}{\partial x^{\dot{t}}} G_{tt} = e^{-\phi} e^{-\phi} G_{tt} = e^{-2\phi} \frac{rb\prime - b}{2r^3} e^{2\phi} = \frac{rb\prime - b}{2r^3} && (12) \\
G_{\dot{r}\dot{r}} &= \frac{\partial x^r}{\partial x^{\dot{r}}} \frac{\partial x^r}{\partial x^{\dot{r}}} G_{rr} = (1 - \frac{b}{r})^{1/2} (1 - \frac{b}{r})^{1/2} G_{rr} = (1 - \frac{b}{r}) \frac{\phi\prime}{r} && (13) \\
G_{\dot{\theta}\dot{\theta}} &= \frac{\partial x^\theta}{\partial x^{\dot{\theta}}} \frac{\partial x^\theta}{\partial x^{\dot{\theta}}} G_{\theta\theta} = \frac{1}{r}\frac{1}{r} r\,(r - b)\left[\phi\prime^2 + \phi\prime\prime + \frac{b - b\prime r}{2r\,(r - b)}\phi\prime\right] \\
&= \left(1 - \frac{b}{r}\right)\left[\phi\prime^2 + \phi\prime\prime + \frac{b - b\prime r}{2r\,(r - b)}\phi\prime\right] && (14)
\end{aligned}$$

The Einstein field equations in the orthonormal frame are

$$G_{\dot{\mu}\dot{\nu}} = 8\pi G T_{\dot{\mu}\dot{\nu}} \tag{15}$$

where $T_{\dot{\mu}\dot{\nu}}$ is the stress-energy tensor and the above equations imply that the stress-energy tensor $T_{\dot{\mu}\dot{\nu}}$ must have the same algebraic structure as the Einstein tensor $G_{\dot{\mu}\dot{\nu}}$.

$$T_{\dot{\mu}\dot{\nu}} = \begin{pmatrix} \rho(r) & 0 & 0 \\ 0 & p_r(r) & 0 \\ 0 & 0 & p_l(r) \end{pmatrix} \tag{16}$$

$$T^{\dot{\mu}}{}_{\dot{\nu}} = g^{\dot{\mu}\dot{\alpha}}T_{\dot{\alpha}\dot{\nu}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho(r) & 0 & 0 \\ 0 & p_r(r) & 0 \\ 0 & 0 & p_l(r) \end{pmatrix} = \begin{pmatrix} -\rho(r) & 0 & 0 \\ 0 & p_r(r) & 0 \\ 0 & 0 & p_l(r) \end{pmatrix} \tag{17}$$

where $\rho(r)$ is the total density of mass energy; $p_r(r)$ is the radial pressure ; $p_l(r)$ is the lateral pressure. If we set the radial tension $\tau(r)$, $\tau(r) = -p_r(r)$. In this case the energy-momentum tensor in the orthonormal flame is represented as

$$T_{\dot{\mu}\dot{\nu}} = \begin{pmatrix} \rho(r) & 0 & 0 \\ 0 & p_r(r) & 0 \\ 0 & 0 & p_l(r) \end{pmatrix} = \begin{pmatrix} \rho(r) & 0 & 0 \\ 0 & -\tau(r) & 0 \\ 0 & 0 & p_l(r) \end{pmatrix} \tag{18}$$

The Einstein field equations yield

$$G_{\dot{t}\dot{t}} = 8\pi G T_{\dot{t}\dot{t}} = 8\pi G\rho(r) \tag{19}$$

$$G_{\dot{r}\dot{r}} = 8\pi G T_{\dot{r}\dot{r}} = -8\pi G\tau(r) = 8\pi G p_r(r) \tag{20}$$

$$G_{\dot{\theta}\dot{\theta}} = 8\pi G T_{\dot{\theta}\dot{\theta}} = 8\pi G p_l(r) \tag{21}$$

more explicitly

$$8\pi G\rho(r) = G_{\dot{t}\dot{t}} = \frac{rb\prime - b}{2r^3} \tag{22}$$

$$8\pi G p_r(r) = G_{\dot{r}\dot{r}} = \left(1 - \frac{b}{r}\right)\frac{\phi\prime}{r} \tag{23}$$

$$8\pi G p_l(r) = G_{\dot{\theta}\dot{\theta}} = \left(1 - \frac{b}{r}\right)\left[\phi\prime^2 + \phi\prime\prime + \frac{b - b\prime r}{2r\,(r-b)}\phi\prime\right] \tag{24}$$

So the Einstein field Equations are obtained as a component representation. Next we have to need the equation of state in order to solve the above equations. Now we assume it as

$$p_r = \omega\rho \tag{25}$$

it is possible to compute the proper radial distance.

$$l(r) = \pm\int_{r_0}^{r} \frac{dr}{\sqrt{1 - \frac{b}{r}}} \tag{26}$$

Some conditions for forming a traversable wormhole
1.flaring-out condition,(details see appendix)

$$r\acute{b}(r) - b(r) < 0 \tag{27}$$

2.Throat definition

$$b(r_0) = r_0 \tag{28}$$

3. $\phi(r)$ should be finite everywhere, namely there should be no horizons present in order to permit two way travel. If horizons exist, a wormhole permits one way travel.

$$|\phi(r)| \ll 1 \tag{29}$$

4.comfortable journey through the wormhole, more precisely , the tidal gravitational forces experienced by a traveller must be reasonably small.

$$|\acute{\phi}(r)| \lesssim |\textbf{our earth gravity}| \tag{30}$$

5.the metric should be spherically symmetric and static for simple calculations

6.The time to traverse the wormhole must be reasonably short (one year)as measured by both the traveller and any observers who wait on the outside of the wormhole.

2.1 Casimir effect

The Casimir effect which appears between two plane parallel, closely spaced, uncharged, metallic plates in vacuum , was predicted theoretically by H. Casimir.[8] This effect has the intriguing feature that an attractive force appears which is generated by negative energy. Casimir effect is one of the candidates which play a part as a negative energy source to construct a traversable wormhole. However wormholes in (2+1) dimensions cannot be sourced solely by both Casimir energy and tension [5] . It was shown that Casimir energy can be a source of the traversable wormhole for all spacetime dimensions with $D > 3$ [6]. Why low dimensional case (three dimensionl space-time) is impossible? Explicitly we will check the above words.
Equation (22) shows if $\rho(r)$ is negative , the flare out condition $r\acute{b}(r) - b(r) < 0$ is satisfied. Namely we have to find out the candidate of the negative ρ in order to build a wormhole .The Casimir system is a suitable example which provides a negative energy. For arbitrary D dimensions, the Casimir energy density is given by [7]

$$\rho_D = -\frac{(D-2)\Gamma(D/2)\zeta_R(D)}{(4\pi)^{D/2}}\frac{1}{r^D} \tag{31}$$

where $\zeta_R(z)$ is the Riemann zeta function. Taking $D = 3$

$$\rho = \rho_3 = -\frac{(3-2)\Gamma(3/2)\zeta_R(3)}{(4\pi)^{3/2}}\frac{1}{r^3} = -\lambda\frac{1}{r^3} \tag{32}$$

where

$$\lambda = \frac{\Gamma(3/2)\zeta_R(3)}{(4\pi)^{3/2}}$$

We substitute (32) into (22) we obtain the equation:

$$\frac{db}{dr}r - b = -16\pi G\lambda \tag{33}$$

$$\int \frac{db}{b - 16G\pi\lambda} = \int \frac{dr}{r} = \ln r$$

$$\ln |b - 16G\pi\lambda| = \ln r + C$$

Using $b(r_0) = r_0$,

$$\ln |b(r_0) - 16G\pi\lambda| = \ln r_0 + C$$

$$C = \ln |\frac{r_0 - 16G\pi\lambda}{r_0}|$$

We obtain

$$b(r) = \frac{r_0 - 16\pi G\lambda}{r_0} r + 16\pi G\lambda \tag{34}$$

Next we determine $\phi(r)$, from following (20),(23),(25) and (32), we obtain the equation:

$$(1 - \frac{b}{r})\frac{\acute{\phi}}{r} = -8\pi G(\omega\lambda\frac{1}{r^3}) \tag{35}$$

continue to calculate,

$$\frac{d\phi}{dr} = -\frac{8\pi G\omega\lambda}{r(r-b)} \tag{36}$$

We obtain

$$\phi = \int d\phi = -8\pi G\lambda\omega \int \frac{dr}{r(r-b)} = -\frac{\omega}{2}\ln |\frac{r - r_0}{r}| \tag{37}$$

where we used the result of $b(r)$ (34).
So we obtain the metric:

$$ds^2 = -|1 - \frac{r_0}{r}|^{-\omega}dt^2 + \frac{r_0}{16\pi G\lambda}\frac{dr^2}{(1 - \frac{r_0}{r})} + r^2 d\theta^2 \tag{38}$$

This metric represents one way because there exists a horizon at $r = r_0$.

2.2 Cosmological Constant

Wormholes with a cosmological constant were investigated [5][13]. In the orthonormal reference frame, the Einstein field equation with a cosmological constant, is given by

$$
\begin{aligned}
G_{\hat{\mu}\hat{\nu}} + \Lambda g_{\hat{\mu}\hat{\nu}} &= 8\pi G T_{\hat{\mu}\hat{\nu}} && (39)\\
G_{\hat{\mu}\hat{\nu}} &= 8\pi G T_{\hat{\mu}\hat{\nu}} - \Lambda g_{\hat{\mu}\hat{\nu}} && (40)\\
&= 8\pi G \left(T_{\hat{\mu}\hat{\nu}} - \frac{\Lambda}{8\pi G} g_{\hat{\mu}\hat{\nu}} \right) && (41)\\
&= 8\pi G \left(T_{\hat{\mu}\hat{\nu}} + T^{(vac)}_{\hat{\mu}\hat{\nu}} \right) && (42)
\end{aligned}
$$

where $T^{(vac)}_{\hat{\mu}\hat{\nu}} = -\frac{\Lambda}{8\pi G} g_{\hat{\mu}\hat{\nu}}$ may be interpreted as the stress-energy tensor associated with the vacuum energy.

$$
T^{(vac)}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \frac{\Lambda}{8\pi G} & 0 & 0 \\ 0 & -\frac{\Lambda}{8\pi G} & 0 \\ 0 & 0 & -\frac{\Lambda}{8\pi G} \end{pmatrix} \tag{43}
$$

Thus the Einstein equations with a cosmological constant:

$$
\begin{aligned}
G_{\hat{t}\hat{t}} &= 8\pi G \rho(r) + \Lambda && (44)\\
G_{\hat{r}\hat{r}} &= -8\pi G \tau(r) - \Lambda && (45)\\
G_{\hat{\theta}\hat{\theta}} &= 8\pi G p_l(r) - \Lambda && (46)
\end{aligned}
$$

More explicitly

$$
\begin{aligned}
\frac{rb\prime - b}{2r^3} &= 8\pi G \rho(r) + \Lambda && (47)\\
\left(1 - \frac{b}{r}\right) \frac{\phi\prime}{r} &= -8\pi G \tau(r) - \Lambda && (48)\\
\left(1 - \frac{b}{r}\right) \left[\phi\prime^2 + \phi\prime\prime + \frac{b - b\prime r}{2r\,(r-b)} \phi\prime \right] &= 8\pi G p_l(r) - \Lambda && (49)
\end{aligned}
$$

Here we consider pure gravity with no matter(no exotic matter); $\rho(r) = \tau(r) = p_l(r) = 0$, namely only the cosmological constant is considered as the stress-energy tensor factor.

$$
\begin{aligned}
\frac{rb\prime - b}{2r^3} &= \Lambda && (50)\\
\left(1 - \frac{b}{r}\right) \frac{\phi\prime}{r} &= -\Lambda && (51)\\
\left(1 - \frac{b}{r}\right) \left[\phi\prime^2 + \phi\prime\prime + \frac{b - b\prime r}{2r\,(r-b)} \phi\prime \right] &= -\Lambda && (52)
\end{aligned}
$$

From the above equations we calculate to obtain;

$$b(r) = r(1 - \Lambda(r_0^2 - r^2)) \tag{53}$$

$$\phi(r) = \frac{1}{2}\log|r^2 - r_0^2| \tag{54}$$

Here we have to notice that the flare-out conditon says $r\acute{b}(r) - b(r) < 0$, namely from (43) $\Lambda < 0$ in other word as far as we consider the pure gravity, $\Lambda < 0$.

The last metric line element is

$$ds^2 = -|r^2 - r_0^2|dt^2 + \frac{dr^2}{\Lambda(r_0^2 - r^2)} + r^2 d\theta^2 \tag{55}$$

This solution has the event horizon at $r = r_0$, which means one way travel, not two way travel. So this solution was rejected by the article [5]. However here we may admit the one way solution as wormhole solution ?

2.3 Casimir energy + Cosmological constant

We consider wormholes with Casimir energy and cosmological constant . We use the equations (25), (32),(47),(48)

$$\frac{rb\prime - b}{2r^3} = 8\pi G\rho(r) + \Lambda \tag{56}$$

$$\left(1 - \frac{b}{r}\right)\frac{\phi\prime}{r} = -8\pi G\tau(r) - \Lambda = 8\pi G p_r(r) - \Lambda \tag{57}$$

$$p_r(r) = \omega\rho \tag{58}$$

$$\rho = -\lambda\frac{1}{r^3} \tag{59}$$

For simplicity, we assume $\phi = constant$. $\acute{\phi} = 0$ We calculate to obtain the metric:

$$ds^2 = -dt^2 + \frac{dr^2}{16\pi G(1+\omega)\lambda(\frac{1}{r_0} - \frac{1}{r})} + r^2 d\theta^2 \tag{60}$$

3 Traversable Casimir Wormholes in (2+1) dimensions with gravitational Rainbow

In this section we will study the traversable Wormhole in the context of Gravity Rainbow[11].

The gravity Rainbow , doubly general relativity, is the extension of doubly special relativity to the case of curved space-time. The doubly special relativity is motivated by the following generalized energy-momentum dispersion relation;

$$E^2 f^2(\epsilon) - p^2 g^2(\epsilon) = m^2 \tag{61}$$

where $\epsilon = E/E_p$, E_p is the Planck energy and E is the energy probe of the test particle. The functions $f(\epsilon)$ and $g(\epsilon)$ are called Rainbow functions. The Rainbow functions should satisfy the following constraint:

$$\lim_{\epsilon\to 0} f(\epsilon) = 1, \quad \lim_{\epsilon\to 0} g(\epsilon) = 1 \tag{62}$$

The space-time metric may be distorted at the Planck scale energy.

The traversable wormhole metric in (2+1) dimensions with gravitational Rainbow is represented as

$$\begin{aligned} ds^2_{RB} &= -e^{2\phi(r)}\frac{dt^2}{f^2(\epsilon)} + \frac{dr^2}{(1-b(r)/r)g^2(\epsilon)} + r^2\frac{d\theta^2}{g^2(\epsilon)} && (63)\\ &= g_{\mu\nu}dx^\mu dx^\nu && (64)\end{aligned}$$

The above space-time metric is modified by the Rainbow functions. if ϵ tends to zero, rainbow effects disappear. On the other hand if ϵ tends to E_p, the metric is deformed in the quantum gravity level.

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\phi(r)}/f^2(\epsilon) & 0 & 0 \\ 0 & \frac{1}{(1-b(r)/r)g^2(\epsilon)} & 0 \\ 0 & 0 & r^2/g^2(\epsilon) \end{pmatrix} \tag{65}$$

$g^{\mu\nu}$ as the inverse of $g_{\mu\nu}$ is

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\phi(r)}f^2(\epsilon) & 0 & 0 \\ 0 & (1-b(r)/r)g^2(\epsilon) & 0 \\ 0 & 0 & r^{-2}g^2(\epsilon) \end{pmatrix} \tag{66}$$

Non-vanishing Christoffel symbols are

$$\begin{aligned} \Gamma^0_{10} &= \Gamma^0_{01} = \phi\prime \\ \Gamma^1_{00} &= \left(1-\frac{b}{r}\right)\frac{g^2}{f^2}e^{2\phi}\phi\prime \\ \Gamma^1_{11} &= \frac{1}{2}\left(1-\frac{b}{r}\right)^{-1}\frac{b\prime r-b}{r^2} \\ \Gamma^1_{22} &= -r\left(1-\frac{b}{r}\right) \\ \Gamma^2_{12} &= \Gamma^2_{21} = 1/r \end{aligned}$$

The Ricci tensor components are calculated similarly as:

$$
\begin{aligned}
R_{00} &= \frac{g^2}{f^2}\left[\left(-\frac{1}{2}\frac{\acute{b}r-b}{r^2}+\frac{1}{r}\left(1-\frac{b}{r}\right)\right)e^{2\phi}\acute{\phi}+\left(1-\frac{b}{r}\right)e^{2\phi}\acute{\phi}^2+\left(1-\frac{b}{r}\right)e^{2\phi}\phi\prime\prime\right] \\
R_{11} &= -\phi\prime\prime-\phi\prime^2+\frac{1}{2}\left(\phi\prime+\frac{1}{r}\right)\left(1-\frac{b}{r}\right)^{-1}\frac{b\prime r-b}{r^2} \\
R_{22} &= -\phi\prime r\left(1-\frac{b}{r}\right)+\frac{b\prime}{2}-\frac{b}{2r}
\end{aligned}
$$

The Ricci scalar ;

$$
\begin{aligned}
R &= R_{\mu\nu}g^{\mu\nu}=R_{00}g^{00}+R_{11}g^{11}+R_{22}g^{22} \\
&= g^2\left[\left(\frac{b\prime}{r}+\frac{b}{r^2}-\frac{2}{r}\right)\phi\prime-2\left(1-\frac{b}{r}\right)\phi\prime^2-2\left(1-\frac{b}{r}\right)\phi\prime\prime+\frac{b\prime}{r^2}-\frac{b}{r^3}\right]
\end{aligned}
$$

The Einstein tensor: $G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$ has no vanishing components

$$
\begin{aligned}
G_{tt} &= R_{tt}-\frac{1}{2}Rg_{tt}=\frac{g^2}{f^2}\frac{rb\prime-b}{2r^3}e^{2\phi} \\
G_{rr} &= \frac{\phi\prime}{r} \\
G_{\theta\theta} &= r\left(r-b\right)\left[\phi\prime^2+\phi\prime\prime+\frac{b-b\prime r}{2r\left(r-b\right)}\phi\prime\right]
\end{aligned}
$$

The Einstein tensors in the orthonormal frame;

$$
G_{\dot{\mu}\dot{\nu}}=\frac{\partial x^{\mu}}{\partial x^{\dot{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\dot{\nu}}}G_{\mu\nu} \tag{67}
$$

$$
\begin{aligned}
G_{\dot{t}\dot{t}} &= \frac{\partial x^{t}}{\partial x^{\dot{t}}}\frac{\partial x^{t}}{\partial x^{\dot{t}}}G_{tt}=e^{-\phi}e^{-\phi}G_{tt}=e^{-2\phi}\frac{g^2}{f^2}\frac{rb\prime-b}{2r^3}e^{2\phi}=\frac{g^2}{f^2}\frac{rb\prime-b}{2r^3} && (68) \\
G_{\dot{r}\dot{r}} &= \frac{\partial x^{r}}{\partial x^{\dot{r}}}\frac{\partial x^{r}}{\partial x^{\dot{r}}}G_{rr}=(1-\frac{b}{r})^{1/2}(1-\frac{b}{r})^{1/2}G_{rr}=(1-\frac{b}{r})\frac{\phi\prime}{r} && (69) \\
G_{\dot{\theta}\dot{\theta}} &= \frac{\partial x^{\theta}}{\partial x^{\dot{\theta}}}\frac{\partial x^{\theta}}{\partial x^{\dot{\theta}}}G_{\theta\theta}=\frac{1}{r}\frac{1}{r}r\left(r-b\right)\left[\phi\prime^2+\phi\prime\prime+\frac{b-b\prime r}{2r\left(r-b\right)}\phi\prime\right] \\
&= \left(1-\frac{b}{r}\right)\left[\phi\prime^2+\phi\prime\prime+\frac{b-b\prime r}{2r\left(r-b\right)}\phi\prime\right] && (70)
\end{aligned}
$$

Then the Einstein field equation in the orthonormal basis ;

$$
G_{\dot{\mu}\dot{\nu}}=8\pi GT_{\dot{\mu}\dot{\nu}}-\Lambda g_{\dot{\mu}\dot{\nu}} \tag{71}
$$

Here we adopt the Casimir energy as the source of negative energy density and we study the case with cosmological constant Λ.

3.1 Casimir energy

Casimir energy density (3-dimensional case)(31) is given as

$$\rho_c = -\lambda \frac{1}{r^3}$$

We obtain the components of the Einstein field equation:

$$\frac{g^2}{f^2}\frac{\acute{b}r - b}{2r^3} = 8\pi G\rho_c = 8\pi G(-\frac{\lambda}{r^3}) \tag{72}$$

$$\frac{r-b}{r^2}\acute{\phi} = 8\pi G p_r = 8\pi G\omega\rho_c = 8\pi G\omega(-\frac{\lambda}{r^3}) \tag{73}$$

where

$$\lambda = \frac{\Gamma(3/2)\zeta_R(3)}{(4\pi)^{3/2}}$$

We used the equation of state : $p_r = \omega\rho_c$
From(73),

$$\frac{db}{dr}r - b = -\frac{16\pi G\lambda f^2}{g^2} \tag{74}$$

The above equation is solved as same as the past section:

$$b(r) = \frac{r_0 - 16\pi G\lambda\frac{f^2}{g^2}}{r_0}r + 16\pi G\lambda\frac{f^2}{g^2} \tag{75}$$

ϕ is obtained in the the same way as the case with no rainbow:

$$\phi == -\frac{\omega}{2}\frac{g^2}{f^2}\ln|\frac{r-r_0}{r}|$$

And we obtain the line element with Gravitational Rainbow:

$$ds^2 = -\left(1 - \frac{r_0}{r}\right)^{-\frac{\omega g^2}{f^2}}\frac{dt^2}{f^2} + \frac{dr^2}{\left(1-\frac{r_0}{r}\right)\frac{16\pi G f^2}{r_0}\frac{\Gamma(3/2)\zeta(3)}{(4\pi)^{3/2}}} + \frac{r^2 d\theta^2}{g^2} \tag{76}$$

3.2 Cosmological Constant

The case of only Cosmological Constant is considered. Einstein field equations are as component:

$$\frac{g^2}{f^2}\frac{\acute{b}r - b}{2r^3} = \Lambda \tag{77}$$

$$\frac{r-b}{r^2}\acute{\phi} = -\Lambda \tag{78}$$

From (78), considering $b(r_0) = r_0$, we obtain

$$b(r) = r\left(1 - \frac{\Lambda f^2}{g^2}(r_0^2 - r^2)\right) \tag{79}$$

Here we will check the flare- out condition:
From (79),we obtain:

$$\frac{d\phi}{dr} = \frac{r^2}{r-b}(-\Lambda) \tag{80}$$

Next we do integration:

$$\phi = \int_{r_0}^{r} dr \frac{\Lambda r^2}{b-r} = \int_{r_0}^{r} dr \frac{\Lambda r^2}{r\left(1 - \frac{\Lambda f^2}{g^2}(r_0^2 - r^2)\right) - r} \tag{81}$$

$$= \frac{g^2}{f^2}\int_{r_0}^{r} dr \frac{r}{r^2 - r_0^2} = \frac{g^2}{2f^2}\log|r^2 - r_0^2|_{r_0}^{r} \tag{82}$$

The above integral is divergent. So the metric is

$$ds^2 = -(r^2 - r_0^2)^{\frac{g^2}{f^2}}\frac{dt^2}{f^2} + \frac{dr^2}{\Lambda f^2(r^2 - r_0^2)} + \frac{r^2 d\theta^2}{g^2} \tag{83}$$

3.3 Casimir energy + Cosmological Constant

Next we consider the case with Casimir energy and Cosmological constant. The components of the Einstein field equation :

$$\frac{g^2}{f^2}\frac{\acute{b}r - b}{2r^3} = 8\pi G\rho_c + \Lambda \tag{84}$$

$$\frac{r-b}{r^2}\acute{\phi} = 8\pi G p_r - \Lambda \tag{85}$$

where

$$\rho = -\frac{\lambda}{r^3} \tag{86}$$

$$p_r = \omega\rho \tag{87}$$

(87) is inserted into (85);

$$\frac{g^2}{f^2}\frac{\acute{b}r - b}{2r^3} = 8\pi G(-\frac{\lambda}{r^3}) + \Lambda \tag{88}$$

(89) is solved explicitly

$$b(r) = \frac{\Lambda f^2}{g^2} r^3 + \left(1 - \frac{16\pi G\lambda f^2}{g^2 r_0} - \frac{\Lambda f^2 r_0^2}{g^2}\right) r + \frac{16\pi G\lambda f^2}{g^2} \tag{89}$$

If we assume, $\phi = constant$, $\acute{\phi} = 0$ Then (61) means

$$8\pi G p_r - \Lambda = 0 \tag{90}$$

Some calculation is performed. we obtain

$$ds^2 = -\frac{dt^2}{f^2} + \frac{dr^2}{16\pi G(1+\omega) f^2 \lambda(\frac{1}{r_0} - \frac{1}{r})} + r^2 \frac{d\theta^2}{g^2} \tag{91}$$

Of course the above metric coincide with equation (60) as $f, g \to 1$.

4 Conclusion

We studied the traversable wormhole in the presence of the Casimir Energy and the cosmological constant in three dimensions. Considering the deformation of the metric when the energy tends to the Planck energy, we found the Rainbow metric for the traversable wormhole in three dimensions in the presence of the Casimir energy and the cosmological constant.

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Appendix A

It is very simplified to use embedding diagrams in order to study wormhole. The line element for a slice is obtained by setting t = constant, in Eq()

$$ds^2 = \frac{dr^2}{1 - b/r} + r^2 d\theta^2 \tag{92}$$

We construct a two dimensional surface with the same geometry as this slice in three dimensional Euclidean space for visualization. In he embedding Euclidean space we introduce cylindrical coordinates z, r, θ. Then the Euclidean metric of the embedding space has the form;

$$ds^2 = dz^2 + dr^2 + r^2 d\theta^2$$

The embedded surface will axially symmetric and thus can be described by the single function $z = z(r)$. On that surface the line element will be

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\theta^2$$

Comparing the above two equations , we have the equation for the embedding surface , given by

$$\frac{dz}{dr} = \pm \left(\frac{r}{b(r)} - 1\right)^{-1/2} \tag{93}$$

On (r,z) plane, r has a minimum point $r = r_0$ and convex downward - clocwise rotation by $\pi/2$ which figures the wormhole shape. Mathematically

$$\frac{d^2r}{dz^2} > 0 \tag{94}$$

$$\frac{dr}{dz} = \left(\frac{r}{b} - 1\right)^{1/2} \tag{95}$$

$$\frac{d^2r}{dz^2} = \frac{\frac{dr}{dz}b - \frac{db}{dr}\frac{dr}{dz}r}{2\left(\frac{r}{b} - 1\right)^{1/2} b^2} = \frac{b - b\prime r}{2b^2} > 0 \tag{96}$$

$$\tag{97}$$

Then,

$$b - b\prime r > 0 \tag{98}$$

This is the flare-out condition. In addition, this condition means that the null energy condition is violated , namely exotic matter is needed for the wormhole. The null energy condition applied to the matter is

$$\rho c^2 - \tau > 0 \tag{99}$$

Usually exoticity is defined through the exoticity parameter ξ as:

$$\xi = \frac{\tau - \rho c^2}{\rho c^2} \tag{100}$$

The parameter ξ is dimensionless and $\xi > 0$ means that space is occupied by exotic matter and NEC(null energy condition) is violated. Using equations(22)-(23) one finds

$$\begin{aligned}
\xi(r) &= \frac{\tau-\rho}{|\rho|} = \frac{\frac{-1}{8\pi G}\left(\frac{r-b}{r^2}\acute{\phi}+\Lambda\right) - \frac{1}{8\pi G}\left(\frac{\acute{b}r-b}{2r^3}-\Lambda\right)}{\left|\frac{1}{8\pi G}\left(\frac{\acute{b}r-b}{2r^3}\right)-\Lambda\right|} \\
&= \frac{\frac{b}{r}-\acute{b}-2r\left(1-\frac{b}{r}\right)\acute{\phi}}{\left|\acute{b}-\frac{b}{r}-2\Lambda r^2\right|} \\
&= \frac{\frac{b-\acute{b}r}{r}}{\left|\acute{b}-\frac{b}{r}-2\Lambda r^2\right|} - \frac{2r\left(1-\frac{b}{r}\right)\acute{\phi}}{\left|\acute{b}-\frac{b}{r}-2\Lambda r^2\right|} \\
&= \frac{\frac{2b^2}{r}\frac{b-\acute{b}r}{2b^2}}{\left|\acute{b}-\frac{b}{r}-2\Lambda r^2\right|} - \frac{2r\left(1-\frac{b}{r}\right)\acute{\phi}}{\left|\acute{b}-\frac{b}{r}-2\Lambda r^2\right|} \\
&= \frac{\frac{2b^2}{r}\frac{d^2r}{dz^2}}{\left|\acute{b}-\frac{b}{r}-2\Lambda r^2\right|} - \frac{2r\left(1-\frac{b}{r}\right)\acute{\phi}}{\left|\acute{b}-\frac{b}{r}-2\Lambda r^2\right|}
\end{aligned}$$

At the throat $(r = r_0, b(r_0) = r_0)$, then

$$\begin{aligned}
\xi(r_0) &= \frac{\frac{2b(r_0)^2}{r_0}\frac{d^2r}{dz^2}}{\left|b(\acute{r_0})-\frac{b(r_0)}{r_0}-2\Lambda r_0^2\right|} \\
&= \frac{\frac{2r_0^2}{r_0}\frac{d^2r}{dz^2}}{\left|b(\acute{r_0})-\frac{r_0}{r_0}-2\Lambda r_0^2\right|} \\
&= \frac{2r_0\frac{d^2r}{dz^2}}{\left|b(\acute{r_0})-1-2\Lambda r_0^2\right|} \\
&= \frac{2r_0\frac{d^2r}{dz^2}}{\left|2(\Lambda+\rho(r_0))r_0^2+1-1-2\Lambda r_0^2\right|} \\
&= \frac{2r_0\frac{d^2r}{dz^2}}{\left|2(\rho(r_0))r_0^2\right|} = \frac{\frac{d^2r}{dz^2}}{|\rho(r_0)r_0|} > 0
\end{aligned} \tag{101}$$

Here we used (22) and (37). We conclude at the throat $\xi(r_0) > 0$ namely NEC is violated , matter is exotic.

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