Displays for Teichmüller spaces

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Abstract

We introduce here a new notion of polarity for hyperbolic and complex analytic spaces. We describe compasses and arithmetic displays for said spaces.

§0 Background

In 2018, Bültel and Pappas defined the notion of a (G,μ)-display[1] for a p-adically complete ring with a miniscule cocharacter μ. This development was groundbreaking, in that it provided a more-or-less direct pipeline for translating from scheme-theoretic discussions to the (often more amenable) case of reductive algebraic groups. This notion, however, was not entirely novel; it is, first and foremost, a *generalization* of the *Witt vector displays* introduced earlier by Zink.[2] Here, we will till the soil first before sowing the conception of a display.

Let $\mathscr{X}$ be a complete Noetherian scheme with a finite basis, and θ_H a Hodge module of highest weight. We define the embedding (§0.0.1):

$$\mathbf{emb}^{\flat} : \mathbb{C}(\theta_H)\backslash\mathbb{A}^f/\mathbb{Q}_p \to \mathscr{X}_{AN}$$

to be *complex analytic* if there is a split epimorphism from the target of the above map to a suitable Shimura variety, $Sh_K(G,X)$. Further, assuming that such a condition is satisfied, we establish a *trivial fibration* $\ell^{f\text{-}k}$, for $0 \leq k < f$, such that all of the adjoints admitted by $\mathscr{X}_{AN}$ (as a lax monoidal category) are retracts of *open topological spaces* $X_k \in \mathscr{X}_{AN}$. We can always (expect to) recover the original, strict, complex space via Hodge decomposition by:

$$\mathbf{E} = \oplus_{f+k=n} \ell^2 X_b;\ b = \int_0^k \sum_{i=0}^{f} \mathbb{Z}_p/i \mapsto \mathbb{A}^f(\theta_H + \theta_H^{-1}),$$

where the right-hand-side is the *de Rham* component of the associated moduli stack.

However, this is inconvenient to us for a few reasons. Firstly, the Zariski topology of $\mathbb{C}(\theta_H)$ is *not restored* when one makes the necessary transfer $\mathscr{T}(\mathscr{X}_{AN},\mathbf{E})$. This is because there is an obstruction to the *sharp* lifting from the Hodge decomposed stack to the complex coanalytic space. Thus, the *Shimura datum* associated with the original root variety is forgotten by the transition map taking the

[1] [O.B.,G.P.], 2018
[2] [T.Z.], 2002

quasi-Noetherian[3] shtuka to its p-adic completion. Thus, we may wish for a more faithful complex, which swaps epimorphisms for monomorphisms, and therefore "remembers" the "pathological" components of the root datum.

Hence, we are now motivated to introduce the concept of a "display."

Definition 0.1.1 (Bültel-Pappas) A (p-adic) display, ϕ, (over a commutative ring) is a quadruple (M, N, F_0, F_1), where[4]:

- M is a finitely projective W(R)-module[5]
- N is a submodule such that $I(R)M \subset N$
- M/N is a projective R-module
- F_0,F_1 are F-linear maps F_0:M→M and F_1:N→M, such that the image $F_1(Q)$ generates M as a W(R)-module, and we have $F_1(V\, w \cdot x) = wF_0(x)$ for $w \in W(R)$ and $x \in M$

Proposition 0.2.1 The map (§0.2.2)

$$\phi_{\mathbb{C}}\colon \mathscr{T}_{\mathrm{GM}} \to {}^{\mathrm{M}}\mathscr{X}_{\mathrm{AN}}$$

is strictly coarser than (§0.0.1).

Proof Let $\xi_1,\ldots,\xi_n$ be a finite set of nilpotent weights that sum to an ideal I of $\mathbb{C}(\theta_{\mathrm{H}})$. Then, we can identify (§0.0.1) with $F_1|_{\xi}$, and we obtain the following modification of (§0.2.2):

$$\phi_{\mathbb{C}}\colon \mathscr{T}_{\mathrm{GM}} \twoheadrightarrow {}^{\mathrm{N}}\mathscr{X}_{\mathrm{AN}} \sim \mathbf{emb}^{\flat}\,;$$

thus, the mapping is *surjective*, and by openness of the codomain, we conclude that it is strictly finer, and further, forgetful.

Proposition 0.3.1 (**Bültel-Pappas**) If R is isometric to a perfect field, then ϕ is a Dieudonne module.

Proof See [O.B.,G.P.], pg. 9, section **2.3**.

Let $\mathscr{T}_{\mathrm{GM}}$ be the canonical Teichmüller space; we define an *outer marking* σ as (**def. 0.4.1**):

$$\sup(\mu+k\varepsilon) \sim \inf(d(\varepsilon,\partial\mathscr{T}_{\mathrm{GM}}))\,;$$

then, the coordinates of $\mathscr{T}_{\mathrm{GM}}$ all obey the same rigidity[6]. That is, given a polar anabelian groupoid g, we define the *immersion* from rep(g) to the *topologically realized* (i.e., shaped, homotopic) Cartesian closed category **Teich** with respect to some canonical basis x as:

$$(x, y, z, \overline{z}) \equiv \sup(d(\varepsilon,\partial\mathscr{T}_{\mathrm{GM}}))+d(\varepsilon,\partial\mathscr{T}_{\mathrm{GM}}),$$

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[3] That is to say, the original closed space, foliated by the Hodge module and its exponentiation, is not necessarily Noetherian. Thus, the addition of this information forces a ramification (equivalently, a tightly laced looping) which cannot be undone.

[4] The terminology is slightly confused in [T.Z.]; displays are called "3n-displays," while "displays" are called "nilpotent displays." We will primarily be interested in nilpotent displays.

[5] Where W(R) is the ring of Witt vectors

[6] See, for example, [K.R.,S.S.], date unknown

thus dualizing the construction of **def 0.4.1** so that the (nilpotent) origin of the space is defined to be the maximal distance from the boundary. Here, the boundary consists of a connected ε-chain[7] with trivial shape.

We shall now move on to a definition, which while non-sequitur in its appearance, is actually quite germane[8]:

Definition 0.4.2 (**Diaf-Seppi**) A **geodesic lamination** λ of $\mathbb{H}^2$ is a collection of disjoint geodesics that foliate a closed subset $X \subseteq$ H2 . The closed set X is called the support of λ. The geodesics in λ are called leaves. The connected components of the complement $\mathbb{H}^2 \setminus X$ are called gaps. The strata of λ are the leaves and the gaps.

This allows us to provide (in our own words),

Definition 0.4.3 An ***earthquake*** (on $\mathscr{T}_{\mathrm{GM}}$) is a bijection $\mathcal{E}{:}\mathbb{H}^2 \to$ **Teich** such that there exists a geodesic lamination λ for which there is restriction to strata s_1, s_2 in the kernel and image of the map which are homographies.

With respect to σ, an earthquake is a map which transfers the diagonal of an *entourage* (of the appropriate ε-chain) to a finite projective module S_2~PSL(2,$\mathbb{R}$) which preserves the miniscule coweight μ up to isomorphism. We say that $\mathcal{E}$ is a *left earthquake* if im($\mathcal{E}$)=-$(x, y, z, \overline{z})^{\mu}\sigma$, and a *right earthquake* otherwise. Thurston famously proved[9] that for any two coordinates $(\widetilde{x}_{\mathrm{i}}, \widetilde{y}_{\mathrm{j}})$ of a complex space, there is a **unique** left[10] earthquake $\widetilde{\boldsymbol{\mathcal{E}}}$: $\widetilde{x}_{\mathrm{i}} \mapsto \widetilde{y}_{\mathrm{j}}$, assuming the underlying manifold to be orientable.[11]

§0.1 Organization of this paper

We have now glimpsed into two very distinct arenas of mathematics; on the one hand, we have the displays of Zink and his ilk; on the other, we have the earthquakes of hyperbolic spaces. The relationship between these two are not so tenuous as they at first may have seemed.

We have used the zeroth section of this paper as an appetizer. For the main entree, we will first begin with a discussion of curve complexes, and their relationship to buildings in the sense of Bruhat-Tits. We will then examine the relationship between these structures to the displays we have already touched upon. Section 2.5 is devoted to the categorification of these concepts, and it is there that we will take preemptive action to ensure they play nicely with one another. In sections 3, we explore this newly discovered territory in a bit more depth before concluding.

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7 [C.P.], pg. 2

8 The definition is fully lifted from the source and remains wholly unchanged

9 [Thurston], 1986

10 It was later demonstrated that this theorem involves no loss of generality; i.e., it can be extended as well to right earthquakes

11 Further, Diaf-Seppi [DS] extended this theorem to the case of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, the anti-de Sitter space. More interestingly, it was shown that two laminations intersect if and only if they are hyperbolic isometries [DS, pg. 13, lemma 5.1].

Remark: No claim to originality is made by writing this paper; in fact, much of it consists in rehashing the works of many more talented authors. Bear in mind that the necessary footwork has been done by giants whose shoulders I stand so humbly upon.

§1 Curve complexes

It is here that we refer the reader immediately to [A.B.,Ji] for an excellent account of Borel-Serre compactification, a technique which the second author makes very clever use of in his testimony of curve complexes, [Ji2]. Both of these references will be essential to our discussion from now on. Once and for all, we will write

$$\mathfrak{G}^{\sharp}\colon \mathbf{G}_{\mathrm{CON}} \to \mathbf{Teich};\ \mathfrak{G}^{\sharp\mathrm{op}}\colon \mathbf{Teich} \to \mathbf{G}_{\mathrm{CON}}$$

for the maps, respectively, from the category of connected and reductive groups to the canonical Teichmüller space, and vice versa.[12] We will use these maps more or less constantly to facilitate cross-pollination between the desired niches.

Definition 1.0.1 A *curve complex*, $\mathscr{C}(S)$, is a profinite simplicial complex $\Delta((t))$ enriched with a homography $\mathcal{E}:\Delta((t)) \to \partial(\mathscr{T}_{GM})$.

Proposition 1.0.2 There is a natural *Borel-Serre* compactification $\overline{\mathcal{T}_{GM}}^{BS}$ of the Teichmüller space which localizes "points at infinity" to separable marked points in neighborhoods of genus >1.

Proposition 1.1.1 For G a simple and reductive group, there is a parabolic subgroup g whose generators are representative $\overline{\mathcal{T}_{GM}}^{BS}$.

We call the geometric complex associated with this subgroup a "building," and its strata are respectively, apartments $\varrho=(\frac{1}{\xi}\mathrm{rep}(g))$, and alcoves $(\frac{1}{\xi}\rho)$. We call this value, $\frac{1}{\xi}$, the *generative factor* of the building, and we make the necessary identification $\frac{1}{\xi} \to SL(\xi\text{-}1,\mathbb{Q}_p)$. We call a hyperbolic n-space *symmetric* if it is an overring of a building, and we call it *compact symmetric* if it has nonnegative sectional curvature, strictly positive Ricci curvature, and is compact.[13] We will call a real semisimple Lie group with finitely many connected components and finite center the ***real locus*** of the building, and we will denote it by $G_{\mathbb{R}}$.

We will always restrict ourselves to the finite case when discussing the *little Borel-Serre site*, otherwise we will explicitly mention which site we are working in. We will call a space with a real locus

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[12] Also, we may write $\mathfrak{G}^{\flat}$ in place of $\mathfrak{G}^{\sharp\mathrm{op}}$ where we are so inspired

[13] [Ji2], pg. 5; we may be less interested in the compact symmetric case than the usual one

a "scene," and if the support of λ is finitely contained within $G_{\mathbb{R}}$, we will say the scene is *maximally flat*, or flat when no confusion seems likely.

Proposition 1.1.2 Every scene covers a fixed point γ, whose stabilizer is a parabolic subgroup of G.

Proof Given two apartments ϱ,ϱ', and two simplices δ,δ', there is an isomorphism $\varrho \to \varrho'$ which keeps δ,δ' pointwise fixed. We then identify $\delta=\gamma$, and the stabilizer G_{stab} is parabolic, given that G/δ is a complete variety.

[Ji2], in particular, gives the following easy way to construct sets of apartments: "For every maximal compact subgroup K of G, let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be the associated Cartan decomposition.For every maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, let $A = \exp \mathfrak{a}$ be the corresponding subgroup… Therefore, the subcomplex of $\varrho(G)$ consisting of the simplices $\delta_{p_1},\ldots,\delta_{p_n}$ is isomorphic to the Coxeter complex of the Weyl group W of G. This subcomplex is defined to be an apartment of $\Delta(G)$"[14]

Proposition 1.1.3 Flat scenes are homeomorphic to objects in **Teich** which are open balls.

Proof Trivial.[15]

Theorem 1.2.1 *Mostow Strong Rigidity*: Assume X is not homeomorphic to the Poincaré upper half-plane $\mathbb{H}^2$. Then, two compact, irreducible, locally symmetric scenes (Y , Y') of noncompact symmetry type are homeomorphic.

Proof A locally symmetric space is said to be irreducible if it does not admit any finite cover which splits as a product. Therefore, there is no unique coordinate

$$\sigma \times_{\sigma} \frac{\partial \bar{z}}{z}$$

such that the map $X \to Y'$ admits a split epimorphism, and therefore, there is a unique left (right) earthquake linking Y and Y'.

Remark This is equivalent to saying that the display over X is Zariski dense; i.e., there are no extremely disconnected simplices whose canonical fiber product is an adjoint of a forgetful functor. Namely, the nilpotent cone of σ is tame, and

$$\lim_{\leftarrow} \partial\sigma = \lim_{\leftarrow} \sigma+k.$$

Li denotes this equality by the map φ_∞; Diaf and Seppi may write it as

$$\partial_\infty Y \to \partial_\infty Y',$$

where ∂_∞ denotes closure of the associated Lie group.

Remark Margulis and Prasad provided later proofs for the more general case of finite volume locally symmetric spaces of rank >1 and >0, respectively.[16]

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14 Ibid, pg. 12; notation my own

15 Loc. cit.

16 See [Ji3]

Let us denote the simplices of top dimension of a building, $\mathfrak{B}$, by $\partial\overline{\mathfrak{B}}_{\delta}^{max}$. Notice that we are tacitly presuming this set to be a boundary of some sort; we shall call this the *Satake boundary* of the building. We also have the canonical preordering

$$\partial\overline{\mathfrak{B}}_{\delta}^{max} < \partial_{\infty}\overline{\mathcal{T}_{GM}}^{BS} < \partial\mathscr{T}_{GM},$$

in decreasing order of specialization. Ji remarks that the Furstenberg boundary

$$f(z) = \frac{1}{2\pi}\int_{0}^{2\pi}\widehat{f}(e^{i\theta})P(z, e^{i\theta})d\theta = G/P_{min}$$

is included in the Satake compactification[17], and so in some sense this represents the minimal amount of information one would want to encode about a spherical building. Indeed, this is the *minimal* boundary containing the boundary symmetric space X_P of every parabolic subgroup P. This gives us the decomposition[18]:

$$\partial\overline{\mathfrak{B}}_{\delta}^{max} = \coprod_{p} X_{p}.$$

§1.3 Tightly laced vs. loose displays

If P is not $\mathbb{Q}$-split, then the neighborhood of the marked point $\sigma \in \mathscr{C}(S)$ has *strictly finer* presentation than its ind-variety, and so again, we require that there is a display:

$$\phi_{\mathbb{Q}}: \sigma \rightarrow {}^{F}\sigma,$$

where F is a nil-truncated Frobenius. By this, we mean that the foliation over the neighborhood $\mathscr{U}({}^{F}\sigma)$ is identical with the foliation of $\mathscr{U}(\sigma)$, up to isomorphism of inertia of the trivial line bundle.

Definition 1.3.1 Whence $\sigma \rightarrow {}^{F}\sigma$ is a perfect map (of schemes), and where $\mathscr{f}(*)$ denotes a foliation, if $\mathscr{f}(\mathscr{U}({}^{F}\sigma))=\mathscr{f}(\mathscr{U}(\sigma))$, we will say the display $\phi_{\{\cdot\}}:\sigma \rightarrow {}^{F}\sigma$ is *tightly laced*; otherwise, we will call it loose.

Remark It is easy to see that for scenes in the little Borel-Serre site, all displays are tightly laced. However, this is not necessarily the case in general. Take for example, a display over the Nisnevich site:

$$\phi_{N}: p \rightarrow {}^{F}p_{ét},$$

where p is a triangulated category. It may be the case that no neighborhood $\mathscr{U}(p)$ contains a normalizer of a parabolic subgroup, and hence, it may be the case that there is no available *Bruhat decomposition*; thus, there is a delooping

$${}^{F}p_{ét} \simeq p^{-1}\prod_{n} p_{i} = \partial_{\infty}^{i}p^{-1}(gfg^{-1})$$

[17] I.e., the compactification whose boundary is the Satake boundary

[18] [Ji2], pg. 15

which does not preserve shape under cross product. This is to say that *some* convergent (θ-adherent) functions over a smooth manifold characterized by a group with central extension will fail to converge in the same manner as the identical function would over an unramified space.

Proposition 1.3.2 Let G be a semisimple linear algebraic group defined over Q. Then for every arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, $\Gamma\backslash X$ is compact if and only if the $\mathbb{Q}$-rank of G is equal to zero, or equivalently there is no proper $\mathbb{Q}$-parabolic subgroup of G

Proof This was conjectured by Godement and proved by Borel and Harish-Chandra [H.C.], and by Mostow and Tamagawa [M.T.].

§2 Compasses

Proposition 2.0.0. A *marking* on $\partial\overline{\mathfrak{B}}_{\delta}^{max}$ fixes a display ϕ:S $\to \Sigma$ between aspherical two-dimensional manifolds with a distinguished homotopy class h$\in\phi$.

Proof See [Ji2], pg. 33.

Definition 2.1.0. Let ϕ_A,ϕ_B be two displays over one or more sites, and denote by $\mathfrak{n}$ the nilpotent cone of the ind-site. Suppose that there is a marking $\mathfrak{m}$ which is consistent across the transition

$$\phi_B \circ \phi_A^{-1};$$

then, we shall call the displays *polar* and say that they have display-polarity $\mathfrak{m}^{\pm}$:$\mathfrak{m}\oplus$-$\mathfrak{m}$. We shall call a Grothendieck universe, $\mathfrak{U}(\phi_{A,B},\mathfrak{m}^{\pm},k)$ a *compass* if it admits stratification into distinct k-manifolds.

Proposition 2.1.1. A compass with the Weil-Petersson metric is a Cat(0)-space.

Proof See [Li2], pg. 33

Compasses (over an underlying Deligne-Mumford stack) naturally produce the map $\mathfrak{G}^{\sharp}$. As we shall see, even in the case where a zero-object is a projective limit of short exact sequence with resolution, we can always select an triangulated category $DM_{-}^{eff}(k)$[19], and a display $\phi(\Omega)$ such that there is an earthquake $\mathcal{E}_\infty$:$\mathfrak{m}^{\pm}(\mathscr{T}_{GM}) \to \partial_\infty\overline{\mathcal{T}_{GM}}^{BS}$, and the centralizers of certain maximal scenes are redshifted.

Let Φ(a,b,...,ω) be a *display block*[20], and let $\mathfrak{F}^{\bullet}$ be a Frobenius isocrystal. We call this block *effectively lensed* if:

$$(\mathfrak{F}^{\bullet}\otimes a^{\bullet})\oplus_a(\mathfrak{F}^{\bullet}\otimes b^{\bullet})\oplus_b\ldots\oplus_{\omega-1}(\mathfrak{F}^{\bullet}\otimes\omega^{\bullet}) = \int_{-\mathfrak{m}}^{\mathfrak{m}} \partial P_{min}.$$

[19] See [V.V.], 2015

[20] [O.B.,G.P.], pg. 26; section 3.7

Suppose that **B** is an effectively lensed display block, and the map **B** → **B** is typified by tightly laced scenes. Then, we have:

$$\frac{\sum_{a}^{\omega} \xi g}{f} = \mathfrak{n},$$

and the nilpotent cone of **B** becomes a nef divisor of the principle line bundle over its compass. That is to say, there is a map $\mathfrak{n} \to d(\varepsilon, \sigma)$, for a simplex $\sigma \in \mathscr{C}(S)$. Here, we take σ to be a $\mathfrak{b}$-adically separated and closed value in a subring $\mathfrak{r}$ of the principal ideal domain of **B**, such that $\{0\} = \bigcap_n \mathfrak{b}^n$ and $A = \bigcap_n (A+\mathfrak{b}^n)$.

Define the map

$$\psi: \phi_x \circ \ldots \circ \phi_{\bar{z}} = \mathbf{B}|_{\Gamma} \times_{\mathfrak{r}} i(\mathbf{B}|_{\mathfrak{n}});$$

with dimension $(\bar{z} + 1) - x$.[21] We say that there is a $\mathfrak{b}$-morphism $\psi \hookrightarrow \mathrm{Co}(\mathbf{Teich})$ if the function $f(\psi)$ has compact support in **Teich**, such that

$$\mathrm{ev}(\psi) \circ \mathrm{coev}(\psi^{-1})$$

is a proper morphism of smooth schemes. $\mathfrak{b}$-morphisms are quasicoherent whenever they exist, and they are coherent if they are exact; they are multi-display transition maps which preserve polarity.[22] They are smooth, stable, normal, and regular immersions. A coherent $\mathfrak{b}$-morphisms from a building $\mathfrak{B}$ to a hyperbolic space $\mathbb{H}$ is fully faithful, and there is an equivalence

$$\begin{array}{c} \mathfrak{B} \to \mathbb{H} \simeq \mathbf{MCG} \to \mathbf{Teich}, \\ \downarrow \\ \partial \overline{\mathfrak{B}}_{\delta}^{max} \end{array}$$

where MCG is the mapping class group. An equivalence of $\mathfrak{b}$-morphisms is an equivalence of *scenes*; an equivalence of *maximally flat scenes* is an equivalence of compasses.

Lemma 2.2.1 An orientation-preserving map preserves polarity.

This is essentially trivial, as *polarity* (of displays) is a strictly stronger property than orientation; thus, if $\widetilde{\mathcal{E}}_L : \widetilde{x}_i \to \widetilde{y}_i$ is an orientation-preserving left earthquake, then the corresponding functor ψ acting on display blocks will preserve polarity. The converse does not always hold. In general, a map which does not preserve orientation may preserve polarity. This is the case with effectively lensed blocks.

Lemma 2.2.2 Effectively lensed blocks preserve scenes of the highest weight, up to tightness homology.

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[21] C.f., "Dirichlet's unit theorem"

[22] Though, not necessarily, orientation

Proof Let $\widehat{S}$ be a scene of the highest weight, and let there be an $\widehat{S}$ -module **q**, such that the functor ψ is a map: ${}^{\mathrm{q}}(\phi_x \circ \dots \circ \phi_{\bar{z}})$-$(\phi_x \circ \dots \circ \phi_{\bar{z}})^{-1}$. Then, we have that the polarity is q($\mathfrak{m}^{\pm}$).

Assuming q to be a subgroup of the Lie group re(PGL(2,ℂ)), we have:

$$\widehat{S} \times_{\mathrm{q}} \mathrm{PGL}(2,\mathbb{C}) \to \left(\int_{-\mathfrak{m}}^{\mathfrak{m}} \partial P_{min}\right)^{q}.$$

Which we rewrite as:

$$\widehat{S} \times_{\mathrm{q}} \partial_{\infty}\overline{\mathcal{T}_{GM}}^{BS} \to \mathbf{Teich};$$

we identify the l.h.s. with the mapping class group. Then, we have that tightness is preserved. □

This amounts to saying that an effectively lensed block is affine if, *and only if*, each of its constituent displays are displays over affine varieties. So, if we let κ be the projector (of a section) of the category of formal rings whose ideals are the generators of **B**, then κ is free if and only if the initial object of $\mathrm{Proj}_{\mathbf{B}}$ is homotopic to the nilpotent cone.

Proposition 2.3.1. Let DRh be the de Rham complex of **B**. Then, if **B** → **B** is a surjective and conservative functor, the *Killing form* K(DRh) is a Killing form of a maximally flat scene if and only if it is nilpotent.

§2.4 Hypercoverings

Let $\Omega_{\phi_n}^{\phi_m}$ be a compass over a hyperbolic n-space, and let $\mathscr{C}(S)$ be the curve complex of the space, as before. We call a covering $(\mathscr{C}(S))' \supset \mathscr{C}(S)$ a *hypercovering* if the following hold:

- There is some compass $\Omega_{\phi_n}^{\phi_{k<m}}=\kappa$ for which the transition map $\Omega_{\phi_n}^{\phi_m} \to \kappa$ is a surjection and
- For a collection of simplices $\varrho_0,\dots,\varrho_k$ in $\mathscr{C}(S)$, there is some hyperplane whose *homotopy group* is a cover (of smooth schemes) over a parcel which is encompassed by $\Omega_{\phi_n}^{\phi_m}$-κ

In other words, if the set-theoretic difference, $\partial(\Omega)$, obtained by taking the complement of κ which is contained in $\Omega_{\phi_n}^{\phi_m}$, is non-empty, then we say that $(\mathscr{C}(S))'$ is a *encompassed by* a hypercovering of $\mathscr{C}(S)$, and that the larger compass is a hypercover of the second.

Notice, a hypercover (of compasses) is not merely a cover in the usual sense. In addition to the usual inclusion one would expect from a covering map, we also have the associated *tilt*:

$$\Omega_{\phi_n}^{\phi_m}{}^{\flat} \to_{\Gamma} \kappa \simeq (\mathscr{C}(S))' \to \mathscr{C}(S)_{DISC}$$

which provides a quasi-separated *covering of co-skeleton* for the image. [23]

Question 2.4.1. Let **P:**$K_1 \to K_2$ be a hypercovering of compasses, and $p^{\pm}$ the polarity of K_1?

Intuitively, it seems satisfying to answer this question in the affirmative. However, upon further inspection, one finds various obstructions to doing so. First off, it is known that if the underlying group is a *Chow group of zero cycles*, then any non-trivial unramified classes obstruct its triviality. Differential forms in positive characteristic, used with mixed characteristic degenerations could potentially prevent the polarity from remaining constant[24]. Thus, we will assume **P** to be *messy* if this is the case.

For the messy hypercovering, it is unclear even if K_1 and K_2 are homologous to one another. We will assume them to be at least diffeomorphic.

Proposition 2.4.2 P is a diffeomorphism of compasses if **B**(P) is effectively lensed.

Proof We have

$$\Omega_{\phi_n}^{\phi_m}{}^{\flat} = (\mathfrak{F}^{\bullet} \otimes k_1^{\bullet}) \oplus_{k1} (\mathfrak{F}^{\bullet} \otimes k_2^{\bullet}) \oplus_{k2} \ldots \oplus_{kn-1} (\mathfrak{F}^{\bullet} \otimes k_n^{\bullet}) \to_{\Gamma} \kappa = \partial_{SET}(P),$$

and by the formula

$$\partial_{SET}(P) < \mathrm{Hom}(\mathscr{C}(S), \mathrm{coord}),$$

we see clearly that if $\mathscr{C}(S)$ is a simplicial complex of a sober space, then said space is diffeomorphic to the space on which the fibrations of $\mathscr{C}(S)$ retract. Thus, the surjection on κ is diffeomorphic.

We are now free to make the following identification:

$$P: \mathscr{V}_{psh} \to \mathscr{V}_{shv};$$

we see nakedly that for curve complexes (and by extension, buildings), the admissible displays of P inherit the fibers of the kernel; that is to say, every earthquake in K_1 corresponds to a quasi-isomorphic earthquake in k_2.

Proposition 2.4.3 If A is a compass of a spectral topos, and P:A→B a hypercovering, then there is a fixed simplex $\hat{\rho}$.

Proof Follows from 2.4.2.

Remark Assume that A has a real locus.

Example Let A be a cobordism, and B be an extremely disconnected projective disc. Let $\hat{z}$ be the marked point at the singularity (i.e., zipper); then there is an isometry $A_{\hat{z}} \leftrightarrow B_{\hat{z}}$. That this is a spectral topos follows from the fact the each semisimple and stable neighborhood of $\hat{z}$ is covered by a frame

[23] [S.T.E.F.], pg.7

[24] See [B.T.]

$\xi\wedge\vee\mathscr{U}(\widehat{z})$, giving us the map P_{spec} from $\partial\Omega \rightarrow \widehat{z}_i$. We then conclude that $\widehat{z}_{i_{stab}}$ is an idempotent, and stabilizer of, the fundamental group of $proj_B$.

We now, finally, define the notion of equivalence of polarities;

Definition 2.4.4 Two displays, ϕ_a, ϕ_a, of compasses Ω_x^y, Ω_a^b are *equivalent* in polarity if {x,y} is isometric to {a,b}, such that linearly independent vectors tangent to the points p_x,p_y have equal determinant to those of p_a,p_b.

Definition 2.4.5 A projective variety X over a field k is CH_0-trivial if, for any field extension $F \supset k$, the degree map $\deg(CH_0)(X_F) \rightarrow \mathbb{Z}$ is an isomorphism, and if $\phi_a(X)$ admits a pushout with equivalent polarity.

Lemma 2.4.6 (**T. Okada**) If X is a smooth, projective, stably rational variety, then X is universally CH_0-trivial.[25]

It is obvious that if $\Omega(\mathscr{T}_{GM})$ has >1 real loci, then its displays admit maps of apartments which are CH_0-trivial. More generally, if a hyperbolic space admits decomposition into quasi-smooth weighted strata, and if $\mathfrak{g}$ is a group with miniscule co-weight μ, then there is a $(\mathfrak{g},\mu)$-display encompassed by $\Omega(-)_{id}$ which is conformal over said space. We then call the map

$$(\mathfrak{g},\mu)_S \rightarrow (\mathfrak{g},\mu)_S$$

a Q^x*-chart* (of rigid varieties).

§2.5 Categorification

We have by now established the notion of a compass as it pertains to analytic spaces. Now, we will seek an essential categorification which distills the lifeblood of these complexities into a more soulful form.

Let **CompS** be the category of compasses. Then, for every ∞-groupoid ι, there is a composition $\iota\cup$**SSets** $\rightarrow$ **CompS** which takes an ideal ι and transforms it into a laced groupoid, **LGrp**. Better yet, for any collection of urelements, $\boldsymbol{f}$, there is a composition $\boldsymbol{f}\bigstar\iota$ ($\bigstar$ being some Hodge filtration) which transforms ι into a semigroup.

If we desire an inclusion

$$\boldsymbol{f}\bigstar\iota \subset \mathbf{CompS},$$

then we must also require that the product

$$\boldsymbol{f_i} \times_{\boldsymbol{u}} \ldots \times_{\boldsymbol{u}} \boldsymbol{f_n}\,\iota = \boldsymbol{f}\Pi\iota$$

obey "spectral fusion rules" (whatever that is taken to mean). We will summarize the main properties we may wish for out of these rules:

- $\boldsymbol{f_i} \times_{\boldsymbol{u}} \boldsymbol{f_i} = \boldsymbol{f_i}$ (idempotence)
- $\boldsymbol{f}\Pi\iota^{-1} \rightarrow$ **Polish**

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[25] [T.O.], lemma 2.2

- rep($\boldsymbol{f}\Pi\iota$) $\subset$ **CompS$_{\mathbf{grps}}$**

The first of these requirements is straightforward. The second states that composing with the opposite category of **Grpd** should leave us with something which is amenable to a Borel σ-algebra construction. Lastly, we require that the concrete realization of the composition should yield us something akin to an *abstract space*, i.e., an analytic variety composed of rigid and atomic units. Essentially, these correspond to the rules for combining quasi-quanta espoused by Emmerson.

§3 Displays, further considered

Suppose we are working with a surface of genus 3g-6; call it $\mathbb{T}_g$. Let σ_i be an outer marking on $\mathbb{T}_g$; let $\mathbb{L}^n$ be a proper subspace of $\mathbb{T}_g$, and let $\mathfrak{D}$ be the divisor class group of $\mathbb{C}_p$-modules of $\mathbb{T}_g$. Suppose, finally, that there is a collection of strata, $\{\mathscr{S}_i\}_{i\in I}$ belonging to $\mathbb{L}^n$. We now define the arithmetic distance on $\mathbb{T}_g$.

Definition 3.0.1 For two ε-chains, ${}^L\varepsilon$, ε^R, let $\phi{:}{}^L\varepsilon \to \varepsilon^R$ be the effectively lensed display acting on $\partial(\mathscr{U}(\varepsilon))$, and let q_{filt} be the principal ultrafilter on the exact category $E \supset {}^L\varepsilon^R$. Then, say that the difference between analytic cosheafs, as an *arithmetic sum of subrings*, is the *arithmetic distance* $d({}^L\varepsilon,\varepsilon^R) := \frac{2q_{filt}+1}{q_i-1}$.

We comment here that for a tilt of rigid and affine varieties, the associated structure sheaf $\mathcal{O}_{\mathscr{T}}{}^{\flat}$ takes as its data the arithmetic distance between subspaces of $\mathbb{T}_g$, and outputs the compass $\Omega_{inf(\mathbb{L}^n)}^{sup(\mathbb{T}_g)}$.

By Kodaira-Spencer[26], we take this to mean that there is an effective equivalence

$$\Omega_{inf(\mathbb{L}^n)}^{sup(\mathbb{T}_g)} \simeq \mathrm{Fac}(\mathrm{Hom}(\mathbb{L}^n, \mathbb{T}_g)$$

between the *factor groups* of the topologies $\mathbb{T}_{g_{top}}$ and $\mathbb{L}^n_{top}$; accordingly, we assign the functor

$$\mathrm{reps}(\mathrm{Fac}(\mathbb{A}^n)) \simeq \pi_1(H^n\mathbb{Q}_p) \to \mathfrak{G}^\#$$

to any abstract affine space and its field of fractions.

Proposition 3.0.2 There is a faithful isomorphism of stacks, $\mathfrak{G}$, if and only if $\{\mathscr{S}_i\}_{i\in I}$ is a hypercovering of schemes.

Proof Trivially, the classifying space $\mathbf{BG}(\mathbb{A}^n \wedge \mathbb{L}^m)$ is a hypercovering of the irreducible strata of $\mathbb{T}_g$ if and only if the arithmetic distance is a notion of completion for the formal ring of power series of $\mathbb{T}_g$. Thus, $\mathfrak{G}$ is a faithful isomorphism of stacks if and only if the Picard variety (as a stratum) of q_{filt} is encompassed by $\Omega_{inf(\mathbb{L}^n)}^{sup(\mathbb{T}_g)}$, and hence our proof is complete.

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[26] [K.S.], 1953

In the same vein,

Proposition 3.0.3 $\mathfrak{G}$ exists and is faithful if the integral lattice of the space on which it acts is a subset of the ∞-topos in which it resides, in which case polarity is preserved, and we assume that there is a clopen set $\mathbf{G}_{\text{perf}}$, and an autoequivalence between said set and the inferior space of the isomorphism.

Proposition 3.1.0 A lifting of displays $\phi_1 \to \phi_n$ is a lifting of a perfect and complete field K whence the residue field k of K is profinite.

Proof If the residue field k is profinite, then it always contains at least one real locus, which is the germ of some harmonic function over the space with exact and identical miniscule co-character.

We shall call here such liftings "desirable," for lack of a better word.

Proposition 3.1.1 An effectively lensed display block, consisting of desirable displays, over a field of equal or mixed characteristic, preserves idempotents.

We shall further wish that our desirable blocks preserve ≤n-dimensional cycles and cocycles of arbitrary characteristic and degree. The lenses of a desirable block automatically constitute regularizers of order q for some coherency class of the associated curve complex.

$$\mathscr{C}(\mathrm{S})_{\mathbf{B}} \cong \mathrm{Coh}(\Delta_q \mathbf{Topo});$$

whence $\mathrm{Coh}(\Delta_q \mathbf{Topo}) \simeq \mathbf{Topo}_{\mathrm{qDISC}}$, such that it is a q-adically separated sheaf of ideals, then we get the following *excision*:

$$\begin{array}{c} Topo_{qDisc_{\tau>0}} \to \mathrm{qCoh}({}^{L}\varepsilon) \to {}^{L}\mathbf{B}^{R} \leftarrow \mathrm{qCoh}({}^{L}\varepsilon^{R}) \leftarrow Topo_{qDisc_{\tau>0}} \\ \downarrow \\ \widehat{B}_{\tau<1}, \end{array}$$

and the divisor class group of the block remains stationary after transferring to the little Nisnevich site. The polarity, $p^{\pm}_{\tau\neq 0}$, acts in a stratified[27] way which is *anisotropic* with respect to the interior and closure operators of $\mathbb{L}^{n-k}$. We see here that while Prop. 3.0.3 is satisfied, the behavior of the underlying "nice" space (topological, metric, etc.) suggests otherwise; this is because the transfer

$$Topo_{qDisc_{\tau>0}} \to \ldots \to \widehat{B}_{\tau<1}$$

is not smooth; yet, while it acts on the curve complex of the space, it is not principally of simplicial type. Thus, the coskelata of the ramified space in which the q-filter resides are not *markedly proper*; they are effectively degenerate under the étale picture, although this is not the case globally. To rectify this peculiarity, we introduce the following easement:

$$\mathscr{C}(\mathrm{S})_{\mathbf{B}} = \mathbf{LocSys}(\Delta_q \mathbf{Topo}) \cup \mathrm{orb}(q_{\text{filt}});$$

[27] Essentially, disjoint

which provides some correction to the anomaly by attaching to every totally disconnected and quasi-separated perfect scheme a new semi-marked outer point which conforms to the charts of the original compass.

Example Let $\mathbb{L}^4$ be the Minkowski lightcone, and $\mathscr{W}_0\ell$ the worldline of a simplicial object (instanton). Then, for every neighborhood of the particle, we assign a real locus Loc$\mathbb{R}$ to a section of the compass over $\mathbb{L}^4$. We then take the arithmetic distance

$$\mathrm{Loc}\mathbb{R}(p_0) - \mathscr{C}(\mathscr{W}_0\ell + p_n) = p_k$$

and we compute the weight of the patch locale over $\mathscr{W}_0\ell$ as

$$Patch = \sum_{k=0}^{n} p_k \int_0^1 \phi(k),$$

and we let there be an immersion Patch$(\mathscr{C}(S)_{\mathbf{B}}) \hookrightarrow \mathbb{T}_g|_{\mathscr{C}(S)\mathbf{B}}$, so that the structure bundle over the original space now has natural pointwise retracts to a frame bundle of the necessary conformal charts. This should give us an immersion from the dominant matrix of the little site to the larger n-by-n matrix of the anomaly-corrected site.

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