below

i-th DERIVATIVE AND HALF-DERIVATIVE OF x

Leonardo ***Rubino***

June 2023

Abstract: here is a proof of the value of the i-th derivative of x, as well as that of the half-derivative of x.

1-EULER GAMMA FUNCTION

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt \tag{1.1}$$

Of course $\Gamma(1)=1$. Then,

$$\Gamma(n+1)=n\Gamma(n); \tag{1.2}$$

in fact, after an integration by parts:

$\Gamma(z)=e^{-t}\frac{t^z}{z}\Big|_0^\infty+\frac{1}{z}\int_0^\infty e^{-t}t^z dt=0+\frac{1}{z}\Gamma(z+1)$ and by iterating the $\Gamma(n+1)=n\Gamma(n)$, we get:

$$\Gamma(n+1)=n!\ . \tag{1.3}$$

2-GAUSS INTEGRAL

We have the following two integrals (identical):

$$I=\int_0^\infty e^{-\alpha x^2}dx \qquad I=\int_0^\infty e^{-\alpha y^2}dy \tag{2.1}$$

After multiplying them each other: $I^2=\int_0^\infty\int_0^\infty e^{-\alpha(x^2+y^2)}dxdy$; now, in polar coordinates: ($x^2+y^2=r^2$, $dS=dxdy=rdrd\theta$), we have:

$I^2=\int_0^{\pi/2}\int_0^\infty e^{-\alpha r^2}rdrd\theta=\frac{\pi}{2}\int_0^\infty e^{-\alpha r^2}rdr=-\frac{\pi}{4\alpha}\left|e^{-\alpha r^2}\right|_0^\infty=\frac{\pi}{4\alpha}$, so:

$$I=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\ . \tag{2.2}$$

3-PARTICULAR VALUES OF THE EULER GAMMA FUNCTION

The (1.3) with n=0 yields Γ(1)=0!=1, while with n=1, gives Γ(2)=1!=1.
Moreover, the (1.1) with z=1/2 becomes:

$\Gamma(\frac{1}{2})=\int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt$; now, by saying that t=w^2, we have: (→ dt/dw=2w)

$$\Gamma(\frac{1}{2})=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt=\int_0^\infty e^{-w^2}w^{-1}2wdw=2\int_0^\infty e^{-w^2}dw \tag{3.1}$$

And according to the (2.2) with α=1, we have: $\Gamma(\frac{1}{2})=2\int_0^\infty e^{-w^2}dw=\sqrt{\pi}$.

At last, according to the (1.2): Γ(3/2)=Γ(1/2 + 1)=1/2 Γ(1/2)=(1/2)!= $\frac{\sqrt{\pi}}{2}$

Results: Γ(1)=1, Γ(2)=1, $\Gamma(1/2)=\sqrt{\pi}$, $\Gamma(3/2)=\frac{\sqrt{\pi}}{2}$. (3.2)

4-HALF-DERIVATIVE OF x

$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x=\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}}$. Proof:

$\frac{d}{dx}x^k=kx^{k-1}$, $\frac{d^2}{dx^2}x^k=k(k-1)x^{k-2}$, $\frac{d^n}{dx^n}x^k=k(k-1)(k-n+1)x^{k-n}=\frac{k!}{(k-n)!}x^{k-n}$; now, according to the Euler Gamma Function, we have:

$$\frac{d^n}{dx^n}x^k=\frac{k!}{(k-n)!}x^{k-n}=\frac{\Gamma(k+1)}{\Gamma(k-n+1)}x^{k-n} \quad (4.1)$$

and with k=1 and n=1/2, we have: $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x=\frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}x^{1-\frac{1}{2}}=\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}}$ and according to the (3.2) we get:

$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x=\frac{1}{\frac{\sqrt{\pi}}{2}}x^{\frac{1}{2}}=\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}$, so:

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x=\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}} . \quad (4.2)$$

Check: as it must be: $(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}})x=\frac{d^{\frac{1}{2}+\frac{1}{2}}}{dx^{\frac{1}{2}+\frac{1}{2}}}x=\frac{d}{dx}x=1$, then:

$$(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}})x=\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x)=\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}})=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(x^{\frac{1}{2}})=\frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)}x^{\frac{1}{2}-\frac{1}{2}}=\frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}x^0=1$$

qed.

5- i-th DERIVATIVE OF x

According to the (4.1) with k=1 and n=i, we have: $\frac{d^i}{dx^i}x=\frac{1!}{(1-i)!}x^{k-n}=\frac{\Gamma(2)}{\Gamma(2-i)}x^{1-i}=\frac{1}{\Gamma(2-i)}x^{1-i}$

Now, considering that: $\frac{1}{\Gamma(2-i)}=\frac{1}{\Gamma[1+(1-i)]}=\frac{1}{(1-i)\Gamma(1-i)}=\frac{1}{(1-i)}\frac{(1+i)}{(1+i)}\frac{1}{\Gamma(1-i)}=\frac{(1+i)}{2\Gamma(1-i)}$,

we have: $$\frac{d^i}{dx^i}x=\frac{(1+i)}{2\Gamma(1-i)}x^{1-i}, \quad (5.1)$$

but according to the Reflection Formula (see the (6.1)), we get: $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$, from which:

$$\Gamma(i)\Gamma(1-i)=\frac{\pi}{\sin i\pi}=\frac{\pi}{i\sinh\pi}, \tag{5.2}$$

but as (see the (1.2) and the (7.1)): $\Gamma(i+1)=i\Gamma(i)=i!$, from which: $\Gamma(i)=\dfrac{i!}{i}$, then the (5.2) becomes:

$\Gamma(i)\Gamma(1-i)=\dfrac{i!}{i}\Gamma(1-i)=\dfrac{\pi}{i\sinh\pi}$, from which: $\Gamma(1-i)=\dfrac{\pi}{i!\sinh\pi}$, and so the (5.1) becomes:

$$\frac{d^i}{dx^i}x=\frac{i!(1+i)\sinh\pi}{2\pi}x^{1-i} \tag{5.3}$$

and as according to the (7.1), $i!=(0.4980-i0.1549)$ (and moreover $\sinh\pi=11.53$), then, from the (5.3):

$$\frac{d^i}{dx^i}x=\frac{11.53(0.4980-i0.1549)(1+i)}{2\pi}x^{1-i}=(1.1983+i0.6297)e^{(1-i)\ln x}=(1.1983+i0.6297)e^{\ln x}e^{-i\ln x}=$$
$$=x(1.1983+i0.6297)[\cos(\ln x)-i\sin(\ln x)]=x[1.1983\cos(\ln x)-i1.1983\sin(\ln x)+$$
$$+i0.6297\cos(\ln x)+0.6297\sin(\ln x)]=x\{[1.1983\cos(\ln x)+0.6297\sin(\ln x)]+$$
$$+i[0.6297\cos(\ln x)-1.1983\sin(\ln x)]\}$$

, that is:

$$\boxed{\frac{d^i}{dx^i}x=x\{[1.1983\cos(\ln x)+0.6297\sin(\ln x)]+i[0.6297\cos(\ln x)-1.1983\sin(\ln x)]\}} \tag{5.4}$$

or again: $\dfrac{d^i}{dx^i}x=a(x)+ib(x)$, where:

$$a(x)=x[1.1983\cos(\ln x)+0.6297\sin(\ln x)]$$
$$b(x)=x[0.6297\cos(\ln x)-1.1983\sin(\ln x)]$$

At last, we evaluate $\dfrac{d^i}{dx^i}x^i$. From the (4.1) with n=k=i, we have:

$\dfrac{d^i}{dx^i}x^i=\dfrac{i!}{(i-i)!}x^{i-i}=i!$, or: $\boxed{\dfrac{d^i}{dx^i}x^i=i!}$.

6-REFLECTION FORMULA

Here it is:

$$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}\ . \tag{6.1}$$

In fact, we know that $\dfrac{\sin x}{x}$ is zero in $-\pi,+\pi,-2\pi,+2\pi,\ldots,-n\pi,+n\pi$, so:

$\dfrac{\sin x}{x}=(1-\dfrac{x}{\pi})(1+\dfrac{x}{\pi})(1-\dfrac{x}{2\pi})(1+\dfrac{x}{2\pi})\ldots\ldots\ldots=(1-\dfrac{x^2}{\pi^2})(1-\dfrac{x^2}{4\pi^2})\ldots\ldots=\prod_{k=1}^{\infty}[1-(\dfrac{x}{k\pi})^2]$, and so:

$$\frac{\pi x}{\sin\pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} \tag{6.2}$$

Moreover, we can see a factorial like this:

$n! = \lim_{k\to\infty} \frac{k!k^n}{(n+1)...(n+k)} \cdot \lim_{k\to\infty} \frac{(k+1)(k+2)...(k+n)}{k^n}$; in fact, the denominator of the first limit is $(n+1)...(n+k) = \frac{(n+k)!}{n!}$, while from both numerators we can collect: $k!(k+1)...(k+n) = (k+n)!$, so just n! is left. Furthermore, we see that the second limit is 1: $\lim_{k\to\infty} \frac{(k+1)(k+2)...(k+n)}{k^n} = 1$, as it gets of the form $\lim_{k\to\infty} \frac{k^n}{k^n}$. Therefore: $n! = \lim_{k\to\infty} \frac{k!k^n}{(n+1)...(n+k)}$. As $\Gamma(x+1) = x!$, we have: $\Gamma(x+1) = x! = \lim_{k\to\infty} \frac{k!k^x}{(x+1)...(x+k)}$ and after dividing both numerator and denominator by k!:

$$\Gamma(x+1) = \lim_{k\to\infty} \frac{k^x}{(1+x)(1+x/2)...(1+x/k)} \quad . \tag{6.3}$$

Now, let's introduce the constant $\gamma_k = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k} - \ln k$, and let's say: $\gamma = \lim_{k\to\infty} \gamma_k$; on the basis of that, the (6.3) becomes: $\Gamma(x+1) = e^{-\gamma x} \lim_{k\to\infty} \frac{e^x}{(1+x)} \frac{e^{x/2}}{(1+x/2)} .... \frac{e^{x/k}}{(1+x/k)}$. (6.4)

In fact, all the e^x, $e^{x/2}$,…, $e^{x/k}$ cancel with terms in $e^{-\gamma x}$ and the term $k^x = e^{(\ln k)x}$ still comes from $e^{-\gamma x}$. Moreover, as $\Gamma(x+1) = x\Gamma(x)$, then the (6.4) becomes:

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \lim_{k\to\infty} \frac{(1+x)}{e^x} \frac{(1+x/2)}{e^{x/2}} .... \frac{(1+x/k)}{e^{x/k}} = xe^{\gamma x} \prod_{k=1}^{\infty} [1 + (\frac{x}{k})] e^{-x/k} \tag{6.5}$$

Now, as $\Gamma(y+1) = y\Gamma(y)$, then $\Gamma(y) = \frac{1}{y}\Gamma(y+1)$ and after replacing y by –x:

$\Gamma(-x) = -\frac{1}{x}\Gamma(1-x)$ and so: $\Gamma(1-x) = -x\Gamma(-x)$. Let's figure out $\Gamma(x)\Gamma(1-x)$:

$$\Gamma(x)\Gamma(1-x) = [x^{-1}e^{-\gamma x} \prod_{k=1}^{\infty} (1+\frac{x}{k})^{-1} e^{x/k}] \cdot [e^{\gamma x} \prod_{k=1}^{\infty} (1-\frac{x}{k})^{-1} e^{-x/k}] = [A] \cdot [B] = \frac{1}{x} \prod_{k=1}^{\infty} [1 - (\frac{x}{k})^2]^{-1} , \tag{6.6}$$

where A is an inverted (6.5) and B is the (6.4) with x changed into –x. So, (6.6) is:

$\Gamma(x)\Gamma(1-x) = \frac{1}{x} \prod_{k=1}^{\infty} [1 - (\frac{x}{k})^2]^{-1}$, and for the (6.2) (($\frac{\pi x}{\sin \pi x} = \prod_{k=1}^{\infty} [1 - (\frac{x}{k})^2]^{-1}$)) we have:

$\frac{2}{\Gamma(n)\Gamma(1-n)} = \frac{2 \sin n\pi}{\pi}$ which is the (6.1) indeed.

7- i! i FACTORIAL

Let's evaluate the modulus of "i factorial", first. ($i = \sqrt{-1}$). About the definition of factorial, according to the Euler Gamma Function, we have:

$$i! = \Gamma(i+1) = \int_0^{\infty} e^{-t} t^{i+1-1} dt = \int_0^{\infty} e^{-t} t^i dt = \int_0^{\infty} e^{-t} e^{i \ln t} dt = \int_0^{\infty} e^{-t} [\cos(\ln t) + i \sin(\ln t)] dt =$$

$$= \int_0^{\infty} e^{-t} \cos(\ln t) dt + i \int_0^{\infty} e^{-t} \sin(\ln t) dt = \boxed{i! \cong (0.4980 - i0.1549)} , \tag{7.1}$$

so the factorial of i is a complex number a+ib and later we will check the official a and b values, before reported, through rough approximated methods.

According to the (1.2), we have: $\Gamma(i+1) = i\Gamma(i)$, from which: $|\Gamma(i+1)| = |\Gamma(i)| = |i!|$. Moeover, the product of a complex number z with its complex conjugate z* gives the square modulus: $zz^* = |z|^2$. As the complex conjugate is obtained through the replacement of i by -i, we also have, according to the definition of Γ: $[\Gamma(z)]^* = \Gamma(z^*)$. From the Reflection Formula (6.1), that is: $\Gamma(1-z)\Gamma(z) = \dfrac{\pi}{\sin z\pi}$, we have, with z=1+i:

$$\Gamma[1-(1+i)]\cdot\Gamma(1+i) = \frac{\pi}{\sin(1+i)\pi} = \Gamma(-i)\cdot\Gamma(1+i) = \Gamma(i^*)\cdot\Gamma(1+i) = \Gamma^*(i)\cdot\Gamma(1+i) = \Gamma^*(i)\cdot i\Gamma(i) =$$

$= i|\Gamma(i)|^2 = i|i!|^2$, from which:

$$|i!| = \sqrt{\frac{\pi}{i\sin(1+i)\pi}} = \sqrt{\frac{2i\pi}{i[e^{i\pi(1+i)} - e^{-i\pi(1+i)}]}} = \sqrt{\frac{2\pi}{[e^{i\pi}e^{-\pi} - e^{-i\pi}e^{\pi}]}} = \sqrt{\frac{2\pi}{[e^{\pi} - e^{-\pi}]}} = \sqrt{\frac{\pi}{\sinh(\pi)}} = |i!| \qquad (7.2)$$

after having used the following equalities: $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, $\sinh(z) = \dfrac{e^{z} - e^{-z}}{2}$ and $e^{i\pi} = e^{-i\pi} = -1$.

So, according to the (7.2): $|i!| = \sqrt{\dfrac{\pi}{\sinh(\pi)}} = \sqrt{\dfrac{\pi}{\frac{(e^{\pi} - e^{-\pi})}{2}}} = \sqrt{\dfrac{3.14}{\frac{(23.1 - 0.043)}{2}}} = |i!| = 0.52$. (7.3)

On a vectorial basis, due to the (7.1), we have: $i! = \int_0^\infty e^{-t}\cos(\ln t)dt + i\int_0^\infty e^{-t}\sin(\ln t)dt$, so, in order to evaluate those two definite integrals, which, as we know, represent the surfaces under the integrand functions, we get the vectorial expression $i! = a + ib$. As those integrals are somewhat difficult, we will carry out a very rough geometric and numerical evaluation! The former integrand function, that is $e^{-t}\cos(\ln t)$, is shown in Fig. 7.1, while in Fig. 7.2 we roughly evaluate the surface under it, through triangles.

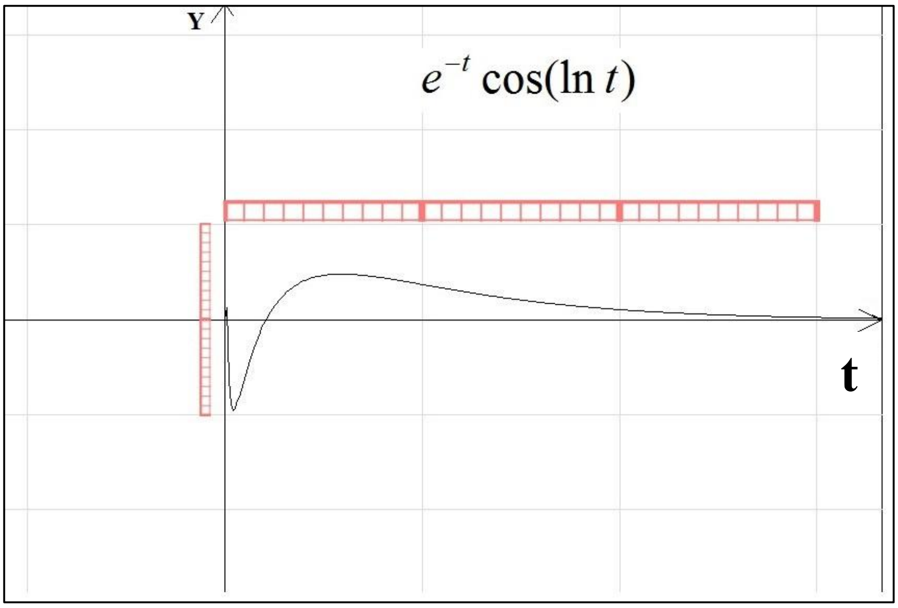

Fig. 7.1

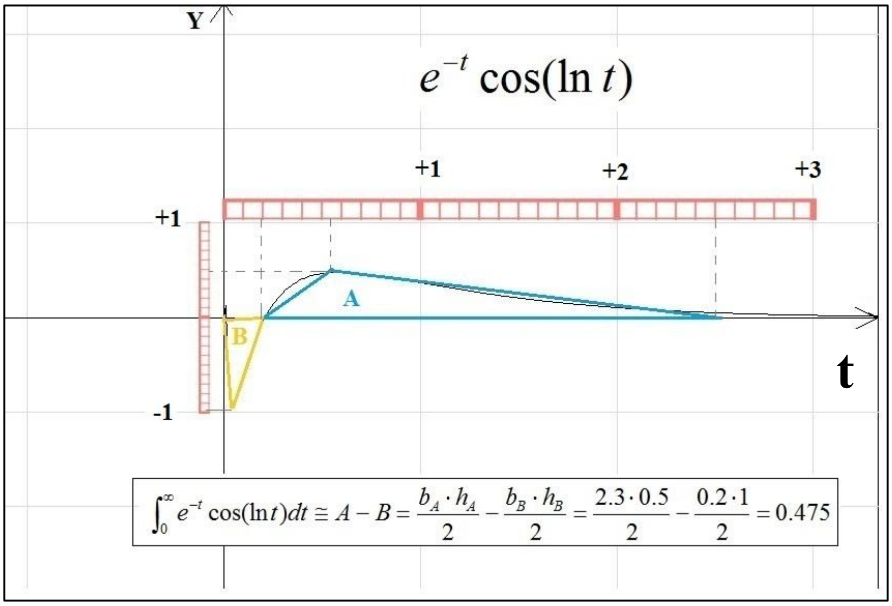

Fig. 7.2

The same goes for the latter integrand function $e^{-t}\sin(\ln t)$, shown in Fig. 7.3, while in Fig. 7.4 we roughly evaluate, as well as before, the surface under it, still through triangles.

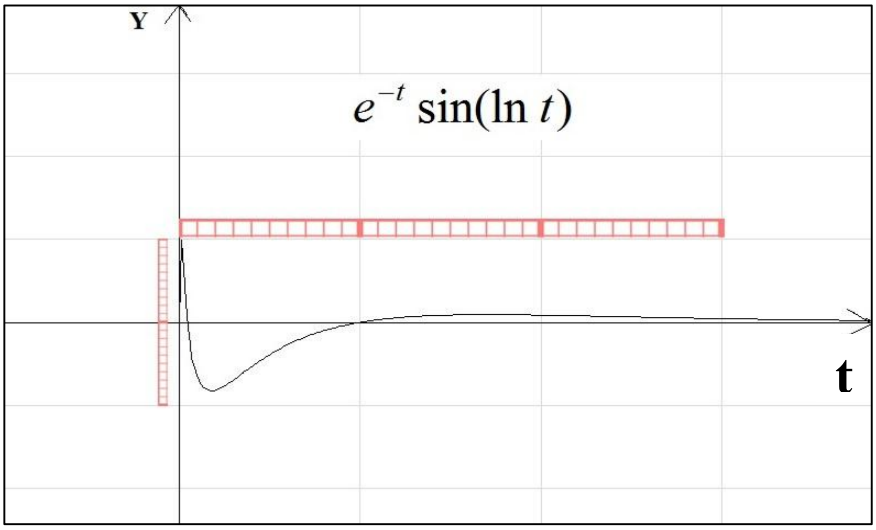

Fig. 7.3

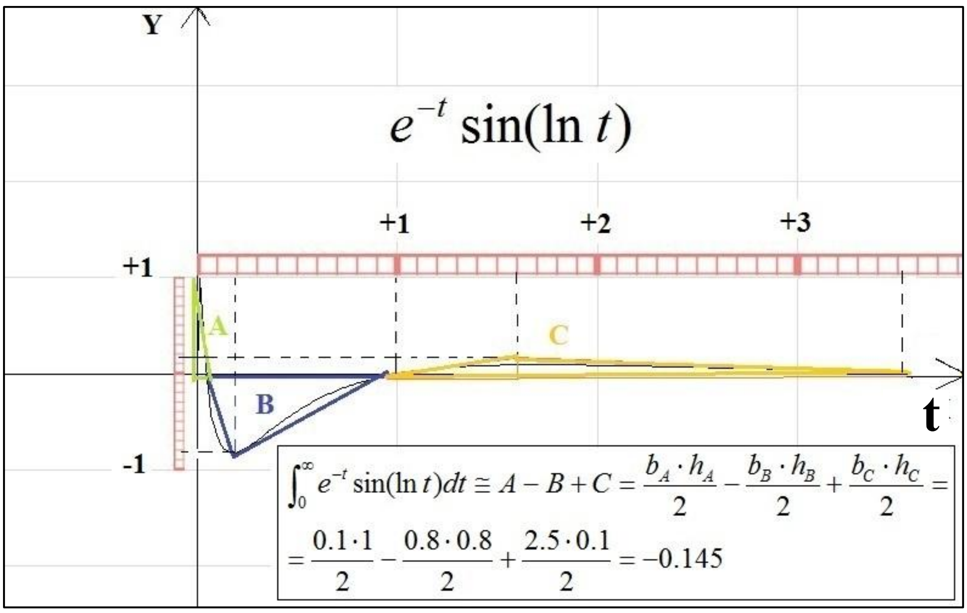

Fig. 7.4

We roughly get: $i! \cong (0.475 - i0.145)$, with $|i!| = 0.50$.

More officially, others calculated a more accurate number, so getting $\boxed{i! \cong (0.4980 - i0.1549)}$, with $|i!| = 0.52$; in both cases, the modulus is in agreement with that previously given by the (7.3), through a different way.

DERIVATA i-ESIMA E SEMIDERIVATA DI x

Leonardo ***Rubino***

Giugno 2023

Abstract: ecco una dimostrazione del valore della derivata i-esima di x, nonché della semiderivata di x.

1-FUNZIONE GAMMA DI EULERO

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{1.1}$$

Naturalmente, $\Gamma(1) = 1$. Poi,

$$\Gamma(n+1) = n\Gamma(n); \tag{1.2}$$

infatti, dopo una integrazione per parti:

$\Gamma(z) = e^{-t} \left.\frac{t^z}{z}\right|_0^\infty + \frac{1}{z}\int_0^\infty e^{-t} t^z dt = 0 + \frac{1}{z}\Gamma(z+1)$ ed iterando la $\Gamma(n+1) = n\Gamma(n)$, otteniamo:

$$\Gamma(n+1) = n! \quad . \tag{1.3}$$

2-INTEGRALE DI GAUSS

Consideriamo i due integrali (uguali tra loro):

$$I = \int_0^\infty e^{-\alpha x^2} dx \qquad I = \int_0^\infty e^{-\alpha y^2} dy \tag{2.1}$$

Moltiplicandoli tra loro: $I^2 = \int_0^\infty \int_0^\infty e^{-\alpha(x^2+y^2)} dxdy$ e, in coordinate polari:

($x^2 + y^2 = r^2$, $dS = dxdy = rdrd\theta$), si ha:

$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-\alpha r^2} rdrd\theta = \frac{\pi}{2}\int_0^\infty e^{-\alpha r^2} rdr = -\frac{\pi}{4\alpha}\left|e^{-\alpha r^2}\right|_0^\infty = \frac{\pi}{4\alpha}$, perciò:

$$I = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad . \tag{2.2}$$

3-VALORI PARTICOLARI DELLA FUNZIONE GAMMA DI EULERO

La (1.3) con n=0 fornisce Γ(1)=0!=1, mentre con n=1, fornisce Γ(2)=1!=1.
Inoltre, la (1.1) con z=1/2 diventa:

$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$; adesso, ponendo t=w^2, si ha: (→ dt/dw=2w)

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = \int_0^\infty e^{-w^2} w^{-1} 2wdw = 2\int_0^\infty e^{-w^2} dw \tag{3.1}$$

e per la (2.2) con α=1, si ha: $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-w^2} dw = \sqrt{\pi}$.

Infine, per la (1.2): Γ(3/2)=Γ(1/2 + 1)=1/2 Γ(1/2)=(1/2)!= $\frac{\sqrt{\pi}}{2}$

Per riassumere: $\Gamma(1)=1$, $\Gamma(2)=1$, $\Gamma(1/2)=\sqrt{\pi}$, $\Gamma(3/2)=\frac{\sqrt{\pi}}{2}$. (3.2)

4-SEMIDERIVATA DI x

$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x=\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}}$. Dimostrazione:

$\frac{d}{dx}x^k=kx^{k-1}$, $\frac{d^2}{dx^2}x^k=k(k-1)x^{k-2}$, $\frac{d^n}{dx^n}x^k=k(k-1)(k-n+1)x^{k-n}=\frac{k!}{(k-n)!}x^{k-n}$; ora, per la Funzione Gamma di Eulero, si ha che:

$$\frac{d^n}{dx^n}x^k=\frac{k!}{(k-n)!}x^{k-n}=\frac{\Gamma(k+1)}{\Gamma(k-n+1)}x^{k-n} \tag{4.1}$$

e per k=1 ed n=1/2, si ha: $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x=\frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}x^{1-\frac{1}{2}}=\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}}$ e per le (3.2) si ha:

$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x=\frac{1}{\frac{\sqrt{\pi}}{2}}x^{\frac{1}{2}}=\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}$, dunque:

$$\boxed{\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x=\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}} . \tag{4.2}$$

Controprova: visto che deve essere: $(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}})x=\frac{d^{\frac{1}{2}+\frac{1}{2}}}{dx^{\frac{1}{2}+\frac{1}{2}}}x=\frac{d}{dx}x=1$, allora:

$$(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}})x=\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x)=\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}})=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(x^{\frac{1}{2}})=\frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)}x^{\frac{1}{2}-\frac{1}{2}}=\frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}x^0=1$$

cvd.

5-DERIVATA i-ESIMA DI x

Per la (4.1) con k=1 ed n=i, si ha: $\frac{d^i}{dx^i}x=\frac{1!}{(1-i)!}x^{k-n}=\frac{\Gamma(2)}{\Gamma(2-i)}x^{1-i}=\frac{1}{\Gamma(2-i)}x^{1-i}$. Ora, considerando che: $\frac{1}{\Gamma(2-i)}=\frac{1}{\Gamma[1+(1-i)]}=\frac{1}{(1-i)\Gamma(1-i)}=\frac{1}{(1-i)}\frac{(1+i)}{(1+i)}\frac{1}{\Gamma(1-i)}=\frac{(1+i)}{2\Gamma(1-i)}$, si ha che:

$$\frac{d^i}{dx^i}x=\frac{(1+i)}{2\Gamma(1-i)}x^{1-i}, \tag{5.1}$$

ma per la Formula di Riflessione (vedere la (6.1)), si ha che: $\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}$, da cui:

$$\Gamma(i)\Gamma(1-i)=\frac{\pi}{\sin i\pi}=\frac{\pi}{i\sinh\pi}, \tag{5.2}$$

ma visto che (vedi le (1.2) e (7.1)): $\Gamma(i+1)=i\Gamma(i)=i!$, da cui: $\Gamma(i)=\frac{i!}{i}$, allora la (5.2) diviene:

$\Gamma(i)\Gamma(1-i)=\frac{i!}{i}\Gamma(1-i)=\frac{\pi}{i\sinh\pi}$, da cui: $\Gamma(1-i)=\frac{\pi}{i!\sinh\pi}$, e allora la (5.1) diventa:

$$\frac{d^i}{dx^i}x=\frac{i!(1+i)\sinh\pi}{2\pi}x^{1-i} \tag{5.3}$$

ed essendo che, per la (7.1), $i!=(0.4980-i0.1549)$ (ed inoltre $\sinh\pi=11.53$), allora, per la (5.3):

$$\frac{d^i}{dx^i}x=\frac{11.53(0.4980-i0.1549)(1+i)}{2\pi}x^{1-i}=(1.1983+i0.6297)e^{(1-i)\ln x}=(1.1983+i0.6297)e^{\ln x}e^{-i\ln x}=$$
$$=x(1.1983+i0.6297)[\cos(\ln x)-i\sin(\ln x)]=x[1.1983\cos(\ln x)-i1.1983\sin(\ln x)+$$
$$+i0.6297\cos(\ln x)+0.6297\sin(\ln x)]=x\{[1.1983\cos(\ln x)+0.6297\sin(\ln x)]+$$
$$+i[0.6297\cos(\ln x)-1.1983\sin(\ln x)]\}$$, ossia:

$$\boxed{\frac{d^i}{dx^i}x=x\{[1.1983\cos(\ln x)+0.6297\sin(\ln x)]+i[0.6297\cos(\ln x)-1.1983\sin(\ln x)]\}} \tag{5.4}$$

ossia ancora: $\frac{d^i}{dx^i}x=a(x)+ib(x)$, con:

$a(x)=x[1.1983\cos(\ln x)+0.6297\sin(\ln x)]$

$b(x)=x[0.6297\cos(\ln x)-1.1983\sin(\ln x)]$

Per ultimo, valutiamo $\frac{d^i}{dx^i}x^i$. Dalla (4.1) con n=k=i, si ha:

$\frac{d^i}{dx^i}x^i=\frac{i!}{(i-i)!}x^{i-i}=i!$, ossia: $\boxed{\frac{d^i}{dx^i}x^i=i!}$.

6-FORMULA DI RIFLESSIONE

Eccola: $$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi} \tag{6.1}$$.

Infatti, sappiamo che $\frac{\sin x}{x}$ vale zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, sicchè:

$\frac{\sin x}{x}=(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi}).........=(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2}).....=\prod_{k=1}^{\infty}[1-(\frac{x}{k\pi})^2]$, e dunque:

$$\frac{\pi x}{\sin\pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} \tag{6.2}$$

Inoltre, un fattoriale possiamo anche vederlo così:

$n!=\lim_{k\to\infty}\frac{k!k^n}{(n+1)...(n+k)}\cdot\lim_{k\to\infty}\frac{(k+1)(k+2)...(k+n)}{k^n}$; infatti, il denominatore del primo limite è:

$(n+1)...(n+k)=\frac{(n+k)!}{n!}$, mentre da entrambi i numeratori possiamo raccogliere:

$k!(k+1)...(k+n)=(k+n)!$, così resta solo n! . Poi, vediamo che il secondo limite vale 1:

$\lim_{k\to\infty}\frac{(k+1)(k+2)...(k+n)}{k^n}=1$, poichè si riduce alla forma $\lim_{k\to\infty}\frac{k^n}{k^n}$. Perciò: $n!=\lim_{k\to\infty}\frac{k!k^n}{(n+1)...(n+k)}$.

Siccome $\Gamma(x+1)=x!$, si ha: $\Gamma(x+1)=x!=\lim_{k\to\infty}\frac{k!k^x}{(x+1)...(x+k)}$ e dopo aver diviso sia numeratore che denominatore per k!:

$$\Gamma(x+1)=\lim_{k\to\infty}\frac{k^x}{(1+x)(1+x/2)...(1+x/k)} \quad . \tag{6.3}$$

Ora, introduciamo la costante $\gamma_k=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}-\ln k$, e poniamo: $\gamma=\lim_{k\to\infty}\gamma_k$; sulla base di ciò, la (6.3) diventa: $\Gamma(x+1)=e^{-\gamma x}\lim_{k\to\infty}\frac{e^x}{(1+x)}\frac{e^{x/2}}{(1+x/2)}....\frac{e^{x/k}}{(1+x/k)}$. (6.4)

Infatti, tutti gli e^x, $e^{x/2}$,..., $e^{x/k}$ si elidono con i termini in $e^{-\gamma x}$ ed il termine $k^x=e^{(\ln k)x}$ pure proviene da $e^{-\gamma x}$. Inoltre, siccome $\Gamma(x+1)=x\Gamma(x)$, la (6.4) diventa:

$$\frac{1}{\Gamma(x)}=xe^{\gamma x}\lim_{k\to\infty}\frac{(1+x)}{e^x}\frac{(1+x/2)}{e^{x/2}}....\frac{(1+x/k)}{e^{x/k}}=xe^{\gamma x}\prod_{k=1}^{\infty}[1+(\frac{x}{k})]e^{-x/k} \tag{6.5}$$

Ora, poichè $\Gamma(y+1)=y\Gamma(y)$, segue che $\Gamma(y)=\frac{1}{y}\Gamma(y+1)$ e dopo aver sostituito y con –x:

$\Gamma(-x)=-\frac{1}{x}\Gamma(1-x)$ e dunque: $\Gamma(1-x)=-x\Gamma(-x)$. Valutiamo $\Gamma(x)\Gamma(1-x)$:

$$\Gamma(x)\Gamma(1-x)=[x^{-1}e^{-\gamma x}\prod_{k=1}^{\infty}(1+\frac{x}{k})^{-1}e^{x/k}]\cdot[e^{\gamma x}\prod_{k=1}^{\infty}(1-\frac{x}{k})^{-1}e^{-x/k}]=[A]\cdot[B]=\frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}, \tag{6.6}$$

dove A è la (6.5) invertita e B è la (6.4) con x scambiato con –x. Perciò, la (6.6) è:

$\Gamma(x)\Gamma(1-x)=\frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$, e per la (6.2) (($\frac{\pi x}{\sin\pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$)) abbiamo:

$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}$ che è proprio la (6.1).

7- i! i FATTORIALE

Valutiamo dapprima il modulo di “i fattoriale”. ($i=\sqrt{-1}$) A livello di definizione di fattoriale, tramite la funzione Gamma di Eulero, si ha:

$$i!=\Gamma(i+1)=\int_0^\infty e^{-t}t^{i+1-1}dt=\int_0^\infty e^{-t}t^i dt=\int_0^\infty e^{-t}e^{i\ln t}dt=\int_0^\infty e^{-t}[\cos(\ln t)+i\sin(\ln t)]dt=$$

$$=\int_0^\infty e^{-t}\cos(\ln t)dt+i\int_0^\infty e^{-t}\sin(\ln t)dt=\boxed{i!\cong(0.4980-i0.1549)}, \tag{7.1}$$

dunque, il fattoriale di i è un numero complesso a+ib e successivamente verificheremo gli a e b più accreditati, sopra riportati, con metodi approssimativi.

Per la (1.2), si ha: $\Gamma(i+1)=i\Gamma(i)$, da cui: $|\Gamma(i+1)|=|\Gamma(i)|=|i!|$. Ricordiamo poi che il prodotto di un numero complesso z con il suo complesso coniugato z* dà il modulo quadro: $zz^*=|z|^2$. Dato che il complesso coniugato si ottiene sostituendo i con -i, si ha anche, dalla definizione di Γ: $[\Gamma(z)]^*=\Gamma(z^*)$. Dalla Formula di Riflessione (6.1), ossia: $\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin z\pi}$, si ha, con z=1+i:

$$\Gamma[1-(1+i)]\cdot\Gamma(1+i)=\frac{\pi}{\sin(1+i)\pi}=\Gamma(-i)\cdot\Gamma(1+i)=\Gamma(i^*)\cdot\Gamma(1+i)=\Gamma^*(i)\cdot\Gamma(1+i)=\Gamma^*(i)\cdot i\Gamma(i)=$$

$=i|\Gamma(i)|^2=i|i!|^2$, da cui:

$$|i!|=\sqrt{\frac{\pi}{i\sin(1+i)\pi}}=\sqrt{\frac{2i\pi}{i[e^{i\pi(1+i)}-e^{-i\pi(1+i)}]}}=\sqrt{\frac{2\pi}{[e^{i\pi}e^{-\pi}-e^{-i\pi}e^{\pi}]}}=\sqrt{\frac{2\pi}{[e^{\pi}-e^{-\pi}]}}=\sqrt{\frac{\pi}{\sinh(\pi)}}=|i!| \qquad (7.2)$$

dopo aver ricordato che: $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$, $\sinh(z)=\dfrac{e^{z}-e^{-z}}{2}$ e $e^{i\pi}=e^{-i\pi}=-1$.

Dunque, per la (7.2): $\boxed{|i!|=\sqrt{\dfrac{\pi}{\sinh(\pi)}}}=\sqrt{\dfrac{\pi}{\frac{(e^{\pi}-e^{-\pi})}{2}}}=\sqrt{\dfrac{3.14}{\frac{(23.1-0.043)}{2}}}=\boxed{|i!|=0.52}$. (7.3)

A livello vettoriale, per la (7.1), si ha: $i!=\int_0^{\infty}e^{-t}\cos(\ln t)dt+i\int_0^{\infty}e^{-t}\sin(\ln t)dt$, dunque, valutando questi due integrali definiti che, come noto, denotano ognuno l'area sottesa dalla curva integranda, si ottiene l'espressione vettoriale $i!=a+ib$. Essendo questi degli integrali piuttosto complicati, effettuiamo una valutazione numerica e geometrica molto molto approssimativa! La prima funzione integranda, ossia $e^{-t}\cos(\ln t)$, è rappresentata in Fig. 7.1, mentre in Fig. 7.2 si valuta grossolanamente l'area da essa sottesa, tramite dei triangoli.

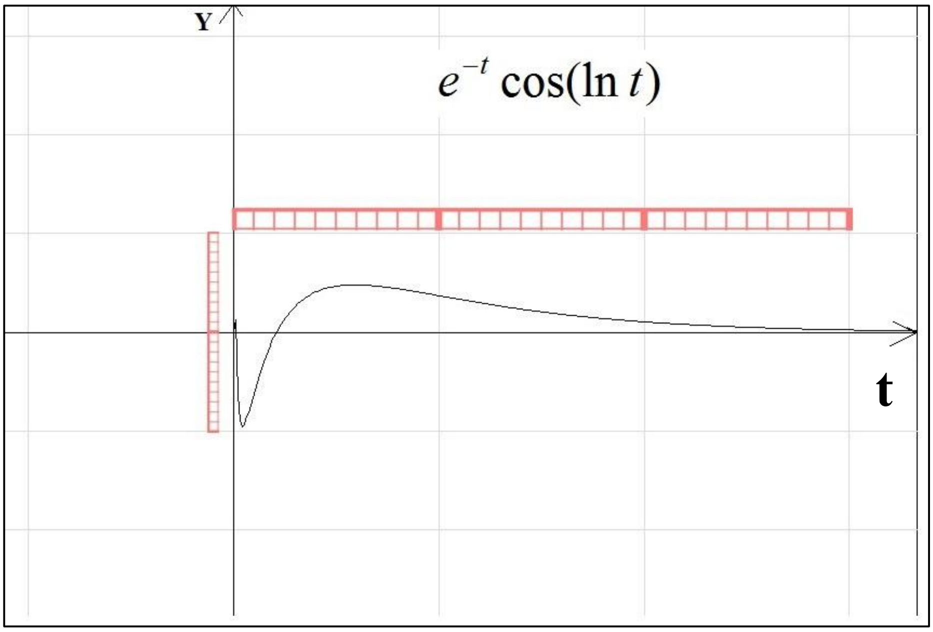

Fig. 7.1

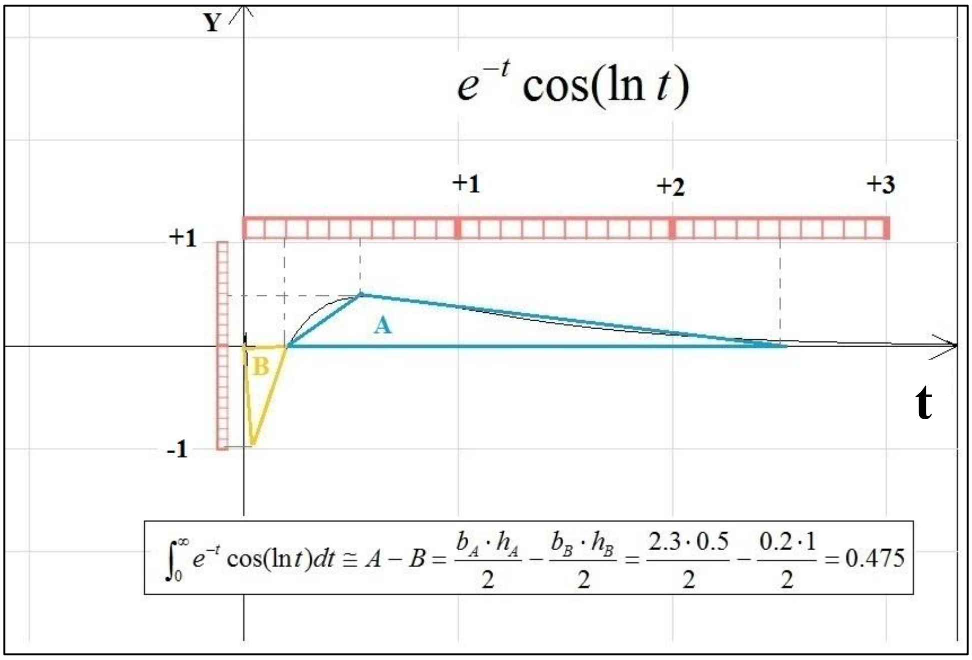

Fig. 7.2

Lo stesso facciamo per la seconda funzione integranda $e^{-t}\sin(\ln t)$, rappresentata in Fig. 7.3, mentre in Fig. 7.4 si valuta grossolanamente, come in precedenza, l'area da essa sottesa, sempre tramite dei triangoli.

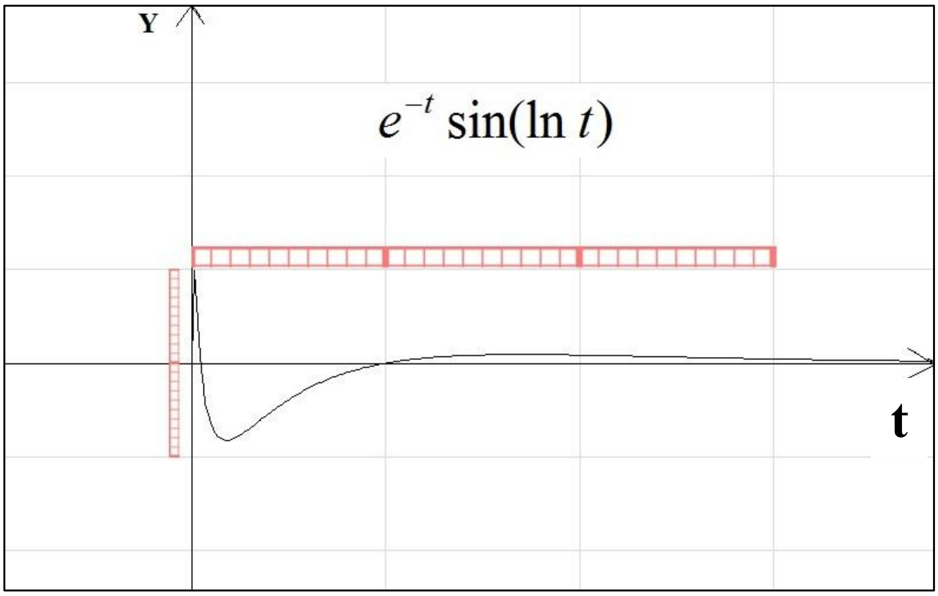

Fig. 7.3

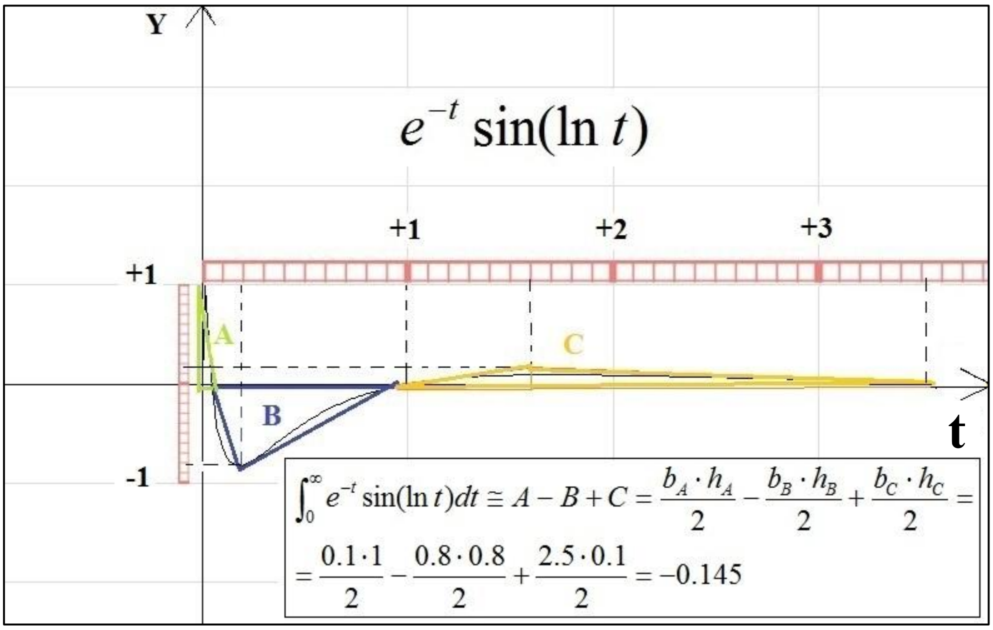

Fig. 7.4

Grossolanamente otteniamo: $i! \cong (0.475 - i0.145)$, con $|i!| = 0.50$.

Più ufficialmente è stato calcolato da altri il numero con un po' di precisione in più, ottenendo $i! \cong (0.4980 - i0.1549)$, con $|i!| = 0.52$; in entrambi i casi, viene confermato il modulo, fornito, per altra via, dalla (7.3).