The Perfect Cuboid Is Nothing More Than a Myth

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Abstract

An impeccable proof of the impossibility of the existence of a perfect cuboid based on the parametrization of Leonhard Euler.

In mathematics, the *Perfect Cuboid* is a rectangular cuboid whose edges, face diagonals and space diagonal all have integer lengths: https://en.wikipedia.org/wiki/Euler_brick

A Perfect Cuboid must satisfy the following system of diophantine equations:

$$\begin{cases} a^2+b^2=d^2 \\ a^2+c^2=e^2 \\ b^2+c^2=f^2 \\ a^2+b^2+c^2=g^2 \end{cases} \qquad \text{(I)}$$

where: a,b,c are the edges, d,e,f are the face diagonals and g is the space diagonal.

Until now, there was no confirmation of the existence of a Perfect Cuboid, but it had not been proven either that such a cuboid cannot exist, so this has remained a problem for several centuries. However, I will show the reason why the Perfect Cuboid is impossible.

Lemma.

If Perfect cuboid exists, the squares of 3 its face diagonals should construct a Heronian triangle.

Proof.

From simple transformations of (I) we have:

$$\begin{cases} g^2=\dfrac{d^2+e^2+f^2}{2} \\ a^2=\dfrac{d^2+e^2+f^2}{2}-f^2 \\ b^2=\dfrac{d^2+e^2+f^2}{2}-e^2 \\ c^2=\dfrac{d^2+e^2+f^2}{2}-d^2 \end{cases} \qquad \text{(II)}$$

By substitution from (IV) и (VI) it is follows:

$$a^2b^2c^2g^2=\left(\frac{d^2+e^2+f^2}{2}-f^2\right)\left(\frac{d^2+e^2+f^2}{2}-e^2\right)\left(\frac{d^2+e^2+f^2}{2}-d^2\right)\left(\frac{d^2+e^2+f^2}{2}\right) \qquad \text{(III)}$$

$$abcg=\frac{1}{4}\sqrt{\left(-d^2+e^2+f^2\right)\left(d^2-e^2+f^2\right)\left(d^2+e^2-f^2\right)\left(d^2+e^2+f^2\right)} \qquad \text{(IV)}$$

Since $abcg\in\mathbb{N}$, then the squares of the face diagonals d^2,e^2,f^2 are the edges of a Heronian triangle with area $abcg$ (https://en.wikipedia.org/wiki/Heronian_triangle). What was required.

Let's parametrize the indicated Heronian triangle by means of the general parametric solution of Leonhard Euler (https://en.wikipedia.org/wiki/Heronian_triangle#Euler's_parametric_equation):

$$\begin{cases} d^2 = mn(p^2+q^2) \\ e^2 = pq(m^2+n^2) \\ f^2 = (mq+np)(mp-nq) \end{cases} \qquad \text{(V)}$$

$$m, n, p, q \in \mathbb{N}, \quad mp > nq$$

Next, we can to express the remaining parameters of the Perfect Cuboid:

$$\begin{cases} a^2 = nq(mq+np) \\ b^2 = np(mp-nq) \\ c^2 = mq(mp-nq) \\ g^2 = mp(mq+np) \end{cases} \qquad \text{(VI)}$$

From (V) и (VI) it is follows:

$$\begin{aligned} & b^2 f^2 g^2 = mnp^2(mp-nq)^2(mq+np)^2 \Rightarrow mn = \square \\ & c^2 f^2 g^2 = pqm^2(mp-nq)^2(mq+np)^2 \Rightarrow pq = \square \quad \Rightarrow m^2+n^2 = \square \end{aligned} \qquad \text{(VII)}$$

Lets: $m = ut^2, \quad n = us^2$, then:

$$mn = u^2 s^2 t^2 = \square$$

$m^2 + n^2 = u^2(s^4 + t^4) \neq \square$, which contradicts (VII).

Conclusion: the assumption of the existence of a Perfect Cuboid is unrealizable.

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