Series with generalized complex harmonic numbers of order 1

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24 Jun 2023

abstract

In this note we give some complex series related to Pi

keywords: complex harmonic numbers, infinite series, number Pi

1. Introduction

Recall that

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1} = 3.1415926535\,... \tag{1}$$

The number π has been in the centre of view of mathematicians for centuries:

(a) Lambert (1761) proved the irrationality of π ,

(b) Legendre (1794) proved that π is not a square root of a rational number,

(c) Lindemann (1882) proved that π is a transcendental number.

The purpose of this note is to develop a class of series summing up to π. These series involve the generalized complex harmonic numbers of order 1.

Definition. For $z = u + i\,v \in \mathbb{C} - \{0\}$, $n \in \mathbb{N} = \{1,\,2,\,3,\,\ldots\}$, we define $H_n(z)$ by

$$H_n(z) = \sum_{k=1}^{n} \frac{1}{k\,z^k} \tag{2}$$

Definition. The Lerch's Transcendent is defined by

$$\Phi(z,\,s,\,a) = \sum_{n=1}^{\infty} \frac{z^n}{(a+n)^s}\ ,\ a \neq 0,\,-1,\,-2,\,\ldots,\,|z|<1;\ \mathbb{R}(s)>1,\,|z|=1 \tag{3}$$

Some properties

$$\Phi(z,\,s,\,a) = z^m\,\Phi(z,\,s,\,a+m) + \sum_{n=0}^{m-1} \frac{z^n}{(a+n)^s}\ ,\ m = 1,\,2,\,3,\,\ldots \tag{4}$$

$$H_n(z) = \sum_{k=1}^{n} \frac{1}{k\,z^k} = -\ln\left(1-\frac{1}{z}\right) - z^{-n-1}\,\Phi\left(\frac{1}{z},\,1,\,n+1\right) \tag{5}$$

Remark: $z = u + i\,v \in \mathbb{C} \Longrightarrow u = \mathrm{Re}(z)$, $v = \mathrm{Im}(z)$, $i = \sqrt{-1}$.

2. Series for π

In this section we give infinite series involving the numbers $H_n(z)$.

Entry 1. for $0 < \theta < \pi/2$, we have

$$\pi = -4\,\frac{\cos(\theta)+\sin(\theta)-1}{\cos(\theta)+\sin(\theta)} \sum_{n=1}^{\infty} \left(\frac{1}{\cos(\theta)+\sin(\theta)}\right)^n \mathrm{Im}(H_n(\cos(\theta)+i\,\sin(\theta))) \tag{6}$$

Entry 2. for $0 < \theta < \pi/2$, we have

$$\pi = -6\,\frac{\cos(\theta) + \sqrt{3}\,\sin(\theta) - 1}{\cos(\theta) + \sqrt{3}\,\sin(\theta)}\sum_{n=1}^{\infty}\left(\frac{1}{\cos(\theta) + \sqrt{3}\,\sin(\theta)}\right)^n \operatorname{Im}(H_n(\cos(\theta) + i\sin(\theta))) \tag{7}$$

Entry 3. for $0 < \theta < \pi/2$, we have

$$\pi = -8\,\frac{\cos(\theta) + \left(\sqrt{2} + 1\right)\sin(\theta) - 1}{\cos(\theta) + \left(\sqrt{2} + 1\right)\sin(\theta)}\sum_{n=1}^{\infty}\left(\frac{1}{\cos(\theta) + \left(\sqrt{2} + 1\right)\sin(\theta)}\right)^n \operatorname{Im}(H_n(\cos(\theta) + i\sin(\theta))) \tag{8}$$

Entry 4. for $\pi/3 < \theta \le \pi/2$, we have

$$\pi = 6\,\frac{1 + \sqrt{3}\,\sin(\theta) - \cos(\theta)}{\sqrt{3}\,\sin(\theta) - \cos(\theta)}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{3}\,\sin(\theta) - \cos(\theta)}\right)^n (-1)^n \operatorname{Im}(H_n(\cos(\theta) + i\sin(\theta))) \tag{9}$$

Entry 5. for $\frac{1}{\sqrt{2}} < u < \frac{\sqrt{3}}{\sqrt{3}-1}$, we have

$$\pi = -12\left(1 - \left(1 - \frac{1}{\sqrt{3}}\right)u\right)\sum_{n=1}^{\infty}\left(1 - \frac{1}{\sqrt{3}}\right)^n u^n \operatorname{Im}(H_n(u(1+i))) \tag{10}$$

Entry 6. for $0 < x < 1$, we have

$$\tan^{-1}(x) = \frac{1-x}{2\,i}\sum_{n=1}^{\infty} x^n (H_n(x-i) - H_n(x+i)) \tag{11}$$

$$\pi = \frac{3}{i}\left(1 - \frac{1}{\sqrt{3}}\right)\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^n \left(H_n\!\left(\frac{1}{\sqrt{3}} - i\right) - H_n\!\left(\frac{1}{\sqrt{3}} + i\right)\right) \tag{12}$$

$$\pi = 6\left(1 - \frac{1}{\sqrt{3}}\right)\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^n \operatorname{Im}\!\left(H_n\!\left(\frac{1}{\sqrt{3}} - i\right)\right) \tag{13}$$

Entry 7. for $u > 1$, we have

$$\pi = 4\left(1 - \frac{1}{u}\right)\sum_{n=1}^{\infty} u^{-n} \operatorname{Im}\!\left(H_n\!\left(\frac{2\,u - 1 - i}{(u-1)^2 + u^2}\right)\right) \tag{14}$$

Entry 8.

$$\pi = -2\sum_{n=1}^{\infty} \operatorname{Im}\!\left((1-i)^{-n+1} H_n(i)\right) \tag{15}$$

Entry 9.

$$\pi = -3\left(\sqrt{3} - 1\right)(1-i)\sum_{n=1}^{\infty}(1+i)^n \left(1 - \frac{\sqrt{3}}{2} - \frac{i}{2}\right)^n H_n(1+i) \tag{16}$$

Entry 10. for $0 < \theta < \pi/6$, we have

$$\theta = \frac{1 - 2\,i\sin(\theta)}{2\,i}\sum_{n=1}^{\infty}(2\,i\sin(\theta))^n H_n\!\left(e^{i\,\theta}\right) \tag{17}$$

$$\pi = 4\left(\frac{1}{i} - \sqrt{2 - \sqrt{2}}\right)\sum_{n=1}^{\infty}\left(i\sqrt{2 - \sqrt{2}}\right)^n H_n\!\left(e^{i\,\pi/8}\right) \tag{18}$$

Entry 11. for $0 < \theta < \pi/3$, we have

$$\theta = (1-\cos(\theta) - i\,(1+\sin(\theta)))\sum_{n=1}^{\infty}(-1)^{n-1}\,(\sin(\theta) + i\,(1-\cos(\theta)))^n\,H_n(-i) \tag{19}$$

$$\pi = 4\left(1 - \frac{1}{\sqrt{2}} - i\left(1 + \frac{1}{\sqrt{2}}\right)\right)\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right)^n H_n(-i) \tag{20}$$

Entry 12. for $0 < \theta < \pi/2$, we have

$$\theta = \frac{2}{i}\sum_{n=1}^{\infty}u^{-n}\left(\frac{(u-1)\,n\,(2\,n+1)+1}{4\,n^2-1}\right)H_n(u) \tag{21}$$

where

$$u = \frac{e^{i\,\theta}+1}{e^{i\,\theta}-1} \tag{22}$$

$$\pi = 8\,i\sum_{n=1}^{\infty}\left(-i\left(1+\sqrt{2}\right)\right)^{-n}\left(\frac{\left(1+i\left(1+\sqrt{2}\right)\right)n\,(2\,n+1)-1}{4\,n^2-1}\right)H_n\left(-i\left(1+\sqrt{2}\right)\right) \tag{23}$$

Entry 13. for $0 < \theta < \pi/3$, we have

$$\theta = (\sin(\theta) - i\,(1-\cos(\theta)))\left(1 - \sum_{n=1}^{\infty}\frac{1}{(n+1)\,(n+2)}H_n\left(\frac{1}{1-e^{i\,\theta}}\right)\right) \tag{24}$$

Entry 14.

$$\pi = 4 - 4\sum_{n=1}^{\infty}\frac{\mathrm{Re}(H_n(1+i))}{(n+1)\,(n+2)} \tag{25}$$

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