RENCONTRES FOR EQUIPARTITE DISTRIBUTIONS OF MULTISETS OF COLORED BALLS INTO URNS

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ABSTRACT. A multiset of ub balls contains u different colors and b balls of each color. Randomly distributing them across u urns with b balls per urn, what is the probability that no urn contains at least two balls of a common color? We reduce this problem to the enumeration of $u \times u$ binary matrices with constant row and column sum b and provide an explicit table of probabilities for small b and u.

1. Statement of the Problem

1.1. **Variables.** The hardware of the problem is a set of labeled balls and urns. The number of balls is a multiple of the number of urns such that one may distribute all balls across the urns, the same number of balls in each urn. The factorial of the number of balls defines the number of permutations of the balls; properties of the urns after filling define statistics and probabilities by sampling all permutations of the balls.

Here we assign a color to each ball, evenly distributed, such that there are u colors and b balls per color, a total of ub balls. Balls are enumerated from 0 to $ub-1$ as in most modern programming languages. Ball number i has color (number) $\lfloor i/u \rfloor$ in the range $0 \dots u-1$, by integer division. A random permutation π of the balls assigns a position π_i to ball number i, $0 \le \pi_i < bu-1$. By dropping the first b permuted balls into urn number 0, the next b balls in the next urn and so on, ball number i ends up in urn number $\lfloor \pi_i/u \rfloor$, again by integer division.

Remark 1. *The way building groups of size b of the balls by integer division of labels is irrelevant to the subsequent statistics. One may as well define groups of equal size by other means, for example considering u families of b members each, randomly seated at u tables, tournaments of u continents each with b countries, etc.*

Definition 1. *$N_{u,b}$ is the number of samples drawn from the $(ub)!$ permutations of the balls where no urn contains two balls of a common color.*

Remark 2. *A related derangment problem testing correlations between i and π_i has been proposed by Kaplansky [2].*

Definition 2. *A rencontre is a multiset of two or more same-colored balls in an urn. For 4 or more balls per urn and at least 2 colors, there may be more than one rencontre even in the same urn.*

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$u\backslash b$	1	2	3	4	5
1	1	1	1	1	1
2	2	6	20	70	252
3	6	90	1680	34650	756756
4	24	2520	369600	63063000	11732745024
5	120	113400	168168000	305540235000	623360743125120
6	720	7484400	137225088000	3246670537110000	88832646059788350720

Table 1. Table of $B_{u,b}$ [1, A060538,A187783].

Definition 3. $P_{u,b} = N_{u,b}/(ub)!$ *is the probability that there are no rencontres, i.e., that a random permutation has in each urn the maximum of* b *different colors.*

The problem considered in this manuscript is to compute $P_{u,b}$. Refined sub-statistics like counting the number of urns with specified numbers of rencontres is not debated here.

1.2. **Basic Limits.** If $b = 1$ there is only a single ball per urn and rencontres cannot occur:

$$P_{u,1} = 1. \tag{1}$$

If $b > u$, one can place only one ball of some color into each urn; the number of urns is too small to place the others without rencontres:

$$P_{u,b} = 0, \quad b > u. \tag{2}$$

2. Reduction to Multisets

A convenient reduction of the base set of $(ub)!$ permutations follows by considering only the permutations of the multiset of the colored balls, of which there are [5, §26.16]

$$B_{u,b} \equiv (ub)!/(b!)^u, \tag{3}$$

a multinomial coefficient illustrated in Table 1.

Because the rencontres are a function of the colors (not of the individual balls), a rencontre in sets of urns of colored balls represents $b!$ permutations within the set of labeled balls of that color, so post-multiplying the number of rencontres of the only-colored balls in urns by $(b!)^u$ raises the count to $N_{u,b}$, which is the same factor that reduces the number of permutations of the labeled balls to the number of permutations of colored balls. In conclusion $P_{u,b}$ can be obtained by dividing the counts of urns with multisets of colored balls without rencontres through $B_{u,b}$, a division of two smaller numbers than in Definition 3.

3. Map to Integer Matrices

Each of the $B_{u,b}$ of the sampling base is a sequence of colors $\lfloor \pi_i/b \rfloor$. Putting a vertical bar after each group of b colors helps to indicate which of them end up in the same urn.

Example 1. *For* $b = 2$ *and* $u = 4$, *permutations of the colored balls are for example* 00|11|22|33*,* 00|11|23|23*,* 00|12|12|33*, and* 00|12|13|23.

$u\backslash b$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	2	3	4	5	6	7
3	6	21	55	120	231	406
4	24	282	2008	10147	40176	132724
5	120	6210	153040	2224955	22069251	164176640
6	720	202410	20933840	1047649905	30767936616	602351808741

TABLE 2. Non-negative integer matrices with all rows and columns sums equal b [1, A257493].

Each of these sequences of ub numbers may be mapped to a $u \times u$ integer matrix where the (nonnegative) entry in rows labeled by color (number) and columns labeled by urn (number) shows how many balls of that color end up in that urn. All row sums and column sums in these matrices equal b, see Table 2.

Example 2. *For $b = 2$ balls per color and $u = 4$ urns/colors,*

$$00|11|22|33 \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \tag{4}$$

$$00|11|23|23 \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tag{5}$$

$$00|12|12|33 \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \tag{6}$$

$$00|12|13|23 \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tag{7}$$

This mapping is lossy in the sense that permutations of colors within the bars in the permutation of the multiset are mapped to the same matrix. However, this multiplicity of permutations can be recovered as the product of the multinomial coefficients of the entries of each column, $\prod_{c=0}^{n-1} \binom{b}{A_{0,c}, A_{1,c}, \ldots, A_{n-1,c}}$, where $A_{r,c}$ are the entries in the matrix.

Example 3. *The* $\binom{2}{2,0,0,0}\binom{2}{0,1,1,0}\binom{2}{0,1,0,1}\binom{2}{0,0,1,1} = 8$ *permutations that map to the matrix (7) are* 00|12|13|23*,* 00|21|13|23*,* 00|12|31|23*,* 00|21|31|23*,* 00|12|13|32*,* 00|21|13|32*,* 00|12|31|32*, and* 00|21|31|32.

$u\backslash b$	1	2	3	4	5	6
1	1					
2	2	1				
3	6	6	1			
4	24	90	24	1		
5	120	2040	2040	120	1	
6	720	67950	297200	67950	720	1

TABLE 3. $E_{u,b}$: Number of binary $u \times u$ matrices with all row and column sums equal to b [1, A008300][4]. The zeros for $b > u$ in the upper right triangle are omitted. Exact formulas for $E_{u,2}$ are known [1, A001499].

The key observation of this manuscript is that the permutations without rencontres are mapped to binary matrices, i.e., matrices which have only 0's and 1's [3].

Definition 4. *$E_{u,b}$ is the number of $u \times u$ matrices with entries $\in \{0,1\}$ and all row sums and column sums equal to b.*

Luckily, the aforementioned multiplicity factor of the map of permutations to matrices is the same $b!$ for each column of the binary matrix, in total $(b!)^u$. Combined with the observation at the end of Section 2:

Theorem 1.

$$P_{u,b} = (b!)^u E_{u,b}/B_{u,b} \tag{8}$$

is the probability of no rencontres defined in Defn. 3.

Thanks to computations by McKay, $E_{u,b}$ have been computed for $u, b \leq 30$, see the OEIS entry in Table 3. A subset of the results is put into Table 4.

b	u	$P_{u,b}$	$P_{u,b}$
2	2	$1 \times (2!)^2/[(2\times 2)!/2!^2]$	0.6666666666666667
2	3	$6 \times (2!)^3/[(2\times 3)!/2!^3]$	0.5333333333333333
2	4	$90 \times (2!)^4/[(2\times 4)!/2!^4]$	0.5714285714285714
2	5	$2040 \times (2!)^5/[(2\times 5)!/2!^5]$	0.5756613756613757
2	6	$67950 \times (2!)^6/[(2\times 6)!/2!^6]$	0.5810485810485810
2	7	$3110940 \times (2!)^7/[(2\times 7)!/2!^7]$	0.5846597846597847
2	8	$187530840 \times (2!)^8/[(2\times 8)!/2!^8]$	0.5873987740654407
2	9	$14398171200 \times (2!)^9/[(2\times 9)!/2!^9]$	0.5895304405108327
2	10	$1371785398200 \times (2!)^{10}/[(2\times 10)!/2!^{10}]$	0.5912368195959527
2	11	$158815387962000 \times (2!)^{11}/[(2\times 11)!/2!^{11}]$	0.5926330833500719
2	12	$21959547410077200 \times (2!)^{12}/[(2\times 12)!/2!^{12}]$	0.5937964690245221
2	13	$3574340599104475200 \times (2!)^{13}/[(2\times 13)!/2!^{13}]$	0.5947805998294573
2	14	$676508133623135814000 \times (2!)^{14}/[(2\times 14)!/2!^{14}]$	0.5956238490130145
2	15	$147320988741542099484000 \times (2!)^{15}/[(2\times 15)!/2!^{15}]$	0.5963543890792641
2	16	$36574751938491748341360000 \times (2!)^{16}/[(2\times 16)!/2!^{16}]$	0.5969933637033929
2	17	$10268902998771351157327104000 \times (2!)^{17}/[(2\times 17)!/2!^{17}]$	0.5975569474528858
2	18	$3237415247416050491577971184000 \times (2!)^{18}/[(2\times 18)!/2!^{18}]$	0.5980577224312149
2	19	$1138803698046507486981918971040000 \times (2!)^{19}/[(2\times 19)!/2!^{19}]$	0.5985056219085982

2	20	$444432474300844787327725684969440000 \times (2!)^{20}/[(2 \times 20)!/2!^{20}]$	0.5989085917227133
3	3	$1 \times (3!)^{3}/[(3 \times 3)!/3!^{3}]$	0.1285714285714286
3	4	$24 \times (3!)^{4}/[(3 \times 4)!/3!^{4}]$	0.0841558441558442
3	5	$2040 \times (3!)^{5}/[(3 \times 5)!/3!^{5}]$	0.0943285286142429
3	6	$297200 \times (3!)^{6}/[(3 \times 6)!/3!^{6}]$	0.1010468523073565
3	7	$68938800 \times (3!)^{7}/[(3 \times 7)!/3!^{7}]$	0.1057395147974538
3	8	$24046189440 \times (3!)^{8}/[(3 \times 8)!/3!^{8}]$	0.1093353501697216
3	9	$12025780892160 \times (3!)^{9}/[(3 \times 9)!/3!^{9}]$	0.1121638744842065
3	10	$8302816499443200 \times (3!)^{10}/[(3 \times 10)!/3!^{10}]$	0.1144433000353959
3	11	$7673688777463632000 \times (3!)^{11}/[(3 \times 11)!/3!^{11}]$	0.1163177527281613
3	12	$9254768770160124288000 \times (3!)^{12}/[(3 \times 12)!/3!^{12}]$	0.1178855115747171
3	13	$14255616537578735986867200 \times (3!)^{13}/[(3 \times 13)!/3!^{13}]$	0.1192156958043755
3	14	$27537152449960680597739468800 \times (3!)^{14}/[(3 \times 14)!/3!^{14}]$	0.1203582388784016
3	15	$65662040698002721810659005184000 \times (3!)^{15}/[(3 \times 15)!/3!^{15}]$	0.1213500686489826
3	16	$190637228506535883540302038364160000 \times (3!)^{16}/[(3 \times 16)!/3!^{16}]$	0.1222190601598070
4	4	$1 \times (4!)^{4}/[(4 \times 4)!/4!^{4}]$	0.0052610246895961
4	5	$120 \times (4!)^{5}/[(4 \times 5)!/4!^{5}]$	0.0031272964099147
4	6	$67950 \times (4!)^{6}/[(4 \times 6)!/4!^{6}]$	0.0039996196321044
4	7	$68938800 \times (4!)^{7}/[(4 \times 7)!/4!^{7}]$	0.0047564209260736
4	8	$116963796250 \times (4!)^{8}/[(4 \times 8)!/4!^{8}]$	0.0053859160013735
4	9	$315031400802720 \times (4!)^{9}/[(4 \times 9)!/4!^{9}]$	0.0059104567812085
4	10	$1289144584143523800 \times (4!)^{10}/[(4 \times 10)!/4!^{10}]$	0.0063515744714243
4	11	$7722015017013984456000 \times (4!)^{11}/[(4 \times 11)!/4!^{11}]$	0.0067263368423600
4	12	$65599839591251908982712750 \times (4!)^{12}/[(4 \times 12)!/4!^{12}]$	0.0070479658385222
4	13	$769237071909157579108571190000 \times (4!)^{13}/[(4 \times 13)!/4!^{13}]$	0.0073266272745447
4	14	$12163525741347497524178307740904300 \times (4!)^{14}/[(4 \times 14)!/4!^{14}]$	0.0075701678383163
5	5	$1 \times (5!)^{5}/[(5 \times 5)!/5!^{5}]$	0.0000399178168892
5	6	$720 \times (5!)^{6}/[(5 \times 6)!/5!^{6}]$	0.0000242017836388
5	7	$3110940 \times (5!)^{7}/[(5 \times 7)!/5!^{7}]$	0.0000386541770339
5	8	$24046189440 \times (5!)^{8}/[(5 \times 8)!/5!^{8}]$	0.0000544880333184
5	9	$315031400802720 \times (5!)^{9}/[(5 \times 9)!/5!^{9}]$	0.0000701139463624
5	10	$6736218287430460752 \times (5!)^{10}/[(5 \times 10)!/5!^{10}]$	0.0000849114389157
5	11	$226885231700215713535680 \times (5!)^{11}/[(5 \times 11)!/5!^{11}]$	0.0000986535981817
6	6	$1 \times (6!)^{6}/[(6 \times 6)!/6!^{6}]$	0.0000000521740810
6	7	$5040 \times (6!)^{7}/[(6 \times 7)!/6!^{7}]$	0.0000000360916944
6	8	$187530840 \times (6!)^{8}/[(6 \times 8)!/6!^{8}]$	0.0000000787923137
6	9	$12025780892160 \times (6!)^{9}/[(6 \times 9)!/6!^{9}]$	0.0000001408575729
7	7	$1 \times (7!)^{7}/[(7 \times 7)!/7!^{7}]$	0.0000000000112182
7	8	$40320 \times (7!)^{8}/[(7 \times 8)!/7!^{8}]$	0.0000000000098297

Table 4: Probabilities $P_{u,b}$: exact rational numbers and floating point approximations.

4. Summary

The probability that no two items of the same group meet if a multiset of ub items (b items per group and u groups) is randomly regrouped in u urns, b items per urn, is derived from the number of $n \times n$ binary matrices with row-column-sums b via (8).

$u\backslash b$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	3	3	4
3	1	5	10	23	40	73
4	1	17	93	465	1746	5741
5	1	73	1417	19834	190131	1398547
6	1	388	32152	1532489	43816115	848597563

TABLE 5. Number of ways of partitioning a multiset of b copies of each element of a set of u into multisets of size b [1, A257463].

APPENDIX A. SAMPLING SORTED COLORS

There is essentially another redundant factor of $u!$ in $B_{u,q}$, because we may consider only color-sorted permutations of the multiset, which means that the first occurrence of a color in the permuted sequence must occur after the first occurrences of all "smaller" colors (which might be called permutations of increasing support leaning on terminology of Gessel). Sorting as a followup the matrices accordingly such that the rows of the matrices are lexicographically sorted leads from Table 2 to Table 5.

Remark 3. *There are similar arguments of sorting the matrices by columns, because the urns are essentially unlabeled here, but mixing these two approaches to introduce a reduction by $u!$ is difficult.*

Example 4. *The 5 matrices represented at $u = 3$, $b = 2$ in Table 5 are*

$$\begin{pmatrix}1&0&1\\1&0&1\\0&2&0\end{pmatrix},\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix},\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix},\begin{pmatrix}2&0&0\\0&1&1\\0&1&1\end{pmatrix},\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}. \tag{9}$$

The multinomial coefficients of assigning multiplicities to these 5 are $\binom{3}{2,1} = 3$ *(3 rows, one occurring twice, one occurring once),* $\binom{3}{1,1,1} = 6$ *(3 rows, all distinct),* $\binom{3}{2,1} = 3$, $\binom{3}{2,1} = 3$, $\binom{3}{1,1,1} = 6$, *and their sum is 21 compatible with Table 2.*

Example 5. *There are 17 color-sorted matrices with $u = 4$, $b = 2$ colors. They have multinomial multiplicities of $4!/(2!2!) = 6$ (2 pairs of 2 rows), $4!/(2!1!1!) = 12$ (one pair of rows and 2 other rows) or $4!/(1!1!1!1!) = 24$ (all 4 rows distinct) and cover the 282 matrices of Table 2.*

In the same fashion Table 3 may be reduced to Table 6 by admitting only matrices with sorted rows.

This road to reduce the numbers of base sample further is not investigated in this manuscript.

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$u\backslash b$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	1	1				
4	1	6	1	1			
5	1	22	22	1	1		
6	1	130	550	130	1	1	
7	1	822	16700	16700	811	1	1

TABLE 6. The number of $u \times u$ binary matrices with sorted rows and all row and columns sums equal b [6, A260340]. The values at $b = 1$ represent the unit matrices; the values at $b = u$ represent the all-1 matrices; the values at $b = u - 1$ represent the matrices with zeros only along the second diagonal.

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