Possible Solutions of the Rational Cuboid Problem

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Abstract

This article covers a number of forms for elliptic equations that were derived from the simultaneous equations describing a rational cuboid. The analysis of these elliptic equations shows that some rational points on the elliptic curves exist, but they are not the points of infinite order, accordingly they do not belong to any of the right triangles.

Keywords: rational cuboid, elliptic equations, basic trigonometric identity for hyperbolic functions, congruent numbers

Fist elliptic equation

Introductory remarks:

A rectangular parallelepiped can be described by the following three square equations:

$$a^2 + b^2 = c^2 \ (x_1, y_1), (x_{11}, y_{11}) \tag{1}$$

$$a^2 + d^2 = e^2 \ (x_2, y_2), (x_{21}, y_{21}) \tag{2}$$

$$b^2 + d^2 = f^2 \ (x_3, y_3), (x_{31}, y_{31}) \tag{3}$$

– triangle-forming numbers (may be irrational or rational)

where a, b, d are the edges (assuming a and b to be “even”), then the space diagonal equals

$$g^2 = a^2 + b^2 + d^2, \tag{4}$$

and c, e, f are face diagonals.

Let us assume that the edge **a** can be derived from the first and the second equation as

$$a = 2x_1y_1 = 2x_2y_2 = (x^2{}_{11} - y^2{}_{11}) = (x^2{}_{21} - y^2{}_{21}), \tag{5}$$

where $x_{11} = (x_1 + y_1)/\sqrt{2}$, $y_{11} = (x_1 - y_1)/\sqrt{2}$, and x_{21}, y_{21} are found in a similar way.

Then Eq. (4) can be rewritten as follows:

$$4x_1^2y_1^2 = 4x_2^2y_2^2 + x_1^4 - 2x_1^2y_1^2 + y_1^4 + x_2^4 - 2x_2^2y_2^2 + y_2^4 = x_1^4 + y_1^4 + x_2^4 + y_2^4 = g^2 \tag{6}$$

or

$$(x_1^2 + y_1^2)^2 + (x_2^2 - y_2^2)^2 = (x_1^2 - y_1^2)^2 + (x_2^2 + y_2^2)^2 = g^2 = c^2 + d^2 = b^2 + e^2 \tag{7}$$

Eq. (5) yields the following relationships:

$$(x_2/x_1 = y_1/y_2 = t \rightarrow x_2k/x_1k = t \rightarrow y_1 = x_2k = x_1tk, \text{ and } x_1k = y_2) \rightarrow$$

$$x_2 = x_1t, \ y_2 = x_1k \ \ y_1 = y_2t = x_2k = x_1kt. \tag{8}$$

Then substitution of $y_1^4 = x_2^4y_2^4/x_1^4$ into Eq. (6) gives:

$$(x_1^4 + x_2^4)(x_1^4 + y_2^4) = g^2x_1^4 \tag{9}$$

and substitution of $y_2^4 = x_1^4y_1^4/x_2^4$ into Eq. (6) gives:

$$(x_1^4 + x_2^4)(x_2^4 + y_1^4) = g^2x_2^4 \tag{10}$$

The equations (9) and (10) are equivalent to each other, therefore any of them may be used.

Taking into account that according to Eq. (8) $x_2^4 = x^4{}_1t^4$, Eq. (10) $(x_1^4 + x_2^4)(x_2^4 + y_1^4) = g^2x_2^4$ gives

$x_1^4t^4 + y_1^4 + x_1^4t^8 + y_1^4t^4 = g^2t^4$.

Dividing it by t^2 and considering that $(y_1^4/t^2 = y_1^2y_2^2)$ results in

$t^2(x_1^4 + y_1^4) + x_1^4t^6 + y_1^2y_2^2 = g^2t^2$.

Translation of the first two terms to the right-hand side, considering that $(g^2 - x_1^4 - y_1^4 = x_2^4 + y_2^4)$, $x_2 = x_1t$, $y_2 = x_1k$, results in:

$y_1^2y_2^2 = t^2(g^2 - x_1^4 - y_1^4) - x_1^4t^6$,

and multiplying this expression by $x^2{}_1 \rightarrow$

$\mathbf{x^2{}_1y_1^2y_2^2 = x^2{}_1t^2(x_2^4 + y_2^4) - x_1^6\,t^6}$.

Then $y^2 = x^2{}_1y_1^2y_2^2$ and $x = x^2{}_1t^2$, consequently

$$\mathbf{y^2 = x\,(x_2^4+y_2^4) - x^3}. \tag{11}$$

When divided by x^6_1 this equation gives the identity:

$$\mathbf{k^4t^2 = t^2(k^4+t^4) - t^6}, \tag{12}$$

that is true for all k and t values. Then $y^2= k^4t^2$, $t^2=x$, so:

$$\mathbf{y^2 = x(k^4+t^4) - x^3}. \tag{13}$$

Second elliptic equation

Introductory remarks:

The idea of the second elliptic equation lies in the fact that, similar to previous expression of the edge a, the edge d can also be expressed in two ways as $d=(x^2_3-y^2_3)=(x^2_2-y^2_2)$ based on Eqs. (2) and (3), and after appropriate transformations we can obtain a new elliptic equation.

If the hyperbola $\mathbf{a}=2x_1y_1$ is rotated by 90 degrees, it will intersect the hyperbola $b= 2x_3 y_3$ at the point (x_{11},y_{11}), then $m=x_3/x_{11}=y_{11}/y_3$ is the hyperbolic rotation factor,→

$x_3= x_{11}\cdot m= (x_1+ y_1)\cdot m/\surd 2$, $y_3= y_{11}/m=(x_1- y_1)/\surd 2\cdot m$.

Also suppose $ch\alpha=x_3/y_3\rightarrow((x_1+y_1)m/\surd 2)/(x_1-y_1)/\surd 2m))$, because $y_1= x_1kt$, where $a_1=(x_1- y_1)/(x_1+ y_1)$ may be reduced by $x_1 \rightarrow(1-kt)/(1+kt)=a_1\rightarrow$

$$\mathbf{m^2=a_1\cdot ch\alpha}. \tag{14}$$

Let us consider the expression $d=(x^2_3-y^2_3)=(x^2_2-y^2_2)$ taking into account that $x_2=x_1t$, $y_2=x_1k$ and $y_1=x_2k=x_1tk$.

Substituting the values of x_2, y_2, X_3, y_3 into the expression of d we obtain:

$(x_1+ y_1)^2\cdot m^2/2 - (x_1- y_1)^2/2\cdot m^2=(x^2_1t^2- x^2_1k^2)$,

reducing it by x^2_1 and multiplying both sides by m^2 we obtain:

$m^4(1+kt)^2 - (1-kt)^2=2m^2(t^2- k^2)$.

Next we divide this expression by $(1+kt)^2$ and translate the second summand to the right-hand side to give:

$$m^4=2\cdot m^2\cdot(t^2- k^2)/ (1+kt)^2 + (1-kt)^2/(1+kt)^2. \tag{15}$$

Multiplying this by m^2 results in

$$m^6 =2\cdot m^4\cdot(t^2- k^2)/ (1+kt)^2 + m^2(1-kt)^2/(1+kt)^2. \tag{16}$$

Let us consider the expression $2\cdot m^4\cdot(t^2- k^2)/(1+kt)^2$.

Multiply the numerator and the denominator by $x^2_1\cdot m^2 \rightarrow$

$2x^2_1\cdot m^6\cdot(t^2- k^2)/ (1+kt)^2\cdot x^2_1\cdot m^2$.

Since $2/(1+kt)^2\cdot x^2_1\cdot m^2=1/x^2_3$, this expression may be rewritten as

$m^6\cdot(x^2_3-y^2_3)/ x^2_3$ but $x^2_3/y^2_3 = ch^2\alpha$, or $\rightarrow m^6\cdot(1- 1/ch^2\alpha)$.

Further notice that $a^2_1=(1-kt)^2/(1+kt)^2$, then the expression (16) may be rewritten as

$m^6= m^6\cdot(1- 1/ch^2\alpha) + a^2_1\cdot m^2. \rightarrow$

$m^6= (m^6\cdot sh^2\alpha)/ch^2\alpha + a^2_1\cdot m^2.\rightarrow$

$m^6- m^2 \cdot sh^2\alpha(m^4/ch^2\alpha)= a^2_1\cdot m^2\rightarrow$

but $m^4/ ch^2\alpha = a^2_1$ (acc. to Eq.(14))

$\rightarrow m^6 - a^2_1\cdot m^2\cdot sh^2\alpha= a^2_1\cdot m^2$. ($m^6= a^3_1ch^3\alpha$, $m^2= a_1ch\alpha$)→

$$\mathbf{a^2_1\cdot m^2= a^3_1ch^3\alpha - a^2_1\cdot sh^2\alpha\cdot a_1ch\alpha} \tag{17}$$

→ ($\mathbf{a^2_1\cdot m^2=y^2}$, $\mathbf{a^2_1\cdot sh^2\alpha=n^2}$, $\mathbf{a_1ch\alpha=x}$),

and finally we obtain

$$\mathbf{y^2=x^3- n^2x}. \tag{18}$$

From this Eq. (17) we can obtain the basic trigonometric identity for hyperbolic functions:

$a^2_1\cdot m^2= a^2_1\cdot a_1 \cdot ch\alpha= a^3_1ch^3\alpha - a^2_1\cdot sh^2\alpha\cdot a_1ch\alpha$,

reducing both sides by $a^3_1\cdot ch\alpha$ gives the identity

$ch^2\alpha - sh^2\alpha=1$.

Now let us show that there is a one-to-one correspondence between Eq. (18) and the identity $ch^2\alpha - sh^2\alpha=1$. We have proved the first part by deriving the identity from Eq. (18). Now let us obtain Eq. (18) from the identity $ch^2\alpha - sh^2\alpha=1$.

Since $ch^2\alpha= m^4/a^2_1\rightarrow m^4/a^2_1-1=sh^2\alpha$

Multiply both sides by $m^2\rightarrow (m^6/a^2_1 -m^2)=m^2\cdot sh^2\alpha\rightarrow$

$a^2{}_1\cdot m^2=m^6- a^2{}_1m^2\cdot sh^2\alpha=a^3{}_1ch^3\alpha - a^2{}_1\cdot sh^2\alpha\cdot a_1ch\alpha\rightarrow$
$y^2=x^3- n^2x$
where ($a^2{}_1\cdot m^2=y^2$, $a^2{}_1\cdot sh^2\alpha=n^2$, $a_1ch\alpha=x$).

Proving the equivalency of the equations $y^2=x(k^4+t^4) – x^3$ and $y^2=x^3- n^2x$.
Previously we have already proved that the equation $y^2=x^3- n^2x$ was equivalent to the basic hyperbolic identity $ch^2\alpha - sh^2\alpha=1$.
Now let us show that the equation $y^2=x(k^4+t^4) – x^3$ is equivalent to it as well.
Preliminaries:
1) $a^2+b^2=c^2$ (x_1, y_1), let $y_1/x_1 = ch\beta$ then $\rightarrow$ $4x_1^4 ch^2\beta +x_1^4sh^4\beta=x_1^4(1+ch^2\beta)^2$
2) $a^2+d^2=e^2$ (x_2, y_2), let $x_2/y_2 = ch\gamma$ then $\rightarrow$ $4y_2^4 ch^2\gamma +y_2^4sh^4 \gamma=y_2^4(1+ch^2 \gamma)^2$
3) $b^2+d^2=f^2$ (x_3 ,y_3), let $x_3/y_3 = ch\alpha$ then $\rightarrow$ $4y_3{}^4ch^2\alpha +y_3{}^4sh^4\alpha= y_3{}^4(1+ch^2\alpha)^2$ (19)
Proof:
Let us divide the equation $k^4t^2=t^2(k^4+t^4)-t^6$ by k^4t^2 $\rightarrow$
$1=(k^4/k^4+t^4/k^4) – t^4/k^4$ $\rightarrow$
($ch\gamma= x_2/y_2=x_1t/x_1k\rightarrow ch^2\gamma=t^2/k^2$).
Then $1=(1+t^4/k^4) - t^4/k^4$,
but since $ch^4\gamma=t^4/k^4$). $\rightarrow$
$1 – ch^4\gamma + ch^4\gamma=1$, and $1 – ch^4\gamma=(1+ch^2\gamma)(1-ch^2\gamma)+ch^4\gamma=1$.
$1-ch^2\gamma= - sh^2\gamma\rightarrow$
$- sh^2\gamma - sh^2\gamma\, ch^2\gamma+ ch^4\gamma=1\rightarrow$
$- sh^2\gamma\, ch^2\gamma+ ch^4\gamma=1+ sh^2\gamma= ch^2\gamma$,
reducing this by $ch^2\gamma$ results in $ch^2\gamma- sh^2\gamma=1= ch^2\alpha- sh^2\alpha$.
The proof is finished.
The reverse proof:
$ch^2\gamma- sh^2\gamma=1\rightarrow$
$ch^2\gamma=sh^2\gamma+1(ch^2\gamma = x^2{}_2/y^2{}_2\rightarrow x^2{}_1t^2/ x^2{}_1k^2)\rightarrow$
$ch^2\gamma= t^2/k^2\rightarrow t^2/k^2-1= t^2/k^2-1\rightarrow$
$t^2 k^6/k^2- k^6=t^2 k^6/k^2-k^6\rightarrow$
$t^2 k^4= t^2 k^4+t^6-t^6\rightarrow$
$t^2 k^4= t^2(t^4+k^4)-t^6$.

Discussion of results
Let us take a look at the first elliptic equation $\mathbf{y^2= x (x_2{}^4+y_2{}^4) – x^3}$, or, in more common form:
$\mathbf{y^2=x^3 - x (x_2{}^4-y_2{}^4)}$, where $\mathbf{y^2}=x^2{}_1y_1{}^2y_2{}^2$, $\mathbf{x}= x^2{}_1t^2=x^2{}_2$, $\mathbf{n^2}= x_2{}^4\pm y_2{}^4$.
This equation was made up based on two square equations $c^2+ d^2= b^2+ e^2=g^2$
These two equations should be solvable by definition. If their solutions (x_1, y_1, x_2, y_2) are rational numbers, the elliptic equation cannot possess a rational point of infinite order, as far as $\mathbf{x_2{}^4\pm y_2{}^4\neq n^2}$, where **n** is considered to be a congruent number, and it should be rational. Still there is a rational point on this curve. If the edges of an Euler brick are known, the problem of finding a rational point becomes quite simple, for example, for the edges (104,153,672) the coordinates of this point will be

$$x=169,\ y=208. \tag{20}$$

Apparently we can find a rational point for any Euler brick (an Euler brick is considered here to be a cuboid that possesses one non-integer element which could be an edge or any of diagonals).
Constructing a brick based on this equation is in theory possible by matching (x_1, y_1, x_2, y_2), but the time to obtain such a brick is indefinite.
It should be noted that sometimes the numbers (x_1, y_1, x_2, y_2) may be irrational, that is when a pair of numbers forms a rational Pythagorean triangle.
All of the aforesaid is also applicable to the numbers in form of (x_{11},y_{11}, x_{21},y_{21}, x_{31},y_{31}).

In our judgment the first elliptic equation is suitable for description of Euler bricks, but it gives no understanding as to existence of a perfect rational cuboid. Therefore let us run through the second elliptic equation.

It may seem that the second elliptic equation was derived somewhat unnaturally, as relations to the basic equation $a^2+b^2+d^2=g^2$ are not apparent. So let me schematically (without detailed manipulations) show the way to obtain the second elliptic equation from the basic Eq. (4).

Let us introduce the following parametrization for Pythagorean triangles:

$a_1= (x- y)/(x+ y)$.

Accordingly any Pythagorean triangle may be represented as follows:

$4(1+ a_1)^2/ (1- a_1)^2 + 16a^2{}_1/(1- a_1)^4=4((1+ a^2{}_1)^2/(1- a_1)^4$.

Multiplying this equation by a specially matched number we can obtain a particular Pythagorean triangle. In this case the first line of the simultaneous equations $a^2 +b^2=c^2$ may be rewritten as

$4(1+ a_1)^2/ (1- a_1)^2 ++16a^2{}_1/(1- a_1)^4=4((1+ a^2{}_1)^2/(1- a_1)^4$,

where $a_1=(x_1- y_1)/(x_1+ y_1)$,

and $x^4{}_1$ is a factor to be used with this equation to give $a^2 +b^2=c^2$, which is implied by relationships (8).

According to Eq. (19) the edge d^2 equals

$y_3{}^4sh^4\alpha= x^4{}_1(1-kt)^4\cdot sh^4\alpha/4m^4$.

Taking into account that $a_1=(1-kt)/(1+kt)\rightarrow$

$(1-kt)^4=16a^4{}_1/(1- a_1)^4\rightarrow$

$d^2 =16\ x^4{}_1\cdot a^4{}_1\cdot sh^4\alpha/4m^4(1- a_1)^4$.

The space diagonal g^2 can also be derived from Eq. (19). For this purpose first we express the edge a through x_3,y_3.

Since $x_3 =(x_1 +y_1)m/\sqrt{2}$, $y_3=(x_1-y_1)/\sqrt{2}m.\rightarrow$

$(x_1,+y_1)=2x_3/\sqrt{2}m$, $(x_1,-y_1)=2y_3m/\sqrt{2}$.

Addition and deduction of these expressions yield:

$x_1=(x_3-y_3m^2)/\sqrt{2}m$, $y_1=(x_3+y_3m^2)/\sqrt{2}m$,

therefore $a=2x_1y_1=2x_2y_2=(x_3{}^2-y_3{}^2m^4)/m^2$.

Now the space diagonal can be described as

$g^2=a^2+f^2=(x_3{}^2+ y_3{}^2)^2+(x_3{}^2-y_3{}^2m^4)^2/m^4$,

Expanding the parentheses and reducing to a common denominator result in

$g^2=(m^4+1)\cdot(x_3{}^4-y_3{}^4m^4)/m^4= g^2/x^4{}_1=4(m^4+1)(m^4+ a_1{}^4)/m^4\cdot(1- a_1)^4$.

Accordingly the equation $a^2+b^2+d^2= g^2$ may be rewritten as

$4\ x^4{}_1\cdot(1+ a_1)^2/ (1- a_1)^2 + x^4{}_1\cdot16a^2{}_1/(1-a_1)^4+16\ x^4{}_1\cdot a^4{}_1\cdot sh^4\alpha/4m^4(1- a_1)^4=4x^4{}_1\cdot(m^4+1)(m^4+a_1{}^4)/m^4\cdot(1-a_1)^4$.

Reducing it by $x^4{}_1$ и solving it relative to $sh^4\alpha$ we obtain

$sh^4\alpha= ((m^4- a_1{}^2)/ a_1{}^2)^2.\rightarrow$

$\mathbf{sh^2\alpha=\pm(m^4- a_1{}^2)/ a_1{}^2)\rightarrow}$

in our case $m^4-a_1{}^2>0$, $ch^2\alpha=m^4/a_1{}^2>1$,

and if $ch^2\alpha=1$, there is no solution, since $x_3{}^2=y_3{}^2$. Therefore

$\mathbf{sh^2\alpha=+(m^4- a_1{}^2)/ a_1{}^2 \rightarrow}$

$a_1{}^2\cdot sh^2\alpha=(m^4- a_1{}^2)= m^4- m^4/ ch^2\alpha\rightarrow$

$a_1{}^2\cdot sh^2\alpha\cdot ch^2\alpha= m^4\cdot ch^2\alpha -m^4$. $(m^4= a_1{}^2\cdot ch^2\alpha) \rightarrow$

$a_1{}^2\cdot sh^2\alpha\cdot ch^2\alpha= m^4\cdot ch^2\alpha - a_1{}^2\cdot ch^2\alpha= a_1{}^2\cdot ch^4\alpha - a_1{}^2\cdot ch^2\alpha$,

reducing it by $ch\alpha$ and multiplying by $a_1\rightarrow$

$a_1{}^2\cdot sh^2\alpha\cdot a_1\cdot ch\alpha= a^3{}_1\cdot ch^3\alpha- a^3{}_1\cdot ch\alpha \rightarrow$

$(a^3{}_1\cdot ch\alpha=a_1{}^2\cdot a_1ch\alpha=a_1{}^2\cdot m^2) \rightarrow$

$$\mathbf{a_1{}^2\cdot a_1ch\alpha=a_1{}^2\cdot m^2= a^3{}_1\cdot ch^3\alpha - a_1{}^2\cdot sh^2\alpha\cdot a_1\cdot ch\alpha.} \qquad (17)$$

$(\mathbf{a^2{}_1\cdot m^2=y^2, a^2{}_1\cdot sh^2\alpha=n^2, a_1ch\alpha=x})$ or $(\mathbf{y=m\cdot a_1\cdot sh\alpha, x=m^2,n=a_1}) \rightarrow$

$$\mathbf{y^2=x^3- n^2x.} \qquad (18)$$

Next, setting the equation $\mathbf{a_1{}^2\cdot a_1ch\alpha= a^3{}_1\cdot ch^3\alpha- a_1{}^2\cdot sh^2\alpha\cdot a_1\cdot ch\alpha}$ equal to zero gives $\rightarrow$

$\mathbf{a^3{}_1\cdot ch\alpha(ch^2\alpha- sh^2\alpha-1)=0.}$

The first multiplier is equal to zero provided $a^3_1=0,\rightarrow$
$(x_1- y_1)^3/(x_1+ y_1)^3=0$.
Accordingly $x_1=y_1,\rightarrow$ the edge b should be zero, but in such case $ch\alpha=m^2/a_1$ doesn't apply, therefore the equation has no solutions.
The second multiplier $(ch^2\alpha- sh^2\alpha-1)$ is always equal to zero at any possible values of $ch\alpha(x_1\neq y_1)\rightarrow$
$ch^2\alpha- sh^2\alpha=1$.
The latter is a hyperbolic equation permitting rational parametrization (in the right-hand side).
Consequently there are infinitely many rational points. But on the other hand, according to the proved statement No.2 in ref. [1], to obtain a coordinate x from a right triangle it shall satisfy three conditions, one of those is that numerator of value x should have no common divisor with n.
Here n is considered to be a congruent number. So we've got $x=a_1ch\alpha$, $n=a_1\cdot sh\alpha$, and a common divisor a_1.
It could be deduced that Eq. (17)-(18) yield no rational point describing a right triangle, still at least one rational point can be found for every n (20). Also it is apparent that if we construct a right triangle from the points $x=a_1ch\alpha$, $y=a_1\cdot m$, $n= a_1\cdot sh\alpha$ we will obtain $ch^2\alpha= sh^2\alpha+1$. There is an infinite number of Euler bricks, but no perfect cuboid in the field of rational numbers.

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