below

i! (i FACTORIAL)

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Abstract: here is a proof of the value of i Factorial, according to my point of view.

1-EULER GAMMA FUNCTION

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt \tag{1.1}$$

Of course $\Gamma(1)=1$. Then,

$$\Gamma(n+1)=n\Gamma(n)\,; \tag{1.2}$$

in fact, after an integration by parts:

$\Gamma(z)=e^{-t}\left.\frac{t^z}{z}\right|_0^\infty+\frac{1}{z}\int_0^\infty e^{-t}t^z dt=0+\frac{1}{z}\Gamma(z+1)$ and by iterating the $\Gamma(n+1)=n\Gamma(n)$, we get:

$$\Gamma(n+1)=n!\ . \tag{1.3}$$

2-REFLECTION FORMULA

Here it is: $$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}\ . \tag{2.1}$$

In fact, we know that $\frac{\sin x}{x}$ is zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, so:

$\frac{\sin x}{x}=(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})..........=(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2}).....=\prod_{k=1}^{\infty}[1-(\frac{x}{k\pi})^2]$, and so:

$$\frac{\pi x}{\sin \pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} \tag{2.2}$$

Moreover, we can see a factorial like this:

$n!=\lim_{k\to\infty}\frac{k!k^n}{(n+1)...(n+k)}\cdot\lim_{k\to\infty}\frac{(k+1)(k+2)...(k+n)}{k^n}$; in fact, the denominator of the first limit is $(n+1)...(n+k)=\frac{(n+k)!}{n!}$, while from both numerators we can collect: $k!(k+1)...(k+n)=(k+n)!$, so just n! is left. Furthermore, we see that the second limit is 1: $\lim_{k\to\infty}\frac{(k+1)(k+2)...(k+n)}{k^n}=1$, as it gets of the form $\lim_{k\to\infty}\frac{k^n}{k^n}$. Therefore: $n!=\lim_{k\to\infty}\frac{k!k^n}{(n+1)...(n+k)}$. As $\Gamma(x+1)=x!$, we have: $\Gamma(x+1)=x!=\lim_{k\to\infty}\frac{k!k^x}{(x+1)...(x+k)}$ and after dividing both numerator and denominator by k!:

$$\Gamma(x+1)=\lim_{k\to\infty}\frac{k^x}{(1+x)(1+x/2)...(1+x/k)}\ . \tag{2.3}$$

Now, let's introduce the constant $\gamma_k = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k} - \ln k$, and let's say: $\gamma = \lim_{k\to\infty} \gamma_k$; on the basis of that, the (2.3) becomes: $\Gamma(x+1) = e^{-\gamma x} \lim_{k\to\infty} \frac{e^x}{(1+x)} \frac{e^{x/2}}{(1+x/2)} .... \frac{e^{x/k}}{(1+x/k)}$. (2.4)

In fact, all the e^x, $e^{x/2}$,…, $e^{x/k}$ cancel with terms in $e^{-\gamma x}$ and the term $k^x = e^{(\ln k)x}$ still comes from $e^{-\gamma x}$. Moreover, as $\Gamma(x+1) = x\Gamma(x)$, then the (2.4) becomes:

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \lim_{k\to\infty} \frac{(1+x)}{e^x} \frac{(1+x/2)}{e^{x/2}} .... \frac{(1+x/k)}{e^{x/k}} = xe^{\gamma x} \prod_{k=1}^{\infty} [1+(\frac{x}{k})] e^{-x/k} \quad (2.5)$$

Now, as $\Gamma(y+1) = y\Gamma(y)$, then $\Gamma(y) = \frac{1}{y}\Gamma(y+1)$ and after replacing y by –x:

$\Gamma(-x) = -\frac{1}{x}\Gamma(1-x)$ and so: $\Gamma(1-x) = -x\Gamma(-x)$. Let's figure out $\Gamma(x)\Gamma(1-x)$:

$$\Gamma(x)\Gamma(1-x) = [x^{-1}e^{-\gamma x} \prod_{k=1}^{\infty} (1+\frac{x}{k})^{-1} e^{x/k}] \cdot [e^{\gamma x} \prod_{k=1}^{\infty} (1-\frac{x}{k})^{-1} e^{-x/k}] = [A]\cdot[B] = \frac{1}{x} \prod_{k=1}^{\infty} [1-(\frac{x}{k})^2]^{-1}, \quad (2.6)$$

where A is an inverted (2.5) and B is the (2.4) with x changed into –x. So, (2.6) is:

$\Gamma(x)\Gamma(1-x) = \frac{1}{x} \prod_{k=1}^{\infty} [1-(\frac{x}{k})^2]^{-1}$, and for the (2.2) (($\frac{\pi x}{\sin \pi x} = \prod_{k=1}^{\infty} [1-(\frac{x}{k})^2]^{-1}$)) we have:

$\frac{2}{\Gamma(n)\Gamma(1-n)} = \frac{2 \sin n\pi}{\pi}$ which is the (2.1) indeed.

3- i! i FACTORIAL

Let's evaluate the modulus of "i factorial", first. ($i = \sqrt{-1}$). About the definition of factorial, according to the Euler Gamma Function, we have:

$$i! = \Gamma(i+1) = \int_0^\infty e^{-t} t^{i+1-1} dt = \int_0^\infty e^{-t} t^i dt = \int_0^\infty e^{-t} e^{i \ln t} dt = \int_0^\infty e^{-t} [\cos(\ln t) + i \sin(\ln t)] dt =$$

$$= \int_0^\infty e^{-t} \cos(\ln t) dt + i \int_0^\infty e^{-t} \sin(\ln t) dt = \boxed{i! \cong (0.4980 - i0.1549)}, \quad (3.1)$$

so the factorial of i is a complex number a+ib and later we will check the official a and b values, before reported, through rough approximated methods.

According to the (1.2), we have: $\Gamma(i+1) = i\Gamma(i)$, from which: $|\Gamma(i+1)| = |\Gamma(i)| = |i!|$. Moeover, the product of a complex number z with its complex conjugate z* gives the square modulus: $zz^* = |z|^2$. As the complex conjugate is obtained through the replacement of i by -i, we also have, according to the definition of Γ: $[\Gamma(z)]^* = \Gamma(z^*)$. From the Reflection Formula (2.1), that is: $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin z\pi}$, we have, with z=1+i:

$$\Gamma[1-(1+i)]\cdot\Gamma(1+i) = \frac{\pi}{\sin(1+i)\pi} = \Gamma(-i)\cdot\Gamma(1+i) = \Gamma(i^*)\cdot\Gamma(1+i) = \Gamma^*(i)\cdot\Gamma(1+i) = \Gamma^*(i)\cdot i\Gamma(i) =$$

$= i|\Gamma(i)|^2 = i|i!|^2$, from which:

$$|i!| = \sqrt{\frac{\pi}{i \sin(1+i)\pi}} = \sqrt{\frac{2i\pi}{i[e^{i\pi(1+i)} - e^{-i\pi(1+i)}]}} = \sqrt{\frac{2\pi}{[e^{i\pi}e^{-\pi} - e^{-i\pi}e^{\pi}]}} = \sqrt{\frac{2\pi}{[e^{\pi} - e^{-\pi}]}} = \sqrt{\frac{\pi}{\sinh(\pi)}} = |i!| \quad (3.2)$$

after having used the following equalities: $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, $\sinh(z) = \frac{e^{z} - e^{-z}}{2}$ and $e^{i\pi} = e^{-i\pi} = -1$.

So, according to the (3.2): $$|i!| = \sqrt{\frac{\pi}{\sinh(\pi)}} = \sqrt{\frac{\pi}{\frac{(e^{\pi} - e^{-\pi})}{2}}} = \sqrt{\frac{3.14}{\frac{(23.1 - 0.043)}{2}}} = |i!| = 0.52 \quad . \qquad (3.3)$$

On a vectorial basis, due to the (3.1), we have: $i! = \int_0^{\infty} e^{-t}\cos(\ln t)dt + i\int_0^{\infty} e^{-t}\sin(\ln t)dt$, so, in order to evaluate those two definite integrals, which, as we know, represent the surfaces under the integrand functions, we get the vectorial expression $i! = a + ib$. As those integrals are somewhat difficult, we will carry out a very rough geometric and numerical evaluation! The former integrand function, that is $e^{-t}\cos(\ln t)$, is shown in Fig. 3.1, while in Fig. 3.2 we roughly evaluate the surface under it, through triangles.

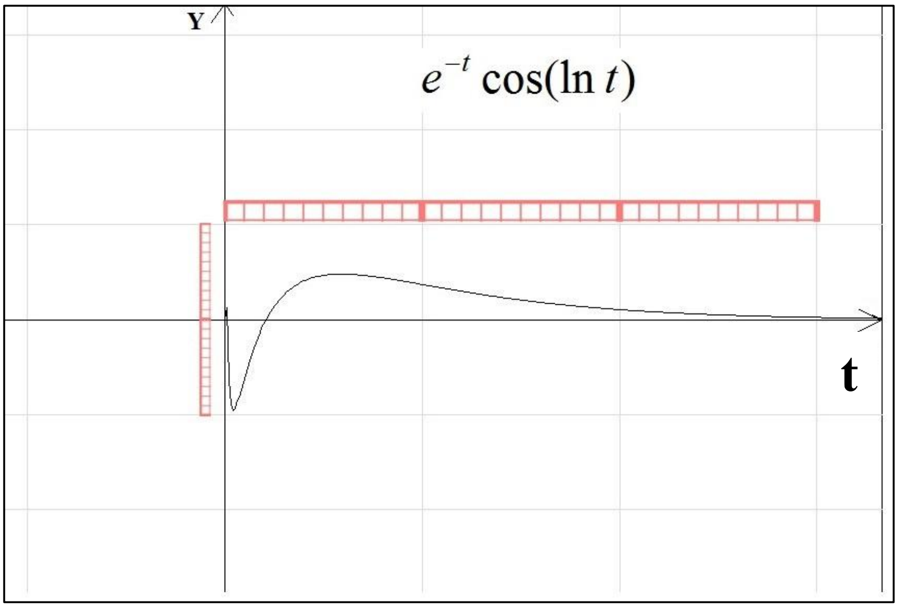

Fig. 3.1

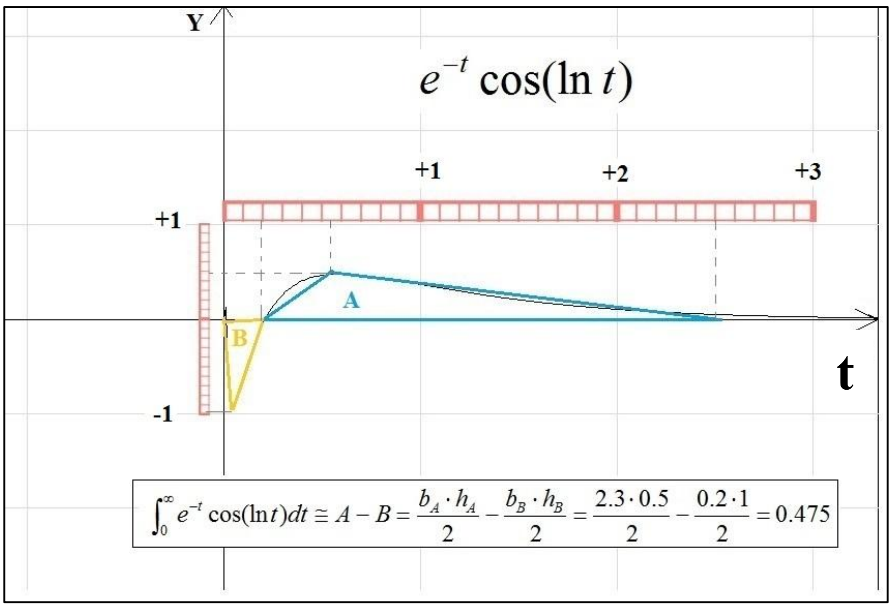

Fig. 3.2

The same goes for the latter integrand function $e^{-t}\sin(\ln t)$, shown in Fig. 3.3, while in Fig. 3.4 we roughly evaluate, as well as before, the surface under it, still through triangles.

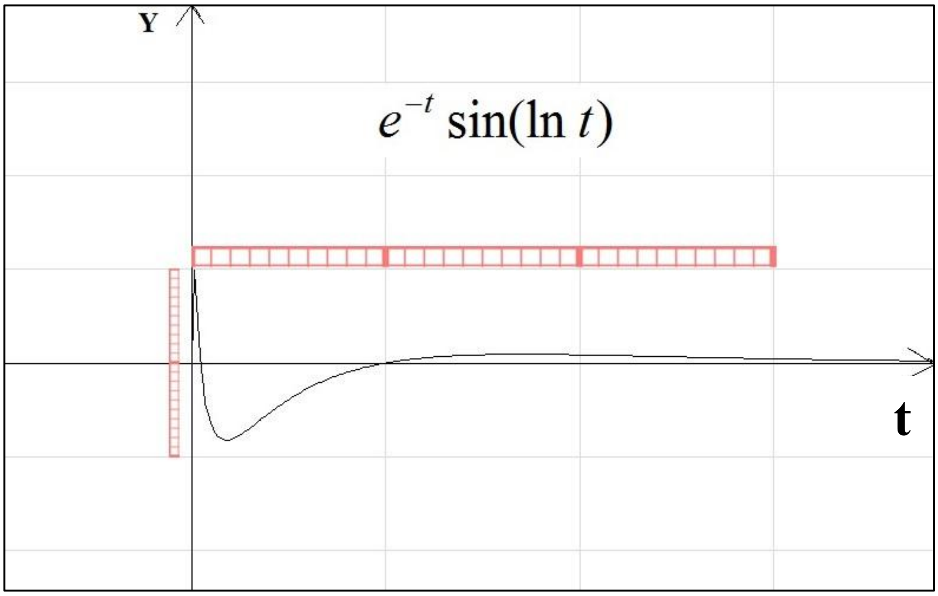

Fig. 3.3

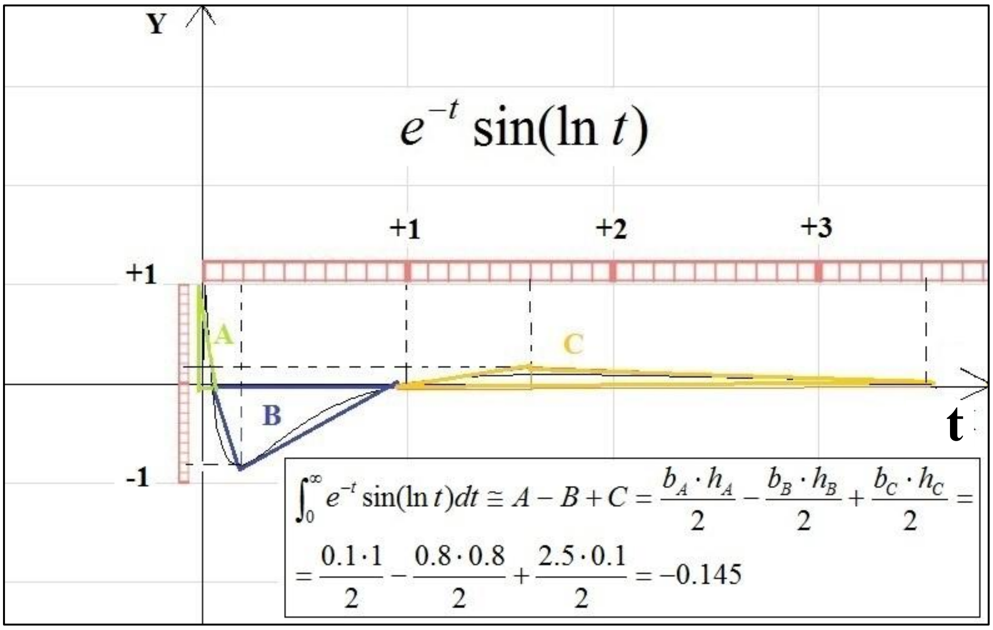

Fig. 3.4

We roughly get: $i! \cong (0.475 - i0.145)$, with $|i!| = 0.50$.

More officially, others calculated a more accurate number, so getting $i! \cong (0.4980 - i0.1549)$, with $|i!| = 0.52$; in both cases, the modulus is in agreement with that previously given by the (3.3), through a different way.

i! (i FATTORIALE)

Leonardo ***Rubino***

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Abstract: ecco una dimostrazione del valore di i fattoriale, secondo il mio punto di vista.

1-FUNZIONE GAMMA DI EULERO

$$\Gamma(z)=\int_0^{\infty} e^{-t}t^{z-1}dt \tag{1.1}$$

Naturalmente, $\Gamma(1)=1$. Poi,

$$\Gamma(n+1)=n\Gamma(n)\,; \tag{1.2}$$

infatti, dopo una integrazione per parti:

$\Gamma(z)=e^{-t}\left.\dfrac{t^z}{z}\right|_0^{\infty}+\dfrac{1}{z}\int_0^{\infty}e^{-t}t^z dt=0+\dfrac{1}{z}\Gamma(z+1)$ ed iterando la $\Gamma(n+1)=n\Gamma(n)$, otteniamo:

$$\Gamma(n+1)=n!\quad . \tag{1.3}$$

2-FORMULA DI RIFLESSIONE

Eccola: $$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}\quad . \tag{2.1}$$

Infatti, sappiamo che $\dfrac{\sin x}{x}$ vale zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, sicchè:

$\dfrac{\sin x}{x}=(1-\dfrac{x}{\pi})(1+\dfrac{x}{\pi})(1-\dfrac{x}{2\pi})(1+\dfrac{x}{2\pi}).........=(1-\dfrac{x^2}{\pi^2})(1-\dfrac{x^2}{4\pi^2}).....=\prod_{k=1}^{\infty}[1-(\dfrac{x}{k\pi})^2]$, e dunque:

$$\frac{\pi x}{\sin \pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} \tag{2.2}$$

Inoltre, un fattoriale possiamo anche vederlo così:

$n!=\lim_{k\to\infty}\dfrac{k!k^n}{(n+1)...(n+k)}\cdot\lim_{k\to\infty}\dfrac{(k+1)(k+2)...(k+n)}{k^n}$; infatti, il denominatore del primo limite è:

$(n+1)...(n+k)=\dfrac{(n+k)!}{n!}$, mentre da entrambi i numeratori possiamo raccogliere:

$k!(k+1)...(k+n)=(k+n)!$, così resta solo n! . Poi, vediamo che il secondo limite vale 1:

$\lim_{k\to\infty}\dfrac{(k+1)(k+2)...(k+n)}{k^n}=1$, poichè si riduce alla forma $\lim_{k\to\infty}\dfrac{k^n}{k^n}$. Perciò: $n!=\lim_{k\to\infty}\dfrac{k!k^n}{(n+1)...(n+k)}$.

Siccome $\Gamma(x+1)=x!$, si ha: $\Gamma(x+1)=x!=\lim_{k\to\infty}\dfrac{k!k^x}{(x+1)...(x+k)}$ e dopo aver diviso sia numeratore che denominatore per k!:

$$\Gamma(x+1)=\lim_{k\to\infty}\frac{k^x}{(1+x)(1+x/2)...(1+x/k)}\quad . \tag{2.3}$$

Ora, introduciamo la costante $\gamma_k = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}-\ln k$, e poniamo: $\gamma = \lim_{k\to\infty}\gamma_k$; sulla base di ciò, la (2.3) diventa: $\Gamma(x+1) = e^{-\gamma x}\lim_{k\to\infty}\frac{e^x}{(1+x)}\frac{e^{x/2}}{(1+x/2)}....\frac{e^{x/k}}{(1+x/k)}$. (2.4)

Infatti, tutti gli e^x, $e^{x/2}$,..., $e^{x/k}$ si elidono con i termini in $e^{-\gamma x}$ ed il termine $k^x = e^{(\ln k)x}$ pure proviene da $e^{-\gamma x}$. Inoltre, siccome $\Gamma(x+1) = x\Gamma(x)$, la (2.4) diventa:

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\lim_{k\to\infty}\frac{(1+x)}{e^x}\frac{(1+x/2)}{e^{x/2}}....\frac{(1+x/k)}{e^{x/k}} = xe^{\gamma x}\prod_{k=1}^{\infty}[1+(\frac{x}{k})]e^{-x/k} \quad (2.5)$$

Ora, poichè $\Gamma(y+1) = y\Gamma(y)$, segue che $\Gamma(y) = \frac{1}{y}\Gamma(y+1)$ e dopo aver sostituito y con –x:

$\Gamma(-x) = -\frac{1}{x}\Gamma(1-x)$ e dunque: $\Gamma(1-x) = -x\Gamma(-x)$. Valutiamo $\Gamma(x)\Gamma(1-x)$:

$$\Gamma(x)\Gamma(1-x) = [x^{-1}e^{-\gamma x}\prod_{k=1}^{\infty}(1+\frac{x}{k})^{-1}e^{x/k}]\cdot[e^{\gamma x}\prod_{k=1}^{\infty}(1-\frac{x}{k})^{-1}e^{-x/k}] = [A]\cdot[B] = \frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}, \quad (2.6)$$

dove A è la (2.5) invertita e B è la (2.4) con x scambiato con –x. Perciò, la (2.6) è:

$\Gamma(x)\Gamma(1-x) = \frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$, e per la (2.2) (($\frac{\pi x}{\sin \pi x} = \prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$)) abbiamo:

$\frac{2}{\Gamma(n)\Gamma(1-n)} = \frac{2\sin n\pi}{\pi}$ che è proprio la (2.1).

3- i! i FATTORIALE

Valutiamo dapprima il modulo di "i fattoriale". ($i=\sqrt{-1}$) A livello di definizione di fattoriale, tramite la funzione Gamma di Eulero, si ha:

$$i! = \Gamma(i+1) = \int_0^{\infty} e^{-t}t^{i+1-1}dt = \int_0^{\infty} e^{-t}t^{i}dt = \int_0^{\infty} e^{-t}e^{i\ln t}dt = \int_0^{\infty} e^{-t}[\cos(\ln t)+i\sin(\ln t)]dt =$$

$$= \int_0^{\infty} e^{-t}\cos(\ln t)dt + i\int_0^{\infty} e^{-t}\sin(\ln t)dt = \boxed{i! \cong (0.4980 - i0.1549)}, \quad (3.1)$$

dunque, il fattoriale di i è un numero complesso a+ib e successivamente verificheremo gli a e b più accreditati, sopra riportati, con metodi approssimativi.

Per la (1.2), si ha: $\Gamma(i+1) = i\Gamma(i)$, da cui: $|\Gamma(i+1)| = |\Gamma(i)| = |i!|$. Ricordiamo poi che il prodotto di un numero complesso z con il suo complesso coniugato z* dà il modulo quadro: $zz^* = |z|^2$. Dato che il complesso coniugato si ottiene sostituendo i con -i, si ha anche, dalla definizione di Γ: $[\Gamma(z)]^* = \Gamma(z^*)$. Dalla Formula di Riflessione (2.1), ossia: $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin z\pi}$, si ha, con z=1+i:

$$\Gamma[1-(1+i)]\cdot\Gamma(1+i) = \frac{\pi}{\sin(1+i)\pi} = \Gamma(-i)\cdot\Gamma(1+i) = \Gamma(i^*)\cdot\Gamma(1+i) = \Gamma^*(i)\cdot\Gamma(1+i) = \Gamma^*(i)\cdot i\Gamma(i) =$$

$= i|\Gamma(i)|^2 = i|i!|^2$, da cui:

$$|i!| = \sqrt{\frac{\pi}{i\sin(1+i)\pi}} = \sqrt{\frac{2i\pi}{i[e^{i\pi(1+i)} - e^{-i\pi(1+i)}]}} = \sqrt{\frac{2\pi}{[e^{i\pi}e^{-\pi} - e^{-i\pi}e^{\pi}]}} = \sqrt{\frac{2\pi}{[e^{\pi} - e^{-\pi}]}} = \sqrt{\frac{\pi}{\sinh(\pi)}} = |i!| \quad (3.2)$$

dopo aver ricordato che: $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, $\sinh(z) = \frac{e^z - e^{-z}}{2}$ e $e^{i\pi} = e^{-i\pi} = -1$.

Dunque, per la (3.2): $|i!| = \sqrt{\dfrac{\pi}{\sinh(\pi)}} = \sqrt{\dfrac{\pi}{\dfrac{(e^{\pi} - e^{-\pi})}{2}}} = \sqrt{\dfrac{3.14}{\dfrac{(23.1 - 0.043)}{2}}} = |i!| = 0.52$. (3.3)

A livello vettoriale, per la (3.1), si ha: $i! = \int_0^{\infty} e^{-t}\cos(\ln t)dt + i\int_0^{\infty} e^{-t}\sin(\ln t)dt$, dunque, valutando questi due integrali definiti che, come noto, denotano ognuno l'area sottesa dalla curva integranda, si ottiene l'espressione vettoriale $i! = a + ib$. Essendo questi degli integrali piuttosto complicati, effettuiamo una valutazione numerica e geometrica molto molto approssimativa! La prima funzione integranda, ossia $e^{-t}\cos(\ln t)$, è rappresentata in Fig. 3.1, mentre in Fig. 3.2 si valuta grossolanamente l'area da essa sottesa, tramite dei triangoli.

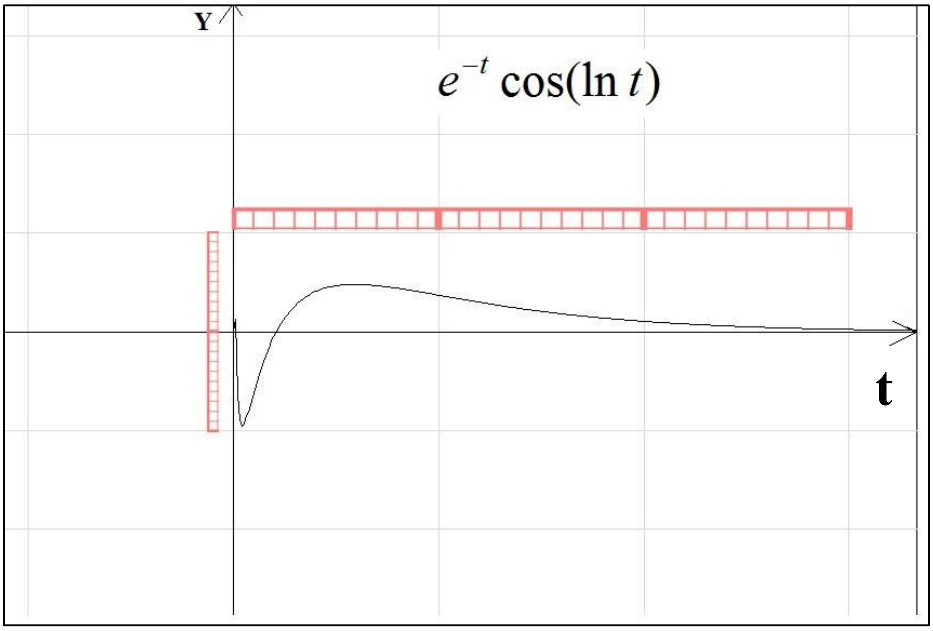

Fig. 3.1

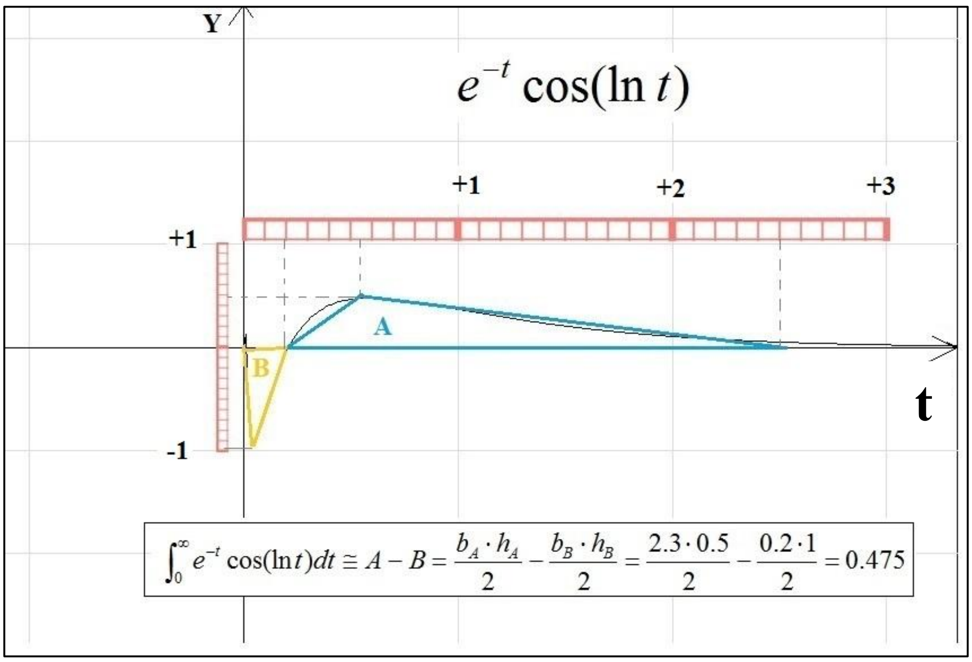

Fig. 3.2

Lo stesso facciamo per la seconda funzione integranda $e^{-t}\sin(\ln t)$, rappresentata in Fig. 3.3, mentre in Fig. 3.4 si valuta grossolanamente, come in precedenza, l'area da essa sottesa, sempre tramite dei triangoli.

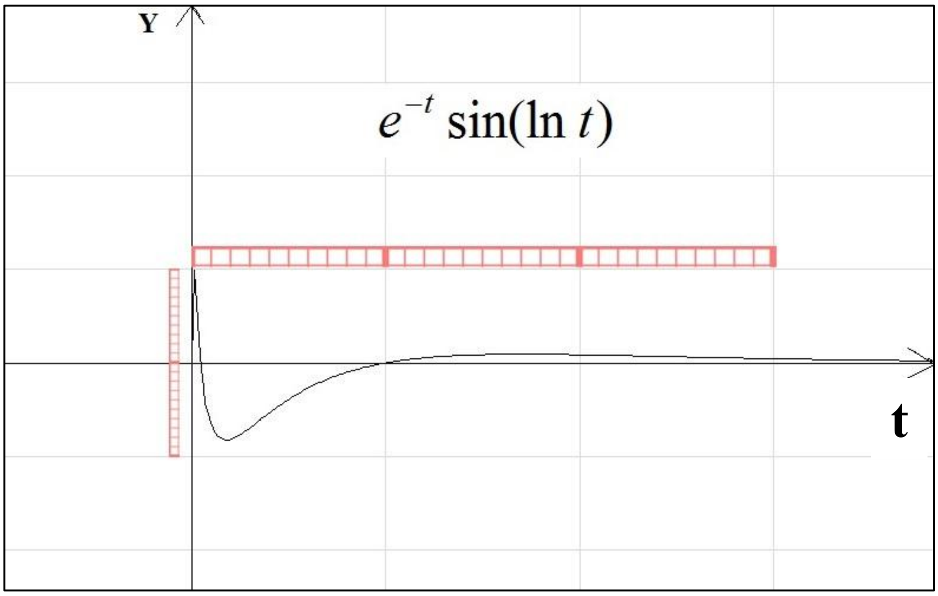

Fig. 3.3

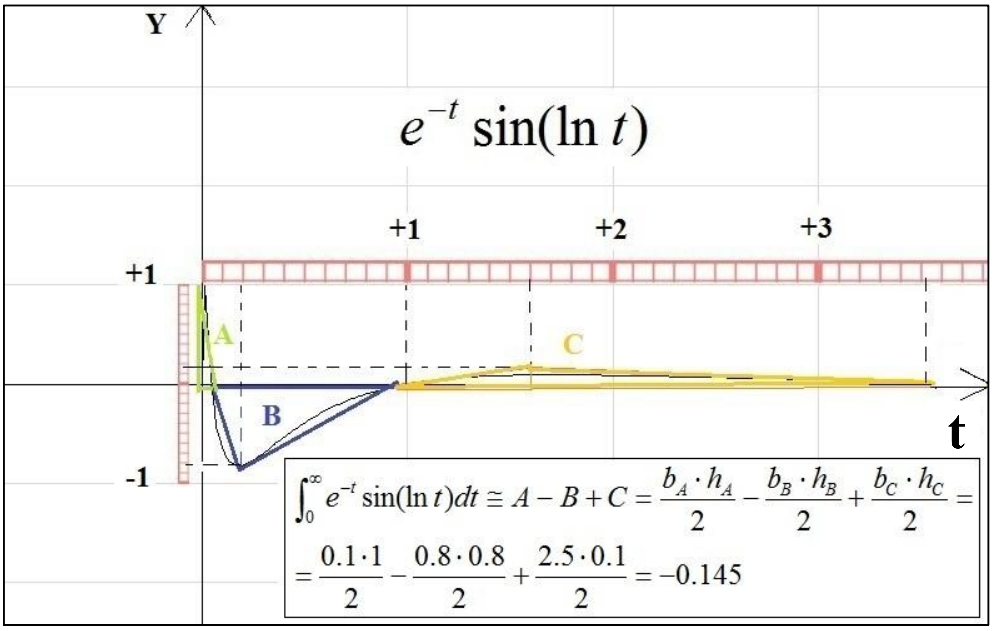

Fig. 3.4

Grossolanamente otteniamo: $i! \cong (0.475 - i0.145)$, con $|i!| = 0.50$.

Più ufficialmente è stato calcolato da altri il numero con un po' di precisione in più, ottenendo $i! \cong (0.4980 - i0.1549)$, con $|i!| = 0.52$; in entrambi i casi, viene confermato il modulo, fornito, per altra via, dalla (3.3).