Note on the function exp(x arctan(x))

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abstract

In this note we give two infinite products for exp(x arctan(x))

keywords: infinite series, infinite products, arctan function

1. Introduction

Let $h(x) = \exp(x \arctan(x)) = e^{x \arctan(x)}$, $\forall\, x \in \mathbb{R}$, we have

$$h(0) = 1 \tag{1}$$

$$h(-x) = h(x) \quad , \quad \forall\, x \tag{2}$$

$$h(x) < h(y) \quad , \quad 0 < x < y \tag{3}$$

$$h'(x) = e^{x \arctan(x)} \left(\frac{x}{1+x^2} + \arctan(x) \right) = h(x) \left(\frac{x}{1+x^2} + \arctan(x) \right) \tag{4}$$

$$h(x) = e^{x \arctan(x)} = F\left(\frac{i\,x}{2},\, 1,\, 1,\, -\frac{2\,i\,x}{1-i\,x} \right) \tag{5}$$

$$h(x) = e^{x \arctan(x)} = 1 - \sum_{n=1}^{\infty} (i\,x)^{n+1}\, F\left(1-n,\, 1+\frac{i\,x}{2},\, 2,\, 2 \right) \tag{6}$$

$$h(x) = e^{x \arctan(x)} = 1 + x^2 + \frac{1}{6} x^4 + \ldots = \sum_{n=0}^{\infty} a_n\, x^{2\,n} \quad , \quad |x| < 1 \tag{7}$$

where

$$a_0 = 1\,,\; a_{n+1} = \frac{(-1)^n}{n+1} \sum_{k=0}^{n} \frac{(-1)^k\,(n-k+1)}{2\,n-2\,k+1}\, a_k \;\;,\;\; n = 0,\, 1,\, 2,\, 3,\, \ldots \tag{8}$$

$$a_n = \left\{ 1,\, 1,\, \frac{1}{6},\, \frac{1}{30},\, -\frac{31}{2520},\, \frac{5}{504},\, -\frac{19\,549}{2\,494\,800},\, \ldots \right\} \tag{9}$$

$$h(x) = e^{-1}\, e^{x\,\pi/2} \sum_{n=0}^{\infty} b_n\, x^{-2\,n} \quad , \quad x > 1 \tag{10}$$

where

$$b_0 = 1\,,\; b_{n+1} = \frac{(-1)^n}{n+1} \sum_{k=0}^{n} \frac{(-1)^k\,(n-k+1)}{2\,n-2\,k+3}\, b_k \;\;,\;\; n = 0,\, 1,\, 2,\, 3,\, \ldots \tag{11}$$

$$b_n = \left\{ 1,\, \frac{1}{3},\, -\frac{13}{90},\, \frac{467}{5670},\, -\frac{18\,401}{340\,200},\, \frac{434\,509}{11\,226\,600},\, -\frac{2\,696\,933\,287}{91\,945\,854\,000},\, \ldots \right\} \tag{12}$$

$$\lim_{x \to \infty} h(x)\, e^{-x\,\pi/2} = e^{-1} \tag{13}$$

$$h(x) = e^{x \arctan(x)} = F\left(i x,\ 1,\ 1,\ 1 - \frac{1}{\sqrt{1+x^2}} - \frac{i x}{\sqrt{1+x^2}}\right) \tag{14}$$

$$h(x) = e^{x \arctan(x)} = \left(\frac{1 - i x}{1 + i x}\right) F\left(1,\ 1 - \frac{i x}{2},\ 1,\ \frac{2 i x}{1 + i x}\right) \tag{15}$$

$$h(x) = e^{x \arctan(x)} = e^{x \pi/2} F\left(\frac{i x}{2},\ 1,\ 1,\ \frac{2}{1 - i x}\right)\ ,\ x > 0 \tag{16}$$

$$\cosh(x \arctan(x)) = F\left(-\frac{i x}{4},\ \frac{i x}{4},\ \frac{1}{2},\ \left(\frac{2 x}{1 + x^2}\right)^2\right) \tag{17}$$

$$e^{x \arctan(x)} = -e^{-x \arctan(x)} + 2 F\left(-\frac{i x}{4},\ \frac{i x}{4},\ \frac{1}{2},\ \left(\frac{2 x}{1 + x^2}\right)^2\right) \tag{18}$$

Remark: $\pi = 4 \sum_{n=0}^{\infty} (2 n + 1)^{-1}$

Remark: $F(a,\ b,\ c,\ x)$ is the Gauss Hypergeometric function.

Remark: $i = \sqrt{-1}$

2. Two infinite products for $e^{x \arctan(x)}$

Entry 1. Define $f(n)$, $n = 1,\ 2,\ 3,\ \ldots$, by

$$f(1) = 1\ ,\ f(2) = -4/3\ ,\ f(n+1) = \frac{(-1)^n - 2 n - 1}{2 n + 1} - \sum_{1 \le k \le n-1,\ \frac{1+n}{1+k} \in \mathbb{N}} f\left(\frac{1+n}{1+k}\right)\ ,\ n = 2,\ 3,\ 4,\ \ldots \tag{19}$$

$$f(n) = \left\{1,\ -\frac{4}{3},\ -\frac{4}{5},\ \frac{4}{21},\ -\frac{8}{9},\ \frac{172}{165},\ -\frac{12}{13},\ \frac{8}{105},\ -\frac{12}{85},\ \frac{200}{171},\ -\frac{20}{21},\ -\frac{760}{5313},\ -\frac{24}{25},\ \ldots\right\} \tag{20}$$

Entry 2.

$$\arctan(x) = \sum_{n=1}^{\infty} \frac{x^{2 n - 1}}{1 - x^{2 n}} f(n) \tag{21}$$

$$x \arctan(x) = \frac{\ln(1 + x^2)}{2} - \sum_{n=1}^{\infty} \frac{f(n)}{2 n} \ln(1 - x^{2 n}) \tag{22}$$

$$\boxed{e^{x \arctan(x)} = \sqrt{1 + x^2}\ \prod_{n=1}^{\infty} (1 - x^{2 n})^{-\frac{f(n)}{2 n}}} \tag{23}$$

Entry 3. Define $g(n)$, $n = 1,\ 2,\ 3,\ \ldots$, by

$$g(1) = 1\ ,\ g(2) = 2/3\ ,\ g(n+1) = \frac{(-1)^n\, 2 n}{2 n + 1} - \sum_{1 \le k \le n-1,\ \frac{1+n}{1+k} \in \mathbb{N}} (-1)^k g\left(\frac{1+n}{1+k}\right)\ ,\ n = 2,\ 3,\ 4,\ \ldots \tag{24}$$

$$g(n) = \left\{1,\ \frac{2}{3},\ -\frac{4}{5},\ \frac{32}{21},\ -\frac{8}{9},\ -\frac{92}{165},\ -\frac{12}{13},\ \frac{328}{105},\ \frac{12}{85},\ -\frac{104}{171},\ -\frac{20}{21},\ -\frac{33\,424}{26\,565},\ -\frac{24}{25},\ \ldots\right\} \tag{25}$$

Entry 4.

$$\arctan(x) = \sum_{n=1}^{\infty} \frac{x^{2 n - 1}}{1 + x^{2 n}} g(n) \tag{26}$$

$$x \arctan(x) = \frac{\ln(1 + x^2)}{2} + \sum_{n=1}^{\infty} \frac{g(n)}{2 n} \ln(1 + x^{2 n}) \tag{27}$$

$$\boxed{e^{x \arctan(x)} = \sqrt{1 + x^2} \prod_{n=1}^{\infty} (1 + x^{2n})^{\frac{g(n)}{2n}}} \tag{28}$$

Entry 5.

$$\arctan(x) = \frac{x}{1+x^2} + \frac{2}{1+x^2} \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} F\left(1,\ 1+\frac{1}{2n},\ 2+\frac{1}{2n},\ x^{2n}\right) f(n) \tag{29}$$

$$\arctan(x) = \frac{x}{1+x^2} + \frac{2}{1+x^2} \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} F\left(1,\ 1+\frac{1}{2n},\ 2+\frac{1}{2n},\ -x^{2n}\right) g(n) \tag{30}$$

Remark: F is the Gauss Hypergeometric function.

Entry 6.

$$\frac{\pi}{6\sqrt{3}} = \sum_{n=1}^{\infty} \frac{f(n)}{3^n - 1} = \sum_{n=1}^{\infty} \frac{g(n)}{3^n + 1} \tag{31}$$

Entry 7.

$$e^{\pi/3\sqrt{3}} = \frac{4}{3} \prod_{n=1}^{\infty} (1 - 3^{-n})^{-f(n)/n} = \frac{4}{3} \prod_{n=1}^{\infty} (1 + 3^{-n})^{g(n)/n} \tag{32}$$

Entry 8.

$$\frac{\pi}{6\sqrt{3}} = \frac{\ln\left(2+\sqrt{3}\right)}{2\sqrt{3}} + 2 \sum_{n=1}^{\infty} \left(\frac{f(2n-1)}{3^{4n-2} - 1} + \frac{f(2n)}{3^{2n} - 1} \right) \tag{33}$$

$$\frac{\pi}{6\sqrt{3}} = \frac{\ln\left(2+\sqrt{3}\right)}{2\sqrt{3}} - 2 \sum_{n=1}^{\infty} \left(\frac{g(2n-1)}{3^{4n-2} - 1} - \frac{g(2n)}{3^{2n} + 1} \right) \tag{34}$$

Entry 9.

$$\pi = 4 \sum_{n=1}^{\infty} \left(\frac{2}{2^{2n} - 1} + \frac{3}{3^{2n} - 1} \right) f(n) \tag{35}$$

$$\pi = 4 \sum_{n=1}^{\infty} \left(\frac{2}{2^{2n} + 1} + \frac{3}{3^{2n} + 1} \right) g(n) \tag{36}$$

References

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