Analytical Proof of 3x+1

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Abstract

The Collatz or 3x + 1 conjecture is perhaps the simplest stated yet unsolved problem in mathematics in the last 70 years. It was circulated orally by Lothar Collatz at the International Congress of Mathematicians in Cambridge, Mass, in 1950 (Lagarias, 2010).

The problem is known as the Thwaites conjecture (after Sir Bryan Thwaites), Hasse's algorithm (after Helmut Hasse), or the Syracuse problem.

In this concise paper I provide a proof of this conjecture, by finding an upper bound to the Collatz sequence and, as a consequence, a contradiction.

Introduction

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The 3x + 1 problem is concerned with the converge to 1 of the following sequence

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \bmod 2 \\ 3n+1 & \text{if } n \equiv 1 \bmod 2 \end{cases}$$

In this concise paper I provide a proof of this conjecture, by finding an upper bound to the Collatz sequence and, as a consequence, a contradiction.

I have divided the proof into four premises and the final proof.

Discussion

Null Premise

Collatz converges for 1 and 2, which can be proved in just a few steps.

First Premise

Let's $\{a_n\}$ be a natural sequence, then it is known that

$\lim a_n = l \Leftrightarrow \lim b_m = l$, where b_m is subsequence of a_n

Second premise

The sequence,

$$g(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \bmod 2 \\ n + 1 & \text{if } n \equiv 1 \bmod 2 \end{cases}$$

converges to 1 as n goes to infinity. The proof is found in the appendix.

Proof by induction in n.

Observe that this is true for k = 1 and k = 2. Let's assume that we have the proof up to k = n-1, and prove it for k = n. If n is even, then we apply the division n/2 < n, which is the hypothesis. So, let's assume that n is odd. In this case there exist m such as n = 2m + 1. Applying the rule for odd numbers we get that the next term is 2m + 1 + 1 = 2(m + 1), which becomes m + 1 in the next iteration. Now we have m + 1 < 2m + 1 = n. But this is the hypothesis, namely all the numbers strictly less than n.

Third premise

$f(n) < 3 * g(f(n))$, for any n.

The proof is made by direct evaluation.Let's assume that the sequence starts with an even number, then

$f(n) = n/2$ and $3 * g(f(n)) = 3 * g(n/2)$

If n/2 is even then $3 * g(n/2) = 3 * n/4 > n/2$

If n/2 is odd then $3 * g(n/2) = 3 * (n/2 + 1) > 3 * n/2 > n/2$

If n is odd, then

$f(n) = 3 * n + 1$ and
$3 * g(f(n)) = 3 * g(3 * n + 1) = 3 * (3 * n + 1)/2 > 3 * n + 1$

So, $f(n) < 3 * g(f(n))$ for all n

Proof

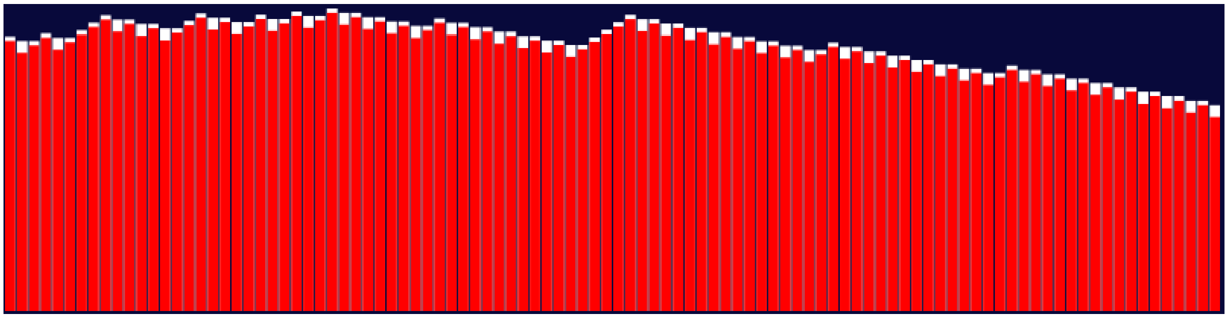
Let's consider the sequence C = $\{c_k\}_n$, generated by f starting at a given n, and assume it diverges.

But any element of C is the fírst element generated by gof(n). Let's call G the union of the sequences gof(n).

Then G converges since any of its subsequences converges (by premise 2). This makes C convergent since this sequence is a subsequence of G (by premise 1), which contradicts the assumption of divergency.

Now using premise 3, we know that the limit of f goes below 3 (limit of 3*g(f(n))), as n goes to infinity. It means that f must converge to 1, by the null premise. Q.E.D.

The following chart gives a hint of what is going on for 63728127 as the initial number



The red bars show $f(n)$ and the white bars show $3 * g(f(n))$, using a logarithmic scale. The code used to generate the chart is shown in the appendix.

Conclusion

The proof obtained here is the effort of two years of work. I first came across this problem during the pandemic, while attending a master in embedded systems at Uppsala University, Sweden in 2021.

I believe the same approach could be used to solve problems of similar nature.

Appendix

Javascript code used to generate the sample in the chart.

```
const startNumber = 63728127n

let nextColas = n => 3n*(n % 2n ?  n + 1n : n >> 1n)
let nextColaz = n => n % 2n ? 3n*n + 1n : n >> 1n

let bigmax = (a,b) => a >= b ? a : b

let biglog  = bigint => {
  if (bigint < 0) return NaN
  const s = bigint.toString(10)

  return s.length + Math.log10("0." + s.substring(0, 15))
}
```

```
let initialize = n => {
	let data = {
		colaz : [n],
		colas : [3n*n],
		max : -1n
	}

	for (let count=0; count < 101; count++ ){
		let clz = nextColaz(data.colaz[0])
		let cls = nextColas(clz)

		data.colaz.unshift(clz)
		data.colas.unshift(cls)
		data.max = bigmax(bigmax(data.max, clz), cls)
	}

	return data

}
```

Reference

Lagarias, 2010. Jeffrey C. Lagarias. 2010. The Ultimate Challenge: The 3x + 1 Problem.