Proof for Collatz Conjecture

Mesut Kavak[a]

The question about the Collatz Problem is pretty clear:

"When a positive integer is chosen, if the number is even, it is divided by 2; otherwise, it is multiplied by 3 and after that 1 is added to the result. Due to the condition that the result is odd or even, the same operation is repeated with the required option of the problem, every positive integer other than 0 and 1 can the integer be reduced to 1?"

[a]kavakmesut@outlook.com.tr

I. Introduction

Theorem *If the input number is not only an even positive integer such as 2^σ, being the number of operations is "σ" and the input number is divided by σ times 2, that is directly by 2^σ, every even number will definitely turn into an odd number as it can be written as $(2\alpha+1)\cdot 2^\sigma$; therefore, this problem should only be worked with all possible odd numbers from the start.*

I.

for the definition of

- $x \in \mathbb{Z}^+$
  $x>0$

$$a_{n+1} = \frac{3a_n + 1}{2} \tag{1}$$

over (1) it becomes

$$a_{n+1} = \{2, 5, 8, 11, ..., 3x-1\}$$

Of these numbers depending on the function f(x)=3x-1, the numbers that make a_n an odd number become

$$a_{n+1} = \{5, 11, 17, ..., 6x-1\}$$

respectively depending on the function f(x)=6x-1. Numbers that make a_{n+1} an odd number become

$$a_n = \{3, 7, 11, 15, ..., 4x-1\}$$

respectively, depending on the function f(x)=4x-1. "O" is odd number and "E" is even number, below is the table of this condition.

a_n	3	7	11	15	19	23	...
a_{n+1}	5	11	17	23	29	35	...
a_{n+2}	E	O	E	O	E	O	...

Table I.

II.

In the line a_{n+1} on the table, there are only numbers that depend on 2 functions such as f(x)=12x-7 and f(x)=12x-1, respectively.

For numbers that depend on the function f(x)=12x-7, (1) occurs over (2), and for every "x" value, the result is an even number.

$$18x - 10 = \frac{3(12x-7)+1}{2} \tag{2}$$

For numbers linked to the f(x)=12x-1 function, (3) occurs over (1), and the result for each "x" value is necessarily an odd number.

$$18x - 1 = \frac{3(12x-1)+1}{2} \tag{3}$$

II. Questions

Questions I. *In this case, the first question should be asked: For* (4)*, Is the "odd number" when* a_{n+2} *is an "even number" and divided by* 2^{σ}*, always smaller than the odd number* a_{n+1} *in the set of odd numbers and before the order on the number line, or can a larger odd number occur?*

$$a_{n+2} = \frac{3a_{n+1}+1}{2} \tag{4}$$

If the answer is; If the result of the (4) operation is even,

$$\frac{\left(3a_{n+1}+1\right)/2}{2^{\sigma}}$$

the condition (5) must be satisfied, since the result of the operation cannot be greater than or equal to the odd number a_{n+1}.

$$1 > \frac{\frac{\left(3a_{n+1}+1\right)/2}{2^{\sigma}}}{a_{n+1}} \tag{5}$$

If (5) is edited it becomes (6),

$$1 > \frac{1}{2^{\sigma+1}}\left(3+\frac{1}{a_{n+1}}\right) \tag{6}$$

The (6) inequality always provides this with the conditions $\sigma > 0$ and $a_{n+1} > 1$.

As a result, when the number a_{n+2} is reduced to an odd number by the rule (1) or, in direct relation by (4), this odd number is always smaller than the odd number a_{n+1}, which creates the number a_{n+2}. This rule also applies to a_{n+3} and other repetitions.

Question II. *In this case, a second question should be asked; in the infinite repetition of the* (1) *rule, is there a number that is constantly growing and cannot be an odd number?*

As for the answer, below is another and a second table for numbers that are not in the first table. If the numbers in this table are added to the first table, a set of odd numbers is obtained in total; therefore, there will not be a single odd number that is not included in the calculations.

5	9	13	**17**	21	25	**29**	...	4x+1

Table II.

Numbers in bold are numbers from row a_{n+1} in the first table. Others are odd numbers that are not in the first table.

If some groups need to be made for the numbers depending on the function f(x)=4x+1 in the second table, there will be "3 groups" that depend only on the following functions:

- f(x)=12x-3
- f(x)=12x+1
- f(x)=12x-7

I.

(7) always returns even number results for $a_{n+1} = 12x-3$.

$$18x-8 = \frac{3(12x-3)+1}{2} \tag{7}$$

II.

(8) always returns even results for $a_{n+1} = 12x+1$.

$$18x+2 = \frac{3(12x+1)+1}{2} \tag{8}$$

III.

It has already been said above that $a_{n+1} = 12x-7$ is an even number as a result of (2).

III. Solution

Theorem *As for even integers, they can always be reduced to an odd number smaller than itself, as shown above, with the* (1) *operation; so if each of the* $a_n = 4x - 1$ *numbers does not always grow to an odd number when* (1) *is repeated, this means that every positive integer except 0 and 1 can be reduced to 1 by the* (1) *rule.*

I.

In Table I., $8x-5$ numbers from row a_n become a_{n+1} numbers via (4); so the only possibility to always result in an odd number in the infinite iteration of (1) is to use the numbers $8x-1$. For this to happen, the numbers a_n must appear in the row a_{n+1} in Table I. Also, an odd number among them should appear on the a_{n+1} line.

Operations $4x-1$ and $6x-1$ yields a_n and a_{n+1} via (1); so the expected loop occurs or not. (9) indicates this on the basis of the equation $4x_1 - 1 = 6x_2 - 1$.

$$x_1 = \frac{6x_2}{4} \tag{9}$$

For (9), and $t \in \mathbb{Z}^+ \wedge t > 0$ definition, it becomes $x_1 = 3t$ and $x_2 = 2t$; therefore the whole problem is reduced to the rule of Table III.

2t	2	4	6	8	10	12	...
3t	3	6	9	12	15	18	...

Table III.

Each number on $4x-1$ and $6x-1$ are also sequence numbers a_n and a_{n+1} on Table III; so in Table III the number with the sequence number 3t on the row a_n or 2t and the number with the sequence number 2t on the row a_{n+1} or 3t are the same numbers .

II.

Table III odd numbers in the 3t row above are eliminated as there are a_{n+1} numbers in Table I. that result in even integers, and Table III becomes Table IV immediately below.

4t	4	8	12	16	20	24	...
6t	6	12	18	24	130	36	...

Table IV.

When a number "4t" is selected to provide an infinite loop on Table IV, the number "6t" in the line "6t" just below "4t" must be even and at the same time this number "6t" should appear again later on line "4t". For Collatz's rule to be broken, this condition must be met for 1 or more numbers again and again; which means that 1 or more numbers cannot be reduced to 1. This is impossible; because with the condition $t > 0 \wedge t, n \in \mathbb{Z}^+$ over $t_{n+1} = \dfrac{6t_n}{4}$ for every integer "t",

$$t_{n+1,t} = \lim_{n \longrightarrow \infty} \frac{6t_{n,t}}{4} \tag{10}$$

$t_{n,t} = 4t$; "n" is the number of iterations of operation (10) for each number of "t"; (10) must be supplied with the condition (11) where t is the ordinal number of the number expected to loop.

$$t_{n,t}, t_{n+1,t} \in \mathbb{Z}^+ \tag{11}$$

As for the condition (11), it cannot be satisfied for every number "t". For example, for $t_{1,1} = 4$, $6t_{1,1} = 4t_{2.1}$, resulting in $t_{2.1} = 6$; For $t_{2.1} = 6$, likewise, $6t_{2,1} = 4t_{3.1}$, resulting in $t_{3.1} = 9$; For $t_{3.1} = 9$, $6t_{3.1} = 4t_{4.1}$, resulting in $t_{4.1} = 27/2$. As you can see, the condition (11) cannot be met since it will be $t_{4,1} \notin \mathbb{Z}^+$.

IV. Result

I.

Operation (10) cannot enter an infinite loop; because this requires imaginary numbers with an infinite number of common divisors, such as 4^{∞} or $(2x+1)\cdot 4^{\infty}$. As can be seen, with the condition $m > 0 \wedge m \in \mathbb{Z}^{+}$, only the number of repetitions can be increased by using a "t" number such as 4^{m}; which increases again if "m" increases; but never an endless repetition.

Result *Every positive integer can be reduced to 1 at the end with the "Collatz Operation", which is the subject of the problem, with the number of repetitions that may change according to the number entered.*

II.

Operation (1) returns m+2 times odd numbers and the last one is even number, where $m > 0 \wedge m \in \mathbb{Z}^{+}$ for (1) repetitions, 2^{m} being the sequence number of a_n selected on Table I. There is a representation of this on Table V.

m		**a_n**			
1	7⟶	11⟶	17		
2	15⟶	23⟶	35⟶	53	
3	31⟶	47⟶	71⟶	107⟶	161
:					

Table V.

Also, different operations can be written for both repetition numbers and other properties. Just below, (12) is one of them. The first number entered for (12) is $a = a_0$ and "m+1" times odd numbers or repetitions occur on (1).

$$a = 7 + \sum_{m=3}^{m} 2^{m} \tag{12}$$

12.06.2023