A easy approach for the *sinc* integral

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ABSTRCT

In this paper, by using simple mathematical method we established a generalized formula of

$$\int_0^\infty \mathrm{sinc}(x)\mathrm{sinc}\left(\frac{x}{a_1}\right)\mathrm{sinc}\left(\frac{x}{a_2}\right)\cdots\mathrm{sinc}\left(\frac{x}{a_n}\right)dx.$$

In fact, the calculation method of this integral is introduced in several papers by using some advanced analysis knowledge like Fourier transform, Poisson summation and so on. But we used only very general mathematical knowledge.

Keywords: *Sinc* function, Integral

1. Introduction

The *sinc* function is well known in people and have many interesting properties. These mathematical properties are well known in electrical engineers and physicists and used in many domains like signal processing. We generally defined *sinc* function as below.

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0. \end{cases}$$

We note that the following equality holds.

$$\int_0^\infty \frac{\sin kx}{x} dx = \begin{cases} \dfrac{\pi}{2} & \text{if} \quad k > 0, \\ -\dfrac{\pi}{2} & \text{if} \quad k < 0. \end{cases}$$

Let's consider following integral.

$$\int_0^\infty \mathrm{sinc}(x)\mathrm{sinc}\left(\frac{x}{a_1}\right)\mathrm{sinc}\left(\frac{x}{a_2}\right)\cdots\mathrm{sinc}\left(\frac{x}{a_{n-1}}\right)dx.$$

When $n = 1, 2, 3, \cdots, 7$, we have

$$\int_0^\infty \prod_{k=1}^{n} \mathrm{sinc}\frac{x}{2k-1} dx = \frac{\pi}{2}. \tag{1}$$

and when $n = 8, 9, \cdots$, equation (1) is not true, in general. The left-side is smaller than right-side.

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Many authors have considered this problem and given answer to why the equation (1) is not true for all n. And they also introduce some properties of integral related to *sinc* and calculation formula using some analysis skills. For example, David and Jonathan Borwein used Fourier transform in [2] and Gert Almkvist and Jan Gustavsson used Poisson summation formula in [5].

In this paper, we focus on using simple method for well-understanding of more readers. We used very low-level skills like trigonometric formula, mathematical induction, L' hospital's rule and some integral formulas.

2. Lemmas

Let n be a natural number, and a_i s be real numbers.
We use a following notation.

$$A_n(x) := \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{a_1}\right)\operatorname{sinc}\left(\frac{x}{a_2}\right)\cdots\operatorname{sinc}\left(\frac{x}{a_n}\right)dx,$$

Lemma 1. For each of the 2^n ordered sets $X := (x_1, x_2, \dots, x_n) \in \theta_n$ define

$$\gamma(X) := \prod_{k=1}^{n} x_k,\quad P(X) := 1 + \sum_{k=1}^{n} \frac{x_k}{a_k},$$

where $\theta_n = \{-1,\ 1\}^n$.
Then

$$A_n(x) = \frac{a_1 a_2 \cdots a_{n-1}}{2^{n-1}} \cdot \frac{H_n(x)}{x^n} \tag{2}$$

where

$$H_n(x) = \begin{cases} (-1)^{\frac{n}{2}} \sum_{X\in\theta_{n-1}} \gamma(X)\cos P(X)x, & \text{if } n = even, \\ (-1)^{\frac{n-1}{2}} \sum_{X\in\theta_{n-1}} \gamma(X)\sin P(X)x, & \text{if } n = odd. \end{cases}$$

Proof. A calculation shows that equation (2) is true for $n = 2,3$.
Let's use mathematical induction. We assume that equation (2) is true when $n = k$ (we also assume that k is odd). Then

$$A_{k+1}(x) = A_k(x)\operatorname{sinc}\left(\frac{x}{a_k}\right) = \frac{a_1 a_2 \cdots a_{k-1} a_k}{2^{k-1}} \cdot \frac{H_k(x)}{x^{k+1}} \cdot \sin\left(\frac{x}{a_k}\right),$$

and

$$H_k(x)\sin\frac{x}{a_k} = (-1)^{\frac{k-1}{2}}\left(-\frac{1}{2}\right)\sum_{X\in\theta_{k-1}} \gamma(X)\left\{\cos\left(P(X)+\frac{1}{a_k}\right)x - \cos\left(P(X)-\frac{1}{a_k}\right)x\right\}$$

$$= \frac{(-1)^{\frac{k+1}{2}}}{2}\sum_{X\in\theta_k}\gamma(X)\cos P(X)x = \frac{1}{2}H_{k+1}(x).$$

From this, we can know that equation (2) is true when $n = k+1$ ($k+1$ is even).
By using the same way, we also know that equation (2) is true when $n = k+2$ ($k+2$ is odd).
From this the lemma is proved. □

Lemma 2. For any real number x,

$$\int_0^\infty \frac{H_n(x)}{x^n}dx = \sum_{X\in\theta_{n-1}} \gamma(X)\frac{P(X)^{n-1}}{(n-1)!}\int_0^\infty \frac{\sin(xP(X))}{x}dx. \tag{3}$$

Proof. We assume that n is even. Then,

$$\int_0^\infty \frac{H_n(x)}{x^n}dx = (-1)^{\frac{n}{2}}\int_0^\infty \frac{1}{x^n}\sum_{X\in\theta_{n-1}}\gamma(X)\cos P(X)xdx = (-1)^{\frac{n}{2}}\sum_{X\in\theta_{n-1}}\gamma(X)\int_0^\infty \frac{\cos P(X)x}{x^n}dx.$$

You can easily get following result using partial integration when n is even. We leave the proof to readers.

We have that

$$\int \frac{\cos mx}{x^n}dx = (-1)^{\frac{n}{2}}\frac{m^{n-1}}{(n-1)!}\int \frac{\sin mx}{x}dx + \sum_{i=1}^{n-1} c_i \frac{m^{n-1-i}}{x^i}\sin mx,$$

where c_i s are coefficients.

From this, we have

$$\int_0^\infty \frac{H_n(x)}{x^n}dx = \sum_{X\in\theta_{n-1}} \gamma(X)\frac{P(X)^{n-1}}{(n-1)!}\int_0^\infty \frac{\sin(xP(X))}{x}dx + \sum_{X\in\theta_{n-1}}\sum_{j=1}^{n-1} c_j\gamma(X)\frac{P(X)^{n-1-j}}{x^j}Q_j(xP(x))\Big|_0^\infty,$$

where

$$Q_j(x) = \begin{cases} \sin x & \text{if } j = even, \\ \cos x & \text{if } j = odd. \end{cases}$$

By using L' hospital's rule we have

$$\sum_{X\in\theta_{n-1}}\sum_{j=1}^{n-1} c_j\gamma(X)\frac{P(X)^{n-1-j}}{x^j}Q_j(xP(x))\Big|_0^\infty = 0.$$

and the equation (3) is true. By using same way, we also know the equation (3) is true when n is odd. From these results, the lemma is proved. □

3. Main Result

Theorem 1. Let $a_1, a_2, a_3, \ldots, a_n$ be real numbers. Then

$$\int_0^\infty A_n(x)dx = \frac{\pi}{2}\left\{1 - 2\frac{a_1a_2\cdots a_{n-1}}{2^{n-1}(n-1)!}\sum_{X\in\theta_{n-1}, P(X)<0}\gamma(X)P(X)^{n-1}\right\} \tag{4}$$

Proof. From the equation (2) and equation (3), we have

$$\int_0^\infty A_n(x)dx = \frac{a_1a_2\cdots a_{n-1}}{2^{n-1}}\int_0^\infty \frac{H_n(x)}{x^n}dx = \frac{a_1a_2\cdots a_{n-1}}{2^{n-1}}\sum_{X\in\theta_{n-1}}\gamma(X)\frac{P(X)^{n-1}}{(n-1)!}\int_0^\infty \frac{\sin(xP(X))}{x}dx$$

$$= \frac{a_1a_2\cdots a_{n-1}}{2^{n-1}(n-1)!}\sum_{X\in\theta_{n-1}}\gamma(X)P(X)^{n-1}\int_0^\infty \frac{\sin(xP(X))}{x}dx = \frac{a_1a_2\cdots a_{n-1}}{2^{n-1}(n-1)!}\sum_{X\in\theta_{n-1}}\gamma(X)P(X)^{n-1}\frac{\pi}{2}.$$

And let's calculate the following expression.

$$\sum_{X\in\theta_{n-1}}\gamma(X)P(X)^{n-1}.$$

In the expansion of

$$\gamma(X)P(X)^{n-1} = x_1 x_2 \cdots x_{n-1}\left(1+\frac{x_1}{a_1}+\frac{x_2}{a_2}+\ldots+\frac{x_{n-1}}{a_{n-1}}\right)^{n-1},$$

there are $(n-1)!$ terms $\dfrac{1}{a_1 a_2 \cdots a_{n-1}}$ and sum of terms $cx_j x_{j+1} \cdots x_{j+r}$ where c is a coefficient. The value of this sum is 0 because of $\sum_{X \in \theta_{n-1}}$. So,

$$\sum_{X \in \theta_{n-1}} \gamma(x)P(X)^{n-1} = \frac{2^{n-1}(n-1)!}{a_1 a_2 \cdots a_{n-1}}.$$

From this, the theorem 1 is completed. □

So now we can say why the equation (1) is not equal to $\frac{\pi}{2}$ all the time.

When $n \le 7$, there is no case such that $P(X)$ is negative. So the value of the equation (1) is always $\frac{\pi}{2}$.

But when $n = 8$, there is a case such that $P(X)$ is negative.

$$1-\frac{1}{3}-\frac{1}{5}-\cdots-\frac{1}{15} = -\frac{982}{45045} < 0.$$

From the equation (4), we have

$$\int \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{15}\right)dx = \frac{1\cdot 3\cdot \ldots \cdot 15}{2^7 7!}\left(\frac{2^7 7!}{1\cdot 3\cdot \ldots \cdot 15} - 2\left(\frac{982}{45045}\right)^7\right)\frac{\pi}{2}$$

$$= \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi.$$

4. Conclusion.

Until now, we've derived the general formula for calculating the singular integrals of product of *sinc* functions.

In short, the reason why the result change suddenly is due to relation of coefficients.

In other words, if $a_0 > a_1 + a_2 + \cdots + a_n$, the result is always equal to $\frac{\pi}{2}$.

But else, the result change chaotically.

Preferences

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