# Illustrative axiomatic derivation of the special Lorentz transformation from merely the properties of empty space and inertial systems

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## Abstract

The Lorentz transformation is derived merely from the properties of space and time when space is empty and Galileo relativity. Additional postulates about the speed of light, reciprocity, and other ones are *not* necessary. Straight world lines are bijectively mapped onto straight world lines. This known fact is exploited in an illustrative manner. This is extremely useful for teaching special relativity, in particular, at an elementary level. Moreover, the approach described here, (i), provides an example of strict physical thinking, (ii), corrects a widespread erroneous belief, see over-next paragraph, and, (iii), presents an elementary introduction to the largely unknown hyperbolic rotations (the common rotations are circular).

The transformation to be found is represented as a kind of rotation times a Lorentzian 'scale factor'. This crucially simplifies the calculations and is much easier to grasp than a rather abstract ansatz with unknown coefficients. The rotation is proven to be hyperbolic rather than circular. After that, the scale factor turns out to equal unity in a most direct manner. The reciprocity property of the transformation is obtained as a by-product.

Not special relativity makes an *additional* assumption for justifying the appearance of a seemingly *additional* natural constant, the speed of light in vacuum $c$, but classical mechanics does whence $c$ disappears. Two common basic assumptions of classical mechanics lead not to the Galileo but to the Lorentz transformation.

The existence of a maximum speed of bodies is shown to be a purely kinematic effect, too. Einstein's second postulate is obtained as a by-product.

## I. INTRODUCTION

The Lorentz transformation is the mathematical heart of the theory of special relativity. For this, it enters all representations of electromagnetism and all other theories which are compatible with the theory of special relativity. Accordingly, there is quite a huge variety of derivations of it. Unfortunately, the ones we are aware of are hard to understand for beginners, involve cumbersome calculations, and contain more assumptions than necessary, respectively. For this, we have developed a derivation which uses no more premises than

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being necessary. Surprisingly enough, that derivations are being used within non-relativistic (!) classical mechanics. In particular, we will reason and—in an illustrative manner—exploit the known fact that straight world lines are mapped onto straight world lines (e.g. [1] p. 554, [2] p. 167, [3] p. 92).

Thus, the goal of this contribution consists in avoiding not Einstein's 1905 second postulate ([4] pp. 892, 899ff.; for a very recent review of such attempts, see [5] and also [6] and references herein) but cuts by Ockham's razor ("more geometrico"). As Newton has putted it,

> "No more causes of natural things should be admitted than are both true and sufficient to explain their phenomena." (Newton 1726 [7] Book 3, Rule 1)

Einstein's second postulate will be obtained as a by-product.

Most calculations, which exploit the linearity of the transformation, make an ansatz with *four* coefficients to be found and involve the lengthy calculation of subsequent and inverse transformations for exploiting the group property. In contrast, the ansatz proposed here contains only *two alternative* rotation angles and *one* unknown function. The group property is used but does not require lengthy calculations; for rotations, it is automatically fulfilled.

As Macdonald [8] stresses, it is often (mostly implicitly) assumed that the relative velocity of frame $S$ w.r.t. frame $S'$ is the opposite of the relative velocity $v$ of $S'$ w.r.t. $S$, i.e. equal $-v$ (reciprocity). One argument consists in that "there is no privileged Galileo [inertial] frame of reference" [9]. Mermin [10] and Datta [5] conclude it from the symmetry of the relative motion of reference frames against the change of the direction of all velocities involved. Actually, its rigorous proof *ab ovo* appears rather cumbersome [11]. An elementary derivation will be presented in Appendix C. In the main text, we will obtain it as a by-product.

Thus we will begin with the two premises in Section II. In section III, we will describe the mapping of straight world lines onto straight world lines as a kind of rotation times a 'scale factor'. In Section IV, we will formulate the general expression of the transformation sought for and list its general properties.

In Section V, the rotation matrix will be calculated. In Section VI, the scale factor will extremely quickly be determined to equal one, using the invariant length found in Subsection

V B. Both results will lead to the special Lorentz transformation in Section VII.

In Section VIII, we will show that the vanishing of the acceleration when the speed of a body approximates that of light in vacuum is a purely kinematic effect. Section IX will shortly deal with the small-speed limit of the special Lorentz transformation for seconding the less-known fact that, for small but finite ratios of the motion of a reference frame and the speed of light, the limit case Galileo transformation contains an additional requirement. In Section X, we will present a straight foundation of Einstein's second postulate ([4] pp. 892, 900f.). Finally, Section XI will summarize and conclude this contribution.

Some details are delegated to the appendices. Appendix A presents some mathematical subtleties of the mapping of straight lines onto straight lines. Appendix B contains additional historical and mathematical material concerning the scale factor. For Appendix C, see above.

## II. ONLY TWO PREMISES

We will use only two premises.

**Premise 1.** There is three-dimensional space and one-dimensional time. They can be equipped with arbitrarily many one- ($t$) and Cartesian three-dimensional ($x$, $y$, $z$) coordinate systems, respectively.

Consequently,

**Corollary 1.** They build one-. . . four-dimensional coordinate spaces. As a consequence, they may move w.r.t. each to another.

**Corollary 2.** By the principle of sufficient reason, in an empty world, space and time are homogeneous and isotropic.[1]

**Corollary 3.** Consequently, empty space is Euclidean.[2]

**Corollary 4.** Moreover, the coordinates be of equal spacing (for an operational realization of that within this approach, see [11] p. 1520 I, fn. 9).

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[1]Often, that is additionally postulated, e.g. [11] p. 1518 II.

[2]Often, that is additionally presupposed. Wehrli [12] argues that a flat space is merely a very special case of a curved space. However, in an empty space, there is nothing that could determine the magnitude of the curvature. Moreover, some authors write from the very beginning 'space-time'. In contrast, we will argue for that notion only when considering the coordinate transformation sought for in more detail.

Next, in agreement with Newton's 1st axiom, we define the notion of a *free, interactionless body.*

**Definition 1.** There are bodies, i.e. collections of matter in space having certain properties. The bodies may interact with each other but not with space and time (as within general relativity). A body without any interaction is called a *free* body.[3]

We will deal only with free point-like bodies, i.e. not include rotations and consider only coordinate transformations between inertial systems (frames).

**Definition 2.** A reference frame (coordinate system in the sense of Premise 1), in which *all* free bodies are at rest or in straight uniform motion, is called an *inertial* or *Galileo* reference frame or system.

This leads us to the second premise, Galileo relativity.[4]

**Premise 2.** All inertial systems are physically equivalent.[5]

That two premises are applied already within non-relativistic classical mechanics. This fact makes it even more surprising that they are *sufficient* to deduce not the Galileo transformation (being usually connected with classical mechanics) but the Lorentz transformation. As a matter of fact, they are more or less implicitly assumed in most derivations of the Lorentz transformation we are aware of. However, it is not special-relativistic mechanics which makes an additional assumption but Newtonian mechanics, see the summary below.

In particular, it is *not* necessary to make any assumption about the propagation of light. Often, it is postulated that the speed of light in an inertial system is

[3]Generally speaking, a body is also called free, if the sums of all forces and all torques applied upon it vanish. However, in this contribution, forces and torques are not needed and thus not defined.

[4]As a matter of fact, one could formulate this premise as an axiom. For the purpose of this contribution, it is not necessary to discriminate between premises and axioms.

[5]"Das Relativitätsprinzip behauptet: Man kann aus der Gesamtheit der Naturerscheinungen durch immer weiter gesteigerte Annäherung ein Bezugssystem $x$, $y$, $z$, $t$ bestimmen, in welchen die Naturgesetze in bestimmten, mathematisch einfachen Formen gelten. Dies Bezugssystem ist aber durch die Erscheinungen keineswegs eindeutig festgelegt. Vielmehr gibt es eine dreifach unendliche Mannigfaltigkeit gleichberechtigter Systeme, welche sich gegeneinander mit gleichförmigen Geschwindigkeiten bewegen." (v. Laue 1911 [13]; latterspaced in the original text) – En.: The principle of relativity asserts: One can determine a reference system $x$, $y$, $z$, $t$ from the totality of the natural phenomena by ever further increased approximation, in which the laws of nature are valid in certain, mathematically simple forms. However, this reference system is by no means uniquely determined by the phenomena. Rather there is a threefold infinite manifold of equal systems which move against each other with uniform velocities.

- independent of the motion of the emitter ([4] p. 892/140)[6], and/or
- independent of the velocity of the detector which moves straight uniformly w.r.t. that inertial system ([4] pp. 900f./p. 149)[7], and/or

[6]Actually, that is a consequence of Maxwell's theory of light [14], in which $c = 1/\sqrt{\varepsilon_0 \mu_0}$. "Das dort [in [4]] benutzte Prinzip der Konstanz der Lichtgeschwindigkeit ist natürlich in den M a x w e l l schen Gleichungen enthalten." ([15] p. 639 fn. 2). En: "The principle of the constancy of the velocity of light used there is of course contained in Maxwell's equations." ([15] p. 172 fn. 2) Actually, this does not refer to the true second postulate, the independence of the motion of the observer. Morover, it is not necessary to presuppose Maxwell's theory.

[7]Actually, that is a consequence of Maxwell's theory of light [14] and Galileo relativity, Premise 2 above. To see that, let us consider the following situation (inspired by Verkhovsky [19]). In an inertial system, there be two persons $A$ and $B$ at positions $\vec{x}_{\rm A} = (0, 0, 0)$ and $\vec{x}_{\rm B} = (L, 0, 0)$, respectively. Moreover, there be a person $N$ in a rocket moving with speed $v$ along the $x$-axis. At time $t = 0$, the rocket crosses the plane $x = 0$, and light signals are sent from both $A$ and $N$ towards $B$. Both light signals reach $B$ at the same time because their speed of propagation is independent of the motions of $A$ and $N$, see the foregoing point. Now, by the principle of Galileo relativity (Premise 2 above), the rest system of $A$ and $B$ is equivalent with that of $N$, i.e. within the rest system of $N$, the speed of the light signal also equals $c$. Hence, $B$ observes the speed of that light signal to equal $c$, although she moves relatively to the rest system of $N$.—See also [20] and references therein.

- the same in all inertial systems. Actually, that is a consequence of Galileo relativity (Premise 2 above). We guess that the authors have in mind 'relatively to all inertial systems'.[8]

For this and other axiomatic reasons (see above), we will derive the (special) Lorentz transformation without the reference to light propagation as first attempted by v. Ignatowsky [22] and Frank & Rothe [23] (for a very recent review of such attempts, see [5] and also [6] and refs. therein).

## III. MAPPING STRAIGHT WORLD LINES ONTO STRAIGHT WORLD LINES

### A. Straight world lines

In agreement with Newton's 1st axiom, Definition 1 and Premise 2 imply that a single body in an otherwise empty space either stays at rest or is in straight uniform motion, and this w.r.t. *all* inertial systems. Rest and straight uniform motion are described by straight world lines, see Figure 1 [24]. For deriving the special Lorentz transformation, it is sufficient to consider motions in the $x$-$t$-plane.

Notice that this description does *not* yet presuppose or imply a space-time as envisaged by Poincaré and Minkowski. In particular, there is not yet a metric tensor.

[8] One may argue that the same could be applied to sound waves. However, in general, sound waves propagate isotropically only in the rest frame of an isotropic medium. But "the vacuum has no rest frame" (Lindquist, quoted after [21] p. 12, fn. 1). Admittedly, the classical vacuum is not really empty as it exhibits the natural constants $\varepsilon_0$ and $\mu_0$. In what follows, we will present a treatment which is free of such uncertainties as light is not used at all.

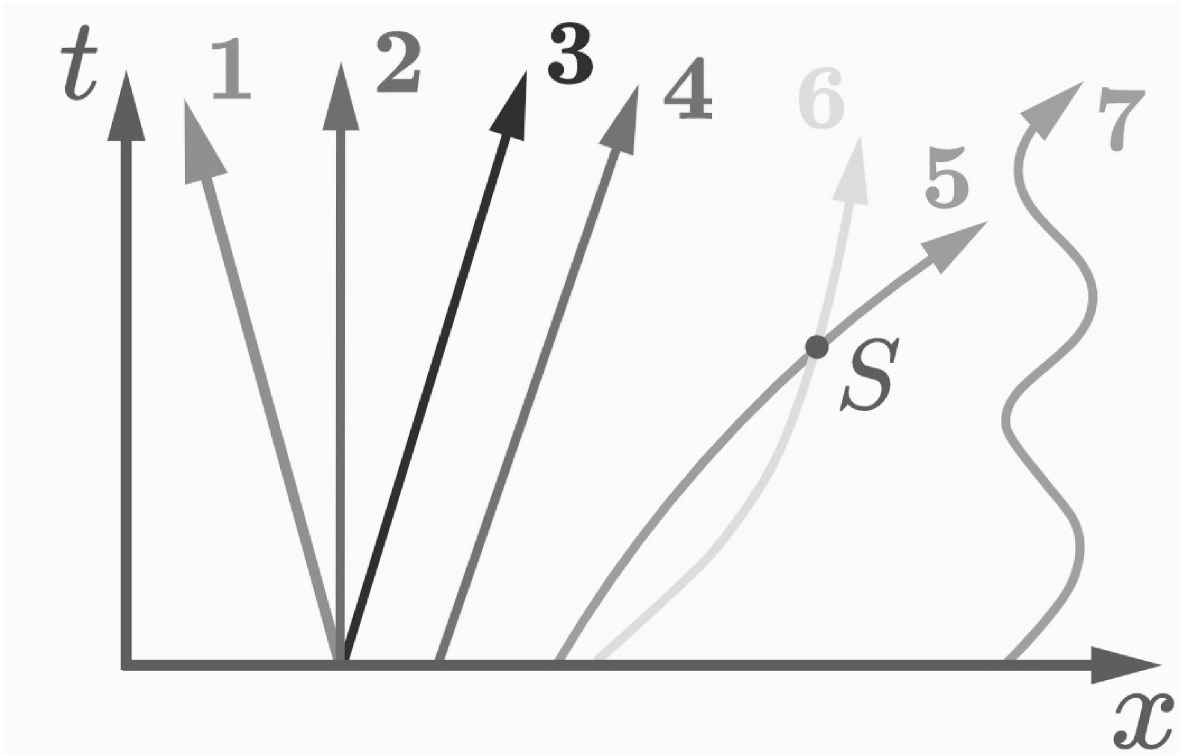


FIG. 1. World lines (not only of straight uniform motion). 1 – body moving with constant negative velocity away from the body with word line 2; 2 – body at rest at a certain position $x$; 3 – body moving with constant positive velocity away from the body with world line 2; 4 - body moving with the same velocity as the body with world line 3 but starting from a larger distance from the origin than that body; 5 - body accelerating and meeting body 6 at event S; 6 – body decelerating until it stops at some position $x$; 7 – body oscillating back and forth from right to left.

### B. The coordinate transformation is linear

Thus, as being well known (e.g. [1] Art. 4, p. 554, [2] p. 167, [3] p. 92), correct coordinate transformations between inertial systems map straight world lines onto straight world lines. As a consequence, these transformations are *linear* (cf. [1] Art. 4, p. 554, [4] § 3, p. 898/146).

Redž writes, "Einstein for the first time gave the explanation of how uniformity of space and time implies the linearity of the transformations in [20], Section 7.4. . . (p. 2)

> Now it is easy to understand what we meant by the homogeneity of time and space in §6, or, in other words, why we assumed a priori that the transformation equations must be linear. For if a rate of a clock at rest with respect to S' is observed from S, this rate does not have to depend on the location of the clock in S' nor on the value of the time of S' in the vicinity of the clock. An analogous

remark applies to the orientation and length of a bar linked with S' and observed from S. (Einstein 1910 [18] p. 137, fn. 15)

Actually, according to the Lorentz transformation seconded by Einstein, the "rate of a clock at rest with respect to S' is observed from S", $t$ does depend on its location $x'$ and time $t'$ in *S'*.

Crediting H. Weyl, Rindler argues as follows ([21] I.6 p. 14; we will rewrite it in some more detail). Let $\tau$ be the reading of a moving clock in $S$. Homogeneity means that equal increments of $\tau$ correspond to equal increments of time and space everywhere, i.e. independent of the initial coordinates. For instance, assume the clock to move along the x-axis from point $(x_1, y_1, z_1) = (0, 0, 0)$ till point $(x_2, y_2, z_2) = (\ell, 0, 0)$. When instead starting at point $(x_1, y_1, z_1) = (\ell, 0, 0)$, it would move till point $(x_2, y_2, z_2) = (\ell + \ell, 0, 0) = (2\ell, 0, 0)$ etc. Thus,

$$\frac{\partial x^\mu}{\partial \tau} = \text{const.}\,, \qquad \frac{\partial^2 x^\mu}{\partial \tau^2} \equiv 0\,; \quad x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)\,. \tag{1}$$

By the principle of relativity, the same applies to frame *S'* (with $\ell$ being replaced by $\ell'$).

$$\frac{\partial x'^\mu}{\partial \tau} = \sum_{\nu=0}^{3} \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial \tau} = \text{const.}\,; \quad x'^\mu = (x'^0, x'^1, x'^2, x'^3) = (t', x', y', z')\,. \tag{2a}$$

$$\frac{\partial^2 x'^\mu}{\partial \tau^2} = \sum_{\nu,\rho=0}^{3} \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\nu} \frac{\partial x^\rho}{\partial \tau} \frac{\partial x^\nu}{\partial \tau} + \sum_{\nu=0}^{3} \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial^2 x^\nu}{\partial \tau^2} = \sum_{\nu,\rho=0}^{3} \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\nu} \frac{\partial x^\rho}{\partial \tau} \frac{\partial x^\nu}{\partial \tau} \equiv 0\,, \tag{2b}$$

because $\partial^2 x^\nu / \partial \tau^2 \equiv 0$, see the second eq. (1). Since $\partial x^\rho / \partial \tau$ and $\partial x^\nu / \partial \tau$ equal arbitrary constants, this happens only, if $\partial^2 x'^\mu / \partial x^\rho \partial x^\nu \equiv 0$.

As a matter of fact, the same arguing applies already to eq. (2a). Since the $\partial x^\nu / \partial \tau$ equal arbitrary constants, it holds true only, if $\partial x'^\mu / \partial x^\nu = \text{const.}$

For a more algebraic proof, see [11] (1)...(7).

### C. Rotation and scaling

We consider two inertial frames $S$ and *S'* in Rindler's "standard configuration" ([21] I.6 p. 13), see Subsection III A. Frame $S\,'$ moves with velocity $\vec{v} = (v, 0, 0)$ relatively to $S$. The world line of the origin $(x\,', t\,') = (0, 0)$ of $S'$ in $S$ reads $(x, t) = (vt, t)$.

In both frames, the motion of a free body in the $x$-$t$- and $x'$-$t'$-planes of $S$ and *S'*, respectively, is described by straight world lines. If the relative velocity $v$ does not vanish, both world lines crosses. For this, we imagine the coordinate transformation from $S$ to $S$' and *vice versa* to consist of a rotation and a scaling. Both depend on $v$ because otherwise the transformation were the identity transformation (from $S$ to $S$). The derivation of them is most simply when that crossing takes place in the origins, $(x, t) = (x', t') = (0, 0)$.

In general, however, the two world lines are

$$x(t) - x_0 = u\,(t - t_0) \quad \text{and} \quad x'(t') - x'_0 = u'\,(t' - t'_0)\,. \tag{3}$$

Some authors argue that suitable coordinate transformations in $S$ and $S$' can be applied to simplify that formulas to

$$x(t) = u\,t \quad \text{and} \quad x'(t') = u't'. \tag{4}$$

However, the coordinate systems of $S$ and $S$' are largely fixed by their setup in that their origins coincide at $t = t' = 0$, see Rindler's "standard configuration" in Subsection III A above.

For this, we propose a the following direct and more rigorous construction.

Since the transformation is linear, it is *in*dependent of the coordinates of the body under consideration. For this, it is sufficient to consider the special case—depicted in Figure 2—that the body moves with velocity $u$ through the origin of $S$ $(x, t) = (0, 0)$. Its world line in $S$ thus reads $(x,\, t) = (u\,t, t)$. Since $(x', t') = (0, 0)$, if $(x, t) = (0, 0)$, in frame $S\acute{}$, its world line is also a straight line through the origin, $(x\acute{},\, t\acute{}) = (u\acute{}t\acute{}, t\acute{})$, where $u\acute{}$ denotes its velocity in $S\acute{}$.

## IV. GENERAL EXPRESSION FOR AND PROPERTIES OF THE TRANSFORMATION

A rotation in the $x$-$t$-plane involves the common transformation of position $x$ and time coordinates $t$. This requires the introduction of—for the time being arbitrary—constant reference speeds $V$ and $V'$.[9] The simplest manner to do so is to formally multiply $t$ with $V$ and $t\acute{}$ with $V'$.[10] This makes the transformation quantities $\hat{R}(v)$ and $\eta(v)$ dimensionless.

[9] It is tempting to use only one reference speed, $V$. This, however, would (more or less implicitly) introduce the existence of a universal speed for all inertial systems. That is close to the postulate of the constance of the speed of light w.r.t. all inertial systems.

[10] Here, students may be asked to justify the wording "simplest".

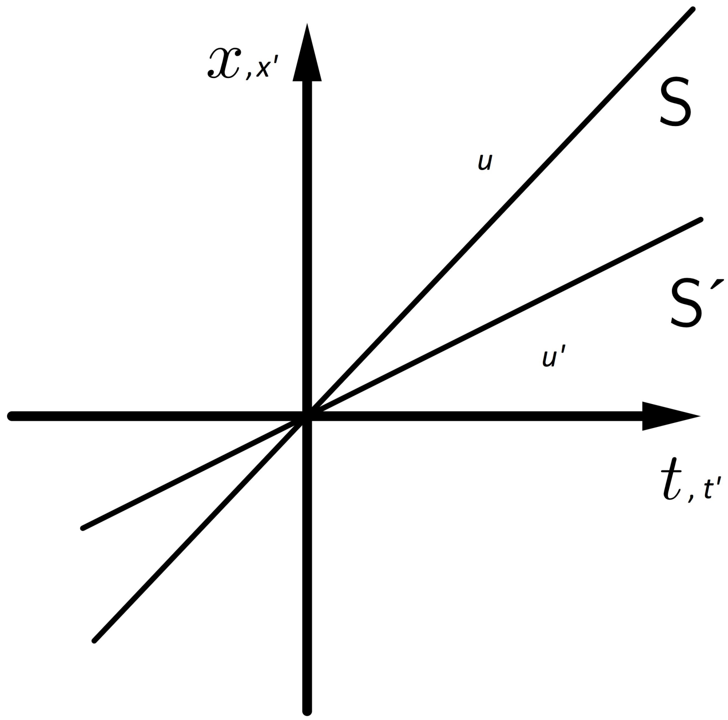


FIG. 2. Crossing world lines of the thought experiment for the case $0 < v < u$. $u > 0$—velocity of the body relatively to frame $S$, $u´ > 0$—velocity of the body relatively to frame $S´$

Then, the most general transformation can be written as

$$\begin{pmatrix} x´ \\ V´t´ \end{pmatrix} = \eta(v)\,\hat{R}(v) \cdot \begin{pmatrix} x \\ Vt \end{pmatrix} \tag{5}$$

(for more mathematical details, see Appendix A). $\hat{R}(v)$ and $\eta(v)$ are subject to the following natural conditions.

1. As already indicated, they do not depend on the coordinates (linearity).

2. As also already indicated, they do depend on the relative velocity $v$ of the two frames.

3. The set of all transformations form a group.[11]

   3.1. If the relative velocity $v$ of the two frames vanishes, the transformation becomes the map of $S$ onto itself. This corresponds to the identity transformation, where

[11]For a most general treatment, see [23] Sect. I. That treatment is largely simplified when one confines oneself to linear transformations as the authors eventually do in № 9.

$\eta(0) = 1$ and $\hat{R}(0) = \mathrm{diag}(1, 1)$.

3.2. If there is a transformation from $S$ to $S'$, then there is also a transformation from $S'$ to $S$. In other words, for each single transformation, an inverse transformation exists such that the subsequent execution of both ones results in the identity transformation. Notice that this does *not* yet imply reciprocity.

3.3. For the subsequent transformations $\eta\hat{R}$ from $S$ to $S'$ and then $\eta'\hat{R}'$ from $S'$ to a third frame $S''$, there is an *immediate* transformation $\eta''\hat{R}''$ from $S$ to $S''$, where

$$\eta'\hat{R}' \cdot (\eta\hat{R}) = \eta''\hat{R}'' . \tag{6}$$

## V. DETERMINATION OF THE ROTATION MATRIX

Since all coordinates are real-valued, the rotation matrix $\hat{R}$ is real-valued, too. Here, a rotation matrix is considered to be an one-parametric $2\times2$ matrix, the determinant of which equals unit. (Recall that a change of the length of any finite segment of a world line during transformation has been delegated to the scale factor $\eta$.) There are two and only two cases (see Subsection V C), namely, (i), the common circular rotation, and, (ii), the "hyperbolic rotation" [25][26]. The simpler interpretation of Figure 2 as a circular rotation suggests to begin with that. The hyperbolic rotation will be dealt with in Subsection V B.

### A. Circular Rotation

A circular rotation is well known to be realized by the matrix

$$\hat{R} = \hat{R}_{\mathrm{circ}} := \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}, \quad \varphi = \varphi(v)\,;$$

$$\begin{pmatrix} x' \\ V't' \end{pmatrix} = \begin{pmatrix} x'_{\mathrm{circ}} \\ V't'_{\mathrm{circ}} \end{pmatrix} := \eta(v)\,\hat{R}_{\mathrm{circ}}(v) \cdot \begin{pmatrix} x \\ Vt \end{pmatrix}. \tag{7}$$

Here, $\varphi$ is the angle between the world lines in frames $S$ and $S'$ in Figure 2 in Section III, p. 11. This rotation of the world lines belongs to an Euclidean vector space because

$$x'^2_{\text{circ}} + (V't'_{\text{circ}})^2 = x^2 + (Vt)^2\,. \tag{8}$$

It has been termed 'circular' because all transformed coordinates are located on one and the same circle with radius $r = \sqrt{x^2 + (Vt)^2}$.

$$\frac{x'^2_{\text{circ}} + (V't'_{\text{circ}})^2}{r^2} = \frac{x''^2_{\text{circ}} + (V''t''_{\text{circ}})^2}{r^2} = \ldots = 1\,. \tag{9}$$

Now, the coordinate origin $x' = 0$ of system $S$' moves with velocity $v$ along the $x$-axis of $S$. As a consequence [11][27],

$$x' = a(v)\,(x - v\,t)\,, \tag{10}$$

where $a(v)$ is an auxiliary coefficient. Formula (7) gives

$$x'_{\text{circ}} = \eta(v)\,(x\cos\varphi(v) - Vt\sin\varphi(v))\,. \tag{11}$$

Comparing both equations yields

$$a = \eta\cos\varphi\,, \quad -a\,v = -\eta\,V\sin\varphi \quad \Rightarrow \quad \tan\varphi = \frac{v}{V}\,. \tag{12}$$

Now, the transformed velocity becomes

$$\frac{u\,'_{\text{circ}}}{V\,'} = \frac{x\,'_{\text{circ}}}{V\,'t\,'_{\text{circ}}} = \frac{\cos(\varphi)\,x - \sin(\varphi)\,Vt}{\sin(\varphi)\,x + \cos(\varphi)\,Vt} = \frac{\frac{u}{V} - \tan(\varphi)}{\tan(\varphi)\,\frac{u}{V} + 1} = \frac{\frac{u}{V} - \frac{v}{V}}{\frac{v}{V}\,\frac{u}{V} + 1}\,, \tag{13a}$$

or

$$u\,'_{\text{circ}} = \frac{V\,'}{V}\frac{u - v}{\frac{v}{V}\,\frac{u}{V} + 1}\,. \tag{13b}$$

For $v = 0$, we have $u' = u$, hence, $V' = V$ and

$$u\,'_{\text{circ}} = \frac{u - v}{\frac{u\,v}{V^2} + 1}\,. \tag{13c}$$

For *finite* values of $V$, however, this formula is physically *not* correct. For every given value of $v \neq 0$, it exhibits a *pole* at $u = -V^2/v$. Obviously, that pole is unphysical because $u'$ should be a monotonously increasing function of $u$. This artefact rules out this coordinate transformation, or confines it to the Galileo limit case $V \to \infty$.

This compels us to examine the second possibility, the hyperbolic rotation.

### B. Hyperbolic rotation

A hyperbolic rotation is described by the matrix

$$\hat{R} = \hat{R}_{\rm hyp}(v) := \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix}, \quad \theta = \theta(v)\,;$$

$$\begin{pmatrix} x' \\ V't' \end{pmatrix} = \begin{pmatrix} x'_{\rm hyp} \\ Vt'_{\rm hyp} \end{pmatrix} := \eta(v)\,\hat{R}_{\rm hyp}(v)\cdot\begin{pmatrix} x \\ Vt \end{pmatrix}. \quad (14)$$

$\theta$ is the angle between, (i), the $x$- and $x'$-axes and, (ii), the $t$- and $t'$-axes in a Minkowski diagram, see Figure 3 [28]. The sign of the non-diagonal matrix elements has been chosen such that the angle $\theta$ becomes the *rapidity*, see below.

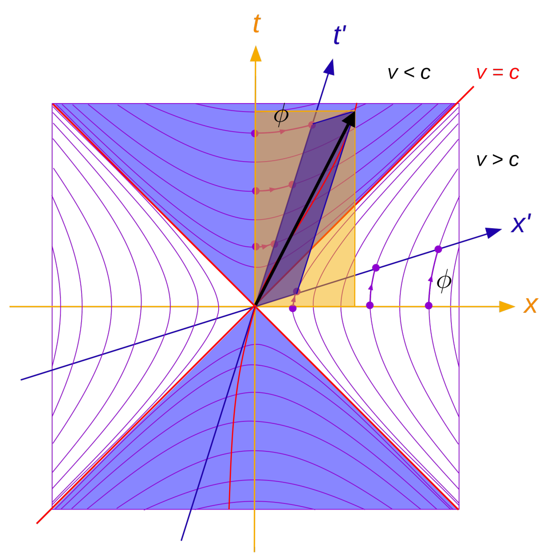


FIG. 3. The hyperbolic rotation angle (here denoted by $\phi$) in a Minkowski diagram.

As indicated in the reference to the Minkowski diagram above and in contrast to the Euclidean rotation of the world lines in the foregoing Subsection V A, this rotation of the coordinate axes corresponds to a rotation in a *non*-Euclidean, *viz.*, Minkowskian vector space. A fundamental invariant quantity is not the sum (8) but the difference of the coordinates squared,

$$(V\,'t_{\rm hyp}\,')^2 - x_{\rm hyp}\,'^2 = (Vt)^2 - x^2\,. \tag{15}$$

It has been termed ‘hyperbolic’ because all transformed coordinates lie on one and the same rectangular hyperbola with equal semi-axes $a = b = |(Vt)^2 - x^2|^{1/2}$.[12]

$$\frac{(V\,'t_{\rm hyp}\,')^2 - x_{\rm hyp}\,'^2}{(Vt)^2 - x^2} = \frac{(V\,''t_{\rm hyp}\,'')^2 - x_{\rm hyp}\,''^2}{(Vt)^2 - x^2} = \ldots = 1 \tag{16}$$

Now, the coordinate origin $x' = 0$ of system *S'* still moves with velocity $v$ along the $x$-axis of $S$ so that eq. (10) holds true. Formula (14) gives

$$x_{\rm hyp}\,' = \eta(v)\,(\cosh\theta(v)x - \sinh\theta(v)Vt)\,. \tag{17}$$

Comparing both equations yields

$$a = \eta\cosh\theta\,, \quad -a\,v = -\eta\,V\sinh\theta \quad \Rightarrow \quad \tanh\theta = \frac{v}{V}\,, \quad \theta(v) = \tanh^{-1}\left(\frac{v}{V}\right)\,. \tag{18}$$

By virtue of $-1 \leq \tanh\theta \leq +1$, we have the fundamental limitation

$$-V \leq v \leq +V\,. \tag{19}$$

The transformed velocity becomes

$$\frac{u\,'_{\rm hyp}}{V\,'} = \frac{x\,'_{\rm hyp}}{V\,'t\,'_{\rm hyp}} = \frac{\cosh(\theta)\,x - \sinh(\theta)\,Vt}{-\sinh(\theta)\,x + \cosh(\theta)\,Vt} = \frac{u - \tanh(\theta)}{-\tanh(\theta)\,u + V} = \frac{\frac{u}{V} - \frac{v}{V}}{-\frac{v\,u}{V^2} + 1}\,. \tag{20a}$$

Again, for $v = 0$, we have $u' = u$, hence, $V' = V$ and

$$u\,'_{\rm hyp} = \frac{u - v}{-\frac{v\,u}{V^2} + 1}\,. \tag{20b}$$

This is a variant of Einstein’s famous velocity addition theorem ([4] § 5), where $V = c$. The Galileo transformation is obtained for $V \to \infty$.

Now, if $u\,'_{\rm hyp} = 0$ (*S'* being the rest system of the body under consideration), we have

[12]Borel [26] discusses hyperbolas using their functional representation $\xi\,\eta = $ const., where our case corresponds to $\xi = Vt + x$, $\eta = Vt - x$.

$u = v$. In view of the ineqs. (19), the velocity of the body in $S$ is limited to the same interval $[-V, V]$ as $v$ is,

$$-V \leq u \leq +V \,. \tag{21}$$

This suggests to apply the rightmost formula (18) to $u$, too.

$$\theta(u) = \tanh^{-1}\left(\frac{u}{V}\right). \tag{22}$$

Further, if $u = 0$ ($S$ being the rest system of the body under consideration), we have $u\acute{}_{\mathrm{hyp}} = -v$. In view of the ineqs. (19), the velocity of the body in $S$' is limited to the interval $[-V, V]$, too.

$$-V \leq u\acute{}_{\mathrm{hyp}} \leq +V \tag{23}$$

This suggests to apply the rightmost formula (18) to $u\acute{}_{\mathrm{hyp}}$, too.

$$\theta(u'_{\mathrm{hyp}}) = \tanh^{-1}\left(\frac{u'_{\mathrm{hyp}}}{V}\right) \tag{24}$$

Together, the formulas (18), (22), and (24) make the velocity addition formula (20b) equivalent to the formula

$$\theta(u'_{\mathrm{hyp}}) = \theta(u) - \theta(v) \,. \tag{25}$$

That common algebraic relation has brought $\theta$ to be termed the *rapidity* of the body.[13]

Thus, *all* physically relevant velocities of a body w.r.t. inertial systems are limited to the interval $[-V, V]$. Because each inertial system can be the rest frame of a body, the same holds true for the relative velocities of inertial systems.

As a consequence, the velocity addition formula (20b) is free of the artefact plaguing formula (13a) with *un*limited velocities.

Moreover, formula (20b) implies this:

1. If $v = \pm V$, then $u' = \mp V$, independent of $u$. However, in this case, the coordinates cannot be transformed but for $u = V$, see formulas (35a) and (35d) (there, $V = c$, the speed of light in vacuum).

[13] Methodically, that arguing for an own name corresponds to Euler's arguing w.r.t. the quantity "Wirksamkeit" (efficacy) $W$, the negative of which is now called potential energy, see [29] § 75. Shortly, $W := \int_{\vec{x}_1}^{\vec{x}_2} \vec{F} \cdot \mathrm{d}\vec{x}$ can be independent of the coordinate system and the path between $\vec{x}_1$ and $\vec{x}_2$, respectively, while the integral $\int_{t_1}^{t_2} \vec{F}\,\mathrm{d}t$ is never independent of the coordinate system and the path between $\vec{x}_1 = \vec{x}(t_1)$ and $\vec{x}_2 = \vec{x}(t_2)$, respectively.

2. If $u = \pm V$, then $u' = \pm V$, independent of $v$. A body, which moves w.r.t. one inertial system with (maximum) speed $V$, moves w.r.t. *all* inertial systems with speed $V$. For bodies with finite mass, this is only asymptotically possible, see Section VIII. It is possible, however, for light, see Section X.

### C. There are no further possibilities. The metric tensor

In this subsection, we will show that the two transformations (mappings) examined in the foregoing subsections are exhausting. To do so, we will consider the metric tensor corresponding to them. Admittedly, this is a preliminary investigation; accounting for the scale factor, the metric tensor will be reconsidered in Section VI.

First, as mentioned in Subsection V A, by virtue of eq. (8), the circular rotation corresponds to a two-dimensional Euclidean vector space. Its metric tensor hence equals[14]

$$\hat{g} = \hat{g}_{\mathrm{E}} \equiv g_{\mathrm{E},\alpha\beta} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha, \beta = 0, 1. \tag{26}$$

Introducing the coordinates

$$x^0 := Vt\,, \quad x^1 := x\,, \tag{27}$$

eq. (8) means that there is the invariant length squared

$$s_{\mathrm{E}}^2 = g_{\mathrm{E},\alpha\beta}\, x^\alpha x^\beta = (x^0)^2 + (x^1)^2 = (Vt)^2 + x^2\,. \tag{28}$$

Second, as mentioned in Subsection V B, by virtue of eq. (15), the hyperbolic rotation corresponds to a two-dimensional Minkowski space. Its metric tensor equals

$$\hat{g} = \hat{g}_{\mathrm{M}} \equiv g_{\mathrm{M},\alpha\beta} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}$$

Both variants are physically equivalent; we are using the first (left) version. Equation (15)

[14]The tensor $\hat{g} = \mathrm{diag}(-1, -1)$ would lead to an imaginary-valued length of a world line as $s_{\mathrm{E}} := \sqrt{-x^2 - (Vt)^2}$, cf. formula (28). This is not meaningful in an Euclidean vector space.

means that there is the invariant length squared

$$s_{\rm M}^2 = g_{{\rm M},\alpha\beta}\, x^\alpha x^\beta = (x^0)^2 - (x^1)^2 = (Vt)^2 - x^2\,. \tag{30}$$

These two metric tensors are the only independent ones for homogeneous and isotropic spaces in 1+1d, i.e. a flat space-time with a common Cartesian coordinate system as in Figures 1 and 2.[15] Consequently, all other physically meaningful coordinate transformations are merely mathematical variants of the two considered ones; examples being $\hat{R}_{\rm P}$ (31) and $\hat{R}_{\rm im}$ (32) in the next subsection.

This conclusion is corroborated by the 'brute-force calculations' in Appendix B 4. By virtue of Galileo relativity, it holds true in all inertial systems.

Notice that the "Doppler transformation" ([23] (129)) does *not* belong to them. This suggests that, for common pedagogical purposes, it is more appropriate to first collect all the assumptions and only afterwards derive the conclusions.

We will reconsider this issue when including the scale factor $\eta(v)$ in the next section.

### D. Additional remarks

A mathematically equivalent representation of the hyperbolic rotation is ([30] p. 256)

$$\hat{R} = \hat{R}_{\rm P}(v) := \begin{pmatrix} \sec(\phi) & -\tan(\phi) \\ -\tan(\phi) & \sec(\phi), \end{pmatrix}; \quad \sin(\phi) = \frac{v}{V}\,. \tag{31}$$

$\hat{R}_{\rm P}$ is seductive in view of the final expression (34) below. However, the angle $\theta$ in $\hat{R}_{\rm hyp}$ (14) is distinguished against $\phi$ by its rapidity property (25).

Moreover, as well known, the hyperbolic rotation can also be seen as a circular rotation in the planes $(x,\, iVt)$, $(x',\, iVt')$, where

$$\hat{R} = \hat{R}_{\rm im}(v) := \begin{pmatrix} \cos(\varphi) & i\sin(\varphi) \\ i\sin(\varphi) & \cos(\varphi) \end{pmatrix} = \begin{pmatrix} \cosh(i\varphi) & \sinh(i\varphi) \\ \sinh(i\varphi) & \cosh(i\varphi) \end{pmatrix}; \quad \varphi = \varphi(v)\,. \tag{32}$$

However, we have no inducement for introducing an imaginary-valued time coordinate.[16]

[15] The determinant of the metric tensor $\hat{g}$ of a flat space equals $|\hat{g}| = \pm 1$.

[16] This is not to criticize Minkowski who first used it for using a pseudo-Euclidean geometry, possibly, not to over-demand his listeners by non-Euclidean geometry with its co- and contravariant coordinates.

As every rotation matrix does, $\hat{R}_{\rm hyp}$ exhibits the reciprocity property as $\hat{R}^{-1}_{\rm hyp}(v) = \hat{R}_{\rm hyp}(-v)$.

## VI. DETERMINATION OF THE SCALE FACTOR

According to formula (15) and the metric tensor $\hat{g}_{\rm M}$ (29), our space-time has the invariant length squared

$$s'^2 := (Vt')^2 - x'^2 = s^2 := (Vt)^2 - x^2\,. \tag{33}$$

Because this invariance is already guaranteed by the hyperbolic rotation (see Subsection V B) and, consequently, would be destroyed by a scale factor $\eta \neq 1$, we have $\eta \equiv 1$.

Appendix B contains additional historical and mathematical material about this topic.

## VII. THE SPECIAL LORENTZ TRANSFORMATION

> "Die Lorentztransformationen entsprechen also ein-eindeutig den hyperbolischen Bewegungen im $R_3$." (Herglotz 1910 [25][17])

Using elementary conversion formulae for the hyperbolic functions and—referring to experiment or the Lorentz force—replacing $V$ with the speed of light in vacuum $c$, the rotation matrix (14) becomes the well-known expression

$$\hat{R} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix}. \tag{34}$$

In agreement with the result $\eta \equiv 1$ of the foregoing section, the spatial coordinates $y$ and $z$ are not affected be the rotation considered above because they are orthogonal to the plane

[17] En: "The Lorentz transformations correspond one-to-one to hyperbolic motions in $R_3$."

of rotation. The final transformation formulae thus reads

$$x' = \gamma\,(x - vt\,); \quad \gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{35a}$$

$$y' = y \tag{35b}$$

$$z' = z \tag{35c}$$

$$t' = \gamma\left(t - \frac{v\,x}{c^2}\right) \tag{35d}$$

$$u' = \frac{u - v}{1 + \frac{v\,u}{c^2}}\,. \tag{35e}$$

The first four formulae are the special Lorentz transformation, where $\gamma$ is the omnipresent Lorentz factor. Formula (35e) is a variant of Einstein's addition formula for velocities ([4] § 5, p. 906).

## VIII. TRANSFORMATION OF THE ACCELERATION

> "It used to be frequently argued that it would be necessary to pass to Einstein's general relativity in order to handle acceleration, but this is completely wrong." (Penrose 2004 [31])

Using the transformation formulae (35), one obtains for the transformation of the acceleration $a := \mathrm{d}u/\mathrm{d}t$ the expression

$$a' = \frac{\mathrm{d}u'}{\mathrm{d}t'} = \frac{a}{\gamma^3\left(1 - \frac{u\,v}{c^2}\right)^3}\,. \tag{36}$$

It confirms Premise 2 on p. 5: If the acceleration vanishes w.r.t. *one* inertial system, it vanishes w.r.t. *all* inertial systems. That reading complies with the crucial point that straight world lines are transformed to straight world lines (Section III).

Gros ([2] p. 166) calls the invariance of the equation $\vec{a} = \vec{0}$ "the principle of inertia".

Now, let us consider the special case that the velocity of the body equals that of frame $S'$, $u = v$, so that $S'$ is the rest frame of the body. In this case, formula (36) simplifies to

$$a' = \gamma^3(u)\,a\,. \tag{37}$$

In turn,

$$a = a'/\gamma^3(u) = a_0/\gamma^3(u)\,, \tag{38}$$

where $a_0$ denotes the acceleration in the rest frame of the body, or when the body just starts to move.[18]

As a matter of fact, with increasing velocity $u$, $1/\gamma^3(u)$ monotonously decreases from $1/\gamma^3(0) = 1$ till $\lim_{u\to c} 1/\gamma^3(u) = 0$. When the speed of a body reaches $c$, it is no longer accelerated. This makes the speed of light in vacuum $c$ to be the maximum speed of a body, again. It is noteworthy that that is a *purely kinematic*, not a dynamical effect ([3] p. 100).

## IX. ON THE LIMIT OF SMALL VELOCITY

It is often stated that the limit of the special Lorentz transformation (35) for small velocities, $v \ll c$, is the *Galileo transformation*,

$$x' = x - vt \tag{39a}$$

$$y' = y \tag{39b}$$

$$z' = z \tag{39c}$$

$$t' = t \tag{39d}$$

$$u' = u - v \tag{39e}$$

$$a' = a\,. \tag{39f}$$

Actually, that is the approximation of *zeroth* order in $v/c$.[19]

The approximation of *first* order in $v/c$ reads ([33] (3); we put $u_y = u_z = 0$ and add the

[18] $\vec{a} = \vec{g}/\gamma^3$, where $\vec{g} =$ const., characterizes relativistically uniform acceleration ([32] (73)).

[19] For a critical examination, see also [16] and the Appendix in [20].

corresponding transformations of velocity and acceleration)

$$x' = x - vt \tag{40a}$$

$$y' = y \tag{40b}$$

$$z' = z \tag{40c}$$

$$ct' = ct - x\frac{v}{c} \tag{40d}$$

$$\frac{u'}{c} = \frac{u}{c} - \frac{v}{c} + \left(\frac{u}{c}\right)^2 \frac{v}{c} \tag{40e}$$

$$a' = \left(1 + 3\frac{u}{c}\frac{v}{c}\right) a\,. \tag{40f}$$

This 'Furry transformation' is close to Zahar's transformation ([32] (28), [34]).[20]

According to formula (40d), the Galileo limit (39d) *additionally* requires

$$\left|x\,\frac{v}{c}\right| \ll |c\,t|\,. \tag{41}$$

This condition can be violated when the spatial distance is much larger than the temporal one. [32] (28), [34]

The Furry (40) and Zahar ([32] (28), [34]) transformations form a group within their accuracy of first order in $v/c$.

Furry [33] and Baierlein [20] have stressed that the low-speed limits of the Galileo and Lorentz transformations are *not* comparable each to another because there is a finite reference speed ($c$) in the latter one but not in the former one. As long as that reference speed is finite, simultaneity is relative.

This jumping transition from relative to absolute simultaneity resembles the jump of symmetry from a rectangular to a square. Physical examples are the transitions from motion to rest and statics to dynamics, and Gibbs' paradox in the sense that the mixing entropy vanishes when the mixing particles become equal.

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[20]Zahar's transformation [34] contains an additional factor $\gamma^2$ on the r.h.s. of formula (40d), with corresponding changes in (40e), (40f). This factor makes the determinant of the corresponding transformation matrix

$$\begin{pmatrix} 1 & -v/c \\ -\gamma^2 v/c & \gamma^2 \end{pmatrix}$$

equal unit.

## X. EINSTEIN'S SECOND POSTULATE

We have shown that the reference velocity $V$ is a Lorentz scalar, i.e. of one and the same value *in* all inertial systems. Moreover, the second implication on p. 17 shows that $V$ has one and the same value *w.r.t.* all inertial systems. With $V = c$, the same holds true for the speed of light in vacuum $c$. This is Einstein's [4] second postulate.

> "Wenn wir anderseits die andern [nicht elektrodynamischen] physikalischen Gleichungen dem Relativitätsprinzip entsprechend umformen und dabei das Vorkommen in denselben der Konstante $n[= -1/c^2]$ ersehen, so brauchen wir durchaus nicht zu schließen, daß hierbei irgendwelche elektrische Kräfte im Spiele sind, sondern folgern vom Standpunkte des Relativitätsprinzips aus nur, daß Raum und Zeit ihr Gepräge auf alle physikalischen Erscheinungen aufdrücken vermittelst der Konstante $n[= -1/c^2]$."[21] (Ignatowsky 1910 [22] p. 974)

In other words,

> "Die Spezielle Relativitätstheorie hat mit Licht nichts zu tun. Es ist genau umgekehrt. Licht hat mit Relativitätstheorie zu tun! Denn auch das Verhalten von Licht unterliegt den Rahmenbedingungen der Kinematik."[22] (Rupp 2022 [3] p. 97)

## XI. SUMMARY AND CONCLUSIONS

Surprisingly enough, despite the numerical value of the reference speed, the speed of light in vacuum $c$, the special Lorentz transformation can be derived using just these two premises being quite common within classical mechanics.

1. There is space and time. They can be equipped with three- and one-dimensional coordinate systems, respectively.

2. All inertial systems are physically equivalent.

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[21]En.: If, on the other hand, we transform the other [non-electrodynamic] physical equations according to the principle of relativity and thereby see the occurrence in them of the constant $n[= 1/c^2]$, then we need not at all conclude that any electrical forces are involved here, but from the standpoint of the principle of relativity only conclude that space and time impress their character on all physical phenomena by means of the constant $n[= -1/c^2]$.

[22]En: The special relativity theory has nothing to do with light. It is exactly the other way round. Light has to do with relativity! Because also the behaviour of light is subject to the basic conditions of kinematics.

The Lorentz transformation for *non*-parallel motion of the body and other reference frames as well as further generalizations to the Lorentz group and so on can be obtained in the usual, straightforward way, *without* additional premises.[23]

As a matter of fact, d'Alembert's force of inertia of a body with constant mass $m$, $-m\vec{a}$, is Galileo-invariant. However, this Galileo invariance results from an *additional* assumption. According to *Definition 2* of the 'Principia' [7], the mass of a body is independent of its velocity. Similarly, Euler [36] assumes forces like gravity to be "absolute forces".

> The absolute force is the force that acts equally on a body either at rest or moving. (№ 111)

As a result,

> ... [the] increment of the speed does not depend on the speed... itself. (№ 131)

That view is corroborated in his 'Anleitung' [29], e.g. in № 57, formula $v = (n/M) \int p\, dt$ ($v$, $M$ – velocity and mass of a body, $n$ – factor for matching the units of measurement, $p$ - external force). Without Euler's assumption, one is led to a *dynamical* foundation of the Lorentz transformation [37].

In contrast, the derivation of the Lorentz transformation presented here is *purely kinematic* in that a dynamical notion occurs only in 'interaction-free'. Therefore, the existence of a maximum speed of a body w.r.t. inertial systems is a purely kinematic effect, too.

Newton's and Euler's assumptions are equivalent with the premise that the reading of a clock is independent of its motion. Macdonald [8] has relaxed that premise and reached an ingeniously short derivation of the Lorentz transformation. However, that assumption is not necessary as having shown in this article, and he also has taken over Einstein's second postulate.

As within any theory of this kind, the numerical values of universal constants in it like the reference speed $V$ cannot be determined by the theory itself. It can be stated, however, that, by definition, $V$ is a space-time constant and Lorentz scalar, and so $c = 1/\sqrt{\mu_0 \varepsilon_0}$ is.

[23] Admittedly, the comprehensive analysis of the special-relativistic velocity space in terms of gyrogroups is *non*-trivial and began much later ([35] Sect. 19). Notably, useful general theorems for the special-relativistic mechanical motion of systems, including the elastic impact, have been obtained in a rigorous manner ([35] Sects. 16... 18).

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## Appendix A: Some mathematical subtleties of the transformation of straight lines onto straight lines

The most general transformations which are represented by differentiable equations and transfer straight lines onto straight lines are the *projective* ones ([45] p. 32, Satz (theorem) 2; see also [23] (38), their $p$ being our $v$).

$$t' = \frac{a_{11}(v)\,t + a_{12}(v)\,x + a_{13}(v)}{a_{31}(v)\,t + a_{32}(v)\,x + a_{33}(v)} \tag{A1a}$$

$$x' = \frac{a_{21}(v)\,t + a_{22}(v)\,x + a_{23}(v)}{a_{31}(v)\,t + a_{32}(v)\,x + a_{33}(v)} \tag{A1b}$$

Let us consider the motion of a free body in $S$ starting from the point $(x_0, t_0)$ with velocity $u$. Its world-line describes the straight line

$$\left\{(x,t) = (x_0 + u\tilde{t}, t_0 + \tilde{t}) \,|\, 0 \le \tilde{t} \le +\infty \right\}. \tag{A2}$$

By the principle of relativity, in *S'*, its world-line describes the straight line

$$\left\{(x',t') = (x'_0 + u'\tilde{t}', t'_0 + \tilde{t}') \,|\, 0 \leq \tilde{t}' \leq +\infty\right\}. \tag{A3}$$

This excludes the occurrence of poles in the transformation formulas (A1). Mathematically speaking, the points at infinity are also left invariant. Subject to that condition, the most general transformations are the *linear* ones ([45] p. 58, Satz (theorem) 11).

$$t' = a_{11}(v)\,t + a_{12}(v)\,x + a_{13}(v) \tag{A4a}$$

$$x' = a_{21}(v)\,t + a_{22}(v)\,x + a_{23}(v) \tag{A4b}$$

(without loss of generality, $a_{33}(v) = 1$).

Now, since the transformation coefficients are *in*dependent of the coordinates, it is sufficient to consider the special case of $(x',t') = (0,0)$, if $(x,t) = (0,0)$, see Fig. 2 and the explanations to it. This reduces the linear transformations (A4) to the *linear homogenous* ones like (5) ([45] p. 496, Satz (theorem) 5 and sentence thereafter).[24]

$$t' = a_{11}(v)\,t + a_{12}(v)\,x \tag{A5a}$$

$$x' = a_{21}(v)\,t + a_{22}(v)\,x \tag{A5b}$$

Finally, the determinant of the special Lorentz transformation (35a)...(35d) equals unit. This makes the linear homogenous transformations (A5) *special* linear homogenous transformations ([45] p. 492) with

$$a_{11}(v)\,a_{22}(v) - a_{12}(v)\,a_{21}(v) = 1\,. \tag{A6}$$

[24]Transformations like (5) combining rotation and expansion or contraction are special cases of *affine* transformations [46].

### Appendix B: Additional historical and mathematical material for the determination of the scale factor

#### 1. Few historical remarks

The scale factor $\eta(v)$ prominently occurs in the pioneering works of Lorentz [39], Einstein [4], and Poincaré [40][41] as

$$x' = \eta(v)\,\gamma(v)\,(x - vt)\,; \quad \gamma(v) := \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{B1a}$$

$$y' = \eta(v)\,y \tag{B1b}$$

$$z' = \eta(v)\,z \tag{B1c}$$

$$t' = \eta(v)\,\gamma(v)\left(t - \frac{v\,x}{c^2}\right). \tag{B1d}$$

All three obtained $\eta(v) \equiv 1$ (see also [42]);

- Lorentz from the transformation properties of force and acceleration ([39] after formula (33));

- Einstein from, (i), transforming additionally from $S$' to a third system $S$" which moves with velocity $-v$ relatively to $S$' and, (ii), the argument that, by symmetry, the transformation of the coordinates perpendicular to the relative motion does not depend on the direction of that motion ([4] pp. 901f.);

- Poincaré from comparing the transformation and its inverse as well as its combination with a 180 degree rotation about the $y$-axis ([41] § IV, eqs. (1)...(3)).

However, Lorentz's transformations of velocity and charge density are not special-relativistic as noted directly by Tyapkin ([38] fn. on p. 70) and indirectly by Poincaré ([40] after formulas (2), [41] after formulas (4a) and (7)). Einstein's second argument is not clear to us, see Subsection B 3. Poincaré [41] implicitly assumes reciprocity. Moreover, Poincaré's [41] formulas (3) are actually not the inverse of (1) as they also transform from $S$ to $S'$.

### 2. $\eta \equiv 1$ from the transformation of the transverse coordinates $y$ and $z$

Straight world lines in the $x$-$t$-plane are transformed to straight world lines in the $x'$-$t'$-plane. Affected are the coordinates $x$ and $t$, while $y$ and $z$ are not because they are orthogonal to $x$ and $t$. Hence, the formulas (B1b) f. imply $\eta(v) \equiv 1$.

That is obvious for the rotational part of the transformation. Why that should be different for the scale factor?

Nevertheless, this may be too heuristic an argument.

### 3. Using the group property

Within our approach, one also could argue as follows. The rotation matrices $\hat{R}$ separately obey the group property (6). This implies the scale factors to do so as well.

$$\eta(v\,\acute{}\,)\,\eta(v) = \eta(v\,\acute{}\,\acute{}\,)\,, \tag{B2}$$

where $v$, $v\,\acute{}$, and $v\,\acute{}\,\acute{}$ are the corresponding relative velocities of the three frames involved. It is tempting to assume that they all are quite arbitrary and thus to conclude that $\eta(v) \equiv 1$. However, this would be premature. For eq. (B2) actually means that

$$\eta(v\,\acute{}\,)\,\eta(v) = \eta\left(\frac{v + v\,\acute{}}{1 + \frac{v\,v\,\acute{}}{c^2}}\right). \tag{B3}$$

As a matter of fact, this equation has got a non-trivial solution $\eta(v) \neq 1$. For there is the following addition theorem for the areatangens hyperbolicus function,

$$\operatorname{atanh}(x) \pm \operatorname{atanh}(y) = \operatorname{atanh}\left(\frac{x \pm y}{1 \pm xy}\right); \quad -1 < x, y < +1\,. \tag{B4}$$

Hence,

$$\eta(v) = e^{k\,\operatorname{atanh}(v/c)} = \left(\frac{c+v}{c-v}\right)^{k/2}, \quad k = \text{const.}, \tag{B5}$$

is a solution to eq. (B3), cf. [19] p. 9/p. 6, eq. (1.3) and [42].[25]

[25] For $k = 1$, it is the longitudinal Doppler factor evoked by Verkhovsky [19]. If $k$ it not constant, the scale factors (B5) loose the group property (B2), see [44] Subsection 3.3. In [19] p. 9/p. 6, formulas (1.4), $k = r := \operatorname{sgn}(x - Vt) \neq \text{const.}$

Nevertheless, this function $\eta(v)$ is not suitable as it depends on the sign of $v$ what it should not do ([4] p. 902, [41] p. 135). Einstein 1905 [4] considers two reference frames as in Section III. There be a rod in system $S'$ with end coordinates $(0,0,0)$ and $(0,l',0)$. Its end coordinates in system $S$ at time $t = t\acute{} = 0$ equal $(0,0,0)$ and $(0, l = l'/\eta(v), 0)$.

> "Aus Symmetriegründen ist nun einleuchtend, daß die im ruhenden System gemessene Länge eines bestimmten Stabes, welcher senkrecht zu seiner Achse bewegt ist, nur von der Geschwindigkeit, nicht aber von der Richtung und dem Sinne der Bewegung abhängig sein kann."[26] (p. 902)

Hence, $\eta(v) = \eta(-v)$.

However, as Verkhovsky [19] stresses, the result of the observation of that rod by means of light depends on whether it is moving toward the observer or away from it.

### 4. Using metric and invariant length

The rotational part of the Lorentz transformation as analysed in this article leaves the length squared (30) invariant. Obviously, this length squared is no longer invariant, if $\eta(v) \neq 1$, and the metric tensor $\hat{g}$ may differ from (29). Nevertheless, there is an invariant length squared as (omitting the subscript M)

$$s^2 := g_{00}(x^0)^2 + 2g_{01}x^0x^1 + g_{11}(x^1)^2 \tag{B6a}$$

$$= s'^2 := g_{00}(x^{0\prime})^2 + 2g_{01}x^{0\prime}x^{1\prime} + g_{11}(x^{1\prime})^2 \,. \tag{B6b}$$

Inserting the transformation formulas (B1a) and (B1d) and collecting the prefactors of $(x^0)^2$, $x^0x^1$, and $(x^1)^2$ yields the following set of equations for the components of $\hat{g}$.

$$\left(\eta^2\gamma^2 - 1\right) g_{00} - \eta^2\gamma^2\beta\, g_{01} + \eta^2\gamma^2\beta^2 g_{11} = 0 \tag{B7a}$$

$$-2\eta^2\gamma^2\beta\, g_{00} + \eta^2\gamma^2\left(1 - \beta^2\right) g_{01} - 2\eta^2\gamma^2\beta\, g_{11} = 0 \tag{B7b}$$

$$\eta^2\gamma^2\beta^2 g_{00} - \eta^2\gamma^2\beta^2 g_{01} + \left(\eta^2\gamma^2 - 1\right) g_{11} = 0 \tag{B7c}$$

[26] En: "For reasons of symmetry it is obvious that the length of a rod measured in the system at rest and moving perpendicular to its own axis can depend only on its velocity and not on the direction and sense of its motion." (p. 151)

For $\beta = 0$, eq. (B7b) yields $g_{01} = 0$. Since $\hat{g}$ is independent of $v$, i.e. of $\beta$, we have $g_{01} \equiv 0$. As a consequence, eqs. (B7a) and (B7c) yield

$$g_{11} = -g_{00} = \mp 1\,, \quad \eta^2\gamma^2 = \frac{1}{1-\beta^2} = \gamma^2 \quad \Rightarrow \quad \eta^2 \equiv 1\,. \tag{B8}$$

Since $\eta > 0$, we finally have $\eta \equiv 1$.[27]

**Appendix C: Independent elementary derivation of the reciprocity property, $v' = -v$**

Let us rewrite the transformation formula (5) as

$$x' = \eta\,(R_{xx}x + R_{xt}Vt)\,; \quad V't' = \eta\,(R_{tx}x + R_{tt}Vt)\,. \tag{C1}$$

The coordinate origin $x' = 0$ of system $S$' moves with velocity $v$ along the $x$-axis of $S$. Hence (cf. [11][27]),

$$0 = R_{xx}x + R_{xt}Vt = R_{xx}\,(x - v\,t) \quad \rightarrow \quad R_{xt} = -\frac{v}{V}R_{xx}\,. \tag{C2}$$

Then the reduced transformed velocity equals

$$\frac{u'}{V'} = \frac{x'}{V't'} = R_{xx}\,\frac{x - vt}{R_{tx}\,x + R_{tt}\,Vt} = R_{xx}\,\frac{\frac{x}{Vt} - \frac{v}{V}}{R_{tx}\frac{x}{Vt} + R_{tt}} = \frac{R_{xx}}{R_{tt}}\,\frac{\frac{u}{V} - \frac{v}{V}}{\frac{u}{V}\,\frac{R_{tx}}{R_{tt}} + 1}\,. \tag{C3}$$

From formula (C3), we obtain the inverse transformation of the velocity as

$$\frac{u}{V} = \frac{\frac{u'}{V'}\,\frac{R_{tt}(v)}{R_{xx}(v)} + \frac{v}{V}}{1 - \frac{u'}{V'}\,\frac{R_{tx}(v)}{R_{xx}(v)}} \stackrel{!}{=} \frac{R_{xx}(v')}{R_{tt}(v')}\,\frac{\frac{u'}{V'} - \frac{v'}{V'}}{\frac{u'}{V'}\,\frac{R_{tx}(v')}{R_{tt}(v')} + 1} \tag{C4}$$

by applying the principle of relativity to formula (C3). Comparing the components yields

$$R_{tt} = R_{xx}\,, \quad \frac{v'}{V'} = -\frac{v}{V}\,, \quad \frac{R_{tx}(v')}{R_{tt}(v')} = -\frac{R_{tx}(v)}{R_{tt}(v)}\,. \tag{C5}$$

The second equation displays the reciprocity property.

[27] The requirement that the determinant of the coefficients of eqs. (B7) vanishes only yields $\eta = (1 \pm \beta^2)/(1+\beta^2)$.

Moreover, since $R_{tt}(0) = 1$, the third eq. (C5) shows that $R_{tt}$ is an even function of $v$, while $R_{tx}$ is an odd function of $v$.

$$R_{tx}(v'/V') = R_{tx}(-v/V) = -R_{tx}(v/V)\,, \quad R_{tt}(v'/V') = R_{tt}(-v/V) = +R_{tt}(v/V) \qquad \text{(C6)}$$

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