Deriving the Spherical Pythagorean Theorem Using Infinitesimal Area

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Abstract

The Spherical Pythagorean Theorem is derived by performing an infinitesimal rotation of a spherical right triangle about a vertex not containing the right angle. The infinitesimal areas swept out by the sides of the spherical triangle are easily computed and used to derive the Spherical Pythagorean Theorem.

The Spherical Pythagorean Theorem is derived by performing an infinitesimal rotation of a spherical right triangle about a vertex that does not contain the right angle. This method is similar to the method used to derive the standard Pythagorean Theorem[1]. Consider the spherical cap shown in Fig. 1, where R is the radius of the sphere and θ is the half cone angle that defines the size of the cap. The surface area of the cap is given by eq. (1).

$$cap\ area = 2\pi R^2[1-\cos(\theta)] \tag{1}$$

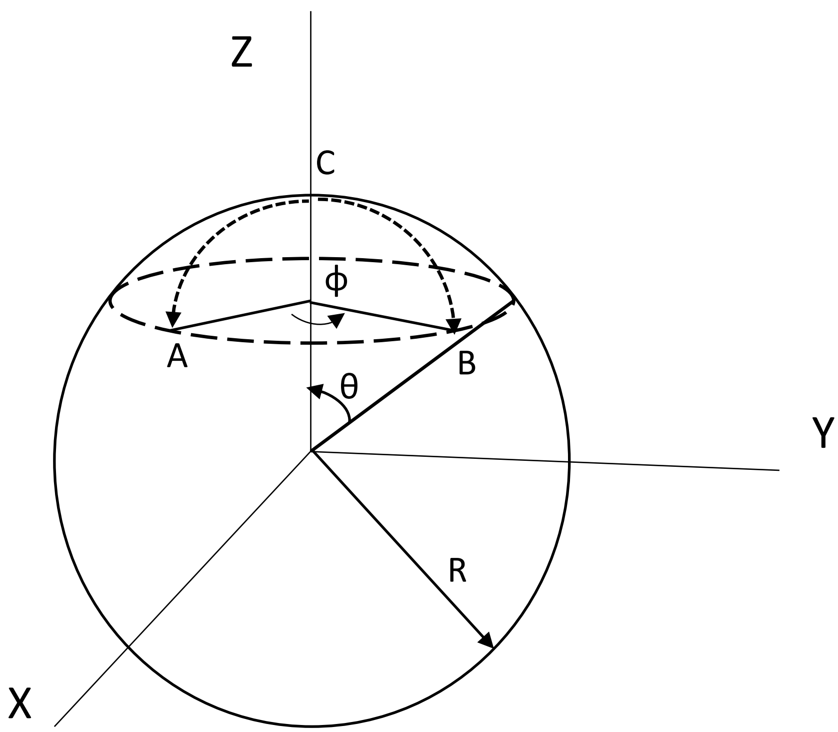

Fig. 1. Surface area of spherical triangle given by AC, AB, BC.

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A portion of the spherical cap defined by angle ϕ is a spherical triangle having sides AC, AB, BC, as shown in Fig. 1. The surface area of the spherical triangle is given by eq. (2). Note that if ϕ equals 2π, eq. (2) becomes eq. (1).

$$spherical\ triangle\ area = \Phi\, R^2[1 - \cos{(\theta)}] \tag{2}$$

The surface area in eq. (2) can be thought of as the area swept out by great circle arc CA as it rotates about the z-axis to great circle arc CB. Equation (2) can be applied to each side of an arbitrary spherical right triangle when the triangle is rotated an infinitesimal amount.

Figure 2 is a 2-dimensional representation of an arbitrary spherical right triangle with the dashed lines representing the spherical triangle after it has been rotated by an infinitesimal angle, δ. Each side of the triangle is a great circle arc on a sphere of radius R. The surface area of the spherical triangle is unchanged by its infinitesimal rotation. Using eq. (2), the surface areas swept out by sides $R\theta_A$ and $R\theta_C$ are given by eq. (3) and eq. (4), respectively. Side $R\theta_B$ rotates by a smaller angle given, which is given by $\delta \cos(\theta_A)$, since its rotation axis makes an angle of $\cos(\theta_A)$ with the original rotation axis shown in Fig. 2. Note that the tangential motion of side $R\theta_B$ due to the infinitesimal rotation does not sweep out surface area. Therefore, using eq. (2), the area swept out by side $R\theta_B$ is given in eq. (5).

$$area\ A = \delta\, R^2[1 - \cos{(\theta_A}] \tag{3}$$

$$area\ C = \delta\, R^2[1 - \cos{(\theta_C}] \tag{4}$$

$$area\ B = \delta \cos{(\theta_A)}\, R^2[1 - \cos{(\theta_B}] \tag{5}$$

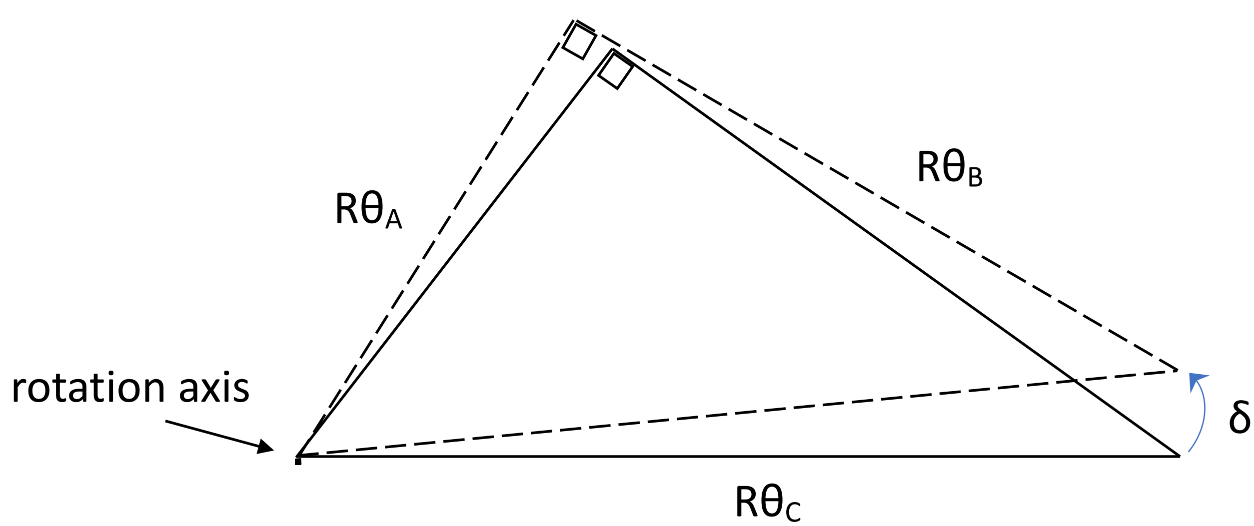

Fig. 2. A spherical right triangle is shown before and after it is rotated by an infinitesimal angle.

The areas swept out by the sides $R\theta_A$ and $R\theta_B$ are considered positive, since they increase the surface area, whereas, the area swept out by side $R\theta_C$ is negative, since it decreases the surface area of the original spherical triangle. Equations (3-5) are used to compute the net change in area, which must be zero, as shown in eq. (6). Equation (6) is simplified to eq. (7), which is the desired Spherical Pythagorean Theorem.

$$area\ A + area\ B - area\ C = \delta\, R^2[1 - \cos(\theta_A) - 1 + \cos(\theta_C) + \{1 - \cos(\theta_B)\}\cos(\theta_A)] = 0 \quad (6)$$

$$\cos(\theta_C) = cos(\theta_B)\cos\,(\theta_A) \quad (7)$$

If the angles in eq. (7) are very small, the Taylor Series approximation given by eq. (8) can be used to reduce eq. (7) to eq. (9). One can multiply each side of eq. (9) by R^2 to obtain the standard Pythagorean Theorem related to Fig. 2, as given by eq. (10). Equation (10) shows that the square of the hypotenuse is equal to the sum of the squares of the adjacent sides, as expected.

$$\cos(\theta) = 1 - \frac{\theta^2}{2} \qquad \text{for } \theta \ll 1 \quad (8)$$

$$\theta_C^2 = \theta_B^2 + \theta_A^2 \quad (9)$$

$$R^2\theta_C^2 = R^2\theta_B^2 + R^2\theta_A^2 \quad (10)$$

References

[1] Patera, R. P., 2021. “Deriving the Pythagorean Theorem using Infinitesimal Area,” VIXRA:2105.0174.