Sequences of prime numbers

Emmanuil Manousos

APM Institute for the Advancement of Physics and Mathematics, Athens, Greece

Abstract. The categorization of odd numbers by "The Octets of the Natural Numbers" theory gives an algorithm for finding prime numbers.

1. Introduction

"The octets of the odd numbers" theory categorizes odd numbers based on the form of the octet they belong to or produce (see [8], equations (5.7), (5.8)):

$$D_1 = 11 + 8m' = 3 + 8m, m \in \mathbb{N} \tag{1}$$

$$Q_1 = 13 + 8m' = 5 + 8m, m \in \mathbb{N} \tag{2}$$

$$D_2 = 7 + 8m, m \in \mathbb{N} \tag{3}$$

$$Q_2 = 9 + 8m' = 1 + 8m, m \in \mathbb{N} . \tag{4}$$

Each of these categories has factors of a different form (see [8], Corollary 3.1). Considering the form of these factors we obtain an algorithm for finding prime numbers. This algorithm does not have the limitations of the so far known formulas for calculating prime numbers (see [1-7] and [9-16]).

2. The sequences W, Θ, U, Φ

The algorithm for finding prime numbers is based on the properties of the sequences W, U and Θ, Φ. We give the definition of sequences W, Θ:

Definition 1. 1. Sequence W:

$$W_n = \prod_{j=1}^{n} \left(1 + 2^{2n}\right) . \tag{5}$$

2. Sequence Θ:

$$\Theta_n = \prod_{j=1}^{n} \left(1 + 2^{2n+1}\right) . \tag{6}$$

W is of the form (2) and Θ is of the form (4). By definition, the sequences W and Θ are products of many factors. Furthermore, have no common factors.

We now give the definition of sequences U and Φ:

Definition 2. 1. Sequence U:

$$\begin{aligned} &U_m(n) = W_n \pm 2^m \\ &W_n - 2^m > 0, m \in \mathbb{N}^* \end{aligned} . \tag{7}$$

2. Sequence Φ:

$$\begin{aligned} &\Phi_m(n) = \Theta_n \pm 2^m \\ &\Theta_n - 2^m > 0, m \in \mathbb{N}^* \end{aligned} . \tag{8}$$

3. The algorithm

We ask for numbers of the form

$$U_m(n) = a \times P$$
$$\Phi_m(n) = a \times P \tag{9}$$

where P is a prime number and a is a composite number (or prime if $a = 1$). These numbers are easily found by any factorization test.

We present the steps of the algorithm for the first of the equations (9). The steps of the algorithm are similar for the second equation.

Step 1. We choose n in equation (7) and calculate the values that m takes from the inequality $W_n - 2^m > 0, m \in \mathbb{N}^*$,

$$m < \frac{\ln W_n}{\ln 2}. \tag{10}$$

Step 2. We factorize the W_n. It follows from its definition that this factorization is easily done by any factorization test. For large numbers W_n we can factorize its factors.

Step 3. A. For the values of m obtained from inequality (10) we run a factorization test on the sequence

$$U_m(n) = W_n \pm 2^m$$

asking for numbers of the form (9).

B. Acceleration of the algorithm: in equations

$$U_m(n) = W_n - 2^m$$
$$U_m(n) = W_n + 2^m$$

we give values $m = 0, 1, 2, \ldots$ to the parameter m and request the periodic iteration of prime factors p that are not factors of W_n,

$$p \notin W_n, m = b + kT$$
$$b, T \in \mathbb{N}, k = 0, 1, 2, \ldots \tag{11}$$

The prime number p appears as a factor of a in equation (9). We ask for small values of the p so that P takes a large value in equation (9).

The algorithm has high prime number efficiency.

4. The application of the algorithm

We present one application of the algorithm.

Step 1. In equation (7) we get $n = 18$,

$$U_m(18) = W_{18} - 2^m \tag{12}$$
$$U_m(18) = W_{18} + 2^m. \tag{13}$$

From inequality (10) we get

$$m = 0, 1, 2, \ldots, 342. \tag{14}$$

Step 2. We factorize the W_{18},

W_{18} = $5^{11} \times 13^3 \times 17^5 \times 29 \times 37 \times 41^2 \times 53 \times 61 \times 97 \times 109 \times 113 \times 137 \times 157 \times 241^2 \times 257^2 \times 397 \times 433 \times 641 \times 673 \times 953 \times 1321 \times 1613 \times 2113 \times 26317 \times 38737 \times 61681 \times 65537 \times$ 6 700417 × 15 790321.

Step 3. A. For $n = 18$, the terms of sequence $U_m(n)$ have 104 digits. For the values of m in equation (14), we run a factorization test on sequences (12), (13) and obtain numbers of the form (9):

3 × 1277 × 3170 860415 441418 276421 331621 580233 256676 293120 817676 685485 260778 995069 751640 957893 388140 207625 949139 (100 digits)
3 × 4 049188 750518 691138 990040 480757 957868 775626 315284 173127 364678 014776 704072 845503 229469 273419 590436 466567 (103 digits)
12 147566 251556 073416 970121 442273 873606 326878 945852 519382 094034 044330 112218 536509 085125 502072 174133 350037 (104 digits) is prime
3^3 × 7 × 64272 837309 820494 269683 182234 253299 504375 020877 526557 577217 111345 661967 693216 698577 575053 419749 736121 (101 digits)
3 × 4 049188 750518 691138 990040 480757 957868 775626 315284 173127 364678 014776 676380 595591 044115 451797 182484 393863 (103 digits)
3^3 × 7 × 64272 837309 820494 269683 182234 253299 504375 020877 526557 577217 111345 211860 459292 460707 526816 019210 469049 (101 digits)
3^2 × 7 × 192818 511929 461482 809049 546702 759898 513125 062632 579672 731651 328506 047118 912627 350842 743077 291397 970987 (102 digits)
7^2 × 251 × 987 687312 103103 782174 983449 245782 064096 827298 630174 756917 308858 422735 087081 762341 870668 773924 618207 (99 digits)
3^3 × 7 × 353 × 182 076026 373429 162237 062839 190519 262052 053883 505717 986218 702402 018552 301085 320973 371176 658728 158297 (99 digits)
3 × 19 × 73 × 2919 386265 694802 551542 927527 583242 875829 579174 680699 206904 727346 526478 930648 379825 247281 163902 904661 (100 digits)
12 147566 251556 073416 970121 442273 873606 326878 933298 315911 320672 516658 533372 121176 857383 700990 560651 677333 (104 digits) is prime
3 × 4 049188 750518 691138 990040 480757 957868 775624 172700 114115 377643 958827 247617 962033 200258 812249 439126 499207 (103 digits)
3 × 21341 × 189 737535 753652 178388 549762 464643 516875 033851 088071 045146 354723 694199 726750 687513 034249 771001 947187 (99 digits)
73 × 166405 017144 603745 437946 666039 540427 806217 325051 595044 795093 545558 521575 569100 173137 228203 652166 120429 (102 digits)
269 × 45158 238853 368302 664413 709407 976421 830713 554674 719626 463805 699589 028233 376877 446611 044791 241033 716137 (101 digits)
3 × 4 049188 750518 688486 599446 295229 199706 736969 532239 692591 598540 428922 211057 037626 466345 486559 651570 724743 (103 digits)
3 × 2939 × 1377 743705 281858 747556 267175 902799 730225 209902 340457 723373 042454 766780 582782 990046 345781 455410 788453 (100 digits)
3 × 4 046272 416218 980456 424210 477788 467706 955155 422709 648844 116110 320634 824135 513237 395298 505942 260209 495943 (103 digits)

3 × 23 × 176051 684805 160484 303914 803511 215559 511983 752838 442309 885420 783251 161046 645456 662167 947998 253459 720337 (102 digits)
2731 × 4448 028653 078020 291823 552340 634885 978149 717666 002387 177625 058236 664266 649043 028081 138195 488644 716479 (100 digits)

11 × 1 104324 204686 915765 179101 949297 624873 302443 540532 047216 554003 094939 101110 776046 335417 128352 680798 891999 (103 digits)
3 × 4 049188 750518 691138 990040 480757 957868 775626 315284 173127 364678 014776 704072 845503 229862 803962 761604 575111 (103 digits)
3 × 4 049188 750518 691138 990040 480757 957868 775626 315284 173127 364678 014776 704072 845503 229862 815969 428579 889031 (103 digits)
11^2 × 100393 109516 992342 289009 268117 965897 572949 412775 640656 050363 917721 736464 616004 212935 093186 821512 761341 (102 digits)
211 × 5059 × 11 379996 844398 255482 903746 635458 812183 370708 058045 414237 208554 267538 882155 996693 542540 852385 458621 (98 digits)
251 × 307 × 157 643903 234697 346340 632537 501769 775702 750936 914913 886890 146697 176507 159875 647217 379702 003189 374357 (99 digits)
11 × 23^2 × 43 × 48 548125 233521 596921 752404 681831 664540 486373 611115 629162 263291 640176 775757 307249 724019 925006 205517 (98 digits)
3^2 × 1 349729 583506 230379 663346 826919 319289 591875 438428 057709 121559 338334 519661 375376 290723 906788 261361 902893 (103 digits)
3 × 174763 × 23 169599 689400 451691 662654 456366 381149 188479 914422 235412 327998 594704 729567 102381 549901 675063 381653 (98 digits)
11 × 1 104324 204686 915765 179101 949297 624873 302443 540532 047216 554011 204300 991485 548097 802951 045679 357704 056799 (103 digits)
3^2 × 1 349729 583506 230379 663346 826919 319289 591875 438428 057709 121579 161143 522273 724626 886148 288118 486752 060717 (103 digits)
11 × 1 104324 204686 915765 179101 949297 624873 302443 540532 047216 562307 081514 844877 356749 090148 470869 837928 771551 (103 digits)
3 × 79 × 101 × 507 480730 733010 545054 523183 451304 407667 079372 764155 256430 453516 399165 681271 287559 341423 292715 590421 (99 digits)
695111 × 17 475721 505710 704357 966024 767661 385888 479507 512102 150937 844030 078749 795439 126972 297366 620719 234371 (98 digits)
23 × 528155 054415 481452 911744 410533 646678 535951 259606 996796 680032 917377 098691 798572 841478 166680 754124 293683 (102 digits)
79 × 153766 661412 102195 151520 524585 745235 544454 121677 840941 383047 120092 010621 970111 787544 268324 804527 801819 (102 digits)
3 × 4 049188 750518 691138 990040 481964 125464 997670 017613 037554 538510 388248 266413 112592 438607 153793 245335 334791 (103 digits)
3 × 4 049188 750518 691138 990040 485582 628253 663801 124599 630836 060007 508662 953433 913860 064840 203293 492620 636039 (103 digits)
3^3 × 167 × 2694 071024 962535 687950 792891 466118 696040 171324 046138 758063 225848 074877 249730 441107 170348 221935 876953 (100 digits)
3^4 × 149969 953722 914486 641252 541831 935038 787489 011102 028133 770558 487208 101012 088424 887585 899555 828287 962373 (102 digits)
247771 × 49 027393 244391 286385 060359 503330 101897 503504 658236 994195 190078 271396 780655 477526 019297 319559 972431 (98 digits)
79 × 153766 661412 108641 467901 329921 461274 908214 953042 946610 749271 525675 857863 970238 585884 979148 274243 880411 (102 digits)
11 × 43 × 25681 958530 772226 558196 620701 823953 999839 082681 863746 362335 797391 163240 396247 870086 423776 439585 308957 (101 digits)
3 × 19 × 701 × 304 015980 142514 525792 323791 617278 246052 528800 240022 641256 241216 500495 862960 531957 158157 034621 192321 (99 digits)

3 × 174763 × 23 169632 281913 907757 459684 644246 823722 686906 968272 647786 461653 059024 650138 176303 402727 907820 277397 (98 digits)

B. Acceleration of the algorithm: in equation (12), we give values $m = 0, 1, 2, \ldots, 100$ to the parameter m and get the equations of the form (11):

$$p = 3, m = 2 + 3 \cdot k$$
$$p = 7, m = 3 + 3 \cdot k$$
$$p = 11, m = 1 + 10 \cdot k$$
$$p = 73, m = 2 + 9 \cdot k$$
$$p = 19, m = 2 + 18 \cdot k \quad .$$
$$p = 47, m = 17 + 23 \cdot k$$
$$p = 43, m = 15 + 28 \cdot k$$
$$k = 0, 1, 2, \ldots$$

Applying the algorithm for these values of m we get the previous prime numbers with fewer trials in the factorization test: the small prime numbers 3, 7, 11, 73, 19, 47, 43 appear as factors in a of equation (9). They are the numbers that run along with the prime numbers in our application. Through the small prime numbers that are not factors of the sequence W_n we can speed up the running of the algorithm. Similarly we find the corresponding small primes and the corresponding equations for the case (13).

For the next value of n, $n = 19$ the terms of the sequence $U_m(n)$ have 115 digits. To find prime numbers of the same form with digits from 104 to 115 we proceed as follows: We multiply the sequence W by the smallest of its factors and get sequences with $104, 105, 106, \ldots, 115$ digits. We then run the algorithm on these sequences. In our application we multiply W_{18} by 5 and get the sequence $5 \cdot W_{18}$ whose terms have 104 digits. The terms of $5 \cdot 13 \cdot W_{18}, 5^2 \cdot W_{18}$ have 105 digits and so on.

To run the algorithm for numbers of the form (1), (3) we multiply the sequences W, Θ by suitable (see [8], corollary 3.1) small prime numbers. In our application, $3W$ has 104 digits and is of the form (3).

References

[1] Atanassov, K. "A formula for the n-th prime number." *Comptes Rendus de l'Academie bulgare des Sciences* 66.4 (2013): 503-506.

[2] Dirichlet, PG Lejeune. "Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält." *Abhandlungen der Königlich Preussischen Akademie der Wissenschaften* 45 (1837): 81.

[3] Erdös, Paul. "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem." *Proceedings of the National Academy of Sciences* 35.7 (1949): 374-384.

[4] Green, Ben, and Terence Tao. "The primes contain arbitrarily long arithmetic progressions." *Annals of mathematics* (2008): 481-547.

[5] Jameson, Graham James Oscar. *The prime number theorem*. No. 53. Cambridge University Press, 2003.

[6] Jones, James P., et al. "Diophantine representation of the set of prime numbers." *The American Mathematical Monthly* 83.6 (1976): 449-464.
[7] Lagarias, Jeffrey. "Euler's constant: Euler's work and modern developments." Bulletin of the American Mathematical Society 50.4 (2013): 527-628.
[8] Manousos, Emmanuil.] **viXra:2103.0159**
[9] Matiyasevich, Yuri V. "Formulas for prime numbers." Kvant Selecta: Algebra and Analysis, II Providence, RI: The American Mathematical Society (1999): 13-24.
[10] Newman, Donald J. "Simple analytic proof of the prime number theorem." *The American Mathematical Monthly* 87.9 (1980): 693-696.
[11] Papadimitriou, Makis. "A recursion formula for the sequence of odd primes." *The American Mathematical Monthly* 82.3 (1975): 289-289.
[12] Regimbal, Stephen. "An explicit formula for the k th prime number." *Mathematics Magazine* 48.4 (1975): 230-232.
[13] Riesel, Hans, and Gunnar Göhl. "Some calculations related to Riemann's prime number formula." Mathematics of Computation 24.112 (1970): 969-983.
[14] Rosser, J. Barkley, and Lowell Schoenfeld. "Approximate formulas for some functions of prime numbers." *Illinois Journal of Mathematics* 6.1 (1962): 64-94.
[15] Rowland E.S., A Natural Prime-Generating Recurrence,
Journal of Integer Sequences, Vol. 11 (2008).
[16] Ruiz, Sebastian Martin. "The general term of the prime number sequence and the Smarandache Prime Function." *Smarandache Notions Journal* 11 (2000): 59-62.