A Proof of the Collatz Conjecture

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Abstract

Take any positive integer N. If it is odd, multiply it by three and add one. If it is even, divide it by two. Repeatedly do the same operations to the results, forming a sequence. It is found that, whatever the initial starting number we choose, the sequence will eventually descend and reach number 1, where it enters an eternal closed loop of 1- 4 - 2 - 1. This has been numerically confirmed for starting numbers up to 2^{60}. This is known as the Collatz conjecture which states that the sequence always reaches number 1. So far no proof has ever been found that this holds for every positive integer. This problem has been stated by some as perhaps the simplest math problem to state, yet perhaps the most difficult to solve. This paper makes significant advances in solving the problem by using new insights. Proving the conjecture requires proving that: 1. The sequence will not diverge to infinity 2. There is no closed loop other than the 1-4-2-1 loop. This paper completely proves the first and makes significant advance in proving the second. The new insight is that the whole of the Collatz sequence up to the point where it enters a closed loop is *encoded* in the starting number. A Collatz sequence diverging to infinity would mean infinite information being encoded in a finite starting number, which is impossible! Therefore, a Collatz sequence generated from a finite starting number can never diverge to infinity. Further reasoning leads to the conclusion that the Collatz sequence necessarily /always ends up in a closed loop.

Introduction

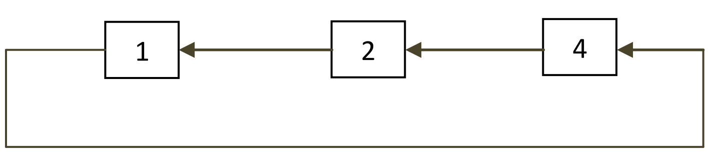
The Collatz function is defined as:

$$C(N) = \begin{cases} 3N + 1\,, & \text{if n is odd} \\ \dfrac{N}{2}, & \text{if n is even} \end{cases}$$

The Collatz conjecture :

Take any positive number *N*. If *N* is odd, multiply it by three and add one. If *N* is even, divide it by two. Repeatedly do this to form a sequence. The Collatz conjecture says that this sequence always reaches number 1.

All sequences finally end in the closed loop:



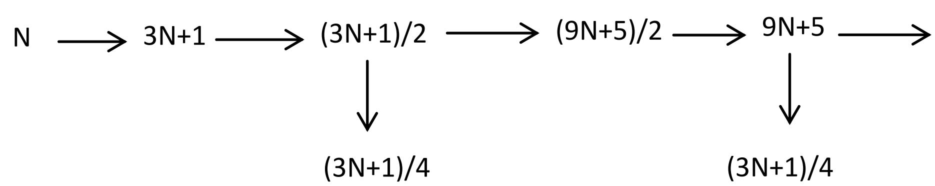
To prove the Collatz conjecture, one has to show that:

1. The sequence will not diverge to infinity.

2. The sequence will not enter some closed loop other than the 1- 4- 2- 1 loop, i.e. no closed loops other than the 1-4-2-1 loop exist.

Intuitive explanation on why the Collatz sequence cannot diverge to infinity

Let the starting number *N* be odd..



We can see that ***no successive Collatz operations can be odd operations, whereas multiple successive even operations can occur before an odd operation occurs***. This is because any even number can have 2^m as a factor. Therefore, after every odd Collatz operation there are one or more even operations. This shows that the Collatz number going to infinity is impossible and in fact, in the long run the sequence necessarily converges and descends.

As the starting number is made larger and larger, the probability that many more even operations occur before an odd operation increases (see Appendix). This leads to the conclusion that the probability that the next number being greater than the current number during any Collatz operation necessarily approaches zero as the starting number approaches infinity.

$$\lim_{N\to\infty} (Probability\ that\ C(N+1) > C(N)) \ = 0$$

where N is the starting number or any number in the sequence.

In other words, the Collatz sequence always wanders around before it eventually (and frequently) lands on a number Q that has 2^m as a factor, where *m* is a positive integer. I refer to such numbers as ‘wells’. Once the sequence lands on these wells, it descends all the way down by a factor of 2^m, more than losing everything it gained by the possible previous odd operations.

$$P \ = \ 2^m * Q$$

Let us call the number *P* a ‘perfect/pure even number’ if it is a power of 2 :

$$P \ = \ 2^m$$

On the other hand, an ‘imperfect’ even number is of the form:

$$P = 2^m * Q$$

where Q does not have 2 or any powers of 2 as a factor. If the Collatz sequence lands on an ‘imperfect’ even number, as opposed to perfect even number, the sequence may descend significantly but the descent will stop somewhere.

One can still ask if there is a greater than zero probability that the sequence will diverge to infinity. This is disproved by the fact that there is always a greater than zero probability that the sequence will land on one of the *perfect* / *pure* even numbers. One can think of the column (shown below) formed by these numbers as ‘eternal well’ because once the sequence lands on such number, it will descend all the way down to one, where it enters the 1-4-2-1 loop.

1- 2 – 4 – 8 – 16 – 32 – 64 – 128 – 256 – 512 – 1024 – 2048 - . . . .

Now let us define one Collatz cycle. For any Collatz operation on an odd number, the next number is necessarily an even number. Therefore, after every Collatz odd operation, there is an even operation. Let us call this the Collatz cycle.

$$N \longrightarrow 3N+1 \longrightarrow (3N+1)/2$$

We can see that an odd number N becomes N’ =(3N+1)/2 after one Collatz cycle.

$$N' = \frac{3N+1}{2} = 1.5N + 0.5 \approx 1.5\,N \quad \text{for} \quad 3N \gg 1$$

Therefore, an odd number is approximately increased by a factor of 1.5 after one Collatz cycle.

Now consider an even number.

$$\begin{array}{ccccc} N & \longrightarrow & N/2 & \xrightarrow{\text{odd}} & (3N+2)/2 \\ & & \Big\downarrow \text{even} & & \\ & & N/4 & & \end{array}$$

One Collatz cycle on an even number *N* has two cases depending on whether the Collatz operation on *N* results in an even integer or an odd integer.

1. If the Collatz operation on *N* results in an even integer, the number at the end of one cycle will be *N/*4.

2. If the Collatz operation on *N* results in an odd integer, the number at the end of one cycle will be 3(*N*/2)+1.

In the first case, the number *N'* after one Collatz operation on *N* will be:

$$N' = \frac{N}{4}$$

In the second case, the number *N'* after one Collatz operation on *N* will be:

$$N' = 3\left(\frac{N}{2}\right) + 1 = \frac{3N}{2} + 1 \approx 1.5\,N \;\;, for\; \frac{3N}{2} \gg 1$$

We defined one Collatz operation to help us compare the probabilities of the sequence increasing or decreasing. We can see that the maximum possible increase of *N* due to any odd operations is approximately:

$$N' - N \approx 1.5N - N = 0.5N$$

which is a fifty percent increase.

The maximum possible decrease of the sequence due to even operations is:

$$N' - N = \frac{N}{4} - N = -0.75N$$

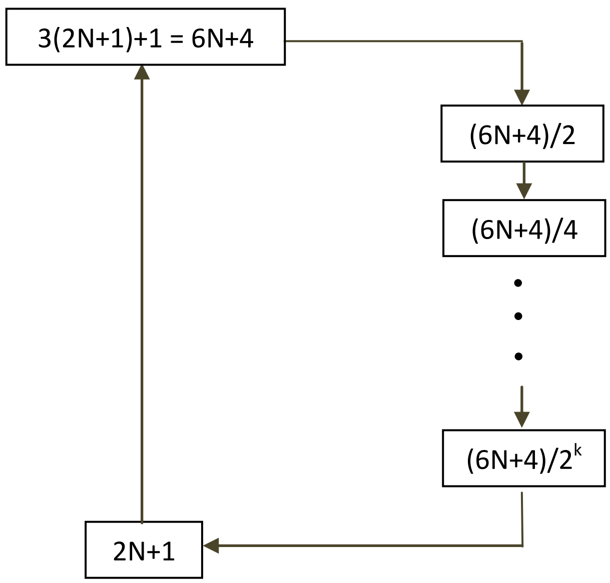
A Collatz cycle on an odd number *N* always increases it to *3N* +1, however a Collatz cycle on an even number *N* may decrease it to *N*/4 or increase it to *(3N/2)+1*. One might think that this will result in slightly higher probability of an ascending Collatz sequence than probability of descending Collatz sequence. However, all this is nullified by the argument we made above: whatever gains (ascents) the Collatz sequence gains due to multiple odd operations is soon more than lost because the sequence inevitably lands on ‘wells’ of varying ‘depth’.

These arguments seal the impossibility of the Collatz sequence diverging to infinity.

Non- existence of closed loop other than the 1-4-2-1 loop

Next we present a new approach to disprove non-existence of any other closed loop other than the 1-4-2-1 loop as follows.

We start with an odd integer, *2N + 1*, where *N* is an integer greater than or equal to zero. Since it is odd, we multiply it by 3 and add 1. To form a loop, successive Collatz operations (divide by 2) on the result must give the original integer,



$$3\ (2N+1)+1=(2N+1)\ 2^k\ ,\quad k \text{ is a positive integer}$$

$$\Rightarrow\ 6N+4=(2N+1)\ 2^k$$

$$\Rightarrow \frac{6N+4}{2N+1}\ =\ 2^k$$

$$\Rightarrow\ \frac{2(\ 3N+2\ )}{2N+1}\ =\ 2^k$$

$$\Rightarrow\ \frac{(\ 3N+2\ )}{2N+1}\ =\ 2^{k-1}$$

$$\Rightarrow \frac{(2N+1)+(N+1)}{2N+1} = 2^{k-1}$$

$$\Rightarrow 1 + \frac{N+1}{2N+1} = 2^{k-1}$$

Since the right hand side is always a positive integer that is a power of 2, the left hand side must also be an integer (not a fraction) and a power of 2 for both sides to be equal. This is possible only if $N = 0$.

$$\Rightarrow N = 0$$

Since $N = 0$, our starting odd number $2N+1$ will be:

$$\Rightarrow 2N + 1 = 2 * 0 + 1 = 1$$

Thus we have proved that no other closed loop exists other than the 4-2-1-4 loop. However, this is by no means a rigorous proof that no other closed loop exists.

Proof that the Collatz sequence will never diverge to infinity and necessarily/always ends in a closed loop

We know that an even number is given by $2n$ and an odd number by $2n+1$, where n is any positive integer. But there are differences between even numbers and between odd numbers themselves.

Although two different even numbers can both be represented as $2n$, they will have different natures depending on whether n itself is even or odd. One might call these "even-even" and "even-odd" numbers, respectively. The same applies to odd numbers. Although two odd numbers can both be represented as $2n+1$, they can have different natures depending on whether n is even or odd. One might call these "odd-even" and "odd-odd" numbers, respectively. By the same argument, "even-even" numbers can be further divided in to two: "even-even-even" and "even-even-odd". "Odd-odd" numbers can be classified as "odd-odd-even" and "odd-odd-odd" numbers, and so on. In this paper, we use this insight as a new approach to the Collatz conjecture.

We can represent any positive integer N as:

$$N = 2n + b$$

where b is 0 or 1 depending on whether N is even or odd. If N is even $b = 0$ and if N is odd $b = 1$

$2n+b_0$

$2(2n+b_1)+b_0 = 4n + 2b_1 + b_0$

$4(2n+b_2)+2b_1+b_0 = 8n+4b_2+2b_1+b_0$

$8(2n+b_3)+4b_2+2b_1+b_0 = 16n+8b_3+4b_2+2b_1+b_0$

$16(2n+b_4)+8b_3+4b_2+2b_1+b_0 = 32n+16b_4+8b_3+4b_2+2b_1+b_0$

$32(2n+b_5)+16b_4+8b_3+4b_2+2b_1+b_0 = 64n+32b_5+16b_4+8b_3+4b_2+2b_1+b_0$

$64(2n+b_6)+32b_5+16b_4+8b_3+4b_2+2b_1+b_0 = 128n+64b_6+32b_5+16b_4+8b_3+4b_2+2b_1+b_0$

If we let $n=1$, we can see that this is equivalent to a binary representation.

Thus any positive integer N can be represented as:

$$N = 2^n b_n + 2^{n-1} b_{n-1} + 2^{n-2} b_{n-2} + \ . \ . \ . \ + 4b_2 + 2b_1 + b_0$$

Let us denote the Collatz sequence as follows:

$$N \longrightarrow C_o \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow C_4 \longrightarrow \ . \ . \ .$$

We know that if N is an odd number, C_0 will be an even number. If N is an odd number, the least significant bit (LSB) b_0 of the binary representation of N is 1. Therefore, if the LSB of the binary representation of any number N is 1, C_0 will be an even number. It can also be shown that if $b_0 = b_1 = 1$, C_1 will be odd, whereas if $b_0 = 1$ and $b_1 = 0$ C_1 will be even.

On the other hand, if N is an even number, $b_0 = 0$, then C_0 will be an even number or an odd number and this depends on the bit b_1. If b_1 is 0 then C_0 will be even and if b_1 is 1 C_0 will be odd, and so on.

It can be shown that the whole Collatz sequence (up to the repeating sequence) from any starting number N is *encoded* in the binary representation of N! The Collatz sequence generated from any starting number N is unique.

Now the question: will the Collatz sequence ever diverge to infinity? No. *This is because a Collatz sequence diverging to infinity would mean infinite information and infinite information cannot be encoded in a finite starting number !!! Therefore, a Collatz sequence generated from a finite starting number N can never diverge to infinity.*

This raises the question: we know that the Collatz sequence is an infinite sequence. However, we have also seen that *an infinite sequence cannot be encoded into a finite starting number*. How can these apparently contradicting statements be reconciled? *The only way out is if the Collatz* *sequence enters a closed loop!* An infinite repeating sequence does not mean an infinite information[2]. The sequence up to the point where it enters a closed loop is unique to the starting number. The repeating sequence (loop) is independent of the starting number.

Thus we have proved that:
1. The Collatz sequence cannot diverge to infinity
2. The Collatz sequence *necessarily* ends up in some closed loop.

Thus, if one can prove that the 1-4-2-1 loop is the only possible closed loop, one completely proves the Collatz conjecture.

Non-existence of closed loop other than the 1-4-2-1 loop

The previous proof of non-existence of closed loop other than 1-4-2-1is only meant to show a possible approach and obviously is far from complete, rigorous proof. Next we present a new approach towards a more rigorous proof that no other closed loop can exist.

We consider two cases: the Collatz sequence starting number N is:

1) Odd

2) Even

Odd starting number

Re-writing the Collatz function:

$$\mathrm{C(N)} = \begin{cases} 3\mathrm{N} + 1\,, & \text{if n is odd} \\ \dfrac{\mathrm{N}}{2}, & \text{if n is even} \end{cases}$$

as

$$\mathrm{C(N)} = \begin{cases} \mathrm{aN} + 1\,, & \text{if n is odd} \\ \mathrm{bN}, & if\ n\ is\ even \end{cases}$$

where

$$a = 3 \quad \text{and} \quad b = \frac{1}{2}$$

Case 1: Starting number N is odd

We assume that the initial/starting number is an odd number that successively results in an odd number after multiplying it by three and adding one, and then dividing the result by two. Note that multiplying an odd number by three and adding one always gives an even number.

$$N \longrightarrow aN+1 \longrightarrow abN + b \xrightarrow{\text{odd}} a^2bN + ab+1 \longrightarrow a^2b^2N + ab^2 + b \xrightarrow{\text{odd}}$$

$$abN + b \;\Big\downarrow{\scriptstyle\text{even}}\; ab^2N + b^2$$

$$a^2b^2N + ab^2 + b \;\Big\downarrow{\scriptstyle\text{even}}\; a^2b^3N + ab^3 + b^2$$

$$\longrightarrow a^3b^2N + a^2b^2 + ab + 1 \longrightarrow a^3b^3N + a^2b^3 + ab^2 + b \xrightarrow{\text{odd}} a^4b^3N + a^3b^3 + a^2b^2 + ab+1 \longrightarrow$$

$$a^3b^3N + a^2b^3 + ab^2 + b \;\Big\downarrow{\scriptstyle\text{even}}\; a^3b^4N + a^2b^4 + ab^3 + b^2$$

$$\longrightarrow a^4b^4N + a^3b^4 + a^2b^3 + ab^2 + b \xrightarrow{\text{odd}} a^5b^4N + a^4b^4 + a^3b^3 + a^2b^2 + ab+1 \longrightarrow$$

$$a^4b^4N + a^3b^4 + a^2b^3 + ab^2 + b \;\Big\downarrow{\scriptstyle\text{even}}\; a^4b^5N + a^3b^5 + a^2b^4 + ab^3 + b^2$$

$$\longrightarrow a^5b^5N + a^4b^5 + a^3b^4 + a^2b^3 + ab^2 + b \xrightarrow{\text{odd}}$$

$$a^5b^5N + a^4b^5 + a^3b^4 + a^2b^3 + ab^2 + b \;\Big\downarrow{\scriptstyle\text{even}}\; a^5b^6N + a^4b^6 + a^3b^5 + a^2b^4 + ab^3 + b^2$$

From the above, the last Collatz number is:

$$a^5b^5N + a^4b^5 + a^3b^4 + a^2b^3 + ab^2 + b$$

We can generalize this as follows:

$$C(N,x) = a^x b^x N + a^{x-1}b^x + a^{x-2}b^{x-1} + a^{x-3}b^{x-2} + \ldots + a^3b^4 + a^2b^3 + ab^2 + b$$

We can see that the part:

$$(a^{x-1}b^x + a^{x-2}b^{x-1} + a^{x-3}b^{x-2} + \ldots + a^3b^4 + a^2b^3 + ab^2 + b)$$

is a geometric sequence with ratio $r = ab$ and first term b .

$$(a^{x-1}b^x + a^{x-2}b^{x-1} + a^{x-3}b^{x-2} + \ldots + a^3b^4 + a^2b^3 + ab^2 + b) = b\frac{1-(ab)^x}{1-ab}$$

Therefore, the x^{th} number will be:

$$\boxed{C(N,x) = a^x b^x N + b\frac{1-(ab)^x}{1-ab} \quad \ldots\ldots\ldots\ldots \ (1)}$$

<u>Case 2: Starting number *N* is even</u>

In this case we assume that the starting number is an even number that successively results in an even number after dividing the result by two.

$$\begin{tikzcd}
N \arrow[r] & bN \arrow[r, "\text{even}"] \arrow[d, "\text{odd}"] & b^2N \arrow[r, "\text{even}"] \arrow[d, "\text{odd}"] & b^3N \arrow[r, "\text{even}"] \arrow[d, "\text{odd}"] & b^4N \arrow[r, "\text{even}"] \arrow[d, "\text{odd}"] & b^5N \arrow[r, "\text{even}"] \arrow[d, "\text{odd}"] & {} \\
 & abN+1 & ab^2N+1 & ab^3N+1 & ab^4N+1 & ab^4N+1 &
\end{tikzcd}$$

The y^{th} number will be:

$$\boxed{C(N,y) = b^y N \ \ldots\ldots\ldots\ldots\ldots \ (2)}$$

Collatz sequence starting with odd number and ending in odd number

A closed loop in a Collatz sequence necessarily contains at least one odd number. So let us assume that the starting number N is odd.

Any number in the sequence can be expressed as:

$$C\,(N) =$$

$$(((\,(\,a^{x1}b^{x1}\,N \;+\; b\frac{1-(ab\,)^{x1}}{1-ab}\;)\,b^{y1}\,a^{x2}b^{x2} \;+\; b\frac{1-(ab\,)^{x2}}{1-ab})b^{y2}\,a^{x3}b^{x3}$$

$$+\,b\frac{1-(ab\,)^{x3}}{1-ab}\;)\;b^{y3}\,a^{x4}b^{x4} \;+\; b\frac{1-(ab\,)^{x4}}{1-ab}\;)b^{y4}\,a^{x5}b^{x5}$$

$$+\;b\frac{1-(ab\,)^{x5}}{1-ab} \qquad \ldots\ldots\ldots\ldots(3)$$

This can be generalized as:

$$C(N) =$$

$$(((\ldots\,(\,a^{x1}b^{x1}\,N \;+\; b\frac{1-(ab\,)^{x1}}{1-ab}\;)\,b^{y1}\,a^{x2}b^{x2} \;+\; b\frac{1-(ab\,)^{x2}}{1-ab})b^{y2}\,a^{x3}b^{x3}$$

$$+\,b\frac{1-(ab\,)^{x3}}{1-ab}\;)\;b^{y3}\,a^{x4}b^{x4} \;+\; b\frac{1-(ab\,)^{x4}}{1-ab}\;)b^{y4}\,a^{x5}b^{x5} \;+\;.\;.\;.$$

$$.\;\;.\;\;.+\,b\frac{1-(ab\,)^{xn-1}}{1-ab})b^{yn-1}\,a^{xn}b^{xn} \;+\; b\frac{1-(ab\,)^{xn}}{1-ab} \qquad \ldots\ldots\ldots \quad (4)$$

Rearranging equation (3)

$$C\,(N) = (ab)^{x1+x2+x3+x4+x5}\, b^{y1+y2+y3+y4}\ N\ +$$

$$b\frac{1-(ab\,)^{x1}}{1-ab}(ab)^{x2+x3+x4+x5}b^{y1+y2+y3+y4}\ +$$

$$b\frac{1-(ab\,)^{x2}}{1-ab}(ab)^{x3+x4+x5}b^{y2+y3+y4}\ +\ \ b\frac{1-(ab\,)^{x3}}{1-ab}(ab)^{x4+x5}b^{y3+y4}\ +$$

$$b\frac{1-(ab\,)^{x4}}{1-ab}(ab)^{x5}b^{y4}\ +\ \ b\frac{1-(ab\,)^{x5}}{1-ab}\ \ \ldots\ldots\ldots\ldots..\ (5)$$

This can be generalized as:

$$C\,(N\,) = (ab)^{x1+x2+\,.\,.\,.+xn}\, b^{y1+y2+\,.\,.\,.+yn-1}\ N\ +$$

$$b\frac{1-(ab\,)^{x1}}{1-ab}(ab)^{x2+x3+\,.\,.\,.+xn}b^{y1+y2+\,.\,.\,.+yn-1}\ +$$

$$b\frac{1-(ab\,)^{x2}}{1-ab}(ab)^{x3+x4+\,.\,.\,.\,.+xn}b^{y2+y3+.\,.\,.+yn-1}\ +$$

$$b\frac{1-(ab\,)^{x3}}{1-ab}(ab)^{x4+x5+\,.\,.\,.+xn}b^{y3+y4+\,.\,.\,.+yn-1}\ +\ \ .\ \ .\ \ .\ +$$

$$b\frac{1-(ab\,)^{(xn-1)}}{1-ab}(ab)^{xn}b^{yn-1}\ +\ \ b\frac{1-(ab\,)^{xn}}{1-ab}\ \ \ldots\ldots..\ (6)$$

Collatz sequence starting with even number and ending in even number

Any number in the sequence can be expressed as:

$$((((\,a^{x1}b^{x1}\,Nb^{y1}\ +\ b\frac{1-(ab\,)^{x1}}{1-ab})b^{y2}a^{x2}b^{x2}\ +\ b\frac{1-(ab\,)^{x2}}{1-ab}\ )b^{y3}a^{x3}b^{x3}\ +$$

$$b\frac{1-(ab\,)^{x3}}{1-ab}\ \ )\ b^{y4}a^{x4}b^{x4} +\ b\frac{1-(ab\,)^{x4}}{1-ab}\ \ )\ b^{y5}\ \ \ldots\ldots\ldots\ (7)$$

Equation (7) can be generalized as:

$$C(N) =$$

$$((..((a^{x1}b^{x1}Nb^{y1} + b\frac{1-(ab)^{x1}}{1-ab})b^{y2}a^{x2}b^{x2} + b\frac{1-(ab)^{x2}}{1-ab})b^{y3}a^{x3}b^{x3} +$$

$$b\frac{1-(ab)^{x3}}{1-ab})b^{y4}a^{x4}b^{x4}+...)b^{yn-1}a^{xn-1}b^{xn-1} + b\frac{1-(ab)^{xn-1}}{1-ab})b^{yn} \quad \ldots\ldots\ldots. \quad (8)$$

To illustrate what *x1, x2, x3*, . . . , *y1, y2, y3*, . . . are let us see what these are for a particular starting number.

Consider an odd starting number 715. The Collatz sequence will be:

715, 2146, 1073, 3220, 1610, 805, 2416, 1208, 604,302,151,454,227,682,341, 1024,512,256,128,

64,32,16,8,4,2,1,4,2,1, 4,2,1,4,2,. . .

We can see that:

	x1		x2		x3		x4		x5		x6		x7		x8		. . .
	2		**1**		**3**		**1**		**1**		**1**		**1**		**1**		. . .
		y1		y2		y3		y4		y5		y6		y7		y8	. . .
		1		**3**		**9**		**1**		**1**		**1**		**1**		**1**	. . .

Consider another odd number 1433.

1433,4300,2150,1075,3226,1613,4840,2420,1210,605,1816,908,454,227,682,341,1024,512,256,

128,64,32,16,8,4,2,1,4,2,1, . . .

	x1		x2		x3		x4		x5		x6		x7		x8		. . .
	1		**2**		**1**		**2**		**1**		**1**		**1**		**1**		. . .
		y1		y2		y3		y4		y5		y6		y7		y8	. . .
		1		**2**		**2**		**9**		**1**		**1**		**1**		**1**	. . .

Consider an even starting number 1738.

1738,869,2608,1304,652,326,163,490,245,736,368,184,92,46,23,70,35,106,53,160,80,40,20,10,

5,16,8,4,2,1,4,2,1, . . .

y1		y2		y3		y4		y5		y6		y7		y8		. . .
1		**3**		**4**		**4**		**3**		**1**		**1**		**1**		. . .
	x1		x2		x3		x4		x5		x6		x7		x8	. . .
	1		**2**		**3**		**1**		**1**		**1**		**1**		**1**	. . .

Test of closed loop

A closed loop is formed if :

$$C(N) = N$$

where C(N) is as expressed in equation (3), (4), (5), or (6).

After substituting a = 3 and b = ½ in the above equations, a valid value of *N* cannot be obtained for any values of *x*'s and *y*'s except for those corresponding to the 1-4-2-1 loop, if there are no loops other than the 1-4-2-1 loop.

Re-writing equation (6) below:

$$C(N) = (ab)^{x1+x2+...+xn} b^{y1+y2+...+yn-1} N +$$

$$b\frac{1-(ab)^{x1}}{1-ab}(ab)^{x2+x3+...+xn} b^{y1+y2+...+yn-1} +$$

$$b\frac{1-(ab)^{x2}}{1-ab}(ab)^{x3+x4+....+xn} b^{y2+y3+...+yn-1} +$$

$$b\frac{1-(ab)^{x3}}{1-ab}(ab)^{x4+x5+...+xn} b^{y3+y4+...+yn-1} + \ . \ . \ . \ +$$

$$b\frac{1-(ab)^{(xn-1)}}{1-ab}(ab)^{xn} b^{yn-1} + \quad b\frac{1-(ab)^{xn}}{1-ab}$$

Let us test it for $x_1 = y_1 = 1$,which is the case of the 1-4-2-1 loop:

$$C\ (N\ ) = \ (ab)^1 b^1\ N \ + \ b\frac{1-(ab\ )^1}{1-ab}b^1$$

$$C(N) = N$$

$$(ab)^1 b^1\ N \ + \ b\frac{1-(ab\ )^1}{1-ab}b^1 \ = \ N$$

$$(\ \frac{3}{2})^1(\frac{1}{2})^1\ N \ + \ \frac{1}{2}\frac{1-\left(\frac{3}{2}\right)^1}{1-\frac{3}{2}}(\frac{1}{2})^1 \ = \ N$$

$$\frac{3}{4}\,N \ + \ \frac{1}{4} \ = \ N$$

$$\frac{1}{4} \ = \ N\left(1-\frac{3}{4}\right) \quad \Longrightarrow \quad N = 1$$

Thus we have reduced the Collatz conjecture to the problem of whether C(N) = N in the above equations, for some positive integer N. We need to show that C(N) = N only for the 1-4-2-1 loop, i.e. if no other loops exist (which is the more likely case).

$$C(N) = N$$

Using C(N) from equation (6):

$$C\ (N\ ) = \ (ab)^{x1+x2+\,.\,.\,.+xn}\, b^{y1+y2+\,.\,.\,.+yn-1}\ \ N \ +$$

$$b\frac{1-(ab\ )^{x1}}{1-ab}(ab)^{x2+x3+\,.\,.\,.+xn} b^{y1+y2+\ .\,.\,.+yn-1} \ +$$

$$b\frac{1-(ab\ )^{x2}}{1-ab}(ab)^{x3+x4+\,.\,.\,.\,.+xn} b^{y2+y3+.\,.\,.+yn-1} +$$

$$b\frac{1-(ab\ )^{x3}}{1-ab}(ab)^{x4+x5+\,.\,.\,.+xn} b^{y3+y4+\,.\,.\,.+yn-1} \ + \ \ . \ \ . \ \ . \ +$$

$$b\frac{1-(ab\,)^{(xn-1)}}{1-ab}(ab)^{xn}b^{yn-1} \;+\; b\frac{1-(ab\,)^{xn}}{1-ab}$$

But

$$\frac{b}{1-ab} = \frac{1/2}{1-(\frac{3}{2})} = -1$$

Therefore,

$$C\,(N\,) = (ab)^{x1+x2+...+xn}\,b^{y1+y2+...+yn-1}\;N \;+$$

$$-\;(1-(ab\,)^{x1}\,)\,(ab)^{x2+x3+...+xn}b^{y1+y2+\,...+yn-1} \;+$$

$$-\;(1-(ab\,)^{x2})\,(ab)^{x3+x4+....+xn}b^{y2+y3+...+yn-1} +$$

$$-\;(1-(ab\,)^{x3})\,(ab)^{x4+x5+...+xn}b^{y3+y4+...+yn-1} \;+\;\;.\;\;.\;\;.\;+$$

$$-\;(1-(ab\,)^{xn-1})(ab)^{xn}b^{yn-1} \;-\;\; (1-(ab\,)^{xn}\,)$$

$$\Longrightarrow\;\; C\,(N\,) = (ab)^{x1+x2+...+xn}\,b^{y1+y2+...+yn-1}\;N \;+$$

$$+\,((ab\,)^{x1}-1\,)\,(ab)^{x2+x3+...+xn}b^{y1+y2+\,...+yn-1} \;+$$

$$+\;((ab\,)^{x2}-1)\,(ab)^{x3+x4+....+xn}b^{y2+y3+...+yn-1} +$$

$$+\,((ab\,)^{x3}-1)\,(ab)^{x4+x5+...+xn}b^{y3+y4+...+yn-1} \;+\;\;.\;\;.\;\;.\;+$$

$$+\;((ab\,)^{xn-1}\;-1)(ab)^{xn}b^{yn-1} \;+\;\; ((ab\,)^{xn}-1\,) \qquad \dots\dots\dots\dots\dots\;\;(9)$$

$$\Rightarrow \quad C\,(N\,) = \,(\frac{3}{2})^{x1+x2+\,.\,.\,.+xn}\,(\frac{1}{2})^{y1+y2+\,.\,.\,.+yn-1}\;\; N \;\; +$$

$$+\;\,(\left(\frac{3}{2}\right)^{x1} - 1\,)\,(\frac{3}{2})^{x2+x3+\,.\,.\,.+xn}(\frac{1}{2})^{y1+y2+\;.\,.\,.+yn-1} \qquad +$$

$$+\;\,(\left(\frac{3}{2}\right)^{x2} - 1)\,(\frac{3}{2})^{x3+x4+\,.\,.\,.\,.+xn}(\frac{1}{2})^{y2+y3+.\,.\,.+yn-1} \qquad +$$

$$+\,(\left(\frac{3}{2}\right)^{x3} - 1)\,(\frac{3}{2})^{x4+x5+\,.\,.\,.+xn}(\frac{1}{2})^{y3+y4+\,.\,.\,.+yn-1} \qquad + \quad . \quad . \quad . \;+$$

$$+\;\,\left(\left(\frac{3}{2}\right)^{xn-1} - 1\right)\left(\frac{3}{2}\right)^{xn}(\frac{1}{2})^{yn-1} \;\; + \;\;\; \left(\left(\frac{3}{2}\right)^{xn} - 1\,\right)$$

For closed loop to occur,

$$C(N) = \;\; N$$

$$\Rightarrow \quad N \;=$$

$$\frac{(\,\left(\frac{3}{2}\right)^{x1} - 1\,)\;\left(\frac{3}{2}\right)^{x2+x3+\,.\,.\,.+xn}\;\left(\frac{1}{2}\right)^{y1+y2+\;.\,.\,.+yn-1} \;+(\left(\frac{3}{2}\right)^{x2} - 1)\,(\frac{3}{2})^{x3+x4+\,.\,.\,.\,.+xn}\left(\frac{1}{2}\right)^{y2+y3+.\,.\,.+yn-1} + \;.\;.\;.}{1 \quad - \quad (\,\frac{3}{2})^{x1+x2+\,.\,.\,.+xn}\,(\frac{1}{2})^{y1+y2+\,.\,.\,.+yn-1}}$$

$$= \;\; \frac{(\,\frac{3^{x1}-2^{x1}}{2^{x1}}\,)\;\left(\frac{3}{2}\right)^{x2+x3+\,.\,.\,.+xn}\;\left(\frac{1}{2}\right)^{y1+y2+\;.\,.\,.+yn-1} \;+(\frac{3^{x2}-2^{x2}}{2^{x2}})\,(\frac{3}{2})^{x3+x4+\,.\,.\,.\,.+xn}\left(\frac{1}{2}\right)^{y2+y3+.\,.\,.+yn-1} + \;.\;.\;.}{\frac{2^{x1+x2+\;.\;.+\,y1+y2+\,.\,.\,.} \;-\; 3^{x1+x2+\,.\,.\,.}}{2^{x1+x2+\;.\;.+\,y1+y2+\,.\,.\,.}}}$$

$$= \; 2^{x1+x2+\;.\;.+\,y1+y2+\,.\,.\,.} \;\; *$$

$$\frac{\left(\frac{3^{x1}-2^{x1}}{2^{x1}}\right)\left(\frac{3}{2}\right)^{x2+x3+\,.\,.\,.}\left(\frac{1}{2}\right)^{y1+y2+\,.\,.\,.} + \left(\frac{3^{x2}-2^{x2}}{2^{x2}}\right)\left(\frac{3}{2}\right)^{x3+x4+\,.\,.\,.\,.}\left(\frac{1}{2}\right)^{y2+y3+.\,.\,.} + \,.\,.\,.}{2^{x1+x2+\;.\;.+\,y1+y2+\,.\,.\,.} \; - \; 3^{x1+x2+\,.\,.\,.}}$$

$$\Longrightarrow$$

$$\boxed{N \; = \; \frac{(3^{x1} - 2^{x1})3^{x2+x3+\,.\,.\,.} + 2^{x1+y1}(3^{x2} - 2^{x2})3^{x3+x4+\,.\,.\,.} + 2^{x1+x2++y1+y2}(3^{x3} - 2^{x3})3^{x4+x5+\,.\,.\,.} \;.\,.\,.}{2^{x1+x2+\;.\;.+\,y1+y2+\,.\,.\,.} \; - \; 3^{x1+x2+\,.\,.\,.}} \qquad .\,.\,.\,.\,.\,.\,. \quad (10)}$$

One solution is:

$$x2 = x3 = x4 = \,.\,.\,. \; = \; 0$$

$$y2 = y3 = y4 = \,.\,.\,. \; = \; 0$$

In this case equation (10) becomes:

$$N = \frac{(3^{x1} - 2^{x1})3^{0}}{2^{x1+y1} \; - \; 3^{x1}}$$

A solution to this equation will be:

$$x_1 = \; y_1 = 1$$

In this case:

$$N = \frac{3^{1} - 2^{1}}{2^{2} \; - \; 3^{1}} = \frac{1}{4 \; - \; 3} = 1$$

So we have proved that number 1 is one of the odd numbers that leads to a closed loop of Collatz sequence. The question is: is N =1 the only solution of equation (10)? A rigorous proof that equation (10) has no integer solution other than N=1 is needed [1].

Conclusion

Proving the Collatz conjecture requires proving that:

1. The sequence will not diverge to infinity
2. There are no other closed loops other than the 1-4-2-1 loop.

We have presented a conclusive proof of the first. For the second, we have made significant progress in proving the second. Proving that equation (10) has no solution other than N=1 is required for a complete proof of the Collatz conjecture.

Thanks to Almighty God Jesus Christ and His Mother Our Lady Saint Virgin Mary

Notes and references

1. In the last version of this paper I made a simple mistake which led to the wrong conclusion that the non-existence of other possible closed loops was completely proved.

2. Also an infinite sequence of only odd numbers (or only even numbers) would not mean infinite information.

APPENDIX

The following are Collatz sequences for four randomly selected starting numbers:
564358543425 , 76736783426, 456734256789, and 87654768546 .

We can see that the number of even operations is always greater than the number of odd operations. For the first number there are 116 odd operations vs. 220 even operations before the sequence reaches number 1, for the second number 84 odd operations vs.168 even operations, for the third number 56 odd operations vs 126 even operations, and for the last number 82 odd operations vs.165 even operations. This explains why the Collatz sequence always converges.

Sequence	0 if even , 1 if odd	Sequence	0 if even , 1 if odd	Sequence	0 if even , 1 if odd	Sequence	0 if even , 1 if odd
564358543425	1	76736783426	0	456734256789	1	87654768546	0
1693075630276	0	38368391713	1	1370202770368	0	43827384273	1
846537815138	0	1.15105E+11	0	685101385184	0	131482152820	0
423268907569	1	57552587570	0	342550692592	0	65741076410	0
1269806722708	0	28776293785	1	171275346296	0	32870538205	1
634903361354	0	86328881356	0	85637673148	0	98611614616	0
317451680677	1	43164440678	0	42818836574	0	49305807308	0
952355042032	0	21582220339	1	21409418287	1	24652903654	0
476177521016	0	64746661018	0	64228254862	0	12326451827	1
238088760508	0	32373330509	1	32114127431	1	36979355482	0
119044380254	0	97119991528	0	96342382294	0	18489677741	1
59522190127	1	48559995764	0	48171191147	1	55469033224	0
178566570382	0	24279997882	0	144513573442	0	27734516612	0
89283285191	1	12139998941	1	72256786721	1	13867258306	0
267849855574	0	36419996824	0	216770360164	0	6933629153	1
133924927787	1	18209998412	0	108385180082	0	20800887460	0

401774783362	0	9104999206	0	54192590041	1	10400443730	0
200887391681	1	4552499603	1	162577770124	0	5200221865	1
602662175044	0	13657498810	0	81288885062	0	15600665596	0
301331087522	0	6828749405	1	40644442531	1	7800332798	0
150665543761	1	20486248216	0	121933327594	0	3900166399	1
451996631284	0	10243124108	0	60966663797	1	11700499198	0
225998315642	0	5121562054	0	182899991392	0	5850249599	1
112999157821	1	2560781027	1	91449995696	0	17550748798	0
338997473464	0	7682343082	0	45724997848	0	8775374399	1
169498736732	0	3841171541	1	22862498924	0	26326123198	0
84749368366	0	11523514624	0	11431249462	0	13163061599	1
42374684183	1	5761757312	0	5715624731	1	39489184798	0
127124052550	0	2880878656	0	17146874194	0	19744592399	1
63562026275	1	1440439328	0	8573437097	1	59233777198	0
190686078826	0	720219664	0	25720311292	0	29616888599	1
95343039413	1	360109832	0	12860155646	0	88850665798	0
286029118240	0	180054916	0	6430077823	1	44425332899	1
143014559120	0	90027458	0	19290233470	0	133275998698	0
71507279560	0	45013729	1	9645116735	1	66637999349	1
35753639780	0	135041188	0	28935350206	0	199913998048	0
17876819890	0	67520594	0	14467675103	1	99956999024	0
8938409945	1	33760297	1	43403025310	0	49978499512	0
26815229836	0	101280892	0	21701512655	1	24989249756	0
13407614918	0	50640446	0	65104537966	0	12494624878	0
6703807459	1	25320223	1	32552268983	1	6247312439	1
20111422378	0	75960670	0	97656806950	0	18741937318	0
10055711189	1	37980335	1	48828403475	1	9370968659	1
30167133568	0	113941006	0	146485210426	0	28112905978	0
15083566784	0	56970503	1	73242605213	1	14056452989	1
7541783392	0	170911510	0	219727815640	0	42169358968	0
3770891696	0	85455755	1	109863907820	0	21084679484	0

1885445848	0	256367266	0	54931953910	0	10542339742	0
942722924	0	128183633	1	27465976955	1	5271169871	1
471361462	0	384550900	0	82397930866	0	15813509614	0
235680731	1	192275450	0	41198965433	1	7906754807	1
707042194	0	96137725	1	123596896300	0	23720264422	0
353521097	1	288413176	0	61798448150	0	11860132211	1
1060563292	0	144206588	0	30899224075	1	35580396634	0
530281646	0	72103294	0	92697672226	0	17790198317	1
265140823	1	36051647	1	46348836113	1	53370594952	0
795422470	0	108154942	0	139046508340	0	26685297476	0
397711235	1	54077471	1	69523254170	0	13342648738	0
1193133706	0	162232414	0	34761627085	1	6671324369	1
596566853	1	81116207	1	104284881256	0	20013973108	0
1789700560	0	243348622	0	52142440628	0	10006986554	0
894850280	0	121674311	1	26071220314	0	5003493277	1
447425140	0	365022934	0	13035610157	1	15010479832	0
223712570	0	182511467	1	39106830472	0	7505239916	0
111856285	1	547534402	0	19553415236	0	3752619958	0
335568856	0	273767201	1	9776707618	0	1876309979	1
167784428	0	821301604	0	4888353809	1	5628929938	0
83892214	0	410650802	0	14665061428	0	2814464969	1
41946107	1	205325401	1	7332530714	0	8443394908	0
125838322	0	615976204	0	3666265357	1	4221697454	0
62919161	1	307988102	0	10998796072	0	2110848727	1
188757484	0	153994051	1	5499398036	0	6332546182	0
94378742	0	461982154	0	2749699018	0	3166273091	1
47189371	1	230991077	1	1374849509	1	9498819274	0
141568114	0	692973232	0	4124548528	0	4749409637	1
70784057	1	346486616	0	2062274264	0	14248228912	0
212352172	0	173243308	0	1031137132	0	7124114456	0
106176086	0	86621654	0	515568566	0	3562057228	0

53088043	1	43310827	1	257784283	1	1781028614	0
159264130	0	129932482	0	773352850	0	890514307	1
79632065	1	64966241	1	386676425	1	2671542922	0
238896196	0	194898724	0	1160029276	0	1335771461	1
119448098	0	97449362	0	580014638	0	4007314384	0
59724049	1	48724681	1	290007319	1	2003657192	0
179172148	0	146174044	0	870021958	0	1001828596	0
89586074	0	73087022	0	435010979	1	500914298	0
44793037	1	36543511	1	1305032938	0	250457149	1
134379112	0	109630534	0	652516469	1	751371448	0
67189556	0	54815267	1	1957549408	0	375685724	0
33594778	0	164445802	0	978774704	0	187842862	0
16797389	1	82222901	1	489387352	0	93921431	1
50392168	0	246668704	0	244693676	0	281764294	0
25196084	0	123334352	0	122346838	0	140882147	1
12598042	0	61667176	0	61173419	1	422646442	0
6299021	1	30833588	0	183520258	0	211323221	1
18897064	0	15416794	0	91760129	1	633969664	0
9448532	0	7708397	1	275280388	0	316984832	0
4724266	0	23125192	0	137640194	0	158492416	0
2362133	1	11562596	0	68820097	1	79246208	0
7086400	0	5781298	0	206460292	0	39623104	0
3543200	0	2890649	1	103230146	0	19811552	0
1771600	0	8671948	0	51615073	1	9905776	0
885800	0	4335974	0	154845220	0	4952888	0
442900	0	2167987	1	77422610	0	2476444	0
221450	0	6503962	0	38711305	1	1238222	0
110725	1	3251981	1	116133916	0	619111	1
332176	0	9755944	0	58066958	0	1857334	0
166088	0	4877972	0	29033479	1	928667	1
83044	0	2438986	0	87100438	0	2786002	0

41522	0	1219493	1	43550219	1	1393001	1
20761	1	3658480	0	130650658	0	4179004	0
62284	0	1829240	0	65325329	1	2089502	0
31142	0	914620	0	195975988	0	1044751	1
15571	1	457310	0	97987994	0	3134254	0
46714	0	228655	1	48993997	1	1567127	1
23357	1	685966	0	146981992	0	4701382	0
70072	0	342983	1	73490996	0	2350691	1
35036	0	1028950	0	36745498	0	7052074	0
17518	0	514475	1	18372749	1	3526037	1
8759	1	1543426	0	55118248	0	10578112	0
26278	0	771713	1	27559124	0	5289056	0
13139	1	2315140	0	13779562	0	2644528	0
39418	0	1157570	0	6889781	1	1322264	0
19709	1	578785	1	20669344	0	661132	0
59128	0	1736356	0	10334672	0	330566	0
29564	0	868178	0	5167336	0	165283	1
14782	0	434089	1	2583668	0	495850	0
7391	1	1302268	0	1291834	0	247925	1
22174	0	651134	0	645917	1	743776	0
11087	1	325567	1	1937752	0	371888	0
33262	0	976702	0	968876	0	185944	0
16631	1	488351	1	484438	0	92972	0
49894	0	1465054	0	242219	1	46486	0
24947	1	732527	1	726658	0	23243	1
74842	0	2197582	0	363329	1	69730	0
37421	1	1098791	1	1089988	0	34865	1
112264	0	3296374	0	544994	0	104596	0
56132	0	1648187	1	272497	1	52298	0
28066	0	4944562	0	817492	0	26149	1
14033	1	2472281	1	408746	0	78448	0

42100	0	7416844	0	204373	1	39224	0
21050	0	3708422	0	613120	0	19612	0
10525	1	1854211	1	306560	0	9806	0
31576	0	5562634	0	153280	0	4903	1
15788	0	2781317	1	76640	0	14710	0
7894	0	8343952	0	38320	0	7355	1
3947	1	4171976	0	19160	0	22066	0
11842	0	2085988	0	9580	0	11033	1
5921	1	1042994	0	4790	0	33100	0
17764	0	521497	1	2395	1	16550	0
8882	0	1564492	0	7186	0	8275	1
4441	1	782246	0	3593	1	24826	0
13324	0	391123	1	10780	0	12413	1
6662	0	1173370	0	5390	0	37240	0
3331	1	586685	1	2695	1	18620	0
9994	0	1760056	0	8086	0	9310	0
4997	1	880028	0	4043	1	4655	1
14992	0	440014	0	12130	0	13966	0
7496	0	220007	1	6065	1	6983	1
3748	0	660022	0	18196	0	20950	0
1874	0	330011	1	9098	0	10475	1
937	1	990034	0	4549	1	31426	0
2812	0	495017	1	13648	0	15713	1
1406	0	1485052	0	6824	0	47140	0
703	1	742526	0	3412	0	23570	0
2110	0	371263	1	1706	0	11785	1
1055	1	1113790	0	853	1	35356	0
3166	0	556895	1	2560	0	17678	0
1583	1	1670686	0	1280	0	8839	1
4750	0	835343	1	640	0	26518	0
2375	1	2506030	0	320	0	13259	1

7126	0	1253015	1	160	0	39778	0
3563	1	3759046	0	80	0	19889	1
10690	0	1879523	1	40	0	59668	0
5345	1	5638570	0	20	0	29834	0
16036	0	2819285	1	10	0	14917	1
8018	0	8457856	0	5	1	44752	0
4009	1	4228928	0	16	0	22376	0
12028	0	2114464	0	8	0	11188	0
6014	0	1057232	0	4	0	5594	0
3007	1	528616	0	2	0	2797	1
9022	0	264308	0	1	1	8392	0
4511	1	132154	0	4	0	4196	0
13534	0	66077	1	2	0	2098	0
6767	1	198232	0	1	1	1049	1
20302	0	99116	0	4	0	3148	0
10151	1	49558	0	2	0	1574	0
30454	0	24779	1	1	1	787	1
15227	1	74338	0	4	0	2362	0
45682	0	37169	1	2	0	1181	1
22841	1	111508	0	1	1	3544	0
68524	0	55754	0	4	0	1772	0
34262	0	27877	1	2	0	886	0
17131	1	83632	0	1	1	443	1
51394	0	41816	0	4	0	1330	0
25697	1	20908	0	2	0	665	1
77092	0	10454	0	1	1	1996	0
38546	0	5227	1	4	0	998	0
19273	1	15682	0	2	0	499	1
57820	0	7841	1	1	1	1498	0
28910	0	23524	0	4	0	749	1
14455	1	11762	0	2	0	2248	0

43366	0	5881	1	1	1	1124	0
21683	1	17644	0	4	0	562	0
65050	0	8822	0	2	0	281	1
32525	1	4411	1	1	1	844	0
97576	0	13234	0	4	0	422	0
48788	0	6617	1	2	0	211	1
24394	0	19852	0	1	1	634	0
12197	1	9926	0	4	0	317	1
36592	0	4963	1	2	0	952	0
18296	0	14890	0	1	1	476	0
9148	0	7445	1	4	0	238	0
4574	0	22336	0	2	0	119	1
2287	1	11168	0	1	1	358	0
6862	0	5584	0	4	0	179	1
3431	1	2792	0	2	0	538	0
10294	0	1396	0	1	1	269	1
5147	1	698	0	4	0	808	0
15442	0	349	1	2	0	404	0
7721	1	1048	0	1	1	202	0
23164	0	524	0	4	0	101	1
11582	0	262	0	2	0	304	0
5791	1	131	1	1	1	152	0
17374	0	394	0	4	0	76	0
8687	1	197	1	2	0	38	0
26062	0	592	0	1	1	19	1
13031	1	296	0	4	0	58	0
39094	0	148	0	2	0	29	1
19547	1	74	0	1	1	88	0
58642	0	37	1	4	0	44	0
29321	1	112	0	2	0	22	0
87964	0	56	0	1	1	11	1

43982	0	28	0	4	0	34	0
21991	1	14	0	2	0	17	1
65974	0	7	1	1	1	52	0
32987	1	22	0	4	0	26	0
98962	0	11	1	2	0	13	1
49481	1	34	0	1	1	40	0
148444	0	17	1	4	0	20	0
74222	0	52	0	2	0	10	0
37111	1	26	0	1	1	5	1
111334	0	13	1	4	0	16	0
55667	1	40	0	2	0	8	0
167002	0	20	0	1	1	4	0
83501	1	10	0	4	0	2	0
250504	0	5	1	2	0	1	1
125252	0	16	0	1	1	4	0
62626	0	8	0	4	0	2	0
31313	1	4	0	2	0	1	1
93940	0	2	0	1	1	4	0
46970	0	1	1	4	0	2	0
23485	1	4	0	2	0	1	1
70456	0	2	0	1	1	4	0
35228	0	1	1	4	0	2	0
17614	0	4	0	2	0	1	1
8807	1	2	0	1	1	4	0
26422	0	1	1	4	0	2	0
13211	1	4	0	2	0	1	1
39634	0	2	0	1	1	4	0
19817	1	1	1	4	0	2	0
59452	0	4	0	2	0	1	1
29726	0	2	0	1	1	4	0
14863	1	1	1	4	0	2	0

44590	0	4	0	2	0	1	1
22295	1	2	0	1	1	4	0
66886	0	1	1	4	0	2	0
33443	1	4	0	2	0	1	1
100330	0	2	0	1	1	4	0
50165	1	1	1	4	0	2	0
150496	0	4	0	2	0	1	1
75248	0	2	0	1	1	4	0
37624	0	1	1	4	0	2	0
18812	0	4	0	2	0	1	1
9406	0	2	0	1	1	4	0
4703	1	1	1	4	0	2	0
14110	0	4	0	2	0	1	1
7055	1	2	0	1	1	4	0
21166	0	1	1	4	0	2	0
10583	1	4	0	2	0	1	1
31750	0	2	0	1	1	4	0
15875	1	1	1	4	0	2	0
47626	0	4	0	2	0	1	1
23813	1	2	0	1	1	4	0
71440	0	1	1	4	0	2	0
35720	0	4	0	2	0	1	1
17860	0	2	0	1	1	4	0
8930	0	1	1	4	0	2	0
4465	1	4	0	2	0	1	1
13396	0	2	0	1	1	4	0
6698	0	1	1	4	0	2	0
3349	1	4	0	2	0	1	1
10048	0	2	0	1	1	4	0
5024	0	1	1	4	0	2	0
2512	0	4	0	2	0	1	1

1256	0	2	0	1	1	4	0
628	0	1	1	4	0	2	0
314	0	4	0	2	0	1	1
157	1	2	0	1	1	4	0
472	0	1	1	4	0	2	0
236	0	4	0	2	0	1	1
118	0	2	0	1	1	4	0
59	1	1	1	4	0	2	0
178	0	4	0	2	0	1	1
89	1	2	0	1	1	4	0
268	0	1	1	4	0	2	0
134	0	4	0	2	0	1	1
67	1	2	0	1	1	4	0
202	0	1	1	4	0	2	0
101	1	4	0	2	0	1	1
304	0	2	0	1	1	4	0
152	0	1	1	4	0	2	0
76	0	4	0	2	0	1	1
38	0	2	0	1	1	4	0
19	1	1	1	4	0	2	0
58	0	4	0	2	0	1	1
29	1	2	0	1	1	4	0
88	0	1	1	4	0	2	0
44	0	4	0	2	0	1	1
22	0	2	0	1	1	4	0
11	1	1	1	4	0	2	0
34	0	4	0	2	0	1	1
17	1	2	0	1	1	4	0
52	0	1	1	4	0	2	0
26	0	4	0	2	0	1	1
13	1	2	0	1	1	4	0

40	0	1	1	4	0	2	0
20	0	4	0	2	0	1	1
10	0	2	0	1	1	4	0
5	1	1	1	4	0	2	0
16	0	4	0	2	0	1	1
8	0	2	0	1	1	4	0
4	0	1	1	4	0	2	0
2	0	4	0	2	0	1	1
1	1	2	0	1	1	4	0
4	0	1	1	4	0	2	0
2	0	4	0	2	0	1	1
1	1	2	0	1	1	4	0