The graphical method of the Kakutani's problem

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Abstract: In this paper we had given an elementary proof of the Kakutani's problem by using the Kakutani's Angle, it holds. By detailed analysis of the properties of both forward and inverse operations of the proposition, we had some important conclusions: 1, there hasn't any triple in the forward path numbers; 2, there have an infinity number of inverse path numbers which had been defined as similar numbers in one time of inverse operation; 3, on the figure of Kakutani's Angle, the operation path of any odd is unique; 4, the inverse operations can start with any odd, and all of the path numbers on the countless paths is getting larger and larger, on the contrary, to do forward operations for any inverse path number, it must go back to the starting point or to 1.

Keywords: conjecture, angle, path number, similar number, cycle

0 Introductions

The Kakutani's problem is the 3x+1 conjecture, also known as the Collatz conjecture. It has not been proved since it was proposed [1]. Its operational rules are: for any given positive integer n, if even, to $n/2$; if odd, and to $3n+1$. To do it repeatedly, n will eventually return to 1.

In this paper, we called the Kakutani's problem as Kakutani's proposition, or proposition for short.

According to the operation rules, an even number will be transformed into an odd firstly, so we take odd numbers directly to analyze and study for the operations.

1 The analysis of the operation properties

In operations, there are many new odds and they form an operation path.

Definition 1

(a) The operation process from an odd to a new odd is called one time of forward operation; times of operations are called continuous forward operations; one time of operation of divided by 2 is called one time of local operation;

(b) The new odd obtained after one operation is called the path number;

(c) The operation which is opposite to the forward operation is called the inverse operation, and its path number is called the inverse path number.

Next, we give the formula and analyze the properties of forward and inverse operations.

1.1 The analysis of the forward operation properties

1.1.1 The operation formula

For any given odd number n, let p be its path number, then according to the operation rules, we have

$$p = \frac{3n+1}{2^k} \tag{1.1}$$

Where $k \in N$ and 2^k is a divisor.

Here, we called formula (1.1) as the forward operation formula of the proposition. To analyze formula (1.1) , it is not difficult to obtain: for any given odd number n, there has only one path number p corresponding to n; the value of k is determined by the odd number n, k can be expressed as the times of local operations, for example, when it equals to 1, that it means in one operation, there is one local operation and when it equals to 3, there are three local operations; for two different odd numbers, the times of local operations are the same or different because the values of k in the divisor can take all of the positive integers, therefore, there are infinite odd numbers that they will all get the same path number after one operation, and these odd numbers have some correlation properties with each other what will be studied below.

1.1.2 Numerical comparison of an odd and its path number

a) Suppose $p = n$, then from formula (1.1), we have

$$n = \frac{1}{2^k - 3}. \tag{1.2}$$

It can be seen that equation (1.2) has only one positive integer solution 1 when k is equal to 2. From this, we can draw a conclusion (conclusion (1)): only when $n = 1$, that is $p = n$; for $n > 1$, that is $p \neq n$. There has only one cycle 1-4-2-1 for an odd and its path number.

If get 1, we stop to do operations.

When $n \neq 1$, we can get the numerical size relationship as fallows

If, $k = 1 \quad p > n$

If, $k \geq 2 \quad p < n$

Obviously, the larger k is, the greater the change rate of the path number is.

b) In the continuous series of odd numbers, we divided four continuous odds into a group starting with 1 and then let $n = 1 + 8(t-1) = 8t - 7$, that is, we take the first in every group, where $t \in N$. From formula (1.1) we can get its path number, that is

$$p = \frac{3(8t-7)+1}{2^k} = \frac{24t}{2^k} - \frac{5}{2^{k-2}}. \tag{1.3}$$

Obviously, if p to be a positive integer above, k must be 2, and then we have

$$p = 6t - 5 \tag{1.4}$$

Now, let $t_1 = t + 1$, that is, we take the next group, where $t_1 \in N$, then we get the path number p_1 of the first, that is

$$p_1 = 6(t+1) - 5 = 6t + 1 \tag{1.5}$$

To compare p_1 and p, then we have

$$p_1 - p = 6t + 1 - (6t - 5) = 6$$

c) Be the same as the above, we divided two continuous odds into a group starting with 3 and then let $n = 3 + 4(t-1) = 4t-1$, that is, we take the first in every group, where $t \in N$. From formula (1.1) we can get its path number, that is

$$p = \frac{3(4t-1)+1}{2^k} = \frac{12t}{2^k} - \frac{1}{2^{k-1}}. \tag{1.6}$$

Obviously, if p to be a positive integer, k must be 1, and then we have

$$p = 6t-1 \tag{1.7}$$

Now, let $t_1 = t+1$, that is, we take the next group, where $t_1 \in N$, then we get the path number p_1 of the first, that is

$$p_1 = 6(t+1)-1 = 6t+5 \tag{1.8}$$

Also to compare p_1 and p, then we have

$$p_1 - p = 6t+5-(6t-1) = 6$$

Now, from b and c, we can draw a conclusion (conclusion (2)): in the continuous odd series, starting with 1, every interval of three odd numbers, that is, 1 added 8 every time, and starting with 3, every interval of an odd number, that is, 3 added 4 every time, for these odd numbers, their path numbers all be with a gap of 6 and the times of local operation are 2 and 1 respectively.

1.1.3 The tendency of continuous operations

Let n be an odd and $n > 1$, let p_1, p_2 and p_3 be three continuous path numbers of n. According to conclusion (1) we have $p_1 \neq n$, $p_2 \neq p_1$ and $p_3 \neq p_2$. From formula (1.1) we have

$$p_1 = \frac{3n+1}{2^{k_1}} \tag{1.9}$$

And

$$p_2 = \frac{3(\frac{3n+1}{2^{k_1}})+1}{2^{k_2}} = \frac{9n+3+2^{k_1}}{2^{k_2+k_1}}. \tag{1.10}$$

Where 2^{k_1} and 2^{k_2} are two divisors, $k_1 \geq 1$ and $k_2 \geq 1$.

Now, suppose $p_2 = n$, from formula (1.10) then we have

$$\frac{9n+3+2^{k_1}}{2^{k_2+k_1}} = n$$

That is

$$n = \frac{3+2^{k_1}}{2^{k_2+k_1}-9}. \tag{1.11}$$

It can be seen from equation (1.11), that if n increases, 2^{k_1} must increase, but at the same time the denominator is also increasing quickly and even bigger than the numerator, so odd number n has a maximum value if it has some positive integer solutions. Now, we take the minimum value 1 of k_2 for analysis. Here are the calculated values

a) when $k_1 = 1$, $n = \frac{3+2}{4-9} = -1$

b) when $k_1 = 2$, $n = \frac{3+4}{8-9} = -7$

c) when $k_1 = 3$, $n = \frac{3+8}{16-9} = \frac{11}{7}$

d) when $k_1 = 4$, $n = \frac{3+16}{32-9} = \frac{19}{23} < 1$

e) when $k_1 > 4$, $n < 1$

From these values above, we can see that equation (1.11) hasn't any positive integer solution for n. In the same way, the same conclusion can be drawn when $k_2 \geq 2$. From this we can obtain $p_2 \neq n$. As $p_1 \neq n$, so we have $p_2 \neq p_1 \neq n$.

In the same way, we can derive that $p_3 \neq p_2 \neq p_1$ when we regard p_1 as n. Now we can draw a conclusion (conclusion (3)): for any given odd except 1, all of its path numbers are different.

For any given odd, there is a finite number of odds less than it, and from conclusion (3), we can get a conclusion (conclusion (4)) here: the path number either goes back to 1 or tends to infinity when keep doing forward operations, there hasn't any cycle except 1-4-2-1.

1.1.4 The triples

An odd number n can be expressed as

$$n = 2x + 1.$$

Where $x = 0$ or $x \in N$.

To do one operation for n, suppose we can get a path number $3p$, where p is an odd, then from formula (1.1) we have the following equation

$$3p = \frac{3(2x+1)+1}{2^k} = \frac{6x+4}{2^k}.$$

To simply, then we have

$$x + \frac{2}{3} = 2^{k-1} p. \tag{1.12}$$

Obviously, the equation (1.12) doesn't hold for integers, so we can get the following conclusion (conclusion (5)): the path number is not a triple, but a non-triple; these triples were skipped in forward operations.

1.1.5 Changes of the values of two adjacent odd numbers

Let n be an odd and expressed as $2x+1$, where $x = 0$ or $x \in N$, thus, one of its adjacent odd numbers can be expressed as

$$2x + 1 + 2.$$

To do one operation for $2x+1$, then we get its path number as fallow

$$\frac{3(2x+1)+1}{2} = 3x + 2.$$

To do one operation for $2x+1+2$, then we get its path number as fallow

$$\frac{3(2x+1+2)+1}{2}=3x+5.$$

Obviously, in these two numbers above, one is odd and the other is even. They both increase firstly, since the even number can be divided by 2 again, so it will decrease finally. From this, we can get a conclusion (conclusion (6)): for two adjacent odd numbers, the two path numbers of them if one becomes larger, the other must become smaller.

1.2 The analysis of the inverse operation properties

1.2.1 The operation formula

The forward operation is reversible for non-triples. Now, to do one inverse operation for formula (1.1), then we have

$$n=\frac{2^k p-1}{3} \tag{1.13}$$

Or

$$3n=2^k p-1. \tag{1.14}$$

Where $k \in N$, and p takes non-triples (conclusion (5)).

Here, formula (1.13) or (1.14) is called the inverse formula, and n is the inverse path number, 2^k is a multiplier.

Obviously, when the multiplier takes 4 or 2, that is, there are two or one local operations in one time of forward operation, we have

$$3n=4p-1 \tag{1.15}$$

And

$$3n=2p-1 \tag{1.16}$$

Bu using formula (1.15) and (1.16), we can find out the First inverse path number. Here, we firstly analyze the first inverse path number.

1.2.2 The properties of the inverse path numbers

Here, we firstly analyze some cases of particular number p.

a) Let $p=1$, from the formula (3.1), then we have

$$n=\frac{2^k-1}{3}. \tag{1.17}$$

Obviously, as k increases and when 2^k take 4 and its 4 multiples, the equation (1.17) has an infinite number of positive integer solutions; they all are inverse path numbers of 1. The minimum is 1, and there is a cycle 1-1-1 when to do inverse operations for 1 repeatedly. The rest inverse path numbers are 5, 21, 85⋯, they can be found one by one.

All inverse path numbers of p can be found one-timely by an iterative formula [2], we will not study further in this paper.

b) Let $n = p$ and $p > 1$, from the formula (1.13), then we have

$$p = \frac{2^k p - 1}{3}.$$

That is

$$p = \frac{1}{2^k - 3}. \tag{1.18}$$

As it can be seen that equation (1.18) has only one positive integer solution ($p = 1$) when $k = 2$. Since $p > 1$, so we can draw a conclusion (conclusion (7)): $n \neq p$ when $p > 1$.

c) Let $p = 3t$, that is, p takes triples, where $t \in N$. From formula (1.13), then we have

$$n = \frac{2^k (3t) - 1}{3} = 2^k t - \frac{1}{3}. \tag{1.19}$$

Obviously, there is no positive integer solution to equation (1.19), so we can draw a conclusion (conclusion (8)): for any triple, it has no inverse path number.

1.2.3 The analysis the cycles in the continuous inverse operations when $p > 1$

Let p be a non-triple odd and $p > 1$, let n_1, n_2 and n_3 be three inverse path numbers which are obtained in turn, here, using the formula (1.13) and conclusion (7), and the same analysis as in section 1.1.3, we can get that $n_3 \neq n_2 \neq n_1$. So we can also draw a conclusion (conclusion (9)): all of continuous inverse path numbers (always the first) is different to each other for $p > 1$ and there hasn't any cycle.

1.2.4 The end of the continuous inverse operations

Since a triple has no its inverse path number (conclusion (8)), so the continuous inverse operations ended at a triple. For the first inverse path numbers obtained in turn by using formula (1.15) and (1.16), there are two trends in general，one is that they gradually increases, the other is that they gradually decreases.

1.2.5 The transition of two continuous inverse operations

As demonstrated above, one time of continuous inverse operations ended at a triple. If getting a triple, we take the next “similar number” of this triple as a transition and continue to do another time of continuous inverse operations.

2 The similar numbers and their relationships

From the analysis at 1.2.2, it’s known that there have an infinite number of inverse path numbers for p, that is, for all of the inverse path numbers, they will get the same forward path number when doing one operation for them respectively. For an example, for 7 and 29, they both get 11.

Definition 2 Suppose, there are two odd numbers n_1 and n_2 whose path numbers are both p, then, we called that n_1 is a similar number of n_2, or n_2 is a similar number of n_1, that is, they are similar each other, and denoted $n_1 \backsim n_2$, or, $n_2 \backsim n_1$.

For an example, 29 is a similar number of 7, or 7 is a similar number of 29, that is $7 \backsim 29$.

Obviously, the similar numbers are caused by different values of k.

Next, we analyze the relationship between two similar numbers.

Suppose, there are two similar numbers n_1 and n_2, where $n_2 > n_1$, to do one operation on each of them, we can get the path numbers p_1 and p_2. According to formula (1.1), we have

$$p_1 = \frac{3n_1 + 1}{2^{k_1}}$$

And

$$p_2 = \frac{3n_2 + 1}{2^{k_2}}.$$

Where $k_1 \in N$, and $k_2 \in N$.

Now, let $p_1 = p_2$, then we have

$$\frac{3n_1 + 1}{2^{k_1}} = \frac{3n_2 + 1}{2^{k_2}}. \tag{2.1}$$

Since $n_2 > n_1$, we can get $k_2 > k_1$, that is, $k_2 - k_1$ are positive integers.

From equation (2.1), n_2 can be obtained, that is

$$n_2 = \frac{2^{k_2}}{2^{k_1}} n_1 + \frac{1}{3}\left(\frac{2^{k_2}}{2^{k_1}} - 1\right) = 2^{k_2 - k_1} n_1 + \frac{1}{3}\left(2^{k_2 - k_1} - 1\right). \tag{2.2}$$

Obviously, for equation (2.2), if n_2 to be an integer, $2^{k_2 - k_1} - 1$ must be a triple and it has an infinite number of values. Here, we take the smallest triple 3, that is, to take the minimum value 4 of $2^{k_2 - k_1}$, and then we have

$$n_2 = 4n_1 + 1 \tag{2.3}$$

Obviously, the gap between n_2 and n_1 is the smallest. We called formula (2.3) as the formula of the similar numbers. Using it, we can find out the numberless similar numbers of n_1 one by one. For examples, we can get some sequences of similar numbers as follows

a) Let n_1=1, we can obtain a sequence generated by 1 in turn

   1, 5, 21, 85, 341⋯

b) Let n_1=3, we can obtain

   3, 13, 53, 213, 853⋯

c) Let n_1=7, we can obtain

   7, 29, 117, 469, 1877⋯

When n_1=5, the sequence generated by 5 is already in the first sequence.
Obviously, each of these sequences is an infinite set of similar numbers. When n_1=5, it's a sub-set. Any odd number can generate a set as a generating number and the similar numbers increase in turn. To do one time of forward operation for n_2 is equal to what to do for n_1 and they can be regarded as connected to each other, and they are also connected to the path numbers. If an odd can go back to 1, then it and all of its path numbers are connected with 1.
All similar numbers which are generated by the first inverse path number are also an inverse path number. For every inverse operation, we always get a set of similar numbers.

To do inverse operation for formula (2.3), then we have

$$n_1 = \frac{n_2 - 1}{4} \tag{2.4}$$

Obviously, if n_1 is an integer, then it is an adjacent similar number of n_2, and $n_1 < n_2$. Here formula (2.4) is called the inverse operation formula of similar numbers.

3 The analysis of the paths and the final conclusion

3.1 The Kakutani's Angle

Now, based on conclusions above we put two groups of continuous odd numbers in a table to form an angular graph and to demonstrate the paths of both forward and inverse operations (see below Fig. 1 the network figure of the path numbers, also see the attachment of the same name which is with more odds).

Notes:

1) There are two groups of continuous odd numbers in the table, one is on the slash in the middle and the other is on the steps below (red); for the steps, the first step is 1 and 3, each complete step contains four odds; in the same column, two odd numbers above and below is the same; the triples is with shadows;
2) A→B: B is from A when doing forward operation for A, on the contrary, A is from B when doing inverse operation for B; the arrow lines (to left and to right) are two sets of parallel lines (non-geometric sense) and they form a diamond-shaped network structure;
3) This is an angle named "Kakutani's Angle".

3.2 The properties of the Kakutani's Angle

This graph shows the relationship between numbers on the path, each number has its next or last (except the triples) unique operation path number; from the angle, we can get the followings about its properties:

1) Two sets of arrow lines are both parallel lines, because the gaps of the forward path numbers as analyzed in 1.1.2 b and c on the steps are all 6 (conclusion (2)); the arrow line 1→1 represents two cycles when doing one time of forward and inverse operation respectively (conclusion (1) and as analyzed at 1.2.2 a);
2) On the slash, for two adjacent odds, theirs lead-out arrow lines go in opposite directions (conclusion (6)); in same columns, an odd corresponding to the first number on the step has no lead-out line, because it has a similar number less than it (the proof is neglected);

3) In a row, the odd on the slash is similar to the first odd on the step, e.g., 1∽5, 3∽13, 9∽37 (as the same above);
4) Every triple on the step hasn't any lead-in line because there isn't any forward triple path number when doing forward operations for any odd on slash (conclusion (5)); it can be proved that the odd numbers on a step except the first have not similar numbers less than themselves by using the formula (2.4) (the proof is neglected);
5) For a triple's location on each step and the directions of the arrow lines, every three steps (12 odds) is in a cycle and it can also be proved (neglected);
6) Except 1, there are no cycles, neither forward nor inverse operations, or an odd number must have two lead-in or lead-out lines (conclusion (1), (9) and see 1.2.2 a).

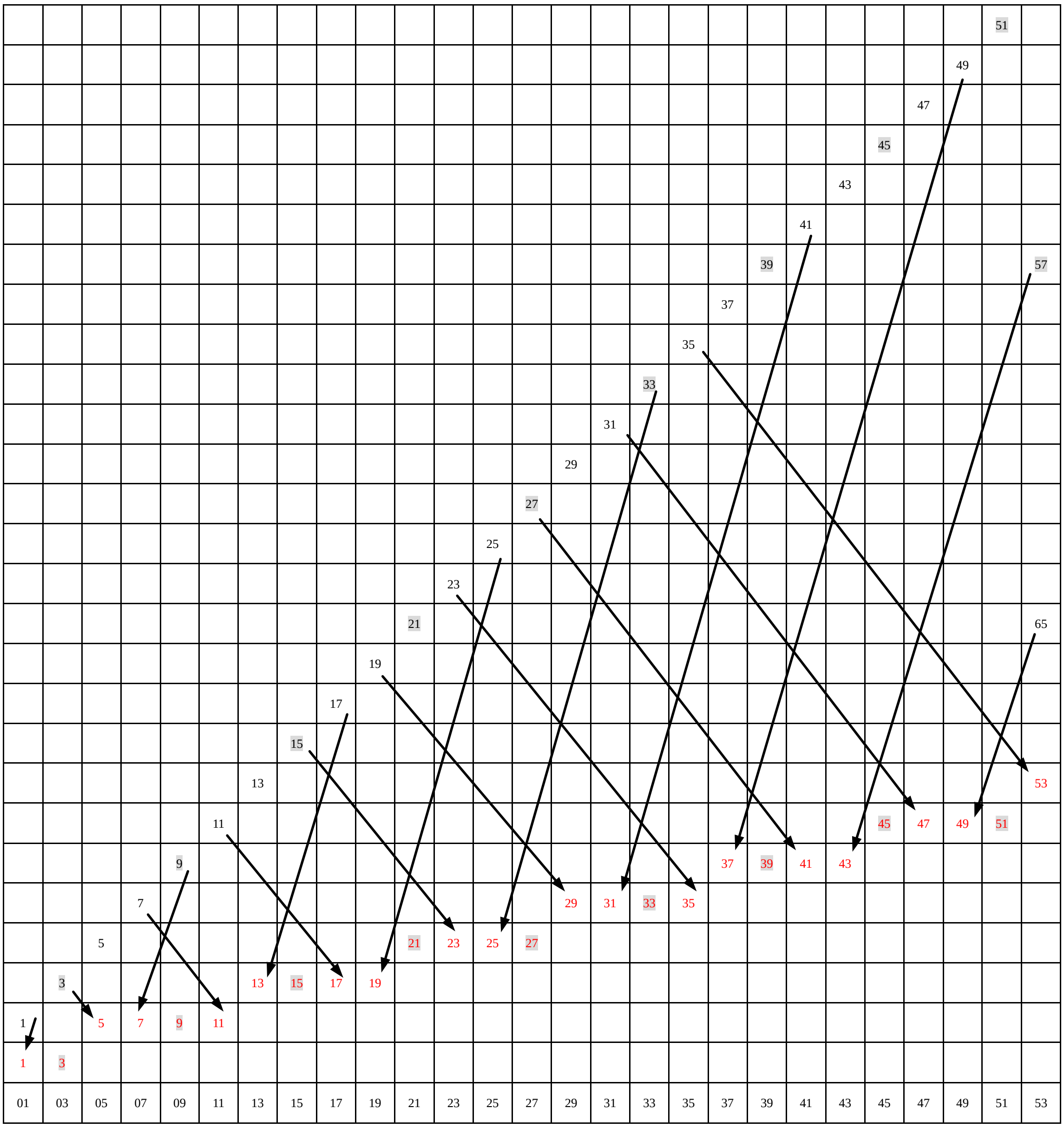

Fig. 1 The network figure of the path numbers

3.3 The forward operations path on the angle

The forward operations path is starting with an odd on the slash (marked slash n). it follows the arrow line to its path number on the step (marked step n) and then goes up to slash again, and does it repeatedly. If a slash n has no lead-out line, it goes down to the step n and takes to left its similar number on the slash (if the similar number has also no lead-out line, it goes to the next again), then to its path number on the step following the arrow line again. For examples, slash 15 to step 23, step 23 up to slash 23, slash 23 to step 35, step 35 up to slash 35, slash 35 to step 53 , then 53 to 13 and 13 to 3; slash 21 down to step 21 and to slash 5, slash 5 down to step 5 and then to slash 1. It can be seen that the path is unique and goes either back to 1 or to infinity (conclusion (4)).

3.4 The inverse operations paths on the angle and the final conclusion

The inverse operations path is starting with an odd on the step. For a given step n, it goes to its first inverse path number on the slash opposite to the arrow line and all of the similar numbers of this inverse path number. For an example, step 5 to slash 3 (the first), slash 3 to right to step 13 (the first similar number), step 13 up to slash 13, slash 13 to right to step 53 (the second similar number), and so on, that is, starting with 5, it goes to 3, 13, 53, 213⋯, it gets the infinite similar number set generated by 3.

The continuity of the inverse operations is to take its similar number for larger when getting a triple. The inverse operations generate more and more bifurcations and theirs paths are all unique (conclusion (9) and even if there are transitions when getting triples). On the Kakutani's Angle, any odd except triples on the step has only one lead-in line and it is connected with its first inverse path number and the similar numbers generated by it; the triples on the slash are connected with theirs forward path numbers or theirs similar numbers.

The inverse operations can be starting with any given odd and continued, and the inverse path numbers on any bifurcation must all tend to infinite as the similar numbers getting large, instead, it must go back to 1 if doing continued forward operations, so the Kakutani's proposition or the Collatz conjecture holds.

In this paper, the operational mechanism of the proposition is analyzed in detail. Why any given odd can regress to 1 is that in operations, we can't get any triple and the similar numbers less than the path number, that is, every operation is excluding some or a large number of odds, thus the operation path becomes narrower and narrower and goes back to 1 quickly at the end.

Attachment 1 the network figure of the path numbers

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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