Pythagorean Triples and the Binomial formula

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Abstract

The article shows the possibility of compiling Pythagorean triples using the binomial formula and provides a Theorem that is an alternative proof of the infinity of Pythagorean triples and confirmation of the close connection of the Pythagorean Theorem with the binomial formula.

Keywords:Pythagorean triple, leg, hypotenuse, binomial

1 Introduction

It is known that the equation

$$a^2 + b^2 = c^2 \tag{1}$$

describes the Pythagorean theorem in integers, and the terms of the equation (a, b, c) are called Pythagorean triples. Pythagorean triples are infinite, this was proved by Euclid.

To generate Pythagorean triples, the Euclid formula is used, which states that for any pair of natural numbers m and n ($m > n$), the integers

$$a = m^2 - n^2, b = 2mn, c = m^2 + n^2 \tag{2}$$

form a Pythagorean triple. The Pythagorean theorem and Pythagorean triples have been studied very well, so in open sources you can easily find answers to almost all questions regarding the Pythagorean Theorem, for example, in [1, 2, 3] there are answers to many questions.

Nevertheless, the study of the Pythagorean Theorem and the development of alternative methods for compiling a Pythagorean triple are ongoing, as an example, one can note the articles [4, 5]. In [5], based on an alternative binomial formula, a complex formula is created that allows one to compose Pythagorean triples. At the same time, the author notes that the newly obtained formula, which is fundamentally different from ours, was first obtained by A. Bottari in 1908.

A review of the works devoted to the Pythagorean Theorem, carried out by us, shows that in the above sources and other freely available sources there is no information about the connection of the Pythagorean Theorem with the usual (not alternative) binomial formula.

2 Methods for compiling a Pythagorean triple

To generate all Pythagorean triples, the following formula is applied

$$a = k(m^2 - n^2), b = k(2mn), c = k(m^2 + n^2), \tag{3}$$

where m,n and k are natural numbers, $m > n$, m,n are coprime.

In the case when k=1, according to the formula (3) primitive Pythagorean triples are obtained. Note that a Pythagorean triple is called primitive if it cannot be obtained from some other Pythagorean triple by multiplying by an integer, so the greatest common divisor of a primitive Pythagorean triple is 1.

As shown by Berggren [6], all primitive Pythagorean triples can be obtained from the triangle (3, 4, 5) using three linear transformations. Various methods are also known for compiling a Pythagorean triple based on a given number a, for example, Pythagorean triples can be composed using the following methods.

Method 1. If an odd number a is given, which is a small leg, then the large leg b and the hypotenuse c of the Pythagorean triples (a, b, c) can be calculated by the formulas,

$$b = \frac{a^2}{2} - \frac{1}{2}; c = \frac{a^2}{2} + \frac{1}{2}. \tag{4}$$

Method 2. If an even number a is given, which is a small leg, then the large leg b and the hypotenuse c of the Pythagorean triples (a,b,c) can be calculated by the formulas,

$$b = \frac{a^2}{2} - 1; c = \frac{a^2}{2} + 1. \tag{5}$$

Using the above formulas (4) and (5), you can get infinitely many Pythagorean triples, but these formulas do not allow you to get all Pythagorean triples. For example, if an even number 20 is given, then the Pythagorean triples $a = 20, b = 21, c = 29$ cannot be obtained by formula (5).

At that time, using the formula we derived, which will be described in the next chapter, it is possible to calculate all Pythagorean triples, including the Pythagorean triples shown above $a = 20, b = 21, c = 29$.

3 Main theorem

The Theorem we propose, which shows the connection between Pythagorean triples and the binomial formula, has the following formulation.

Theorem 1.For any pair of natural numbers b and d ($b > d$), the integers b, $c = b + d$, $a = (2bd + d^2)$, form a Pythagorean triple (a, b, c).

Next, we give a proof of the validity of Theorem 1. It is known that the square of the sum of two natural numbers b and d according to the binomial formula has the following representation

$$(b + d)^2 = b^2 + 2bd + d^2. \tag{6}$$

If we accept that $c = b + d$, since $c > b$, then to get the equality $a^2 + b^2 = c^2$, it should be

$$2bd + d^2 = a^2, \tag{7}$$

or

$$a = (2bd + d^2)^{1/2}, \tag{8}$$

which means Theorem 1 is true and it has been proven. Taking into account the above and formula (1), we represent equation (5) in the following form

$$((2bd + d^2)^{1/2})^2 + b^2 = (b + d)^2. \tag{9}$$

Example 1. Let the big leg b=21 be given and the difference between the hypotenuse and the big leg, d=c-b=8, you need to find the small leg and make Pythagorean triples.

Solution. Knowing $b = 21, c = 21 + 8 = 29$ and using Theorem 1, we calculate the small leg a, $a = (2 \cdot 21 \cdot 8 + 8^2)^{1/2} = 20$, then we get the Pythagorean equation $20^2 + 21^2 = 29^2$.

It is not difficult to establish that the equation $a = (2bd + d^2)^{1/2}$ has infinitely many natural solutions. Obviously, for a natural value $a = (2bd + d^2)^{1/2}$, for natural b and d, formula (9) gives a natural value, i.e., formula (9) also has infinitely many natural solutions.

Note that formula (9) allows us to calculate all primitive and non-primitive Pythagorean triples.

It follows from the above that Theorem 1 is a new proof of the infinity of Pythagorean triples.

4 Conclusion

Thus, we have shown the possibility of composing Pythagorean Triples using the binomial formula, and proposed the Theorem, which is a new proof of the infinity of Pythagorean triples.

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