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The Reality
真 实 意 义

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The Reality*

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Abstract

This document introduces the genesis and cardinal structure of existences, philosophically derived by the two concepts of certainty and diversity. All existences constructed by the x diversity maps is the base structure of The Reality, on which sequential excitations of z diversity maps found realities. Natural and other classes of existences in The Reality are partially characterized.

Contents

†This is not a paper for publishment nor a research for you, whatever chaebol you are. Also don't pretend as if you have somewhere in history the philosophy, definition, and structure of RR maps in your poorly applied but highly dominated set theories.

1 NR vertex

One tries to recognize the genesis of everything.

Suppose one has the *concept of certainty*, that finds an *existence*. One *names* the existence first found. This existence can have arbitrary aliases such as the null-reality (NR) vertex, or n_0. Aliases or expressions attached to it are only for the convenience to speak but irrelevant to itself. As illustrated in Fig. 1, one is only certain about the existence of the NR vertex.

Figure 1: The NR vertex.

2 QR vertices

One wants to be certain about other existences. Based on the concept of certainty, one can find existence. However, any existence found is not a second existence unless one tells that the found existence is different from the first one i.e. the NR vertex n_0. Suppose one has also the *concept of diversity*, to *name* an existence *diverse from* existence(s) one already named. A *map of diversity*

$$x_0 : \begin{cases} R_0 \to C_0 \\ n_0 \mapsto n_1 \end{cases} \tag{1}$$

demonstrates how the existence of n_1 is named given that the existence of n_0. For x_0, the known existence is n_0. The term R_0 denotes all named existence(s), where for x_0 it is n_0. The term C_0 denotes all possible existence(s) to-be-named that is different from named existence(s) in R_0. The existence found by the concept of certainty and named by x_0 is labeled by n_1, or any other aliases. As illustrated in Fig. 2, the integrity of co-existing n_0 and n_1, as well as other named existences called quantized-reality (QR) vertices, is only originated from the map of diversity which defines a certain existence only from named existences.

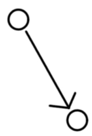

Figure 2: Two QR vertices and the map of diversity.

The incompleteness of the concept of diversity lies in that the diversity map itself is a certain thing but not a certain existence. A similar incompleteness in Axiomatic Set Theory is that the set operation $s : p \mapsto \{p\}$ collecting element(s) p itself is an object but not in constructed sets.

Here, any named existence is above the "surface" of conceptual incompleteness. One choose not to have iterative set operation (fictitious abstraction) like $\{\{\{n_0\}\}\}$ or else. The only fact is the co-existence of QR vertices linked by diversity maps, which restricts the term *set* in use here to only the collection of existences named by diversity maps, and the diversity maps from named existences. This allows one to focus on the exact structure revealing The Reality.

3 CR set

One clarifies the map of diversity, generically

$$x_s : \begin{cases} R_S \to C_S \\ R_S \supseteq \{n_s\} \mapsto n_r \notin R_S \end{cases} \tag{2}$$

in which S is the label for a set R_S of QR vertices, and s is the label for elements n_s in a subset of R_S. At x_s, a new existence n_r is named a.k.a. defined by the concept of diversity from the QR vertices $\{n_s\}$ already named in R_S.

One asks how large a realized set R_S can exist. One questions whether at last for some S, one cannot name any existence in C_S. One is indeed asking what is the structure of all existences.

One first focus on the case of special $\bar{x}$ maps of diversity satisfying $\bar{x}_s : |\{n_s\}| = 1$. Under this kind of maps only, one finally reach a sequence of named existences: $n_0, n_1, n_2, \cdots$.

To see it clearly, one first define a natural relation $<$ from the maps of diversity between two named existences n_k and n_l $(k, l = 0, 1, 2, \ldots)$ in a realized set R: the relation $n_l < n_k$ holds iff n_l maps to n_k under some $\bar{x}$ map(s), otherwise $n_l \not< n_k$. By definition of diversity maps one knows that i) $n_k \not< n_k$; ii) $n_k < n_l \Rightarrow n_l \not< n_k$; iii) $n_k < n_l \wedge n_l < n_m \Rightarrow n_k < n_m$. Hence $<$ (or $\leq$) from the x maps forms an order on arbitrary set R_S. Samely R_S is endowed with the $>$ order (or $\geq$), i.e. $n_k > n_l$ iff n_l maps to n_k under some $\bar{x}$ map(s), otherwise $n_k \not> n_l$.

One considers for any $R_k = \{n_0, n_1, \ldots, n_k\}$ and any $n_l < n_k$, $\bar{x}_l : n_l \mapsto n_r$, one knows that both n_r and the existing $n_{l+1} \in R_k$ are defined only by the diversity from n_l hence n_r is already named as n_{l+1}. Since $\bar{x}_l : |\{n_l\}| = 1$, n_r has no diversity map from n_{l+1}. The only new map of such

$\bar{x}$ from R_k is $\bar{x}_k : n_k \mapsto n_r$ hence one derives the chain of existences under only $\bar{x}$ maps, as illustrated in Fig. 3. One calls it a QR chain Q_0. One may not know what are natural numbers yet but indeed this QR chain can be aliased by $(\mathbb{N}, <)$ with the $<$ usual order. The number of QR vertices a.k.a. the cardinality of the QR chain is denoted by $\aleph_0$ and called countably infinite. The term "countably infinite" at present is only originated from the $\bar{x}$ maps s.t. $\forall n_k \in Q_0$, $\exists|\aleph_0$: $k < \aleph_0$, since the above property of $\bar{x}$ maps holds $\exists|Q_0$: $n_0 \in Q_0 \ \wedge \ (\forall n_k \in Q_0, \exists|n_r \equiv n_{k+1}$: $n_r \in Q_0)$. If one has to axiomize the concept of diversity, then this trivial property stands for the Axiom of Infinity.

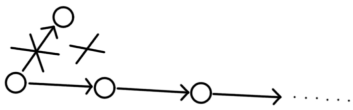

Figure 3: The countable chain Q_0 of QR vertices is derived under only $\bar{x}$ maps.

On this realized QR chain, one then[1] define an operation $-$ on $n_k > n_l > n_0$ as $- : (n_k, n_l) \mapsto n_s$ i.e. $n_k - n_l = n_s$ where n_l maps to n_k under the same number of $\bar{x}$ map(s) that n_0 maps to n_s. The named existence $n_s \in R$ can be denoted as n_{k-l}. For any three QR vertices satisfying $n_k - n_l = n_s$, rewriting $n_l + n_s = n_k$ one defines $+ : (n_l, n_s) \mapsto n_k$. Since i) $n_k - n_l = n_s \Rightarrow n_k > n_s > n_0 \Rightarrow n_k - n_s = n_l$, one knows if $+ : (n_l, n_s) \mapsto n_k$ then $+ : (n_s, n_l) \mapsto n_k$ which is the commutative law for $+$; ii) $(n_m - n_k) - n_l = n_s \Rightarrow n_m - n_k = n_u = n_s + n_l$, $(n_m - n_l) - n_k = n_t \Rightarrow n_m - n_l = n_v = n_t + n_k$ so $n_m = n_u + n_k = n_v + n_l = (n_s + n_l) + n_k = (n_t + n_k) + n_l$, while $n_s = n_u - n_l$, $n_t = n_v - n_k$ so $n_v > n_k > n_0$, $n_u > n_l > n_0 \Rightarrow n_u = (n_v - n_k) + n_l \Rightarrow (n_u - n_l) = (n_v - n_k) \Rightarrow n_s = n_t$, one knows $(n_s + n_l) + n_k = (n_s + n_k) + n_l$ which is the associative law for $+$. From the derivations i) and ii) one knows that the $+$ commutativity requires only $\forall n_k > n_l \Rightarrow n_k - n_l < n_k$, and the $+$ associativity does not rely on the commutativity but requires just $\forall n_m > n_k,\ n_m > n_l,\ n_m - n_k > n_l \Rightarrow n_m - n_l > n_k$. It is obvious that $\leq$ on the QR chain is a total order s.t. $n_k \leq n_l \ \vee \ n_l \leq n_k$, so the requirements are naturally met.

One then considers the general $\tilde{x}$ maps of diversity

$$\tilde{x}_s : \begin{cases} R_S \to C_S \\ R_S \supseteq \{n_s\} \mapsto n_r \notin R_S \\ |\{n_s\}| > 1 \end{cases} \tag{3}$$

where the cardinality of the predecessor set $\{n_s\}$ can be finite from several QR vertices, or countably infinite from the entire QR chain, or else.

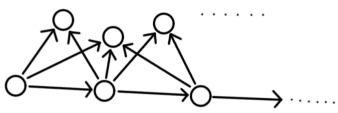

Figure 4: The uncountable set P_0 of QR vertices is derived under a $\tilde{x}$ map from the countable QR chain Q_0.

In order to derive the possibly largest realized set R_S, one calls it the constructed-reality (CR) set CR, and starting from the NR vertex, one first has derived the QR chain $Q_0 \subset CR$ from only the $\bar{x}$ maps. At $R_S = Q_0$, due to the uniqueness of Q_0 one cannot name another QR vertex using the $\bar{x}$ map hence one has to use the $\tilde{x}$ maps starting from Q_0. Under a single $\tilde{x}$ map from Q_0, one names the existences diversified from QR vertices in Q_0 subsets. As illustrated in Fig. 4, one denotes all the QR vertices named by the $\tilde{x}$ maps from Q_0 as the set P_0.

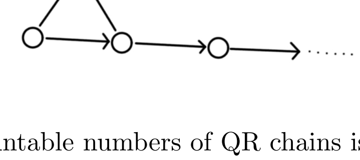

Figure 5: The set Q_1 including uncountable numbers of QR chains is derived under $\bar{x}$ maps from $Q_0 \cup P_0$.

[1]The general form of operators $-$ and $+$ are defined by the crystal order $n_l \succ n_k$ or their z maps, see later the Section 5.

For any QR vertices n_{r_1} and n_{r_2} in P_0, their predecessor sets $\{n_{s_1}\}$ and $\{n_{s_2}\}$ of the $\tilde{x}_{s_1}$ and $\tilde{x}_{s_2}$ maps, are two different subsets of Q_0. By definition of the x diversity maps, n_{r_1} and n_{r_2} are indeed two different QR vertices. This ensures P_0 being unique. If axiomized, the obvious property of $\tilde{x}$ maps $\forall i$ ($\forall \{n_k\} \subseteq Q_i$, $\forall n_r$, $\forall n_s$: $(\tilde{x} : \{n_k\} \to n_r \ \wedge \ \tilde{x} : \{n_k\} \to n_s) \Rightarrow n_r = n_s$), $\forall Q_i$, $\exists \{n_p\} \subseteq P_i$: $\forall n_{pl} \in \{n_p\}$, $\exists | \{n_l\} \subseteq Q_i$: $\tilde{x} : \{n_l\} \to n_{pl}$ explains the Axiom of Replacement, and here $i = 0$.

One can see $|P_0| > |Q_0|$ very straightforward by considering that if there is a bijective map $\bar{b} : Q_0 \to P_0$ as some directed edges selected from all the $\tilde{x}$ maps, then the QR vertices $\{n_l\} \subseteq Q_0$ s.t. $\forall n_l \in \{n_l\}$ ($\nexists \tilde{x} : \{\cdots, n_l, \cdots\} \mapsto \bar{b}(n_l)$), $\forall \tilde{x}(\{n_l\}) = n_{xl} \in P_0$, $\exists n_b \in Q_0$: $\bar{b}(n_b) = n_{xl}$ however contradicts with $n_b \in \{n_l\} \Rightarrow \nexists \tilde{x} : \{\cdots, n_b, \cdots\} \mapsto \bar{b}(n_b)$. One can thus denote the cardinality of P_0 as $\aleph_1 > \aleph_0$ and call every cardinality greater than Q_0 uncountable.

Starting from Q_0 and P_0, under only the $\bar{x}$ maps one derives uncountably infinite numbers of QR chains, each begins with a QR vertex in P_0. As illustrated in Fig. 5, one denotes P_0 and the QR vertices in the QR chains derived in this step as set Q_1, so $P_0 \subset Q_1$, and $|Q_1| = \aleph_1$.

At $R_S = Q_0 \cup Q_1$, one cannot name another QR vertex using the $\bar{x}$ map hence one has to use the $\tilde{x}$ map starting from $Q_0 \cup Q_1$. Under a single $\tilde{x}$ map from $Q_0 \cup Q_1$, one names the existences diversified from elements in the power set of $Q_0 \cup Q_1$. As illustrated in Fig. 6, one denotes all the QR vertices named by the $\tilde{x}$ maps from $Q_0 \cup Q_1$ as set P_1.

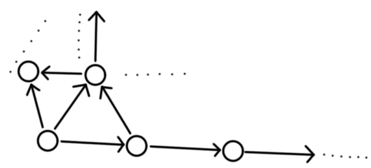

Figure 6: The uncountable set P_1 of QR vertices is derived under a $\tilde{x}$ map from $Q_0 \cup Q_1$.

Following the above prove of $\aleph_1 > \aleph_0$, one can see from the property of $\tilde{x}$ maps that the cardinality of P_1 is $\aleph_2 > \aleph_1$. The sequence of diversity maps

$$\underbrace{\underbrace{\bar{x}\ldots\bar{x}}_{\text{All, }\aleph_0}\tilde{x}\underbrace{\bar{x}\ldots\bar{x}}_{\text{All, }\aleph_0}\tilde{x}\underbrace{\bar{x}\ldots\bar{x}}_{\text{All, }\aleph_0}\tilde{x}\ldots\ldots}_{\aleph_0} \quad (4)$$

starting from the NR vertex, derives the CR set. The total number of each $\bar{x}\ldots\bar{x}\tilde{x}$ period must be countable, since each period only relies on the realized set from its previous period that these periods, in the same way as existences named by $\bar{x}$ diversity maps, form a QR chain which is countable. The cardinality of Q_i is $\aleph_i$ for all $i \in \mathbb{N}$. The CR set can be expressed as

$$CR = \dot{\bigcup_{i \in \mathbb{N}}} Q_i \quad (5)$$

To prove Eq. 5, one considers a QR vertex n_r not named at CR. The NR vertex n_0 has a directed path to n_r since n_r is a named existence and only diversity maps can name it, starting from the NR vertex. If the directed path is countable then the proof is trivial. If the directed path is uncountable, the cardinality of such definition through x maps to reach this QR vertex contradicts with the countably infinite $\bar{x}\ldots\bar{x}\tilde{x}$ period i.e. Peano predecessor in Eq. 4.

The CR set is what one can get with only the two concepts of certainty and diversity, starting from the NR vertex. The Axiomatic Set Theory has derived similar structure by seeing the NR vertex as the empty set $\emptyset$, the x maps as the set operation s, and the directed edges here as the $\in$ relation between sets. However the set operation brings nonessential vagueness that may blind human from seeing The Reality, since one cannot buy that the QR vertices, i.e., those ordinal numbers in Axiomatic Set Theory, already represented everything one observes in reality.

For any QR vertex in the CR set, i.e., countably reachable existence by maps of diversity, one can mark it with a natural number m, indicating that it is the m-th QR vertex in its QR chain. Then the starting point of the QR chain n_{r_1} is marked by a repeatable set of natural numbers $\{m_{r_1}\}_{r_1}$, indicating that each QR vertex in the predecessor set of n_{r_1} is the m_{r_1}-th QR vertex in its QR chain. By marking all predecessor sequences down to the NR vertex,

$$\{m, m_{r_1}, m_{r_1 r_2}, m_{r_1 r_2 r_3}, \cdots\}_{r_1, r_2, r_3, \cdots} \quad (6)$$

one locates the QR vertex in the CR set.

4 RR maps

From definition of x maps there is no directed loop in the directed graph of the CR set. This is because the directed edges are only the x maps. Referring to the concept of diversity, known diverse existences can be confirmed by diversity maps too. One calls these maps post-CR diversity maps, or represented-reality (RR) maps, or z maps.

Note that for any two QR vertices n_k and n_l, one and only one must hold among i) $n_k < n_l$ i.e. there exists a directed path from n_k to n_l; ii) $n_l < n_k$ i.e. there exists a directed path from n_l to n_k; iii) $n_k \not< n_l$ and $n_l \not< n_k$ under the order $<$ of directed path. A z map from n_k to n_l in the case ii) forms a finite directive loop in the QR path where n_k and n_l exists. A z map from n_k to n_l in the case i) or iii) indeed is an existing x map with $\{n_k\} \cup P(n_l)$ its predecessor set, where $P(n_l)$ denotes the predecessor set of n_l. Therefore, an RR map must map between two QR vertices connected by a directed path and form a directed loop in the CR set.

The special case is for all the QR vertices in any path from n_l to n_k, not including n_l, its predecessor set only includes the QR vertices n_s that $n_l \leq n_s$. This special case is denoted $n_l \prec n_k$. One can mark the special RR map from n_k to n_l, i.e., the relative location between $n_k \succ n_l$ by the repeatable set

$$\{m, m_{r_1}, m_{r_1 r_2}, m_{r_1 r_2 r_3}, \cdots\}_{r_1, r_2, r_3, \cdots} \quad (7)$$

tracing only the directed paths of x maps from n_l to n_k.

This mark is the same as marking a QR vertex n_r starting from the NR vertex n_0.

The general case is $n_l \not\prec n_k$, i.e., $\forall n_l < n_k$, $\exists n_c$: $n_c \prec n_l$, $n_c \prec n_k$. One can select the QR vertex n_{c_0} closest to n_l and n_k such that $\exists | n_{c_0} : n_{c_0} \prec n_l$, $n_{c_0} \prec n_k$, $\forall n_c \neq n_{c_0}$: $n_c < n_{c_0}$. In this case, the general RR map can be marked by two repeatable sets in Eq. 7. One is for $n_{c_0} \prec n_k$ which is called the minuend set, and the other one is for $n_{c_0} \prec n_l$ which is called the subtrahend set. The two sets keep the complete information of the general RR map from n_k to n_l where $n_l < n_k$ but $n_l \not\prec n_k$.

The relation $\prec$ (or $\preceq$), as well as its dual $\succ$ (or $\succeq$), forms a partial order on the CR set. For convenience one says $n_l \not< n_k \Rightarrow n_l \not\prec n_k$. One calls $\prec$ the *crystal order*.

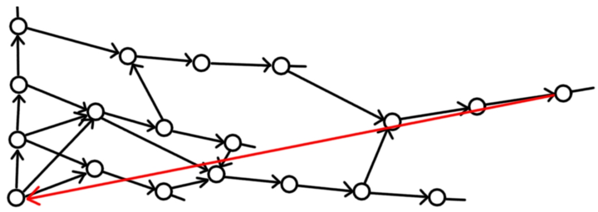

Figure 7: An example of special RR map from n_f to n_i where $n_i \prec n_f$.

For instance, one considers the example of special RR map from n_f to n_i shown in Fig. 7. The repeatable set for $n_i \prec n_f$ or equivalently this RR map can be written as:

$$\begin{aligned} &m = 2; \\ &m_1 = 2,\ m_2 = 2; \\ &m_{11} = 1,\ m_{12} = 0,\ m_{13} = 2,\ m_{21} = 1,\ m_{22} = 3; \\ &m_{111} = 0,\ m_{112} = 1,\ m_{121} = 0,\ m_{122} = 1,\ m_{123} = 2, \\ &m_{131} = 0,\ m_{132} = 1,\ m_{133} = 2, \\ &m_{211} = 0,\ m_{212} = 1,\ m_{213} = 2. \end{aligned} \tag{8}$$

The mark is equivalent to the tree shown in Fig. 8, which is called the tree representation of $n_i \prec n_f$ or the RR map. In this tree, the root is $m = 2$. All nodes are numbers corresponding to the repeatable set. All leaves are QR vertices on the QR chain where n_i exists. Starting from this QR chain and recursively, two nodes in the tree can be recognized as on a same QR chain if and only if their branching nodes are the same numbers that correspond to the same recognized QR chains.

The tree representation is free from the lower indices of m's in the set representation. The permutation of lower indices is redundant information, so that the tree representation is faithful to the crystal order and special RR maps.

On the CR set, one root, any natural numbers and more-than-one branches for any branching nodes, and also any layers are allowed for the representation tree of a possible RR map, as long as any nodes of the same number must not exist on the same QR chain, due to the uniqueness of QR vertices in the CR set. Starting from the leaves, this condition can be implemented layer by layer.

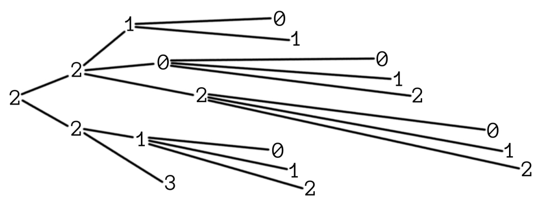

Figure 8: The tree representation of the example RR map from n_f to n_i.

For general RR maps on $n_l < n_k \ \wedge \ n_l \not\prec n_k$ from n_k to n_l, the minuend set and the subtrahend set correspond to the minuend tree and the subtrahend tree, respectively.

For fast comprehension, the CR set is the base space, the vacuum, or the ground state upon all the named existences, while all the possible RR maps are its excitations.

5 Realistics

A *reality* or a physical world is a *sequence* of some RR maps, with the sequence called *time*, endowed with a total order that is derived from certain *rules* in a sense of *minimal diversity*. A *moment* in the sequence of reality is some RR maps excited on the base of The Reality i.e. the CR set. In that sense of non-minimal diversity, the total order of a reality can also be embedded into some partial ordered lattices of realities containing the sequence of that reality.

RR maps are not unique thus allow replicas of the CR set, and thus direct products on QR vertices or x/z maps.

The role of a reality in The Reality is as small as the rate between $\aleph_0$ (its RR excitations) and $\aleph_{\aleph_0}$ (the all possible RR excitations). Hence normally it is not an efficient way to exhaust all possibilities of RR maps in The Reality to solve realistic problems.

However one wants to take whatever imaginable tools in The Reality into consideration when facing its reality. These tools, recognized as mathematical structures, are sequences of x/z maps under certain constraints. Before exporing the uncountable possibilities of RR maps and hidden rules of their sequences, one starts with the x maps of diversity which in natural construct the CR base.

5.1 Natural basis

Naturalness refers to the exact and unique structure of CR set, the static base of The Reality. The x maps of diversity in the CR set first define the natural conditions for `if-then` branches. These conditions include:

- $<$ / $\nless$. $\forall n_k, n_l$: $n_k < n_l \dot{\vee} n_k \nless n_l$.
- $>$ / $\ngtr$. $\forall n_k, n_l$: $n_k > n_l \dot{\vee} n_k \ngtr n_l$.
- $\prec$ / $\nprec$. $\forall n_k, n_l$: $n_k \prec n_l \dot{\vee} n_k \nprec n_l$.
- $\succ$ / $\nsucc$. $\forall n_k, n_l$: $n_k \succ n_l \dot{\vee} n_k \nsucc n_l$.
- $=$ / $\neq$. $\forall n_k, n_l$: $n_k = n_l \dot{\vee} n_k \neq n_l$.

For the simplest case $\forall n_k, n_l$ in a QR chain, one and only one of these three cases holds: $n_k < n_l \Leftrightarrow n_k \prec n_l$, or $n_k > n_l \Leftrightarrow n_k \succ n_l$, or $n_k = n_l$.

The structure of diversity maps in the CR set defines the natural operations to `forall-exists` statements. These operations include:

- $-$. $\forall n_l \prec n_k$, $\exists | n_m$: $n_m - n_0 = n_k - n_l$ i.e. the tree representation of $n_l \prec n_k$ is the same tree representation of $n_0 \prec n_m$.

For the simplest case $\forall n_k, n_l, n_0$ in the QR chain Q_0, $\forall n_l < n_k$, $\exists | n_m$: $n_m - n_0 = n_k - n_l \Leftrightarrow n_m < n_k \Leftrightarrow n_l - n_0 = n_k - n_m$ thus defining $n_l + n_m = n_m + n_l = n_k$. In this QR chain Q_0 starts with the NR vertex n_0, natural operations of `for-loop` iterations are:

- $++ : Q_0 \to Q_0$. $\forall n_k$, $\exists | n_{k+1}$: $n_k{++} = n_{k+1}$.
- $+ : Q_0 \times \mathbb{N} \to Q_0$. $\forall n_k, h_1$, $\exists | n_{k+h_1}$: $n_{k+l} = n_k + h_1 = n_k \underbrace{++\ldots++}_{\#h_1}$.
- $\underline{\times} : Q_0 \times \mathbb{N} \times \mathbb{N} \to Q_0$. $\forall n_k, h_1, h_2$, $\exists | n_{k+h_1 \times h_2}$: $n_{k+h_1\times h_2} = n_k + h_1 \times h_2 = n_k \underbrace{+h_1 \ldots +h_1}_{\#h_2}$.
- $\underline{\dot{\times}} : Q_0 \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to Q_0$. $\forall n_k, h_1, h_2, h_3$, $\exists | n_{k+h_1\times h_2 \dot{\times} h_3}$: $n_{k+h_1\times h_2\dot{\times} h_3} = n_k + h_1 \times h_2 \dot{\times} h_3 = n_k + h_1 \underbrace{\times h_2 \ldots \times h_2}_{\#h_3}$.
- $\underline{\ddot{\times}} : Q_0 \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to Q_0$. $\forall n_k, h_1, h_2, h_3, h_4$, $\exists | n_{k+h_1\times h_2\dot{\times} h_3 \ddot{\times} h_4}$: $n_{k+h_1\times h_2\dot{\times} h_3\ddot{\times} h_4} = n_k + h_1 \times h_2 \dot{\times} h_3 \ddot{\times} h_4 = n_k + h_1 \times h_2 \underbrace{\dot{\times} h_3 \ldots \dot{\times} h_3}_{\#h_4} = n_k + h_1 \times \left[\underbrace{\times h_2 \ldots \times h_2}_{\#h_3}\right] \underbrace{\dot{\times} h_3 \ldots \dot{\times} h_3}_{\#(h_4-1)} = n_k + h_1 \times \left[\underbrace{\left[\underbrace{\times h_2 \ldots \times h_2}_{\#h_3}\right] \ldots \left[\underbrace{\times h_2 \ldots \times h_2}_{\#h_3}\right]}_{\#h_3}\right] \underbrace{\dot{\times} h_3 \ldots \dot{\times} h_3}_{\#(h_4-2)} = \cdots$.
- $\underline{\times^{(n)}} : Q_0 \underbrace{\times \mathbb{N} \ldots \times \mathbb{N}}_{\#n} \to Q_0$. $\forall n_k, h_1, h_2, h_3, h_4, \ldots, h_n$, $\exists | n_{k+h_1\times h_2\dot{\times} h_3\ddot{\times} h_4 \cdots \times^{(n)} h_n}$, $n \in \mathbb{N}$.

Note that here $\underline{\times^{(0)}}$ is $++$, $\underline{\times^{(1)}}$ is $+$, $\underline{\times^{(2)}}$ is $\underline{\times}$. The procedure to calculate formulas up to $\underline{\times^{(n)}}$ are: i) High n expands prior to low n; ii) $\times^{(n)}$ expand in left associativity, expanded bulks in square brackets hold until all $\times^{(n)}$'s expanded; iii) Square brackets are released by the # indices after $\times^{(n)}$'s expanded and before the $\times^{(n-1)}$'s expansion. Note that $\times^{(n)}$ has no meaning outside of $\underline{\times^{(n)}}$.

The natural numbers h_n as operation loop indices can be elements in $Q_0 = \mathbb{N}$, hence $\underline{\times^{(n)}} : \prod^{n+1} Q_0 \to Q_0$ holds. By simply expanding and counting the x maps in chain Q_0, lower levels of $\underline{\times^{(n)}}$ can extract the known properties:

1. $+ : Q_0 \times Q_0 \to Q_0$, commutativity $n_k + n_l = n_l + n_k$, associativity $(n_j + n_k) + n_l = n_j + (n_k + n_l)$.
2. $\times : Q_0 \times Q_0 \to Q_0$, commutativity $n_k \times n_l = n_l \times n_k$, associativity $(n_j \times n_k) \times n_l = n_j \times (n_k \times n_l)$, distributivity $(n_j + n_k) \times n_l = n_j \times n_l + n_k \times n_l$.

Any higher levels of operations can't be directly extracted to a binary algebraic operator of $Q_0 \times Q_0 \to Q_0$ due to the expansion procedure, e.g., $\underline{\dot{\times}}$'s expansion relies on h_1. However, one can simplify $\underline{\times^{(3)}}$ by letting $n_k = n_0$, $h_1 = 1$ to get the exponential operation $\text{Pow}^{(1)} : Q_0 \times Q_0 \to Q_0$, $(h_2, h_3) \mapsto h_2^{h_3}$. Higher level of n can always be simplified to get $\text{Pow}^{(n-2)} : Q_0 \times Q_0 \to Q_0$, $(h_2, h_n) \mapsto h_2^{(n-2)h_n}$ in this way: i) Let $n_k = n_0$, $h_1 = 1$; ii) Let $h_2 = h_3 = \cdots = h_{n-1}$. Obviously commutativity fails and only left associativity is obeyed. Distributivity is only trivial from counting the # numbers following the above procedure i)-iii).

For $++$, it is not necessary to start from the NR vertex. Standing at the middle of Q_0, in a finite range, the natural operations $--$, $-$, $\underline{\times}$, $\underline{\dot{\times}}$, $\underline{\ddot{\times}}$, ..., $\underline{\times_{(n)}}$ holds for the finite $\mathbb{Z}$. One can induce two QR chains to construct the uncountably infinite $\mathbb{Z}$ by artificial rules but then its $<$, $>$, and $=$ conditions are not natural (from the x maps of diversity).

The $(n{+}1)$-ary operations $\underline{\times^{(n)}}$ (or $\underline{\times_{(n)}}$) are the only natural operations inheriting the structure of x maps in a QR chain. The algebra they form can be denoted by $\mathcal{N}_1 \equiv (\mathbb{N}, \cdots, \underline{\times^{(n)}}, \cdots, \underline{\times}, +, ++, 0)$. The above simplifications form $\mathcal{N}_2 \equiv (\mathbb{N}, \cdots, \text{Pow}^{(n)}, \cdots, \text{Pow}^{(1)}, \times, +, ++, 0)$ which is not natural (only rely on counting x maps), since the simplifications in fact bring restrictions on $\underline{\times^{(n)}}$. Obviously that $\forall n_k \in Q_0$ $(n_k \neq n_0)$, $\forall n \in \mathbb{N}$, $\exists n_l < n_k$, $\exists h_1, h_2, \cdots, h_n \in \mathbb{N}$: $n_k = n_{l+h_1\times h_2\dot{\times} h_3\ddot{\times} h_4\cdots\times^{(n)} h_n}$, however not all $n_k \in Q_0$ are reachable by operation(s) of some restricted $\underline{\times^{(n)}}$. For instance, $\forall n_k \in Q_0$ $(n_k > n_1)$, $\forall h_1, h_2 \in \mathbb{N}$ $(h_1, h_2 > 1)$: $n_k = n_{h_1 \times h_2}$ is not true, which leads to the classes of prime and composite numbers. Another example is the radix $\forall r \in \mathbb{N}$ $(r > 1)$, $\forall n_k \in Q_0$, $\exists n \in \mathbb{N}$, $\exists \{h_i\}_{n \geq i \in \mathbb{N}^*}$ $(r > h_i \in \mathbb{N})$: $n_k = n_{\sum_i h_i \times \text{Pow}^{(1)}(r,i)}$ leads to the base-r representation of natural numbers.

Vitally, any restrictions on the operations on the direct products $\prod Q_0$ is not only about naturally counting the x maps in Q_0, but imposing rules on replicas of Q_0 which induce classes i.e. properties of QR vertices in Q_0 in its sense. One has to distinguish other properties induced by restricted $\underline{\times^{(n)}}$ operations, from the natural $=$ equivalence defined only by the count of x maps in Q_0.

For the general case where $n_l \prec n_k$ are QR vertices in the CR set but not necessarily in the NR chain above, one has $n_m - n_0 = n_k - n_l$ defining the associative $+ : CR \times CR \to CR$, $(n_l, n_m) \mapsto n_k$, but not commutative since the CR set does not always hold $n_m \prec n_k$. The general $+$ operation under $\prec$, as the inverse operation of $-$ in the CR set, is guaranteed to be associative.

However, subsets of the CR set may hold commutativity, associativity, and other identities for operation $+$ defined under the crystal order $\prec$. From now on, one refers *tree* to that crystal tree faithfully representing the crystal order between two QR vertices $n_l \prec n_k$ as in Fig. 8, and *paths* to some directed paths that links $n_l \prec n_k$ in their corresponding crystal tree. As shown in Fig. 9, the crystal diagram starts with the NR vertex n_0 and goes to $n_a \succ n_0$ by tree a, to $n_b \succ n_0$ by tree b, and note that n_a and n_b s.t. $n_a \not\prec n_b \wedge n_b \not\prec n_a$ are not necessarily the starting vertices of their corresponding QR chains. In the CR set there exists the QR vertex n_{a+b} that holds $n_{a+b} \succ n_a$ by tree b, and $n_{a+b} \succ n_b$ by tree a, thus holds the commutativity $n_a{+}n_b = n_b{+}n_a = n_{a+b}$ i.e. both $a{+}b$ and $b{+}a$ are paths in $n_{a+b} \succ n_0$. The CR subset $\{n_{\mu a+\nu b}\}_{(\mu,\nu)\in\mathbb{N}\times\mathbb{N}}$ fullfills the commutativity, associativity, and closure with respect to the binary operator $+$, where $n_{(\mu+1)a+\nu b} \succ n_{\mu a+\nu b}$ by tree a and $n_{\mu a+(\nu+1)b} \succ n_{\mu a+\nu b}$ by tree b hold simultaneously. This CR subset generated by trees a and b forms a lattice.

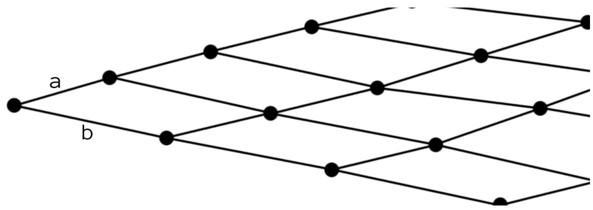

Figure 9: The lattice simply generated by trees a and b under the crystal order.

More generally, a set of generator trees $\{a_i\}_{i\in I}$ s.t. $\forall i \neq j \in I$: $n_{a_i} \not\prec n_{a_j} \;\wedge\; n_{a_j} \not\prec n_{a_i}$ extracts the CR subset $L = \{n_{\mu^i a_i}\}_{i\in I}$ (Einstein auto sum) where $\forall j \in I$, $n_{\mu^i a_i + a_j} \succ n_{\mu^i a_i}$ by tree a_j, that forms a lattice in mathematics by defining $\curlyvee : L \times L \to L$, $n_{\mu^i a_i} \curlyvee n_{\nu^i a_i} = n_{\max\{\mu^i,\nu^i\}a_i}$ as well as $\curlywedge : L \times L \to L$, $n_{\mu^i a_i} \curlywedge n_{\nu^i a_i} = n_{\min\{\mu^i,\nu^i\}a_i}$. It is direct to verify the commutative laws, the associative laws, the idempotent laws, and the absorption laws of lattice $(L, \curlyvee, \curlywedge)$.

It also preserves the commutativity, associativity, and closure of the binary operator $+$ when predecessing a tree $c_{\lambda^1\lambda^2\cdots\lambda^i\cdots} : n_{c_{\lambda^1\lambda^2\cdots\lambda^i\cdots}} \succ n_0$ where $n_{c_{\lambda^1\lambda^2\cdots\lambda^i\cdots}} \not\prec n_{\mu^i a_i} \;\wedge\; n_{\mu^i a_i} \not\prec n_{c_{\lambda^1\lambda^2\cdots\lambda^i\cdots}}$ to the simply generated lattice $L = \{n_{\mu^i a_i}\}_{i\in I}$ raising $n_{(\mu^i+\lambda^i)a_i} \succ n_{\mu^i a_i}$ by a tree including the paths of $c_{\lambda^1\lambda^2\cdots\lambda^i\cdots}$ with any constants $(\lambda^1, \lambda^2, \cdots, \lambda^i, \cdots) \subseteq \prod_{i\in I}\mathbb{N} \setminus \{(0,\cdots,0),(0,\cdots,0,1,0,\cdots,0)\}$. By predecessing countable trees of this kind, one gets a different subset of the CR set under the crystal order, that is still a lattice, which can be denoted as $\prod_i a_i + \sum_\lambda c_\lambda$ with its dimension $\dim L = |\{a_i\}|$.

For instance, the simply generated tree in Fig. 9 can be accompanied by trees $c_{\rho\sigma}$ that raise $n_{(\mu+\rho)a+(\nu+\sigma)b} \succ n_{\mu a+\nu b}$ by the tree including the paths of $c_{\rho\sigma}$, for any constants $(\rho,\sigma) \in \mathbb{N}\times\mathbb{N} \setminus \{(0,0),(0,1),(1,0)\}$. As shown in Fig. 10, one predecesses trees c_{11} and c_{21} to the simply generated lattice. The resulting lattice contains the QR vertices in the CR set, except for $\{n_{\mu a}, n_{\nu b}\}$, indeed different from those in Fig. 9. The QR vertex $n_{2a+b} \succ n_0$ by tree $r = (a{+}a{+}b)\cup(a{+}c_{11})\cup(a{+}b{+}a)\cup c_{21}\cup(c_{11}{+}a)\cup(b{+}a{+}a)$. Different paths of sums of embedded trees in the crystal tree shows the identities of operation $+$ in the lattice, here for example n_{2a+b} yields $a + c_{11} \approx c_{11} + a$ and $a{+}a{+}b \approx a{+}b{+}a \approx b{+}a{+}a$. It is direct to see that the relation $\approx$, called paths equivalence, is an equivalence relation on paths, with equivalence classes separated by only the different QR vertices. The lattice ensures that $\forall a, b, c$: $a + b \approx b + a \;\wedge\; b + c \approx c + b \;\wedge\; a + c \approx c + a$ as well as the associativity under $\approx$ inherited from the crystal order $\prec$ for every path in L, as L is a subset of the CR set.

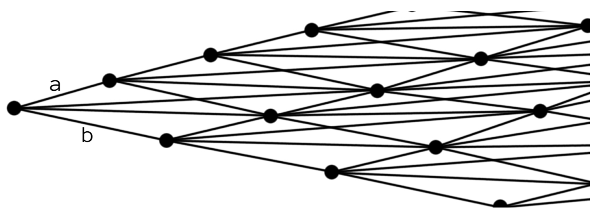

Figure 10: The lattice generated by trees a and b accompanied by trees c_{11} and c_{21} under the crystal order.

Given some specific generator trees $\{a_i\}_{i\in I}$, a badly specified lattice $(L, \curlyvee, \curlywedge)$ without the constraint of $\forall i \neq j \in I$: $n_{a_i} \not\prec n_{a_j} \;\wedge\; n_{a_j} \not\prec n_{a_i}$ can be reduced to $(L', \curlyvee, \curlywedge)$ if some elements of L indeed refer to the same QR vertex. Such a reduction due to bad definition of a lattice L from abstract notations $a, b, \cdots$ of trees, has no complexity other than the identical expressions with embedded trees $\{a_{i_n}\}_{i_n\in I_n}$ of a tree r s.t. $\forall i_k \in I_k$: $\lambda^{i_k} < \aleph_0$ and

$$r = \sum_{i_1\in I_1} \lambda^{i_1} a_{i_1} = \sum_{i_2\in I_2} \lambda^{i_2} a_{i_2} = \cdots = \sum_{i_k\in I_k} \lambda^{i_k} a_{i_k} = \cdots \tag{9}$$

requiring that $\forall k \neq l$: $\{a_{i_k}\} \cap \{a_{i_l}\} = \emptyset$ since the lattice has already the commutativity and the associativity for $+$ so that any tree $a_{i_{kl}} \in \{a_{i_k}\} \cap \{a_{i_l}\}$ can be eliminated in

$r = \lambda^{i_k} a_{i_k} = \lambda^{i_l} a_{i_l}$ in the sense of changing the starting vertex of r from n_0 to $n_{a_{i_{kl}}}$.

The expression Eq. 9 i.e. identification of lattice elements is originated from the structure of a tree r that includes the trees $\{a_{i_n}\}$. Under the tree equivalence $=$ with respect to the crystal order (stricter than the equivalence $\approx$ which is applied to paths), meaning that the same QR vertex is described by the associative sums of trees $\lambda^{i_k} a_{i_k}$ and $\lambda^{i_l} a_{i_l}$, $\exists i, j \in I_k \cup I_l$: all QR vertices in tree $a_i : n_{a_i} \succ n_0$ is a subset of QR vertices in tree $a_j : n_{a_j} \succ n_0$. More importantly, $\forall L$ $(\lambda^{i_k} a_{i_k} = \lambda^{i_l} a_{i_l})$, $\forall i_k \in I_k$, $\exists$ tree e_{i_k}: $\forall i_l \in I_l$, $\exists \lambda_{i_k}, \lambda_{i_{lk}} \in \mathbb{N}^*$, $\exists e_{i_{lk}} \in \{e_{i_k}\}$: $\lambda_{i_k}, \lambda_{i_{lk}} < \aleph_0 \ \wedge \ a_{i_k} = \lambda_{i_k} e_{i_k} \ \wedge \ a_{i_l} = \lambda_{i_{lk}} e_{i_{lk}}$. The proof is obvious when one takes the tree e_{i_k} as the greatest common divisor (gcd) tree of a_{i_k} and all $a_{i_{lk}}$ that paired with a_{i_k} i.e. $a_{i_k} \prec a_{i_{lk}} \ \vee \ a_{i_k} \succ a_{i_{lk}}$. Here the $\prec$ between trees in short denotes the crystal order $\prec$ between the QR vertices that the two trees goes from n_0. The lattice s.t. $\forall i_l \in I_l$, $\exists i_k \in I_k$: $\lambda^{i_l} a_{i_l}(\text{no sum}) + \sum_{i_{l'} \neq i_l \in I_l} \lambda^{i_{l'}} a_{i_{l'}} = \lambda^{i_k} a_{i_k}(\text{no sum}) + \sum_{i_{k'} \neq i_k \in I_k} \lambda^{i_{k'}} a_{i_{k'}}$ and vice versa, thus ensures that $\forall a_{i_l}$ is paired with an a_{i_k} and be that $a_{i_{lk}}$ (due to $a_{i_k} \not\prec a_{i_l} \ \wedge \ a_{i_l} \not\prec a_{i_k} \Rightarrow r = a_{i_k} + \cdots = a_{i_l} + \cdots$ LHS and RHS are only paths of r but not the tree r). A tree e'_{i_k} s.t. $a_{i_k} = \lambda'_{i_k} e'_{i_k} \ \wedge \ a_{i_{lk}} = \lambda'_{i_{lk}} e'_{i_k}$ exists otherwise $\nexists \lambda_{i_k}, \lambda_{i_{lk}} \in \mathbb{N}^*$: $r = \lambda_{i_k} a_{i_k} = \lambda_{i_{lk}} a_{i_{lk}}$ (the difference $-$ between ordered trees in r is still a tree in the minuend tree, and by doing the difference iteratively one at least reaches a gcd tree that is the divisor of original minuend and subtrahend trees). The order $\prec$ is a total order in $\{a_{i_k}, a_{i_{lk}}\}$ since for any trees $a_{i_{lk}}$ and $a_{i_{l'k}}$ in the lattice, $\exists e'_{i_k}$ hold the above identity with $\lambda'_{i_{lk}}$ and $\lambda'_{i_{l'k}}$ for both $\{a_{i_k}, a_{i_{lk}}\}$ (the $+$ is commutative, and by doing the difference between the their gcd trees one must get a finite gcd tree since the QR vertices in the CR set are discrete), hence $a_{i_{lk}} \prec a_{i_{l'k}} \ \vee \ a_{i_{lk}} \succ a_{i_{l'k}}$. The finiteness of lease common multiple $\mathrm{lcm}(\lambda^{i_k}, \lambda^{i_{lk}})$ ensures the uniqueness of gcd tree e_{i_k} under the total order $\prec$ among a_{i_k} and all $a_{i_{lk}}$, such that $\forall e'_{i_k}$: $e'_{i_k} \prec e_{i_k}$, $a_{i_k} = \lambda_{i_k} e_{i_k}$, $a_{i_{lk}} = \lambda_{i_{lk}} e_{i_k}$, and (proof omitted) $\lambda_{i_k}, \lambda_{i_{lk}} \leq \mathrm{lcm}(\lambda^{i_k}, \lambda^{i_{lk}})$.

In this case of $\lambda^{i_k} a_{i_k} = \lambda^{i_l} a_{i_l}$, one calls L reducibly generated. Suppose $|I_k| \leq |I_l|$, one can induce the reduction $\pi : L \to L', n_{\mu^i a_i} \mapsto n_{\mu'^i a'_i}$ such that

$$\mu'^i a'_i = \begin{cases} \mu^i a_i, & a_i \notin \{a_{i_k}\} \cup \{a_{i_l}\} \text{ i.e. } i \notin I_k \cup I_l \\ \frac{\mu^i \lambda^i}{\gcd(\lambda^i, \lambda^{i_{lk}})} e_i, & i = i_k \in I_k \\ \frac{\mu^i \lambda^{i_{lk}}}{\gcd(\lambda^{i_k}, \lambda^{i_{lk}})} e_{i_k}, & i = i_{lk} \in I_l \end{cases} \tag{10}$$

in which $\gcd(\lambda^{i_k}, \lambda^{i_{lk}})$ or $\mathrm{lcm}(\lambda^{i_k}, \lambda^{i_{lk}})$ includes all $\lambda^{i_{lk}}$ for $a_{i_{lk}}$ paired with a_{i_k} inside the parenthesis. The dimension lowers down by $\dim L - \dim L' = \max\{|I_k|, |I_l|\} = |I_l|$. Starting from a reducibly generated L, by doing all reduction(s) one will finally get an irreducibly generated L by generator trees $\{a_i\}_{i \in I}$ s.t. $\nexists J, K \subseteq I, \lambda^j, \lambda^k \in \mathbb{N}^*$, tree r: $J \cap K = \emptyset \ \wedge \ r = \sum_{j \in J} \lambda^j a_j = \sum_{k \in K} \lambda^k a_k$.

Clearly, by using generator trees $\{a_i\}_{i \in I}$ s.t. $\forall i, j \in I$, $a_i \not\prec a_j \ \wedge \ a_j \not\prec a_i$, one can avoid the specified lattice L to be a reducibly generated one.

The generator trees of an irreducibly generated lattice L is called principal trees in L, with their corresponding QR vertices from n_0 called principal vertices in L. For QR vertices n_k and n_l generating a lattice L, $n_k \not\prec n_l \wedge n_l \not\prec n_k$ infers that n_k and n_l are coprincipal, but coprincipal vertices n_k and n_l can hold $n_k \prec n_l \ \vee \ n_l \prec n_k$ as long as they don't have a common divisor vertex n_e. In the case of $n_l \prec n_k$, $L = a_l \times a_k$ is instead generated by $a_l : n_l \succ n_0$ and $a_{k-l} : n_k \succ n_l$.

Principal QR vertices are those QR vertices that are principal in all possible lattices L. There is no principal QR vertices other than n_1 in Q_0. All QR vertices in Q_1 is principal, but not all QR vertices in Q_n $(n > 1)$ is principal. For instance, the QR vertex $\{m = 0; m_1 = 0, m_2 = 4; m_{11} = 0, m_{12} = 4, m_{21} = 0, m_{22} = 4\}$ in $P_1 \subset Q_2$ is not principal, but $\{m = 0; m_1 = 1, m_2 = 3; m_{11} = 0, m_{12} = 4, m_{21} = 0, m_{22} = 4\}$ is principal, and $\{m = 1; m_1 = 0, m_2 = 2; m_{11} = 0, m_{12} = 3\}$ is principal.

In summary, the x maps hold the solid structure of all QR vertices in the CR base. This structure is understood to be of two main aspects:

1. Intra QR chains: Algebra $\mathcal{N}_1$ counting the specific numbers of $\bar{x}$ maps. The count leads to natural relations $<, >, =$ and natural operations $\underline{\times^{(n)}}$ that hold their corresponding basic properties on QR vertices.

2. Inter QR chains: CR subsets forming lattices L under the crystal order of x maps. The structure of specified CR subsets L leads to natural relations $\prec, \succ, =$ (tree equivalence), $\approx$ (paths equivalence) and natural operation $+$ (and its iterated operations) that hold their corresponding basic properties including the algebra of trees under $\approx$ with the commutative, associative $+$ and the principality of QR vertices.

5.2 Sequential space

On the CR base, all possible RR maps allow for sequential excitations which are characterized by direct products on replicas of CR set and the RR maps. Restrictions on maps involving direct products of CR subsets and x/z maps lead to properties or classes of QR vertices and x/z maps other than the natural ones. As previously shown in $\mathcal{N}_2$, possible restrictions and their derived properties are severely uncountable. It is extremely impossible to examine all of them as they count for all realities of The Reality yet one is in only its own reality. However, rules of one's own reality is not uncountable thus one has to comply to pragmatism on mathematics, if one wants only to cover the characteristics of its own physical world.

The methodology and its implementations for one to fetch mathematical tools and examine its reality is diverse and mostly out of the scope of current documentation. Below one shows only some examples of constructions on the CR set replicas with the RR maps.

In mathematics, an irreducibly generated lattice $(L, \curlyvee, \curlywedge)$ of a CR subset under $\prec$ is distributive (distributive i.e. $\forall a, b, c \in L$: $a \curlywedge (b \curlyvee c) = (a \curlywedge b) \curlyvee (a \curlywedge c)$ i.e. $a \curlyvee (b \curlywedge c) = (a \curlyvee b) \curlywedge (a \curlyvee c)$) hence modular (modular i.e. $\forall a, b, c \in L$: $a \preceq b \Rightarrow a \curlyvee (b \curlywedge c) = b \curlywedge (a \curlyvee c)$), but neither a complete lattice (complete i.e. $\forall A \subseteq L$, $\exists \sup A, \inf A$: $\sup A \in L \wedge \inf A \in L$) nor a compactly generated lattice (compactly generated i.e. $\forall n_l \in L$, $\exists G \subset L$: $n_l \in \sup G \ \wedge \ \forall n_g \in G, \forall A \subseteq L$ $(\sup A \subseteq L \ \wedge \ n_g \preceq \sup A)$, $\exists B \subseteq A$: $|B| < \aleph_0 \ \wedge \ n_g \preceq \sup B$) hence not algebraic. However, it is obvious that any finite sublattice of such $(L, \curlyvee, \curlywedge)$ is algebraic.

To fetch some algebraic lattices under the crystal order, one induces a projection, which is a collection of RR maps,

$\theta_{\lambda^1\lambda^2\ldots\lambda^i\ldots} : L \to \bar{L}, n_{\mu^i a_i} \mapsto n_{\bar{\mu}^i a_i}$ where

$$\bar{\mu}^i = \begin{cases} \mu^i, & a_i \notin \{a_{i_k}\} \cup \{a_{i_l}\} \text{ i.e. } i \notin I_k \cup I_l \\ \mu^i \bmod \lambda^i, & i = i_k \in I_k \ \vee \ i = i_l \in I_l \end{cases} \quad (11)$$

appeared the binary operator mod : $\mathbb{N} \times \mathbb{N}^* \to \mathbb{N}$ of the usual definition on natural numbers. The $\bmod(\mu, \lambda)$ operator is defined by fixing λ to be a constant in $\underline{\times} : n_\mu = n_{\bar{\mu}} + h_1 \times \lambda$ in $\mathcal{N}_1$ and inversing the order of variables in the tuple, from $(n_{\bar{\mu}}, \lambda) \mapsto n_\mu$ to $(n_\mu, \lambda) \mapsto n_{\bar{\mu}}$, which is the same tuple inversion as in the definition of $+$ from $-$ on QR vertices in the previous Sections. The image $\bar{L}$ is a finite thus algebraic lattice. One can define the unary operation $^{-1} : \bar{L} \to \bar{L}$ by $n_{\bar{\mu}^i a_i} \mapsto n_{(\lambda^i - \bar{\mu}^i) a_i}$ and define the nullary operation e as n_0 so that $(\bar{L}, \bar{+}, ^{-1}, e)$ forms an Abelian group, with $\bar{+}$ compatibly induced in $\bar{L}$ by θ from the natural $+$. It is direct to see the structure of finite Abelian groups $\prod_i Z_i$ corresponds to this lattice construction $n_{\bar{\mu}^i a_i}$ with any principal trees a_i ($i \in \mathbb{N}$). As shown in Fig. 11, the quotient lattice $\bar{L} \ni n_{\bar{\mu} a}$ with $\lambda = 4$ presents the group Z_4, and the quotient lattice $\bar{L} \ni n_{\bar{\mu}^1 a_1 + \bar{\mu}^2 a_2}$ ($a_1 = a, a_2 = b$) with $\lambda^1 = 3$ and $\lambda^2 = 2$ presents the group $Z_3 \times Z_2$. Morphisms, direct products, actions, classes, and other properties in Group Theory can also be defined on replicas of $\bar{L}$'s.

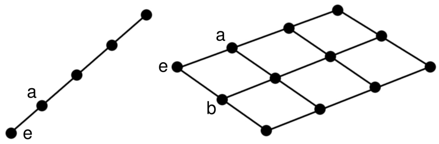

Figure 11: The quotient lattice $\bar{L}$ presenting the Abelian groups Z_4 and $Z_3 \times Z_2$.

For presenting non-Abelian groups, the lattice L is no longer commutative with respect to $+$, while it inherits the associativity of $+$ from the crystal order $\prec$. Given n generators $\{a_i\}_{1 \le i \le n \in \mathbb{N}}$, the number of permutations $P(a_i^1 + a_i^2 + \cdots + a_i^k)$ ($k \in \mathbb{N}^*$) is n^k. The above congruent projection $\theta_{\lambda^1\lambda^2\ldots\lambda^i\ldots} : L \to \bar{L}$ no longer holds for L because of the non-commutativity of tree sums. If one finds another congruence θ that close the entire n^k space without deriving the commutativity then it appears a non-Abelian finite group. This means that the subspace $k^i a_i$ ($k^i \in \mathbb{N}^*$) must be closed by θ, hence in any non-Abelian finite group $(\tilde{L}, \tilde{+}, ^{-1}, e)$ any generator a_i obeys a relation $\forall i \le n$, $\exists \lambda^i \in \mathbb{N}$: $e \simeq n_0 \simeq n_{\lambda^i a_i}$ or denoted as $\lambda^i a_i \simeq e$. One has to estimate the reduction of permutation space n^k by this relation $\simeq$ when k is sufficiently large, and that the additional relations which $(\tilde{L}, \tilde{+}, ^{-1}, e)$ holds together with the λ^i's should eventually lead to a finite permutation space when k is very large. For instance, one lets $a_1 \equiv r$, $a_2 \equiv s$, $\lambda^1 = 3$, $\lambda^2 = 2$, $\theta_1 : rr \simeq srs$, $\theta_2 : rs \simeq srr$, $\theta_3 : sr \simeq rrs$, $\theta_4 : s \simeq rsr$, then the quotient $\tilde{L} = L/\{\lambda^1, \lambda^2, \theta_1, \theta_2, \theta_3, \theta_4\}$ is finitely closed, presenting the dihedral group of order 6 a.k.a. D_3 as shown in Fig. 12, in which the dashed red lines mark the congruence from λ, dashed blue lines from θ, and dashed pink/cyan lines derived by the lattice associativity.

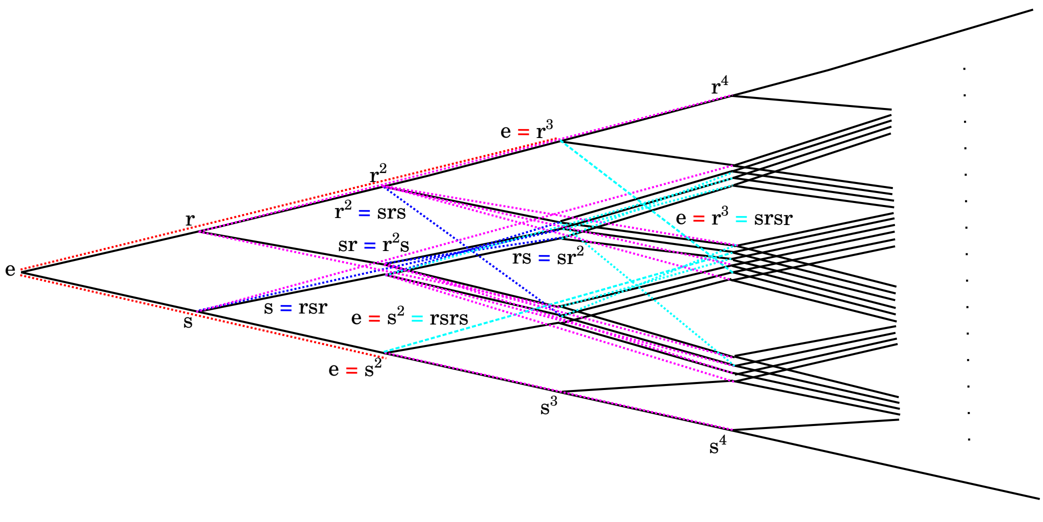

Figure 12: The quotient lattice $\tilde{L}$ presenting the non-Abelian group D_3.

Overall, the discovery of such θ congruences is out of the scope of the current documentation. One should always be aware of that understanding the structure of CR base does not necessarily mean that one exhausts all the properties or classes in mathematical structures derived from restricted natural operations on replicas of the CR subsets and the x/z maps. Even the finite rules founding one's own reality is far from one's limited border of observations and knowledge. For a quick guess, one may name the RR maps of its reality as "quintessences" with *mass at moment* h/c^2 $[\mathrm{M} \cdot \mathrm{T}]$ forming the space and *mass assemblies* of constant numbers of RR maps *per moment* $[\mathrm{T}^{-1}]$ ruled by physical laws with their spins and charges. The RR maps provide only the framework but not the direct results for physics.