Zeros of a sigma-additive set complex function. The case of the Fourier Transform

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Abstract

A non-trivial interpretation of Fourier integral theorem in the framework of measure spaces.

Let $f \in L^1(-\infty,+\infty)$ have constant sign and no zeros. By the Fourier integral theorem:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)\, e^{i\omega t} d\omega \tag{1}$$
$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt$$

We define:

$$\mu : \mathbb{R} \longrightarrow (0,+\infty) \tag{2}$$
$$\mu(t) = \int_{-\infty}^{t} |f(\tau)|\, d\tau > 0, \quad \forall t \in \mathbb{R}$$

$$\Sigma := \{A = [t_0, t] \mid t_0, t_1 \in \mathbb{R}\} \tag{3}$$

Σ is manifestly a σ-algebra on $\mathbb{R}$. (2) defines a countably additive and positive set function on Σ

$$\mu : \Sigma \longrightarrow (0,+\infty) \tag{4}$$
$$\mu : A \longrightarrow \mu(A) = \int_A f(t)\, dt, \quad \forall A \in \Sigma$$

and is complete on Σ [1]. So $(\mathbb{R},\Sigma,\mu)$ is a measurement space. The second of (1) becomes:

$$\hat{f}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} d\mu \tag{5}$$

We define:

$$\nu_\omega(A) := \int_A e^{-i\omega t} d\mu, \quad \forall A \in \Sigma, \quad \omega \in \mathbb{R} \tag{6}$$

which is broken down into a real part and an imaginary part:

$$\nu_\omega(A) = \underbrace{\mathrm{Re}\,\nu_\omega(A)}_{\xi_\omega(A)} + i\underbrace{\mathrm{Im}\,\nu_\omega(A)}_{\eta_\omega(A)} \tag{7}$$

where

$$\xi_\omega(A) = \int_A \cos(\omega t)\, d\mu, \quad \eta_\omega(A) = \int_A [-\sin(\omega t)]\, d\mu, \quad \forall A \in \Sigma \tag{8}$$

are countably additive set functions.

Lemma 1 *The set functions $\xi_\omega(A), \eta_\omega(A)$ are absolutely continuous with respect to μ.*

Proof.

$$\mu(A) \equiv 0 \Longleftrightarrow f(t) \equiv 0) \Longrightarrow (\xi_\omega(A) \equiv 0 \Longleftrightarrow \mu(A) \equiv 0$$

Likewise for $\eta_\omega(A)$. ■

Lemma 2

$$\frac{d\xi_\omega}{d\mu} = \cos(\omega t), \quad \frac{d\eta_\omega}{d\mu} = \sin(\omega t) \tag{9}$$

where $\frac{d}{d\mu}$ *is the Radon-Nikodym derivation operator.*

Proof. The statement follows immediately from the Radon-Nikodym theorem [1]. ■

Definition 3 *The* $\nu_\omega(A)$ *defined by (6) is called* **set complex function** .

From this it follows that the function:

$$\rho_\omega(t) = e^{-i\omega t} \tag{10}$$

is the Radon-Nikodym derivative of the set function $\nu_\omega(A)$ with respect to the measure μ:

$$\frac{d\nu_\omega}{d\mu} = \rho_\omega(t) \tag{11}$$

Notation 4 $\nu_\omega(\mathbb{R})$ *is the Fourier transforma of* $f(t)$.

From the absolute continuity of $\nu_\omega(A)$ with respect to μ, it follows that $\nu_\omega(\mathbb{R})$ can vanish only with respect to ω:

$$\exists\omega_0 \in \mathbb{R} \mid \nu_{\omega_0}(\mathbb{R}) = 0$$

We will therefore say that ω_0 is a zero of $\nu_\omega(\mathbb{R})$. It follows

Lemma 5 *If* $f(t)$ *has definite parity* $\nu_\omega(\mathbb{R})$ *is devoid of zeros.*

For example for the Gaussian

$$f(t) = e^{-\frac{t^2}{2\alpha}}(\alpha > 0) \tag{12}$$

we have $\nu_\omega(\mathbb{R}) = e^{-\frac{\alpha\omega^2}{2}}$ which is devoid of zeros.

Notation 6 *If* $f(t)$ *has parity* $(+1)$, $\nu_\omega(\mathbb{R})$ *has zero imaginary part. If* $f(t)$ *has parity* (-1), $\nu_\omega(\mathbb{R})$ *has zero real part. In both cases, the function* $\hat{f}(\omega)$ *preserves parity. Furthermore*

$$\left|\hat{f}(\omega)\right| \text{ has definite parity} \nRightarrow f(t) \text{ has definite parity}$$

The introduction of a parameter α in f (eq. (12)) suggests extending the previous arguments to a real function of the two real variables (a, t) defined on the strip

$$S = [a, b] \times (-\infty, +\infty) \tag{13}$$

for a given interval $[a, b]$ of $\mathbb{R}$, limited or unlimited. We keep the previous hypothesis, i.e. $f(\alpha, t)$ of class $L^1(-\infty, +\infty)$ with respect to t and of constant sign. Fourier's integral theorem returns the function of the complex variable $\alpha + i\omega$

$$\hat{f}(\alpha + i\omega) = \int_{-\infty}^{+\infty} f(\alpha, t)\, e^{-i\omega t} dt$$

which for a given $f(\alpha, t)$ can be holomorphic on the strip (13).

The function

$$\mu_{\alpha}(t) := \int_{-\infty}^{t} |f(\alpha, t')| \, dt', \quad \forall \alpha \in [a, b] \tag{14}$$

defines a one-parameter measure:

$$\begin{aligned} &\mu_{\alpha} : \Sigma \longrightarrow (0, +\infty) \\ &\mu_{\alpha} : A \longrightarrow \mu_{\alpha}(A) = \int_{A} f(\alpha, t) \, dt, \ \ \forall A \in \Sigma \end{aligned} \tag{15}$$

The generalization of the previous definition follows

$$\nu_{\alpha,\omega}(A) := \int_{A} e^{-i\omega t} d\mu_{\alpha}, \quad \forall A \in \Sigma, \quad (\alpha, \omega) \in S \tag{16}$$

If $\hat{f}(\alpha + i\omega)$ is holomorphic on S, any accumulation points of the set of zeros of $\nu_{\alpha,\omega}(\mathbb{R})$ belong to ∂S.

References

[1] Kantorovic L., Akilov G. *Analisi funzionale.* Editori Riuniti