A Tribute to the Memory of Professor Helmut MORITZ (1933-2022)

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ABSTRACT: This paper is a tribute to the memory of professor and geodesist Helmut Moritz who passed away in December 2022. We present a paper about the theory of geodetic refraction written by him and presented during the International Symposium " Figure of the Earth and Refraction ", Vienna, March 14th-l7th, 1967.

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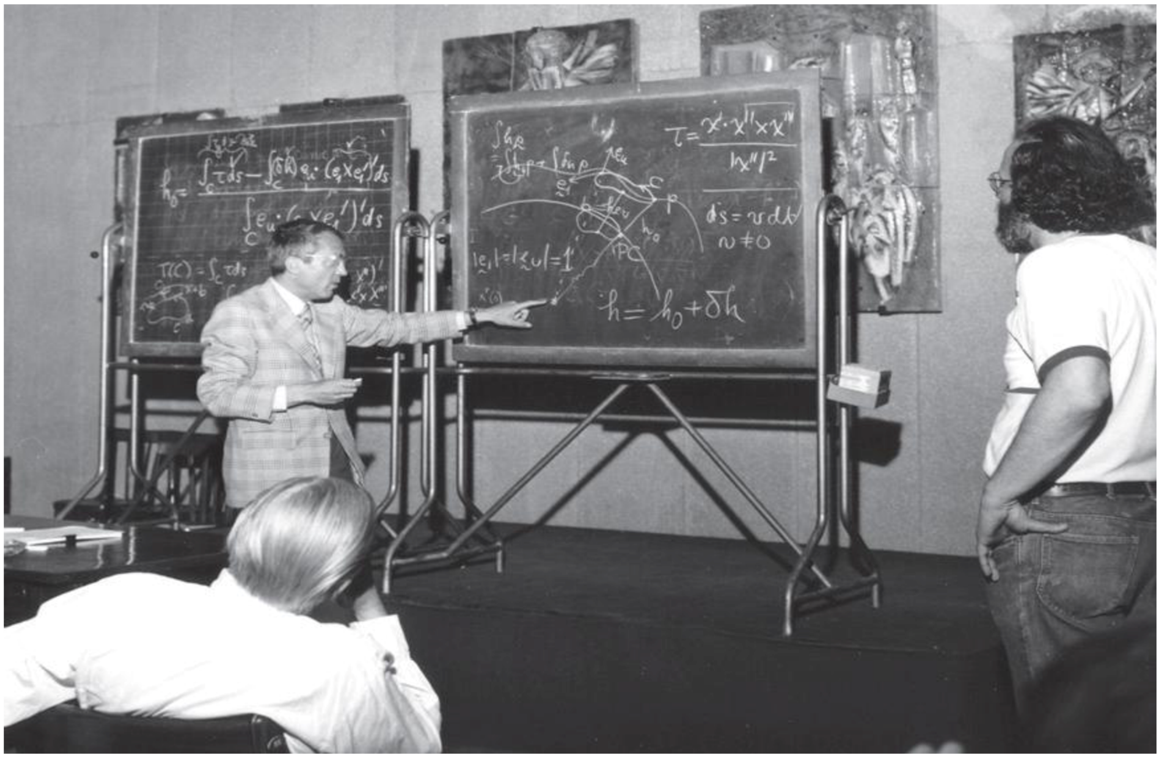

Figure 1. Prof. Helmut Moritz at the blackboard, Prof. Antonio Marussi (left) and Prof. Nathaniel Grossmann (right) [10]

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Application of the Conformal Theory of Refraction

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HELMUT MORITZ

1 INTRODUCTION

A light ray or an electromagnetic wave of high frequency describes a slightly curved path in the atmosphere, rather than a straight line. In electronic distance measurement the straight distance $s = AB$ between two points A and B is to be computed from the measured travel time T. This is usually done in two steps :

1 . Computation of the length S of the curved light path between A and B from the travel time T.

2. Computation of the chord $s = AB$ from the curved arc S.

It is, however, possible to give a method of directly obtaining the straight distance s from the travel time T, without needing the curved arc S. By an extension of this method vertical and lateral refraction affecting measured directions can be treated as well ; we thus obtain a unified theory of all geodetically important phenomena of refraction.

A convenient geometrical visualization of this method is furnished by the theory of conformal mapping in space. Conformal mapping between two surfaces being familiar to geodesists, it is gratifying that the reduction of electronically measured distances and observed directions for atmospheric refraction is the precise three-dimensional analogue of the reduction of distances and directions in the conformal mapping of a surface such as an ellipsoid onto a plane.

2 REFRACTION AND CONFORMAL MAPPING

Essentially the same laws hold for the propagation of light and of radio waves of high frequency. Henceforth we shall speak only of light, implying high-frequency radio waves as well. According to the well-known Fermat principle, light traveling from point A to Point B describes the shortest path; the travel time:

$$T = \int_A^B dt = \int_A^B \frac{ds}{v} \tag{2.1}$$

is a minimum.The instantaneous light velocity v is related to the constant light velocity in vacuum c by:

$$v = \frac{c}{n}$$

where n is the index of refraction. Hence (2.1) becomes:

$$T = \frac{1}{c}\int_A^B n ds$$

where:

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

is the ordinary line element. If we define the element of "optical length" $\bar{s}$ by:

$$d\bar{s} = nds = n\sqrt{dx^2 + dy^2 + dz^2} \tag{2.2}$$

then:

$$T = \frac{1}{c}\int_A^B d\bar{s} \tag{2.3}$$

Since c is a constant, Fermat's principle is equivalent to

$$\bar{s} = minimum \tag{2.4}$$

The optical length has indeed the dimension of a length. It is obtained from the measured travel time T by simple multiplication by c according to:

$$\bar{s} = cT$$

hence *the optical length s can be considered the direct result of electronic distance measurement*.

For the moment, assume for simplicity that the light is propagated along the xy-plane, which we shall denote by S. Then $z = 0$, and we have by (2.2):

$$ds^2 = n^2(dx^2 + dy^2) \tag{2.5}$$

whereas the ordinary line element is given by:

$$ds^2 = dx^2 + dy^2 \tag{2.6}$$

Obviously $d\bar{s}$ in (2.5) may be considered the line element in isothermic coordinates of a certain curved surface $\bar{S}$. The condition (2.4), $\bar{s} = minimum$, defines *geodesic lines* on this surface $\bar{S}$.

The length of such a geodesic on $\bar{S}$, the geodesic distance, is identical with the optical length and can therefore be considered the direct result of measurement. The reduction for refraction consists in computing the straight distance AB in the plane:

$$s = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

from the measured optical length s. The plane S is related to the surface $\bar{S}$ by a *conformal mapping*, since the line elements (2.5) and (2.6) have the form corresponding to such a mapping; hence the relation between the geodesic distance $\bar{s}$ and the straight distance s is given by the conventional reduction of distances in conformal mapping:

$$\Delta\bar{s} = \bar{s} - s$$

which, physically, is precisely the reduction of the measured optical length $\bar{s}$ for refraction.

Consider now the measurement of directions, again in the plane, disregarding the third dimension . The direct result of our measurement is the angle between light rays in the plane S. These light rays are geodesics in our auxiliary surface $\bar{S}$; in the plane S they are consequently the image curves of these geodesics. The angle between image curve and chord is well-known as the arc-to-chord or angle correction of conformal mapping (Bomford, 1962, p.169); it is thus identical with the angle between light path and straight line which is needed for the reduction of measured angles for refraction.

Hence we see that the introduction of our auxiliary surface $\bar{S}$ helps to reduce the problem of refraction to the theory of conformal mapping familiar to geodesists. In this way we achieve two purposes : first, we obtain a uniform treatment of the influence of refraction on observed angles and electronically measured distances; and second, there results a conceptual simplification: the relatively complicated light paths are represented by the simplest curves, the geodesics, in the auxiliary surface, and the travel time of the light waves gets a simple geometrical interpretation as geodesic distance.

Clearly the light ray moves in three-dimensional space and not in a plane. This means that we must restore the z-coordinate, which we have omitted for simplicity. The essential relations, however, which we have just found, remain intact. The plane S is replaced by three-dimensional ordinary space R, and the auxiliary surface $\bar{S}$ is replaced by an auxiliary space $\bar{R}$. Since $\bar{S}$ is a curved surface, $\bar{R}$ will in general be a curved "Riemannian" space (it is no longer Euclidean). Hence the light rays are geodesics in this auxiliary space $\bar{R}$, and the measured optical length (proportional to the travel time of light) is the geodesic distance in $\bar{R}$. We may thus say that there is a certain (fictitious) curved space $\bar{R}$ in which we measure directly by means of its geodesics, both when observing angles and measuring distances electronically.

The transition from this "refraction space" with linear element given by:

$$d\bar{s}^2 = n^2(dx^2 + dy^2 + dz^2) \tag{2.7}$$

to ordinary Euclidean space with:

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{2.8}$$

is effected through a three-dimensional conformal mapping ; the reduction of observed horizontal and vertical angles and electronically measured distances is identical with angle and distance correction of this conformal mapping.

The mathematical properties of three-dimensional conformal mappings and their application to the problem of refraction have been studied extensively ; we mention (Marussi, 1953), (Moritz, 1962), and (Hotine, 1965). Hence we need not go into the details here. We shall instead use the principles just explained to give explicit, practically applicable formulas for the reduction of angles and distances for refraction.

3 THE EICONAL EQUATION

The geodesics in Riemannian space are described by two differential equations:

1. the ordinary differential equation for the geodesic curve; and
2. the partial differential equation for the geodesic distance.

These two equations occur in many different contexts. (In mechanics, for instance, we have Newtons equation of motion, which is a system of ordinary differential equations corresponding to 1., and the Hamilton-Jacobi equation, which is a partial differential equation corresponding to 2.). They therefore deserve closer attention.

Let the square of the linear element of a three-dimensional space in curvilinear coordinates x_1, x_2, x_3 :

$$ds^2 = \sum_{i,j=1} a_{ij} dx_i dx_j \tag{3.1}$$

Then the ordinary differential equation for the geodesic line is:

$$\frac{d^2 x_i}{ds^2} + \frac{1}{2} \sum_{k,l,r=1}^{3} a^{ir} \left[\frac{\partial a_{rk}}{\partial x_l} + \frac{\partial a_{rl}}{\partial x_k} - \frac{\partial a_{kl}}{\partial x_r} \right] \frac{dx_k}{ds} \frac{dx_l}{ds} = 0 \quad (i = 1, 2, 3) \tag{3.2}$$

and the partial differential equation for the geodesic distance s is:

$$\sum_{i,j=1}^{3} a^{ij} \frac{\partial s}{\partial x_i} \frac{\partial s}{\partial x_j} = 1 \tag{3.3}$$

Here the matrix (a^{ij}) is simply the inverse to the matrix (a_{ij}).

The reader familiar with Ricci calculus will notice that the formulas (3.1) through (3.3) could be simplified by the use of certain notational conventions peculiar to this calculus. We have purposely dispensed with these conventions here in order to be more generally intelligible.

It should be mentioned that the formulas (3.1) through (3.3) are as well valid for a surface if all subscripts are assumed to take the values 1 ,2 only and if consequently the summation goes from 1 to 2 instead of from 1 to 3. To get the familiar form, substitute:

$$\begin{gathered} x_1 = u, \quad x_2 = v, \\ a_{11} = E, \quad a_{12} = a_{21} = F, \quad a_{22} = G \end{gathered} \tag{3.4}$$

Then (3.1) becomes:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{3.5}$$

Furthermore, assume that the coordinates u, v are orthogonal ; then $F \equiv 0$. In this case it is readily shown that:

$$a^{11} = \frac{1}{E}, \quad a^{12} = a^{21} = 0, \quad a^{22} = \frac{1}{G}$$

Then (3.2) becomes the system:

$$\begin{gathered} u" + \frac{1}{2E}(E_u u'^2 + 2E_v u'v' - G_u v'v'^2) = 0 \\ v" + \frac{1}{2G}(-E_v u'^2 + 2G_u u'v' + G_v v'^2) = 0 \end{gathered} \tag{3.6}$$

where: $u' = \dfrac{du}{ds}; E_u = \dfrac{\partial E}{\partial u}, etc.$ The distance equation (3.3)) takes the form:

$$\frac{1}{E}\left(\frac{\partial s}{\partial u}\right)^2 + \frac{1}{G}\left(\frac{\partial s}{\partial v}\right)^2 = 1 \tag{3.7}$$

These equations are important in geometrical geodesy, for computations on the reference ellipsoid on which u and v are orthogonal coordinates (usually, geographical coordinates φ and λ). The system (3.6), in a somewhat modified form, is the usual starting point for solving the "direct geodetic problem", the computation of coordinates from distance and azimuth.

Similarly, (3.7) is the best starting point for the solution of the "inverse geodetic problem", the computation of geodesic distance s and azimuth α from coordinates. Curiously enough, this simple equation seems to have never been used for this purposes, except by Gauss (1828). He needed the quantities $s.cos\alpha$ and $s.sin\alpha$ for obtaining his well-known formulas for small geodesic triangles on an arbitrary surface. Although Gauss'work belongs to general differential geometry, it may be properly quoted in connection with geodesy since the problem of geodesic triangles has important geodetic applications and since Gauss was inspired by his practical experience with triangulation[1]

After Gauss, the partial differential equation for the geodesic distance, (3.3) or (3.7), was neglected in geodesy as well as in differential geometry and its most important physical application, the General Theory of Relativity. This is the more surprising as the Hamilton-Jacobi equation (Bergmann, 1949, sec. 2.4) and its equivalent in optics, the "eiconal equation" (Bergmann, 1949, sec. 10.3), have had very successful physical applications. Only recently Synge (1964) has made extensive use of the distance equation (3.3) in General Relativity and has obtained important results in this way.

After this digression, intended to point out related problems, we shall return to atmospheric refraction. Here we have by (2.7):

$$d\bar{s}^2 = n^2(dx^2 + dy^2 + dz^2)$$

The comparison with (3.1) shows that: $x_1 = x,\ x_2 = y,\ x_3 = z$ and: $a_{11} = a_{22} = a_{33} = n^2,\ a_{12} = a_{13} = a_{23} = 0$; hence we have : $a^{11} = a^{22} = a^{33} = \dfrac{1}{n^2},\ a^{12} = a^{13} = a^{23} = 0$ because the matrix (a^{ij}) is the inverse to the (a_{ij}). Thus (3.3) becomes:

$$\left(\frac{\partial \bar{s}}{\partial x}\right)^2 + \left(\frac{\partial \bar{s}}{\partial y}\right)^2 + \left(\frac{\partial \bar{s}}{\partial z}\right)^2 = n^2 \tag{3.8}$$

This is the eiconal equation already mentioned. It is a first-order partial differential equation for the optical distance $\bar{s}$. The following developments will be based on the eiconal equation.

[1]The spirit of his work on differential geometry is shown by the concluding sentence of (Gauss, 1828) : "Si eadem formula triangulis in superficie curva non sphaerica applicatur, error generaliter loquendo erit quinti ordinis, sed insensibilis in omnibus triangulis, qualia in superficie telluris dimetiri licet."

4 SOLUTION OF THE EICONAL EQUATION

We shall now solve the eiconal equation (3.8) by a suitable series expansion. Since for the atmosphere the index of refraction, n, is very nearly 1 (it is approximately 1.0003) we may put:

$$n^2 = 1 + \epsilon\mu \tag{4.1}$$

where ϵ is a small constant parameter (e.g., $\epsilon = 0.0006$) and $\mu = \mu(x, y, z)$ is a function of position. Hence the measured optical length s (see section 2) will deviate little from the ordinary straight distance s, so that we may expand $\bar{s}$ as a power series with respect to the small parameter ϵ:

$$\bar{s} = s + \epsilon s' + \epsilon^2 s" + \ldots \tag{4.2}$$

Here:

$$s = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = s(x, y, z) \tag{4.3}$$

is the straight distance of a variable point $P(x, y, z)$ from a fixed point $P_1(x_1, y_1, z_1)$ as a function of the coordinates of P. The functions $s', s", \ldots$ will be obtained from the eiconal equation (3.8); we may safely neglect terms of order ϵ^3 and higher. We keep in mind that (4.2) is the desired *direct relation between measured optical length $\bar{s}$ and straight distance s* mentioned at the beginning.

The straight distance (4.3) satisfies the partial differential equation:

$$\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2 + \left(\frac{\partial s}{\partial z}\right)^2 = 1 \tag{4.4}$$

which is obtained from (3.8) by replacing $\bar{s}$ by s and n by 1. This is readily verified by substituting (4.3) into (4.4).

By introducing the vector:

$$grad\bar{s} = \left(\frac{\partial \bar{s}}{\partial x}, \frac{\partial \bar{s}}{\partial y}, \frac{\partial \bar{s}}{\partial z}\right) \tag{4.5}$$

we may abbreviate the eiconal equation as:

$$(grad\bar{s})^2 = n^2 \tag{4.6}$$

We substitute (4.1) and (4.2) into this equation, obtaining:

$$(grads + \epsilon grads' + \epsilon^2 grads")^2 = 1 + \epsilon\mu$$

Working out the square and comparing the terms independent on ϵ, those multiplied by ϵ, and those multiplied by ϵ^2 we find:

$$(grads)^2 = 1 \tag{4.7}$$
$$2grads'.grads = \mu \tag{4.8}$$
$$2grads".grads + (grads')^2 = 0 \tag{4.9}$$

With (4.7) we have recovered (4.4), whose solution (4.3) can be written as:

$$s = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} = ||P_1P||, \quad OP = (x, y, z) \tag{4.10}$$

where $||P_1P||$ is the norm of the vector P_1P. For later application, we evaluate:

$$grads = \left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}, \frac{\partial s}{\partial z}\right) = \frac{\boldsymbol{x} - x_1}{s} = \boldsymbol{e} \tag{4.11}$$

where $\boldsymbol{e}$ denotes the unit vector of the direction P_1P; see Fig. 1.

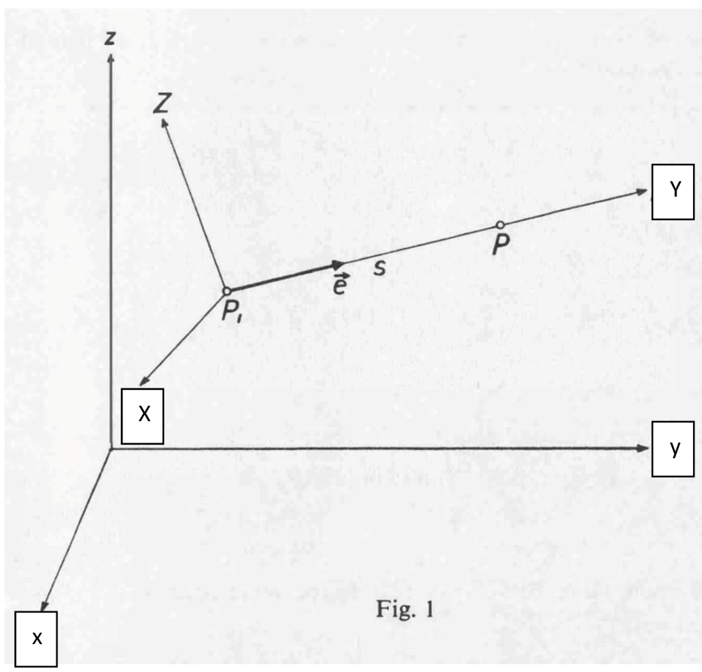

Fig. 1

In agreement with this figure, we introduce an additional rectangular coordinate system XYZ with origin at P_1, whose X-axis contains P and whose Y-axis is parallel to the original xy-plane.

Now, we consider (4.8). In view of (4.11), it may be written as:

$$2\boldsymbol{e}.grads' = \mu$$

Here $\boldsymbol{e}.grads'$ is the projection of $grads'$ onto the direction of $\boldsymbol{e}$; it is therefore identical with the derivative of s' along the direction of X, $\partial s'/\partial X$. Hence we obtain:

$$2\frac{\partial s'}{\partial X} = \mu$$
$$s' = \frac{1}{2}\int_0^s \mu dX \tag{4.12}$$

This integral is extended over the straight line P_1P.

To evaluate s" by (4.9), we need $grads'$. For this purpose we must express (4.12) as an explicit function of the coordinates $x, y, z)$ of P. This is simply achieved by introducing a parameter:

$$t = \frac{X}{s} \tag{4.13}$$

which runs from 0 to 1 as the current point of integration moves along the straight line from P_1 to P. Since the coordinates of this current point are given by:

$$\boldsymbol{x}_1 + t(\boldsymbol{x} - \boldsymbol{x}_1)$$

we have along P_1P explicitly:

$$\mu = \mu\left[\boldsymbol{x}_1 + t(\boldsymbol{x} - \boldsymbol{x}_1)\right]$$

Substituting this into (4.12), taking (4.13) into account $(dX = sdt)$, we find:

$$s' = \frac{s}{2}\int_0^1 \mu\left[\boldsymbol{x}_1 + t(\boldsymbol{x} - \boldsymbol{x}_1)\right]dt$$

Having thus obtained an explicit expression of s' as a function of $\boldsymbol{x}$, we may at once perform the differentiation with respect to x, y, z to get:

$$grads' = \frac{1}{2}grads.\int_0^1 \mu dt + \frac{s}{2}\int_0^1 grad\mu t dt$$

Returning to X by (4.13) we have:

$$grads' = \frac{s'}{s}\boldsymbol{e} + \frac{1}{2s}\int_0^1 grad\mu X dX = \frac{s'}{s}\boldsymbol{e} + \frac{1}{2s}\boldsymbol{a} \tag{4.14}$$

as the desired result; we shall find the abbreviation:

$$\int_0^1 grad\mu X dX = \boldsymbol{a} \tag{4.15}$$

quite useful.

Now we can attack (4.9). By (4.11) and (4.15), this equation becomes:

$$2\boldsymbol{e}.grads" + \left(\frac{s'}{s}\boldsymbol{e} + \frac{1}{2s}\boldsymbol{a}\right)^2 = 0$$

or:

$$2\boldsymbol{e}.grads" + \left(\frac{s'}{s}\right)^2 + \frac{s'}{s^2}\boldsymbol{a}.\boldsymbol{e} + \frac{||\boldsymbol{a}||^2}{4s^2} \tag{4.16}$$

This equation is considerably simplified by using the system XYZ. The components of the vectors $\boldsymbol{e}$ and $\boldsymbol{a}$ in this system are denoted by capital letters. Thus:

$$\boldsymbol{e} = (E_1, E_2, E_3) = (1, 0, 0)$$
$$\boldsymbol{a} = (A_1, A_2, A_3)$$

in the system XYZ. Then we have:

$$\boldsymbol{a}.\boldsymbol{e} = A_1, \quad \boldsymbol{a}^2 = A_1^2 + A_2^2 + A_3^2$$

For A we obtain the simple expression:

$$A_1 = \int_0^s \frac{\partial \mu}{\partial X} X dX = \mu s' - \int_0^s \mu dX = \mu s - 2s' \tag{4.17}$$

by partial integration; A_2 and A_3 are obviously given by:

$$A_2 = \int_0^s \frac{\partial \mu}{\partial Y} X dX, \quad A_3 = \int_0^s \frac{\partial \mu}{\partial Z} X dX \tag{4.18}$$

Thus (4.16) reduces to:

$$2\frac{\partial s"}{\partial X} + \frac{\mu^2}{4} + \frac{A_2^2 + A_3^2}{4s^2} = 0 \tag{4.19}$$

for the end point P. To integrate this equation we write it for a current point along the straight line $P_1 P$ by replacing s by X and (4.18) by:

$$A_2 = \int_0^X \frac{\partial \mu}{\partial Y} \xi d\xi, \quad A_3 = \int_0^X \frac{\partial \mu}{\partial Z} \xi d\xi \tag{4.20}$$

(we have now denoted the integration variable by ξ to avoid confusion with the upper limit X). We thus obtain:

$$2\frac{\partial s"}{\partial X} + \frac{\mu^2}{4} + \frac{A_2^2 + A_3^2}{4X^2} = 0 \tag{4.21}$$

with the solution:

$$s" = -\frac{1}{8}\int_0^s \mu^2 dX - \frac{1}{8}\int_0^s \frac{A-2^2+A_3^2}{X^2}dX \tag{4.22}$$

A_2 and A_3 being given by (4.20).

By (4.10), (4.12), and (4.22) we have expressed $s = s + \epsilon s' + \epsilon^2 s"$ as a function of the index of refraction and its partial derivatives in a practically exact way. If these quantities have been determined by suitable measurements, we can evaluate:

$$\Delta s = \bar{s} - s = \epsilon s' + \epsilon^2 s" \tag{4.23}$$

using (4.12) and (4.22) and computing these integrals by numerical or graphical integration. It may be pointed out again that these integrals are taken along the straight line $P1P$ and not along the light path.

The quantity Δs defined by (4.23) represents the desired reduction of the measured optical length for refraction. According to sec. 2, it corresponds to the distance reduction in three dimensional conformal mapping. Estimates show that the first integral of (4.22) is of the order of $5 \times 10^{-8}s$ is consequently negligible. We may also neglect A_2, which is caused by lateral refraction, so that there remains as a practical approximation:

$$\Delta s = \frac{\epsilon}{2}\int_0^s \mu dX - \frac{\epsilon^2}{8}\int_0^s \frac{A_3^2}{X^2}dx \tag{4.24}$$

The term with ϵ^2 reaches the order of some 10 meters for $s = 1000\,km$.

5 Effect on Horizontal and Vertical Angles

This method also furnishes the effect of vertical and lateral refraction on measured angles. The principle is as follows; see Fig. 2.

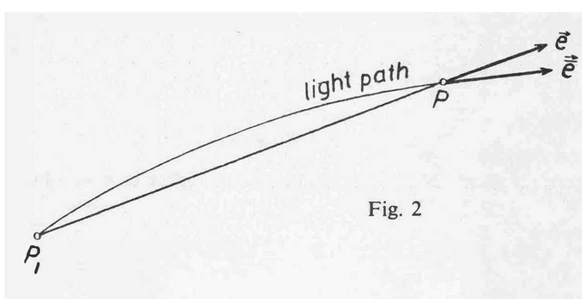

Fig. 2

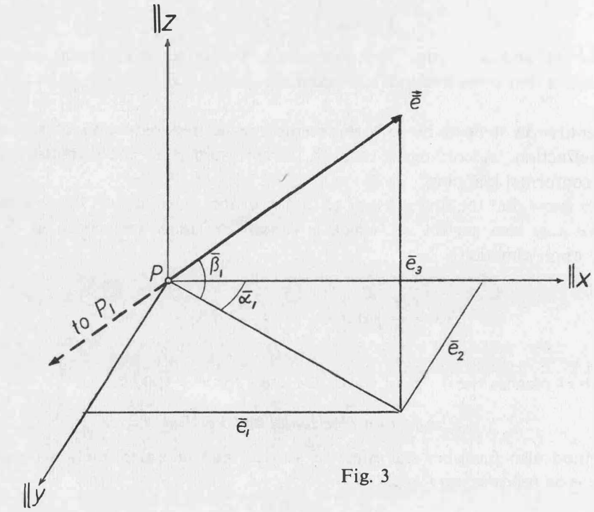

Fig. 3

We consider the two unit vectors $\boldsymbol{e}$ and $\vec{\boldsymbol{e}}$, the first directed along the chord $P1P$, the second tangent to the light path at P. It may be shown that this tangent has the direction of $grad\bar{s}$ which is not, however, a unit vector. Hence:

$$\vec{\boldsymbol{e}} = \frac{grad\bar{s}}{||grad\bar{s}||} = \frac{grad\bar{s}}{n} = \frac{grad\bar{s}}{\sqrt{1+\epsilon\mu}} \tag{5.1}$$

Here we have used (4.6) and (4.1). We again use an expansion with respect to :

$$\vec{\boldsymbol{e}} = \boldsymbol{e} + \epsilon\boldsymbol{e}' + \epsilon^2\boldsymbol{e}'' + \dots \tag{5.2}$$

with:

$$\boldsymbol{e} = grads, \quad \boldsymbol{e}' = -\frac{1}{2}\mu\boldsymbol{e} + grads', \quad \boldsymbol{e}'' = -\frac{3}{8}\mu^2\boldsymbol{e} - \frac{1}{2}\mu grads' + grads" \tag{5.3}$$

Thus we know $\vec{\boldsymbol{e}}$; it is obvious that all refractional changes of directions or angles can be obtained through $\boldsymbol{e}$. We shall outline the derivation. Consider the vector $\vec{\boldsymbol{e}}$ according to Fig. 3. Its components in the system xyz (the z-axis being parallel to the vertical of P) are e_1, e_2, e_3. By Fig.3 we have:

$$tg\alpha_1 = \frac{e_2}{e_1}, \quad sin\beta_1 = e_3$$

Here α_1 and β_1 are taken in the direction P_1P, whereas our angles are measured at P, thus referring to the opposite direction PP_1. Hence the measured horizontal angle is $\bar{\alpha} = \alpha_1 \pm 180°$, and the vertical angle is $\bar{\beta} = -\beta_1$, so that:

$$\bar{\alpha} = arctg\left(\frac{e_2}{e_1}\right)$$
$$\bar{\beta} = -arcsine_3 \tag{5.4}$$

Inserting e_1, e_2, e_3 from (5.2) and (5.3) and expanding with respect to ϵ we obtain after some calculations:

$$\bar{\alpha} = \alpha + \epsilon\alpha' + \epsilon^2\alpha" + \ldots, \quad \bar{\beta} = \beta + \epsilon\beta' + \epsilon^2\beta" + \ldots \tag{5.5}$$

with:

$$\alpha' = \frac{A^2}{2scos\beta} = \frac{1}{2scos\beta}\int_0^s \frac{\partial\mu}{\partial Y} XdX$$
$$\beta' = -\frac{A_3}{2s} = -\frac{1}{2s}\int_0^s \frac{\partial\mu}{\partial Z} XdX$$
$$\beta" = -\frac{A_2^2}{8s^2}tg\beta + \frac{\mu}{4s}A_3 + \frac{1}{8s}\int_0^s \frac{\partial\mu}{\partial Z} XdX +$$
$$+\frac{1}{4s}\int_{X=0}^s \frac{A_2}{X_2}\left|\int_{\xi=0}^X \frac{\partial^2\mu}{\partial Y\partial Z}\xi^2 d\xi\right| dX + \frac{1}{4s}\int_{X=0}^s \frac{A_3}{X_2}\left|\int_{\xi=0}^X \frac{\partial^2\mu}{\partial Z^2}\xi^2 d\xi\right| dX \tag{5.6}$$

The notations are those of the preceding section. The angles α and β refer to the straight line PP_1. We have omitted $\alpha"$ because the effect of lateral refraction is small as compared to the vertical effect.

Estimates indicate that $\epsilon^2\beta"$ is usually only of the order of a few tenths of a second of arc even for $s = 50\,km$. Consequently it may often be neglected. In this case we have with $\epsilon\mu \doteq 2(n-1)$ simply:

$$\Delta\alpha = \bar{\alpha} - \alpha = \frac{1}{scos\beta}\int_0^s \frac{\partial n}{\partial Y} XdX$$
$$\Delta\beta = \bar{\beta} - \beta = -\frac{1}{s}\int_0^s \frac{\partial n}{\partial Z} XdX \tag{5.7}$$

These equations have been derived in an elementary geometric way in (Moritz, 1962), using the theory of conformal mapping. We remind the reader that $\Delta\alpha$ and $\Delta\beta$ correspond to the angle corrections of three-dimensional conformal mapping;

see sec. 2. As a matter of fact, a formula such as the second of (5.7) can also be used for evaluating the angle correction in the conformal mapping of a surface such as the ellipsoid onto a plane.

lnspecting our results such as (4.24) for Δs and (5.6) or (5.7) for $\Delta\alpha$ and $\Delta\beta$ we see that these formulas require the index of refraction n and certain of its partial derivatives to be known along the straight line $P_1 P$. These values may be obtained by performing measurements in the neighborhood of this line. Formulas for practical computation and a numerical example will be found in (Jordan-Eggert-Kneissl, 1966, p. 527-531).

References

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