Rutherford Cross Section in the laboratory frame-Part III

Anindya Kumar Biswas*
Department of Physics;
North-Eastern Hill University,
Mawkynroh-Umshing, Shillong-793022.

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In this pedagogical article, we elucidate on the direct derivation of the classical non-relativistic Rutherford scattering cross section, differential, in the laboratory frame, of two electrons, a la relativistic quantum mechanics as presented in the book of Bjorken and Drell.

* anindya@nehu.ac.in

I. INTRODUCTION

The classical non-relativistic Rutherford scattering cross section, differential, in the laboratory frame is computed in the classical mechanics in two steps. First the Rutherford scattering cross section, differential, in the center of mass frame is calculated, [1], as $\frac{k^2}{16E^2_{c.o.m}}\frac{1}{\sin^4\frac{\Theta}{2}}$, where, Θ is the c.o.m scattering angle. Then using transformation from the the center of mass to the laboratory frame, [1], the differential scattering cross section is calculated in the laboratory frame, for two particles of equal mass and elastic scattering, as $\frac{k^2}{E^2_{lab}}\frac{cos\theta}{\sin^4\theta}$, where, θ is the laboratory frame scattering angle. On the other hand, in the relativistic quantum mechanics of Dirac, the differential scattering cross section can be calculated in the laboratory frame directly, [2]. Taking non-relativistic limit yields us the classical non-relativistic differential cross-section. In our previous two papers, [3], [4], we have done that and have gotten correction to that laboratory frame cross-section up to the order of $(\frac{v}{c})^2$. In this paper, we derive the differential cross-section for two electrons, in the laboratory frame, in the zeroth order of $\frac{v}{c}$, a la [2], assuming plane wave coming and going to asymptotic without any Coulomb modification, [5].

II.

Starting from the Dirac equation of a spin half particle, of rest mass m_0 and charge q, in the presence of electromagnetic field, A^μ, given by $(i\hbar\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m_0c)\psi(x) = 0$ and considering a scattering between an and an electron, invoking Pauli's exclusion principle, we get the transition amplitude as, [2],[3], [4],

$$S_{fi} = i\mu_0 e^2(c\hbar)^5\frac{m_0^2}{\sqrt{E_{f1}E_{i1}E_{f2}E_{i2}}V^2}[\frac{\bar{u}_{f1}\gamma_\mu u_{i1}\bar{u}_{f2}\gamma^\mu u_{i2}}{(p_{f1}-p_{i1})^2} - \frac{\bar{u}_{f2}\gamma_\mu u_{i1}\bar{u}_{f1}\gamma^\mu u_{i2}}{(p_{f2}-p_{i1})^2}](2\pi)^4\delta^4(p_{f1}+p_{f2}-p_{i1}-p_{i2})$$

The transition probability from an initial state i to a final state f, $S_{fi}S^*_{fi}$, is obtained as

$$|S_{fi}|^2 = \mu_0^2 e^4(c\hbar)^{10}\frac{m_0^4}{E_{f1}E_{i1}E_{f2}E_{i2}V^4}[\frac{\bar{u}_{f1}\gamma_\mu u_{i1}\bar{u}_{f2}\gamma^\mu u_{i2}}{(p_{f1}-p_{i1})^2} - \frac{\bar{u}_{f2}\gamma_\mu u_{i1}\bar{u}_{f1}\gamma^\mu u_{i2}}{(p_{f2}-p_{i1})^2}][\frac{\bar{u}_{i1}\gamma_\lambda u_{f1}\bar{u}_{i2}\gamma^\lambda u_{f2}}{(p_{f1}-p_{i1})^2} - \frac{\bar{u}_{i1}\gamma_\lambda u_{f2}\bar{u}_{i2}\gamma^\lambda u_{f1}}{(p_{f2}-p_{i1})^2}]$$
$$\frac{cVT}{\hbar^4}(2\pi)^4\delta^4(p_{f1}+p_{f2}-p_{i1}-p_{i2})$$

Summing over the final spin states and averaging over the initial spin states, one gets,

$$\frac{1}{4}\Sigma_{fi}|S_{fi}|^2 = \frac{\mu_0^2 e^4\hbar^6 c^{11}}{4}\frac{m_0^4}{E_{f1}E_{i1}E_{f2}E_{i2}}\frac{T}{V^3}[I - II - III + IV](2\pi)^4\delta^4(p_{f1}+p_{f2}-p_{i1}-p_{i2})$$

where,

$$I = \frac{Tr[\gamma_\mu\frac{\not{p}_{i1}+m_0c}{2m_0c}\gamma_\lambda\frac{\not{p}_{f1}+m_0c}{2m_0c}]Tr[\gamma^\mu\frac{\not{p}_{i2}+m_0c}{2m_0c}\gamma^\lambda\frac{\not{p}_{f2}+m_0c}{2m_0c}]}{(p_{f1}-p_{i1})^4},$$
$$IV = \frac{Tr[\gamma_\mu\frac{\not{p}_{i1}+m_0c}{2m_0c}\gamma_\lambda\frac{\not{p}_{f2}+m_0c}{2m_0c}]Tr[\gamma^\mu\frac{\not{p}_{i2}+m_0c}{2m_0c}\gamma^\lambda\frac{\not{p}_{f1}+m_0c}{2m_0c}]}{(p_{f2}-p_{i1})^4},$$
$$II = \frac{Tr[\gamma_\mu\frac{\not{p}_{i1}+m_0c}{2m_0c}\gamma_\lambda\frac{\not{p}_{f2}+m_0c}{2m_0c}\gamma^\mu\frac{\not{p}_{i2}+m_0c}{2m_0c}\gamma^\lambda\frac{\not{p}_{f1}+m_0c}{2m_0c}]}{(p_{f1}-p_{i1})^2(p_{f2}-p_{i1})^2},$$
$$III = \frac{Tr[\gamma_\mu\frac{\not{p}_{i1}+m_0c}{2m_0c}\gamma_\lambda\frac{\not{p}_{f1}+m_0c}{2m_0c}\gamma^\mu\frac{\not{p}_{i2}+m_0c}{2m_0c}\gamma^\lambda\frac{\not{p}_{f2}+m_0c}{2m_0c}]}{(p_{f1}-p_{i1})^2(p_{f2}-p_{i1})^2}$$

By integrating over the full final states phase space and dividing by the incident flux of the first electron, $\frac{v_{i1}}{V}$ and the the time of travel, T, of the first electron from the initial state to the final state, one gets a quantity of the dimension of length square referred to as total scattering cross-section in the laboratory frame and denoted as $\sigma|_{lab}$, as below,

$$\sigma|_{lab} = \int\frac{\frac{1}{4}\Sigma_{fi}|S_{fi}|^2\frac{Vd^3p_{f1}}{(2\pi)^3\hbar^3}\frac{Vd^3p_{f2}}{(2\pi)^3\hbar^3}}{\frac{v_{i1}}{V}T} = \mu_0^2 e^4\hbar^6 c^{11}\frac{T}{V^3}\int\frac{m_0^4}{4E_{f1}E_{i1}E_{f2}E_{i2}}\frac{V^2}{(2\pi)^6\hbar^6}\frac{1}{\frac{v_{i1}}{V}T}|\vec{p}_{f1}|^2 d|\vec{p}_{f1}|d\Omega_{f1}d^3p_{f2}$$
$$[I - II - III + IV](2\pi)^4\delta^4(p_{f1}+p_{f2}-p_{i1}-p_{i2})$$

Dividing by the differential solid angle swept out by the final state electron, one derives the differential scattering cross-section, denoted as $\frac{d\sigma}{d\Omega_{f1}}|_{lab}$ and given by

$$\frac{d\sigma}{d\Omega_{f1}}|_{lab} = \mu_0^2 e^4 \hbar^6 c^{11} \frac{T}{V^3} \int \frac{m_0^4}{4E_{f1}E_{i1}E_{f2}E_{i2}} \frac{V^2}{(2\pi)^6\hbar^6} \frac{1}{\frac{v_{i1}}{V}T} |\vec{p}_{f1}|^2 d|\vec{p}_{f1}| d^3p_{f2} \\ [I - II - III + IV](2\pi)^4 \delta^4(p_{f1} + p_{f2} - p_{i1} - p_{i2}) \\ = \mu_0^2 e^4 c^{11} \int \frac{m_0^4}{4v_{i1}E_{f1}E_{i1}E_{f2}E_{i2}} \frac{1}{(2\pi)^2} |\vec{p}_{f1}|^2 d|\vec{p}_{f1}| d^3p_{f2} \\ [I - II - III + IV]\delta^4(p_{f1} + p_{f2} - p_{i1} - p_{i2})$$

Introducing the identity $\frac{c}{2E_{f2}} = \int \theta(p^0_{f2})\delta(p^2_{f2} - m_0^2c^2)dp^0_{f2}$,

$$\frac{d\sigma}{d\Omega_{f1}}|_{lab} = \mu_0^2 e^4 c^{10} \int \frac{m_0^4}{2v_{i1}E_{f1}E_{i1}E_{i2}} \frac{1}{(2\pi)^2} |\vec{p}_{f1}|^2 d|\vec{p}_{f1}| d^4p_{f2} \\ [I - II - III + IV]\delta^4(p_{f1} + p_{f2} - p_{i1} - p_{i2})\theta(p^0_{f2})\delta(p^2_{f2} - m_0^2c^2), \\ = \mu_0^2 e^4 c^8 \int \frac{m_0^4}{2v_{i1}E_{i1}E_{i2}} \frac{1}{(2\pi)^2} |\vec{p}_{f1}| dE_{f1} d^4p_{f2} \\ [I - II - III + IV]\delta^4(p_{f1} + p_{f2} - p_{i1} - p_{i2})\theta(p^0_{f2})\delta(p^2_{f2} - m_0^2c^2), \\ = 2(\frac{e^2}{4\pi\epsilon_0})^2 c^4 \frac{m_0^4}{v_{i1}E_{i1}E_{i2}} \int |\vec{p}_{f1}| dE_{f1} d^4p_{f2} \\ [I - II - III + IV]\delta^4(p_{f1} + p_{f2} - p_{i1} - p_{i2})\theta(p^0_{f2})\delta(p^2_{f2} - m_0^2c^2), \\ = 2(\frac{e^2}{4\pi\epsilon_0})^2 c^4 \frac{m_0^4}{v_{i1}E_{i1}E_{i2}} \int |\vec{p}_{f1}| dE_{f1} d^4p_{f2} \\ [I - II - III + IV]\delta^4(p_{f1} + p_{f2} - p_{i1} - p_{i2})\theta(p^0_{f2})\delta(p^2_{f2} - m_0^2c^2)$$

In the laboratory frame, $\vec{p}_{i2} = 0$. The differential scattering cross-section, $\frac{d\sigma}{d\Omega_f}|_{lab}$, is thereby expressed as

$$\frac{d\sigma}{d\Omega_{f1}}|_{lab} = 2(\frac{e^2}{4\pi\epsilon_0})^2 \frac{c^4 m_0^4}{m_0c^2 E_{i1}v_{i1}} \int_{m_0c^2}^{E_{i1}+m_0c^2} dE_{f1} \delta((-p_{f1} + p_{i1} + p_{i2})^2 - m_0^2c^2)|\vec{p}_{f1}|[I - II - III + IV]|_{p_{f2}=-p_{f1}+p_{i1}+p_{i2}}$$

III. NON-RELATIVISTIC LIMIT

$$\delta((-p_{f1} + p_{i1} + p_{i2})^2 - m_0^2c^2) = \delta((-p_{f1} + p_{i1})^2 + 2(-p_{f1} + p_{i1}).p_{i2}) = \delta(\frac{(-E_{f1} + E_{i1})^2}{c^2} - (-\vec{p}_{f1} + \vec{p}_{i1})^2 + 2(-E_{f1} + E_{i1})m_0) \\ = \delta(\frac{(\frac{m_0}{2}(-v^2_{f1} + v^2_{i1}))^2}{c^2} - m_0^2(-\vec{v}_{f1} + \vec{v}_{i1})^2 + (-v^2_{f1} + v^2_{i1})m_0^2) = \delta(-m_0^2(-\vec{v}_{f1} + \vec{v}_{i1})^2 + (-v^2_{f1} + v^2_{i1})m_0^2) \\ = \frac{1}{m_0^2}\delta(-(v^2_{f1} + v^2_{i1} - 2v_{f1}v_{i1}cos\theta) - v^2_{f1} + v^2_{i1}) = \frac{1}{m_0^2}\delta(-2v^2_{f1} + 2v_{f1}v_{i1}cos\theta) = \frac{1}{2m_0^2v_{f1}}\delta(-v_{f1} + v_{i1}cos\theta) \\ = \frac{1}{2m_0^2v_{f1}}\delta(v_{f1} - v_{i1}cos\theta)$$

$$|\vec{p}_{f1}| = m_0v_{f1}, E_{f1} = m_0c^2 + \frac{1}{2}m_0v^2_{f1}$$

$$\int |\vec{p}_{f1}| dE_{f1} \delta((-p_{f1} + p_{i1} + p_{i2})^2 - m_0^2c^2) = \int m_0v_{f1}m_0v_{f1}dv_{f1} \frac{1}{2m_0^2v_{f1}}\delta(v_{f1} - v_{i1}cos\theta) = \frac{1}{2}v_{i1}cos\theta$$

Hence,

$$\frac{d\sigma}{d\Omega_{f1}}|^{N.R}_{lab} = 2(\frac{e^2}{4\pi\epsilon_0})^2 \frac{c^4m_0^4}{m_0c^2m_0c^2v_{i1}} \frac{1}{2} v_{i1}cos\theta[I - II - III + IV]|^{v_{f1}=v_{i1}cos\theta}_{p_{f2}=-p_{f1}+p_{i1}+p_{i2}} \\ = (\frac{e^2}{4\pi\epsilon_0})^2 m_0^2 cos\theta[I - II - III + IV]|^{v_{f1}=v_{i1}cos\theta}_{p_{f2}=-p_{f1}+p_{i1}+p_{i2}}$$

IV. EVALUATION OF I, II, III, IV IN THE NON-RELATIVISTIC LIMIT

Here,

$$E_{i1} = m_0c^2(1+\frac{v_{i1}^2}{2c^2}), \frac{E_{i1}}{c}\frac{E_{f1}}{c} = m_0^2c^2 + \frac{1}{2}m_0^2(v_{i1}^2+v_{f1}^2), E_{i1}-E_{f1} = \frac{1}{2}m_0(v_{i1}^2-v_{f1}^2), \vec{p_{i1}}.\vec{p_{f1}} = m_0^2 v_{i1}v_{f1}cos\theta,$$

$$(p_{f1}-p_{i1})^2 = 2m_0^2c^2 - 2p_{i1}.p_{f1} = 2[m_0^2c^2 - (\frac{E_{i1}}{c}\frac{E_{f1}}{c} - \vec{p_{i1}}.\vec{p_{f1}})] = m_0^2[-v_{i1}^2 - v_{f1}^2 + 2v_{i1}v_{f1}cos\theta]$$

$$\frac{1}{(p_{f1}-p_{i1})^2}|_{v_{f1}=v_{i1}cos\theta} = -\frac{1}{m_0^2v_{i1}^2sin^2\theta}$$

$$(p_{f2}-p_{i1})^2 = (p_{f1}-p_{i2})^2 = 2m_0^2c^2 - 2p_{i2}.p_{f1} = 2[m_0^2c^2 - (\frac{E_{i2}}{c}\frac{E_{f1}}{c} - \vec{p_{i2}}.\vec{p_{f1}})] = 2[m_0^2c^2 - m_0c\frac{E_{f1}}{c}]$$

$$= 2[m_0^2c^2 - m_0(m_0c^2 + \frac{1}{2}m_0v_{f1}^2)] = -m_0^2v_{f1}^2$$

$$\frac{1}{(p_{f2}-p_{i1})^2}|_{v_{f1}=v_{i1}cos\theta} = -\frac{1}{m_0^2v_{i1}^2cos^2\theta}$$

$$I|^{v_{f1}=v_{i1}cos\theta}_{p_{f2}=-p_{f1}+p_{i1}+p_{i2}} = \frac{1}{m_0^4v_{i1}^4sin^4\theta}Tr[\gamma_\mu\frac{\not p_{i1}+m_0c}{2m_0c}\gamma_\lambda\frac{\not p_{f1}+m_0c}{2m_0c}]Tr[\gamma^\mu\frac{\not p_{i2}+m_0c}{2m_0c}\gamma^\lambda\frac{\not p_{f2}+m_0c}{2m_0c}]$$

$$= \frac{1}{16m_0^4c^4}\frac{1}{m_0^4v_{i1}^4sin^4\theta}Tr[\gamma_\mu(\not p_{i1}+m_0c)\gamma_\lambda(\not p_{f1}+m_0c)]Tr[\gamma^\mu(\not p_{i2}+m_0c)\gamma^\lambda(\not p_{f2}+m_0c)]$$

$$= \frac{1}{16m_0^4c^4}\frac{1}{m_0^4v_{i1}^4sin^4\theta}(4.4.2.2m_0^4c^4) = \frac{4}{m_0^4v_{i1}^4sin^4\theta}$$

$$IV|^{v_{f1}=v_{i1}cos\theta}_{p_{f2}=-p_{f1}+p_{i1}+p_{i2}} = \frac{1}{m_0^4v_{i1}^4cos^4\theta}Tr[\gamma_\mu\frac{\not p_{i1}+m_0c}{2m_0c}\gamma_\lambda\frac{\not p_{f2}+m_0c}{2m_0c}]Tr[\gamma^\mu\frac{\not p_{i2}+m_0c}{2m_0c}\gamma^\lambda\frac{\not p_{f1}+m_0c}{2m_0c}]$$

$$= \frac{1}{16m_0^4c^4}\frac{1}{m_0^4v_{i1}^4cos^4\theta}Tr[\gamma_\mu(\not p_{i1}+m_0c)\gamma_\lambda(\not p_{f2}+m_0c)]Tr[\gamma^\mu(\not p_{i2}+m_0c)\gamma^\lambda(\not p_{f1}+m_0c)]$$

$$= \frac{1}{16m_0^4c^4}\frac{1}{m_0^4v_{i1}^4cos^4\theta}(4.4.2.2m_0^4c^4) = \frac{4}{m_0^4v_{i1}^4cos^4\theta}$$

$$II|^{v_{f1}=v_{i1}cos\theta}_{p_{f2}=-p_{f1}+p_{i1}+p_{i2}} = \frac{1}{m_0^4v_{i1}^4cos^2\theta sin^2\theta}Tr[\gamma_\mu\frac{\not p_{i1}+m_0c}{2m_0c}\gamma_\lambda\frac{\not p_{f2}+m_0c}{2m_0c}\gamma^\mu\frac{\not p_{i2}+m_0c}{2m_0c}\gamma^\lambda\frac{\not p_{f1}+m_0c}{2m_0c}]$$

$$= \frac{1}{16m_0^4c^4}\frac{1}{m_0^4v_{i1}^4cos^2\theta sin^2\theta}Tr[\gamma_\mu(\not p_{i1}+m_0c)\gamma_\lambda(\not p_{f2}+m_0c)\gamma^\mu(\not p_{i2}+m_0c)\gamma^\lambda(\not p_{f1}+m_0c)]$$

$$= \frac{1}{16m_0^4c^4}\frac{1}{m_0^4v_{i1}^4cos^2\theta sin^2\theta}(16.2.m_0^4c^4) = \frac{1}{16m_0^4c^4}\frac{1}{m_0^4v_{i1}^4cos^2\theta sin^2\theta}(16.2.m_0^4c^4) = \frac{2}{m_0^4v_{i1}^4cos^2\theta sin^2\theta}$$

$$III = II$$

V. FINAL

$$\frac{d\sigma}{d\Omega_{f1}}|^{N.R}_{lab} = (\frac{e^2}{4\pi\epsilon_0})^2 m_0^2 cos\theta[I - II - III + IV]|^{v_{f1}=v_{i1}cos\theta}_{p_{f2}=-p_{f1}+p_{i1}+p_{i2}}$$

$$= (\frac{e^2}{4\pi\epsilon_0})^2 m_0^2cos\theta[\frac{4}{m_0^4v_{i1}^4sin^4\theta} + \frac{4}{m_0^4v_{i1}^4cos^4\theta} - 4\frac{1}{m_0^4v_{i1}^4cos^2\theta sin^2\theta}]$$

$$= (\frac{e^2}{4\pi\epsilon_0})^2 cos\theta(\frac{1}{\frac{1}{2}m_0v_{i1}^2})^2[\frac{1}{sin^4\theta} + \frac{1}{cos^4\theta} - \frac{1}{cos^2\theta sin^2\theta}] = (\frac{e^2}{4\pi\epsilon_0})^2cos\theta(\frac{1}{E_{lab}})^2[\frac{1}{sin^4\theta} + \frac{1}{cos^4\theta} - \frac{1}{cos^2\theta sin^2\theta}]$$

$$\frac{d\sigma}{d\Omega_{f1}}|^{N.R}_{lab} = \frac{k^2}{E_{lab}^2}cos\theta[\frac{1}{sin^4\theta} + \frac{1}{cos^4\theta} - \frac{1}{cos^2\theta sin^2\theta}]$$

This is the result sort of quoted in the reference, [5], when one considers non-Coulomb-modified plane wave in the asymptotic region. Moreover, the result is independent of $\hbar$ and c. Hence it should be applicable to two classical objects, if identical, say two hypothetical electrically charged fermionic black holes with their gravitational interaction switched off.

VI. ACKNOWLEDGMENT

The reference where it had been done, has not reached the author. Hopefully, nothing new has been presented in the paper.

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