Linear dynamical systems and transient terms

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Abstract

Under very general hypotheses, the behavior of dynamical systems described by a linear first order differential equation is independent of the initial condition.

Theorem 1 ***Hp.***

$$\dot{y} + \alpha(t)\, y = \beta(t)\,, \tag{1}$$

where the coefficients $\alpha(t)$ and $\beta(t)$ are functions of class $C^1(X)$ being $X = [t_0, +\infty)$. Moreover $\alpha(t)$ and each of its primitives diverges positively for $t \to +\infty$.

Th. *The general integral of the (1) is*

$$y(t, K) = y_0(t, K) + y_1(t)\,, \tag{2}$$

where K where K is a constant of integration, and

$$y_1(t) = \frac{\beta(t)}{\alpha(t)}, \quad y_0(t, K) \underset{t \to +\infty}{\longrightarrow} 0$$

for which the asymptotic behavior of the general integral is

$$y(t, K) \underset{t \to +\infty}{\longrightarrow} \frac{\beta(t)}{\alpha(t)} \tag{3}$$

Dimostrazione. We apply the standard procedure for integrating (1). Precisely, an integral factor is

$$I(t) = e^{\int \alpha(t) dt}$$

Multiplying the first and second sides of (1) by $I(t)$:

$$\dot{y} e^{\int \alpha(t) dt} + \alpha(t)\, y(t)\, e^{\int \alpha(t) dt} = \beta(t)\, e^{\int \alpha(t) dt} dt$$

i.e.

$$\frac{d}{dt}\left[y(t)\, e^{\int \alpha(t) dt} \right] = \beta(t)\, e^{\int \alpha(t) dt} dt$$

from which

$$y(t)\, e^{\int \alpha(t) dt} = K + \int \beta(t)\, e^{\int \alpha(t) dt} dt$$

where K is a constant of integration. It follows that by integrating, the constant of integration will not appear as incorporated in K. Therefore the general integral is

$$y(t, K) = K e^{-\gamma(t)} + e^{-\gamma(t)} \int \beta(t)\, e^{\gamma(t)} dt \tag{4}$$

having defined $\gamma(t) \equiv \int \alpha(t)\, dt$. Performing an integration by parts in the integral a second member of the (4):

$$\int \beta(t)\, e^{\gamma(t)} dt \underset{\dot{\gamma}(t)=\alpha(t)}{=} \frac{\beta(t)}{\alpha(t)} e^{\gamma(t)} - \int \frac{\dot{\beta}(t)}{\alpha(t)} e^{\gamma(t)} dt$$

It follows

$$y(t, K) = K e^{-\gamma(t)} + \frac{\beta(t)}{\alpha(t)} - e^{-\gamma(t)} \int \frac{\dot{\beta}(t)}{\alpha(t)} e^{\gamma(t)} dt \tag{5}$$

hence the assertion:

$$y(t, K) \underset{t\to+\infty}{\longrightarrow} \frac{\beta(t)}{\alpha(t)} \tag{6}$$

since by hypothesis $\gamma(t) \underset{t\to+\infty}{\longrightarrow} +\infty$. ■

From the theorem just proved it follows that $y_0(t)$ is the so-called *transitional term*, while $y_1(t)$ expresses the steady state behaviour. The latter does not depend on K, and therefore on the initial condition $y(t_0) = y_0$.