Bandlimited Functions and Timelimited Functions on Adeles

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Abstruct: Let $(\mathcal{F}f)(\eta)$ be the Fourier transform of $f(t)$. We will call the member of

$$\mathfrak{B} = \{\, f(x) \mid (\mathcal{F}f)(\eta) = 0,\ \forall\eta,\ |\eta| > \Omega \,\}$$

"bandlimited". On the other hand, let

$$Df(t) = \begin{cases} f(t) & \cdots\ |t| \leq T/2 \\ 0 & \cdots\ T/2 < |t| \end{cases}$$

for $f(t)$. We will call $Df(t)$ "timelimited". We will think of bandlimited functions and timelimited functions on adeles.

0.

Let $K/\mathbb{Q}$ be a number field of degree n. Denote the completion of K at the place $\mathfrak{p}$ of K by $K_{\mathfrak{p}}$.

Let $x \in \mathbb{R}$. Then $-x = \{-x\} + n$ where $\{-x\} \in [0, 1)$ and $n \in \mathbb{Z}$. Put

$$\lambda(x) = \{-x\}, \quad x \in \mathbb{R}.$$

Let $x \in \mathbb{Q}_p$. Then $x = \{x\}_p + n$ where $n \in \mathbb{Z}_p$. Namely $\{x\}_p$ is the fractional part of a p-adic number x. Put

$$\lambda(x) = \{x\}_p, \quad x \in \mathbb{Q}_p.$$

Denote the trace of the element ξ of K by

$$S\xi = \xi + \xi^{(1)} + \cdots + \xi^{(n-1)},$$

where $\xi,\ \xi^{(1)},\ \cdots,\ \xi^{(n-1)}$ are conjugates of ξ. If $K = \mathbb{R}$ then $S\xi = \xi$, if $K = \mathbb{C}$ then $S\xi = S(x+iy) = 2x$ and if $K = K_{\mathfrak{p}}/\mathbb{Q}_p$ then $S\xi \in \mathbb{Q}_p$.

Definition 0.1. Let k be a local field, namely k is $\mathbb{R}$, $\mathbb{C}$ or $K_{\mathfrak{p}}$. Put

$$\Lambda(\xi) =_{\text{def}} \lambda(S\xi) \qquad \xi \in k.$$

Proposition 0.1. k and $\hat{k}$ are isomorphic by the map

$$\begin{array}{ccc} k & \longrightarrow & \hat{k} \\ \cup & & \cup \\ \eta & \longmapsto & e^{2\pi i \Lambda(\eta\xi)} \end{array}.$$

Proposition 0.2. Let $d\xi$ be a Haar measure on k. The Fourier transform of $f(\xi) \in L^1(k)$ is defined by

$$(\mathcal{F}f)(\eta) = \int_k f(\xi) e^{-2\pi i \Lambda(\eta\xi)}\, d\xi\,.$$

The inverse Fourier transform is that

$$f(\xi) = \int_k (\mathcal{F}f)(\eta) e^{2\pi i \Lambda(\xi\eta)}\, d\eta\,.$$

We will think of the function space $C_c^\infty(K_\mathfrak{p})$ of compactly supported, locally constant functions. The space $C_c^\infty(K_\mathfrak{p})$ is regarded as the $\mathfrak{p}$-adic Schwartz-Bruhat space $\mathcal{S}(K_\mathfrak{p})$. We will regard $L^2(K_\mathfrak{p})$ as the completion of $\mathcal{S}(K_\mathfrak{p})$ and we shall think of the Fourier transform of $f(x)\in L^2(K_\mathfrak{p})$. Any function in $C_c^\infty(K_\mathfrak{p})$ can be written as the sum of characteristic functions of balls. Set

$$\mathrm{B}_{\leq N\mathfrak{p}^n}(a) = \{\, x\in K_\mathfrak{p} \mid |x-a|_\mathfrak{p} \leq N\mathfrak{p}^n \,\}.$$

Denote $\mathrm{B}_{\leq N\mathfrak{p}^n}(0)$ by $\mathrm{B}_{\leq N\mathfrak{p}^n}$. Let $\mathrm{Supp}(f)\subseteq \mathrm{B}_{\leq N\mathfrak{p}^f}$. Choose a suitable n such that $\mathrm{B}_{\leq N\mathfrak{p}^n} \subseteq \mathrm{B}_{\leq N\mathfrak{p}^f}$. Then we can choose a finite set of points $\{\, a_i \,\} \subseteq \mathrm{B}_{\leq N\mathfrak{p}^f}$ and we obtain

$$\mathrm{B}_{\leq N\mathfrak{p}^f} = \coprod_{i=1}^{k} a_i + \mathrm{B}_{\leq N\mathfrak{p}^n}\,.$$

We can write $f(x)$ as

$$f(x) = \sum_{i=1}^{k} c_i \vartheta_{\mathrm{B}_{\leq N\mathfrak{p}^n}(a_i)}(x);\;\; c_i\in\mathbb{C},\, a_i\in K_\mathfrak{p} \text{ and } n_i\in\mathbb{Z}$$

where $\vartheta_{\mathrm{B}_{\leq N\mathfrak{p}^n}(a_i)}(x)$ is the characteristic function of $\mathrm{B}_{\leq N\mathfrak{p}^n}(a_i)$. We can regard $f(x)$ as the function of the form

$$f(x) = \sum_{i=1}^{k} c_i\, \xi_{N\mathfrak{p}^n}(x-a_i)\,.$$

where $\xi_{N\mathfrak{p}^n}$ is the the characteristic function of $\mathrm{B}_{\leq N\mathfrak{p}^n}$.

Let

$$\mathfrak{B} = \{\, f(x)\in L^2(K_\mathfrak{p}) \mid (\mathcal{F}f)(\omega) = 0,\; \forall\omega,\; |\omega|_\mathfrak{p} > \Omega \,\}.$$

Proposition 1.1. Put $N\mathfrak{p}^{-n} \leq \Omega$. Then $f(x)\in\mathfrak{B}$ has the form

$$f(x) = \sum_{i=1}^{k} c_i\, \xi_{N\mathfrak{p}^n}(x-a_i)\,.$$

Proof. Let $f(x) = \sum_{i=1}^{k} c_i\, \xi_{N\mathfrak{p}^n}(x-a_i)$. Now, $(\mathcal{F}\xi_{N\mathfrak{p}^n})(\omega) = N\mathfrak{p}^n \xi_{N\mathfrak{p}^{-n}}(\omega)$ and $(\mathcal{F}\xi_{N\mathfrak{p}^n}(x-a))(\omega) = e^{-2\pi i\Lambda(a\omega)}(\mathcal{F}\xi_{N\mathfrak{p}^n})(\omega)$. We see that

$$(\mathcal{F}f)(\omega) = \sum_{i=1}^{k} c_i\, e^{-2\pi i\Lambda(a_i\omega)} N\mathfrak{p}^n \xi_{N\mathfrak{p}^{-n}}(\omega)\,.$$

Then $(\mathcal{F}f)(\omega)$ vanishes for $|\omega|_\mathfrak{p} > N\mathfrak{p}^{-n}$.

□

Let $N\mathfrak{p}^d \leq T < N\mathfrak{p}^{d+1}$ and put

$$Df(x) = \begin{cases} f(x) \;\cdots\; |x|_\mathfrak{p} \leq T \\ \;0 \quad\;\cdots\; |x|_\mathfrak{p} > T \end{cases}$$

for $f(x)\in L^2(K_\mathfrak{p})$. Let

$$\mathfrak{D} = \{\, Df(x) \mid f(x)\in L^2(K_\mathfrak{p})\}.$$

Proposition 1.2. $Df(x)$ has the form $\sum_{i=1}^{l} c_i \xi_{N\mathfrak{p}^m}(x-a_i)$. Here

$$\mathrm{B}_{\leq N\mathfrak{p}^d} = \coprod_{i=1}^{l} a_i + \mathrm{B}_{\leq N\mathfrak{p}^m} .$$

Proof. We see that

$$Df(x) = f(x)\xi_{N\mathfrak{p}^d}(x) = \sum_{i=1}^{k} c_i \vartheta_{\mathrm{B}_{\leq N\mathfrak{p}^n}(a_i)\cap \mathrm{B}_{\leq N\mathfrak{p}^d}}(x) .$$

Choose a suitable m such that $\mathrm{B}_{\leq N\mathfrak{p}^m} \subseteq \mathrm{B}_{\leq N\mathfrak{p}^d}$ and choose a finite set of points $\{a'_i\} \subseteq \mathrm{B}_{\leq N\mathfrak{p}^d}$. It will be enable us to write down

$$c_g \vartheta_{\mathrm{B}_{\leq N\mathfrak{p}^n}(a_g)\cap \mathrm{B}_{\leq N\mathfrak{p}^d}}(x) = \sum_{i=j}^{h} c'_i \vartheta_{\mathrm{B}_{\leq N\mathfrak{p}^m}(a'_i)}(x) .$$

So we can write $\sum_{i=1}^{k} c_i \vartheta_{\mathrm{B}_{\leq N\mathfrak{p}^n}(a_i)\cap \mathrm{B}_{\leq N\mathfrak{p}^d}}(x)$ like $\sum_{i=1}^{l} c'_i \vartheta_{\mathrm{B}_{\leq N\mathfrak{p}^m}(a'_i)}(x)$. We may say that $Df(x)$ has a form

$$\sum_{i=1}^{l} c_i \vartheta_{\mathrm{B}_{\leq N\mathfrak{p}^m}(a_i)}(x) = \sum_{i=1}^{l} c_i \xi_{N\mathfrak{p}^m}(x-a_i) .$$

Here

$$\mathrm{B}_{\leq N\mathfrak{p}^d} = \coprod_{i=1}^{l} a_i + \mathrm{B}_{\leq N\mathfrak{p}^m} .$$

□

Theorem 1.1. Suppose that $Df(x)$ has the form $\sum_{i=1}^{l} c_i \xi_{N\mathfrak{p}^m}(x-a_i)$ and $-d \leq -m \leq d$. Then the Fourier transform $(\mathcal{F}Df)(\omega)$ vanishes for $|\omega|_{\mathfrak{p}} > T$.

Proof. The Fourier transform $(\mathcal{F}Df)(\omega)$ vanishes for $|\omega|_{\mathfrak{p}} > N\mathfrak{p}^{-m}$. Here $m \leq d$. So $-d \leq -m$. Moreover, let $-d \leq -m \leq d$. Then we see that $(\mathcal{F}Df)(\omega)$ vanishes for $|\omega|_{\mathfrak{p}} > T$. Namely, $Df(x)$ is a member of $\mathfrak{B}$ of $\Omega = T$.

□

2.

Definition 2.1. Let L^2_A be the class of all complex valued functions $f(t)$ defined for $-A \le t \le A$ and integrable in absolute square in the interval $(-A, A)$.

Given any $T>0$ and any $\Omega>0$, we can find a countably infinite set of real functions $\psi_0(t), \psi_1(t), \psi_2(t), \cdots$ and a set of real positive numbers

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots$$

with the following properties:

i. The $\psi_i(t)$ are bandlimited, i.e. its Fourier transform $(\mathcal{F}\psi_i)(\omega)$ vanishes for $|\omega|> \Omega$; orthogonal on the real line and complete in $\mathfrak{B} = \{ f(t) \in L^2(\mathbb{R}) \mid (\mathcal{F}f)(\omega) = 0, \forall\omega, |\omega|>\Omega \}$:

$$\int_{-\infty}^{\infty} \psi_i(t)\psi_j(t)\,dt = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases} \qquad i, j = 0, 1, 2, \cdots .$$

ii. In the interval $-T/2 \le t \le T/2$, the ψ_i are orthogonal and complete in $L^2_{T/2}$:

$$\int_{-T/2}^{T/2} \psi_i(t)\psi_j(t)\,dt = \begin{cases} 0 & i \ne j \\ \lambda_i & i = j \end{cases} \qquad i, j = 0, 1, 2, \cdots .$$

iii. For all values of t, real or complex,

$$\lambda_i \psi_i(t) = \int_{-T/2}^{T/2} \frac{\sin(\Omega(t-s))}{\pi(t-s)}\,\psi_i(s)\,ds \quad i = 0, 1, 2, \cdots .$$

Both the ψ's and the λ's are functions of $c = \Omega T/2$. In order to make this dependence explicit, we write

$$\lambda_i = \lambda_i(c), \ \psi_i(t) = \psi_i(c, t), \quad i = 0, 1, 2, \cdots.$$

Put

$$a_n = (f, \psi_n(c, t))_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(t)\psi_n(c, t)\,dt .$$

We shall call $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$ *the Fourier series expansion of* $f(t)$. Let $f(t) \in L^2(\mathbb{R})$ and let $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$ be the Fourier series expansion of $f(t)$:

$$f(t) \sim \sum_{n=0}^{\infty} a_n\psi_n(c, t) \ \ t \in \mathbb{R}.$$

Since $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$ doesn't always converge and it doesn't always coincide with $f(t)$, we shall use "$\sim$". We can calculate as follows;

$$0 \le \| f(t) - \sum_{n=0}^{N} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}^2$$

$$= \| f(t) \|_{L^2(\mathbb{R})}^2 - 2(f(t), \sum_{n=0}^{N} a_n\psi_n(c, t))_{L^2(\mathbb{R})} + (\sum_{n=0}^{N} a_n\psi_n(c, t), \sum_{n=0}^{N} a_n\psi_n(c, t))_{L^2(\mathbb{R})}$$

$$= \| f(t) \|_{L^2(\mathbb{R})}{}^2 - 2\sum_{n=0}^{N} (f(t), a_n\psi_n(c, t))_{L^2(\mathbb{R})} + \sum_{m,n=0}^{N} (a_m\psi_m(c, t), a_n\psi_n(c, t))_{L^2(\mathbb{R})}$$
$$= \| f(t) \|_{L^2(\mathbb{R})}{}^2 - 2\sum_{n=0}^{N} |a_n|^2 + \sum_{n=0}^{N} |a_n|^2$$
$$= \| f(t) \|_{L^2(\mathbb{R})}{}^2 - \sum_{n=0}^{N} |a_n|^2 .$$

Thus

$$\| f(t) \|_{L^2(\mathbb{R})}{}^2 \geq \sum_{n=0}^{N} |a_n|^2$$

and

$$\| f(t) - \sum_{n=0}^{N} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 = \| f(t) \|_{L^2(\mathbb{R})}{}^2 - \sum_{n=0}^{N} |a_n|^2 .$$

When $N \longrightarrow \infty$,

$$\| f(t) \|_{L^2(\mathbb{R})}{}^2 \geq \sum_{n=0}^{\infty} |a_n|^2$$

and

$$\lim_{N\to\infty} \| f(t) - \sum_{n=0}^{N} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 = \| f(t) \|_{L^2(\mathbb{R})}{}^2 - \sum_{n=0}^{\infty} |a_n|^2 .$$

We can consider

$$\lim_{N\to\infty} \| f(t) - \sum_{n=0}^{N} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 = \| f(t) - \sum_{n=0}^{\infty} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 .$$

It must be instructive that we can't show $\| f(t) - \sum_{n=0}^{\infty} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 = \| f(t) \|_{L^2(\mathbb{R})}{}^2 - \sum_{n=0}^{\infty} |a_n|^2$ directly. Now, we see that finite sums $f_N(t) = \sum_{n=0}^{N} a_n\psi_n(c, t)$ permit approximations to $f(t)$ by bandlimited functions, i.e. $f_N(t)$. Let $f(t) \in \mathfrak{B}$

$$\lim_{N\to\infty} \| f(t) - \sum_{n=0}^{N} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 = \| f(t) \|_{L^2(\mathbb{R})}{}^2 - \sum_{n=0}^{\infty} |a_n|^2 = 0$$

since the $\psi_n(c, t)$ are complete in $\mathfrak{B}$. So, $\{f_N(t)\}$ converges to $f(t)$ in L^2 norm. Then $f(t)$ can be integrable term by term, and

$$\int_{-T/2}^{T/2} f(t)\psi_n(c, t)dt = \int_{-T/2}^{T/2} \sum_{i=0}^{\infty} a_i\psi_i(c, t)\, \psi_n(c, t)dt$$
$$= \sum_{i=0}^{\infty} \int_{-T/2}^{T/2} a_i\psi_i(c, t)\psi_n(c, t)dt = \lambda_n(c)a_n .$$

Proposition 2.1. Let $f(t) \in L^2(\mathbb{R})$ and suppose that $f(t)$ is not a bandlimited function. Let $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$ be the Fourier series expansion of $f(t)$:

$$f(t) \sim \sum_{n=0}^{\infty} a_n\psi_n(c, t), \quad a_n = \int_{-\infty}^{\infty} f(t)\psi_n(c, t)dt \text{ and } t \in \mathbb{R}.$$

Then $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ and there exists a function $h(t) = \sum_{n=0}^{\infty} a_n\psi_n(c, t)$ of $\mathfrak{B}$ but $f(t) \neq h(t)$.

Proof. It holds that $\| f(t) \|_{L^2(\mathbb{R})}{}^2 \geq \sum_{n=0}^{\infty} |a_n|^2$. So $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ because $\| f(t) \|_{L^2(\mathbb{R})}{}^2 < \infty$. Thus there exists a function $h(t) = \sum_{n=0}^{\infty} a_n\psi_n(c, t)$ of $\mathfrak{B}$. But $f(t) \neq h(t)$ since $f(t)$ isn't bandlimited.

□

An interesting argument is given by D. Slepian and H.O. Pollak. Let $f(t) \in L^2{}_{T/2}$. Then

$$f(t) = \sum_{n=0}^{\infty} a_n\psi_n(c, t), \quad a_n = 1/\lambda_n(c) \int_{-T/2}^{T/2} f(t)\psi_n(c, t)dt .$$

The ψ_i are orthogonal and complete in $L^2{}_{T/2}$,

$$\| f(t) \|_{L^2{}_{T/2}}{}^2 = \sum_{n=0}^{\infty} \lambda_n(c)|a_n|^2 < \infty.$$

Let

$$h(t) = \sum_{n=0}^{\infty} a_n \psi_n(c, t), \quad t \in \mathbb{R}.$$

Namely $f(t)$ is a piece of a function $h(t)$. Suppose that $\sum_{n=0}^{\infty} a_n \psi_n(c, t)$ converges. It means $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. We can consider that $h(t)$ is integrable term by term, then $a_n = \int_{-\infty}^{\infty} h(t)\psi_n(c, t)dt$. The series $\sum_{n=0}^{\infty} a_n \psi_n(c, t)$ is the Fourier series expansion of $h(t)$ and $h(t)$ is bandlimited. On the other hand, if $\sum_{n=0}^{\infty} a_n \psi_n(c, t)$ doesn't converge then $\sum_{n=0}^{N} |a_n|^2$ grows without bound for increasing N. The function $h(t)$ can not be band-limited. We shall consider that $\sum_{n=0}^{\infty} a_n \psi_n(c, t)$ is also the Fourier series expansion of $h(t)$. Namely, $\sum_{n=0}^{\infty} a_n \psi_n(c, t)$ is the "formal" Fourier series expansion of non-band-limited function $h(t)$. Here $\lambda_0(c) > \lambda_1(c) > \lambda_2(c) > \cdots$. The $\lambda_n(c)$ approach zero rapidly for sufficient large n. Thus it may be happen that $\sum_{n=0}^{N} |a_n|^2$ grows without bound for increasing N but $\sum_{n=0}^{\infty} \lambda_n(c)|a_n|^2$ converges.

For any function $f(t) \in L^2(\mathbb{R})$, put

$$Df(t) = \begin{cases} f(t) & \cdots \ |t| \le T/2 \\ 0 & \cdots \ T/2 < |t| \end{cases}.$$

$Df(t)$ isn't bandlimited in general. We will think of approximations to $Df(t)$ by band-limited functions $f_N(t) = \sum_{n=0}^{N} a_n \psi_n(c, t)$. Here

$$\begin{aligned} &\| Df(t) - \sum_{n=0}^{N} a_n \psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 \\ &= \| Df(t) \|_{L^2(\mathbb{R})}{}^2 - 2(Df(t), \sum_{n=0}^{N} a_n \psi_n(c, t))_{L^2(\mathbb{R})} + (\sum_{n=0}^{N} a_n \psi_n(c, t), \sum_{n=0}^{N} a_n \psi_n(c, t))_{L^2(\mathbb{R})} \\ &= \| Df(t) \|_{L^2(\mathbb{R})}{}^2 - 2\sum_{n=0}^{N} (Df(t), a_n \psi_n(c, t))_{L^2(\mathbb{R})} + \sum_{m,n=0}^{N} (a_m \psi_m(c, t), a_n \psi_n(c, t))_{L^2(\mathbb{R})} \\ &= \| Df(t) \|_{L^2(\mathbb{R})}{}^2 - 2\sum_{n=0}^{N} \bar{a}_n (Df(t), \psi_n(c, t))_{L^2(\mathbb{R})} + \sum_{n=0}^{N} |a_n|^2. \end{aligned}$$

Let $f(t) \in L^2{}_{T/2}$. Then

$$f(t) = \sum_{n=0}^{\infty} a_n \psi_n(c, t), \quad a_n = 1/\lambda_n(c) \int_{-T/2}^{T/2} f(t)\psi_n(c, t)dt.$$

Now

$$\int_{-T/2}^{T/2} f(t)\psi_n(c, t)dt = \int_{-\infty}^{\infty} Df(t)\psi_n(c, t)dt.$$

Thus

$$\int_{-\infty}^{\infty} Df(t)\psi_n(c, t)dt = \lambda_n(c)a_n.$$

We can obtain the Fourier series expansion of $Df(t)$:

$$Df(t) \sim \sum_{n=0}^{\infty} \lambda_n(c)a_n \cdot \psi_n(c, t).$$

We shall adopt $\sum_{n=0}^{\infty} \lambda_n(c)a_n \cdot \psi_n(c, t)$. Then

$$\| Df(t) - \sum_{n=0}^{N} \lambda_n(c)a_n \cdot \psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 = \| Df(t) \|_{L^2(\mathbb{R})}{}^2 - \sum_{n=0}^{N} \lambda_n(c)^2 |a_n|^2.$$

When $N \longrightarrow \infty$,

$$\| Df(t) - \sum_{n=0}^{\infty} \lambda_n(c)a_n \cdot \psi_n(c, t) \|_{L^2(\mathbb{R})}{}^2 = \| Df(t) \|_{L^2(\mathbb{R})}{}^2 - \sum_{n=0}^{\infty} \lambda_n(c)^2 |a_n|^2.$$

Here,

$$\| Df(t) \|_{L^2(\mathbb{R})}^{\ 2} = \int_{-\infty}^{\infty} Df(t)\overline{Df(t)}\,dt = \int_{-T/2}^{T/2} f(t)\overline{f(t)}\,dt = \| f(t) \|_{L^2_{T/2}}^{\ 2}.$$

The ψ_i are orthogonal and complete in $L^2{}_{T/2}$, so $\| f(t) \|_{L^2_{T/2}}^{\ 2} = \sum_{n=0}^{\infty} \lambda_n(c)|a_n|^2 < \infty$. Thus

$$\| Df(t) \|_{L^2(\mathbb{R})}^{\ 2} = \sum_{n=0}^{\infty} \lambda_n(c)|a_n|^2 .$$

We can say that

$$\| Df(t) - \sum_{n=0}^{\infty} \lambda_n(c)a_n \cdot \psi_n(c, t) \|_{L^2(\mathbb{R})}^{\ 2} = \sum_{n=0}^{\infty} \lambda_n(c)|a_n|^2 - \sum_{n=0}^{\infty} \lambda_n(c)^2|a_n|^2 ,$$

from the proposition 2.1, $\sum_{n=0}^{\infty} \lambda_n(c)^2|a_n|^2 < \infty$, so

$$= \sum_{n=0}^{\infty} \lambda_n(c)(1-\lambda_n(c))|a_n|^2 .$$

Consider the proposition 2.1, we see that $\sum_{n=0}^{\infty} \lambda_n(c)a_n \cdot \psi_n(c, t)$ is bandlimited but $Df(t) \neq \sum_{n=0}^{\infty} \lambda_n(c)a_n \cdot \psi_n(c, t)$.

On the other hand, there exists another function

$$h(t) = \sum_{n=0}^{\infty} a_n\psi_n(c, t) \quad t \in \mathbb{R}.$$

We will adopt it. Then

$$\| Df(t) - \sum_{n=0}^{N} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}^{\ 2} = \| Df(t) \|_{L^2(\mathbb{R})}^{\ 2} - 2\sum_{n=0}^{N} \lambda_n(c)|a_n|^2 + \sum_{n=0}^{N} |a_n|^2 .$$

Therefore,

$$\| Df(t) - \sum_{n=0}^{\infty} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}^{\ 2} = \| Df(t) \|_{L^2(\mathbb{R})}^{\ 2} - 2\sum_{n=0}^{\infty} \lambda_n(c)|a_n|^2 + \sum_{n=0}^{\infty} |a_n|^2$$

$$= \sum_{n=0}^{\infty} |a_n|^2 - \sum_{n=0}^{\infty} \lambda_n(c)|a_n|^2 .$$

(i) If $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$ converges then

$$\| Df(t) - \sum_{n=0}^{\infty} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}^{\ 2} = \sum_{n=0}^{\infty} (1-\lambda_n(c))|a_n|^2 .$$

Here $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$ is bandlimited but $Df(t) \neq \sum_{n=0}^{\infty} a_n\psi_n(c, t)$.

(ii) If $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$ doesn't converge then $\sum_{n=0}^{N} |a_n|^2$ grows without bound for increasing N and

$$\| Df(t) - \sum_{n=0}^{\infty} a_n\psi_n(c, t) \|_{L^2(\mathbb{R})}^{\ 2} \text{ diverges.}$$

So $Df(t) \neq \sum_{n=0}^{\infty} a_n\psi_n(c, t)$.

Theorem 2.1. $Df(t)$ can't have the form $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$. Namely $Df(t)$ can't be bandlimited even in a sense "formally".

Proof. Suppose that $Df(t)$ has the form $\sum_{n=0}^{\infty} a_n\psi_n(c, t)$. Then

$$f(t) = \sum_{n=0}^{\infty} a_n\psi_n(c, t), \quad a_n = 1/\lambda_n(c)\int_{-T/2}^{T/2} f(t)\psi_n(c, t)\,dt .$$

for $f(t) \in L^2{}_{T/2}$ since the restricted $Df(t)$ to the interval $[-T/2, T/2]$ is $f(t)$. However it is impossible for $Df(t)$ to have such a form according to the above argument. □

Let $f(z)\in L^2(\mathbb{C})$. We will think of the Fourier transform

$$(\mathcal{F}f)(\omega) = \int_{\mathbb{C}} f(z)e^{-2\pi i\Lambda(\omega z)}\,dz\,.$$

Set $z = x + iy$ and $dz = 2dxdy$. Then

$$(\mathcal{F}f)(\omega) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+iy)e^{-2\pi i\Lambda(\omega(x+iy))}dxdy\,.$$

Let $\omega = \mu + i\nu$. $\Lambda(\omega(x+iy)) = -2(\mu x - \nu y) \bmod 1$. Now $\Lambda(\omega x) = -2\mu x \bmod 1$ and $\Lambda(\omega iy) = 2\nu y \bmod 1$. It holds that $e^{-2\pi i\Lambda(\omega(x+iy))} = e^{-2\pi i\Lambda(\omega x)}e^{-2\pi i\Lambda(\omega iy))}$. We can compute as follows;

$$\begin{aligned}(\mathcal{F}f)(\omega) &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+iy)e^{-2\pi i\Lambda(\omega x)}e^{-2\pi i\Lambda(\omega iy)}dxdy \\ &= 2\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x+iy)e^{-2\pi i\Lambda(i\omega y)}\,dy\right)e^{-2\pi i\Lambda(\omega x)}\,dx\,.\end{aligned}$$

Denote $\int_{-\infty}^{\infty} f(x+iy)e^{-2\pi i\Lambda(i\omega y)}\,dy$ by $(\mathcal{F}_y f)(i\omega)$. We denote $(\mathcal{F}f)(\omega)$ as follows;

$$\begin{aligned}(\mathcal{F}f)(\omega) &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+iy)e^{-2\pi i\Lambda(\omega x)}e^{-2\pi i\Lambda(\omega iy)}dxdy \\ &= 2\int_{-\infty}^{\infty}(\mathcal{F}_y f)(i\omega)e^{-2\pi i\Lambda(\omega x)}\,dx = 2(\mathcal{F}_x(\mathcal{F}_y f)(i\omega))(\omega).\end{aligned}$$

Definition 3.1.

$$\mathfrak{B} = \{\, f(x+iy)\in L^2(\mathbb{C}) \mid (\mathcal{F}_x(\mathcal{F}_y f)(i\omega))(\omega) = 0,\ \forall\omega,\ |\omega|>\Omega\}.$$

We shall call the member of $\mathfrak{B}$ “bandlimited”.

Lemma 3.1. Fix a positive real number Ω. Let $c = T/2\cdot 2\Omega$. The Fourier transform $\int_{-\infty}^{\infty}\psi_n(c,t)e^{-2\pi i\Lambda(\omega t)}\,dt$ of $\psi_n(c, t)$ vanishes for $|\omega| > \Omega$.

Proof. Let $\omega = \mu + i\nu$. Then

$$\int_{-\infty}^{\infty}\psi_n(c,t)e^{-2\pi i\Lambda(\omega t)}\,dt = \int_{-\infty}^{\infty}\psi_n(c,t)e^{-2\pi i\Lambda((\mu+i\nu)t)}\,dt = \int_{-\infty}^{\infty}\psi_n(c,t)e^{-2\pi i(-2\mu)t}\,dt\,.$$

Thus $\int_{-\infty}^{\infty}\psi_n(c,t)e^{-2\pi i\Lambda(\omega t)}\,dt$ vanishes for $|2\mathrm{Re}\omega|>2\Omega$. If $|\omega|\le\Omega$ then $|\mathrm{Re}\omega|\le\Omega$. Thus if $|\mathrm{Re}\omega| > \Omega$ then $|\omega| > \Omega$. Since $\int_{-\infty}^{\infty}\psi_n(c,t)e^{-2\pi i\Lambda(\omega t)}\,dt$ vanishes for $|\mathrm{Re}\omega|>\Omega$, it vanishes for $|\omega| > \Omega$.

□

Lemma 3.2. If $(\mathcal{F}_x f)(\omega)$ vanishes for $|\omega| > \Omega$ then $f(z) = f(x+iy)$ has the Fourier series expansion $\sum_{n=0}^{\infty} a_n\psi_n(c, x)$. If $f(z) = f(x+iy)$ has the Fourier series expansion $\sum_{n=0}^{\infty} a_n\psi_n(c, x)$ then $(\mathcal{F}_x f)(\omega)$ vanishes for $|\omega| > \Omega$.

Proof. Since $f(z)\in L^2(\mathbb{C})$; the function $f(x+iy)$, as a function of x, is considered to be integrable in absolute square. Suppose that $(\mathcal{F}_x f)(\omega)$ vanishes for $|\omega| > \Omega$, namely $f(x+iy)$ is “bandlimited”. From the lemma, $\psi_n(c, x)$ is also “bandlimited”. Therefore $f(x+iy)$ has the Fourier series expansion:

$$\sum_{n=0}^{\infty} a_n \psi_n(c, x), \quad a_n = \int_{-\infty}^{\infty} f(x+iy)\psi_n(c, x)dx .$$

Suppose that $f(z) = f(x + iy)$ has the Fourier series expansion $\sum_{n=0}^{\infty} a_n \psi_n(c, x)$. Then $(\mathcal{F}_x f)(\omega)$ vanishes for $|\omega| > \Omega$ since $\int_{-\infty}^{\infty} \psi_n(c, x) e^{-2\pi i \Lambda(\omega x)} dx$ of $\psi_n(c, x)$ vanishes for $|\omega| > \Omega$.

□

Since $f(z) \in L^2(\mathbb{C})$; the function $(\mathcal{F}_y f)(i\omega)$, as a function of x, is considered to be integrable in absolute square. According to the above arguments, we can say as follows;

Proposition 3.1. Let $f(z) \in L^2(\mathbb{C})$.
$f(z) \in \mathfrak{B}$ if and only if $(\mathcal{F}_y f)(i\omega)$ has its Fourier series expansion $\sum_{n=0}^{\infty} a_n \psi_n(c, x)$.

For any function $f(z) \in L^2(\mathbb{C})$, put

$$Df(z) = \begin{cases} f(z) & \cdots \ |z| \leq T/2 \\ 0 & \cdots \ T/2 < |z| \end{cases} .$$

Set $z = x + iy$ and think of $Df(x + iy)$. Consider it as a function of x. If $T/2 < |x|$ then $T/2 < |z|$. Thus $Df(x + iy)$ vanishes for $|x| > T/2$. Here

$$(\mathcal{F}_y Df)(i\omega) = \int_{-\infty}^{\infty} Df(x+iy) e^{-2\pi i \Lambda(i\omega y)} dy .$$

It also vanishes for $|x| > T/2$. We can apply the case of $\mathbb{R}$ to this case.

Theorem 3.1. $Df(z)$ can't be bandlimited even in a sense "formally".

The ring of adeles is defined as

$$\mathbb{A}_K = {\prod}'_{\mathfrak{p}<\infty} K_{\mathfrak{p}} \times \prod_{\mathfrak{p}\mid\infty} K_{\mathfrak{p}}$$

Denote the ring of integers of $K_{\mathfrak{p}}$ by $\mathcal{O}_{\mathfrak{p}}$.

$${\prod}'_{\mathfrak{p}<\infty} K_{\mathfrak{p}} = \{(r_{\mathfrak{p}}) \in \prod_{\mathfrak{p}<\infty} K_{\mathfrak{p}} \mid r_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}} \textit{ for almost all } \mathfrak{p}\}.$$

The number field K has d_1 real conjugate fields and $2d_2$ imaginary conjugate fields. Here $n = d_1 + 2d_2$. The field K has $d_1 + d_2$ infinite places. Set

$K_{\mathfrak{p}} = \mathbb{R}$ for d_1 infinite places and $K_{\mathfrak{p}} = \mathbb{C}$ for d_2 infinite places.

Therefore

$$\prod_{\mathfrak{p}\mid\infty} K_{\mathfrak{p}} = \mathbb{R}^{d_1} \times \mathbb{C}^{d_2} \cong \mathbb{R}^n.$$

Denote the set of infinite places by $S_\infty = \{\mathfrak{p}_{\infty 1}, \cdots, \mathfrak{p}_{\infty d_1}; \mathfrak{p}_{\infty d_1+1}, \cdots, \mathfrak{p}_{\infty d}\}$.

For each of places $\mathfrak{p}$, let $dr_{\mathfrak{p}}$ be a Haar measure on $K_{\mathfrak{p}}$ such that

$$\int_{\mathcal{O}_{\mathfrak{p}}} dr_{\mathfrak{p}} = 1 \textit{ for almost all } \mathfrak{p}.$$

Then we can write a Haar measure dr on $\mathbb{A}_K$ like $dr = \Pi_{\mathfrak{p}} dr_{\mathfrak{p}}$. Let $f(r)$ be a complex valued function on $\mathbb{A}_K$. For each of places $\mathfrak{p}$, if $f_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}) = \{1\}$ for almost all $\mathfrak{p}$, then we can write $f(r)$ like $f(r) = \Pi_{\mathfrak{p}} f_{\mathfrak{p}}(r_{\mathfrak{p}})$ similarly.

Definition 4.1.

$$L^1(\mathbb{A}_K) = \{f(r) = \Pi_{\mathfrak{p}} f_{\mathfrak{p}}(r_{\mathfrak{p}}) \mid f_{\mathfrak{p}}(r_{\mathfrak{p}}) \in L^1(K_{\mathfrak{p}}) \text{ and } f_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}) = \{1\} \textit{ for almost all } \mathfrak{p}\}.$$

Proposition 4.1. $\mathbb{A}_K$ and $\hat{\mathbb{A}}_K$ are isomorphic by the map

$$\begin{array}{ccc} \mathbb{A}_K & \longrightarrow & \hat{\mathbb{A}}_K \\ \Cup & & \Cup \\ \eta & \longmapsto & e^{2\pi i \Lambda(\eta r)} \end{array}.$$

Since $\Lambda_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}) = \{0\}$,

$$e^{2\pi i \Lambda(\eta r)} = \exp(2\pi i \Sigma_{\mathfrak{p}} \Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}} r_{\mathfrak{p}})) = \Pi_{\mathfrak{p}}\, e^{2\pi i \Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}} r_{\mathfrak{p}})}.$$

Proposition 4.2. The Fourier transform of $f(r) \in L^1(\mathbb{A}_K)$ is defined by

$$(\mathcal{F}f)(\eta) = \int_{\mathbb{A}_K} f(r) e^{-2\pi i \Lambda(\eta r)}\, dr.$$

The inverse Fourier transform is that

$$f(r) = \int_{\mathbb{A}_K} (\mathcal{F}f)(\eta) e^{2\pi i \Lambda(r\eta)}\, d\eta.$$

It holds that

$$\int_{\mathbb{A}_K} f(r)e^{-2\pi i\Lambda(\eta r)}\,dr \;=\prod\nolimits_{\mathfrak{p}}\int_{K_{\mathfrak{p}}} f_{\mathfrak{p}}(r_{\mathfrak{p}})e^{-2\pi i\Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}} r_{\mathfrak{p}})}\,dr_{\mathfrak{p}}\,.$$

Denote the Schwartz-Bruhat space on $\mathbb{A}_K$ by $\mathcal{S}(\mathbb{A}_K)$. We define a function of the space as linear combinations of the product $\prod_{\mathfrak{p}} f_{\mathfrak{p}}(r_{\mathfrak{p}})$ where $f_{\mathfrak{p}_{\infty i}}\in\mathcal{S}(\mathbb{R}^m)$, $f_{\mathfrak{p}}\in\mathcal{S}(K_{\mathfrak{p}})$ and $f_{\mathfrak{p}}$ is the characteristic function $\xi_{N\mathfrak{p}^0}$ of $\mathcal{O}_{\mathfrak{p}}$ for all but finitely many $\mathfrak{p}$. We will regard $L^2(\mathbb{A}_K)$ as the completion of $\mathcal{S}(\mathbb{A}_K)$. Let S be some finite set $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}\cup\mathrm{S}_\infty$. Set

$$\mathbb{A}_{\mathrm{S}} = \prod\nolimits_{\mathfrak{p}\in\mathrm{S}} K_{\mathfrak{p}} \times \prod\nolimits_{\mathfrak{p}\notin\mathrm{S}} \mathcal{O}_{\mathfrak{p}}\,.$$

Let $\mathbb{A}^{\mathrm{S}} = \prod_{\mathfrak{p}\in\mathrm{S}}\{1\}\times\prod_{\mathfrak{p}\notin\mathrm{S}}\mathcal{O}_{\mathfrak{p}}$. $\mathbb{A}^{\mathrm{S}}$ is a compact subgroup of $\mathbb{A}_{\mathrm{S}}$. We shall identify $K_{\mathfrak{p}}$ ($\mathfrak{p}<\infty$) with $K_{\mathfrak{p}}\times\prod_{\mathfrak{p}'\in\mathfrak{p}}\{1\}$. Then we can decompose $\mathbb{A}_{\mathrm{S}}$ as follows;

$$\mathbb{A}_{\mathrm{S}} = \prod\nolimits_{\mathfrak{p}\in\mathrm{S}} K_{\mathfrak{p}} \times \mathbb{A}^{\mathrm{S}}.$$

We see that

$$\mathbb{A}_K = \cup_{\mathrm{S}}\, \mathbb{A}_{\mathrm{S}}\,.$$

For any function $f(r)\in L^2(\mathbb{A}_K)$, we will consider it as a function on $\mathbb{A}_{\mathrm{S}}$.

Let $f(r)\in L^2(\mathbb{A}_K)$, as a function on $\mathbb{A}_{\mathrm{S}}$,

$$f(r) = \prod\nolimits_{\mathfrak{p}\in\mathrm{S}} f_{\mathfrak{p}}(r_{\mathfrak{p}})\times\prod\nolimits_{\mathfrak{p}\notin\mathrm{S}}\xi_{N\mathfrak{p}^0}(r_{\mathfrak{p}})\,.$$

The Fourier transform of $f(r)$ will be

$$(\mathcal{F}f)(\eta) = \prod\nolimits_{\mathfrak{p}\in\mathrm{S}}(\mathcal{F}f_{\mathfrak{p}})(\eta_{\mathfrak{p}})\times\prod\nolimits_{\mathfrak{p}\notin\mathrm{S}}(\mathcal{F}\xi_{N\mathfrak{p}^0})(\eta_{\mathfrak{p}})\,.$$

Let $r\in\mathbb{A}_K$. We will think of $r = (\,r_{\mathfrak{p}}\,)_{\mathfrak{p}\leq\infty}\in\mathbb{A}_{\mathrm{S}}$ where $\mathrm{S} = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}\cup\mathrm{S}_\infty$. Its absolute value will be

$$\begin{aligned}|r| &= |r_{\mathfrak{p}_1}|_{\mathfrak{p}_1}\cdots\,|r_{\mathfrak{p}_k}|_{\mathfrak{p}_k}\cdot\prod\nolimits_{\mathfrak{p}\notin\mathrm{S}}|r_{\mathfrak{p}}|_{\mathfrak{p}}\cdot\prod\nolimits_{\mathfrak{p}_\infty\in\mathrm{S}_\infty}|r_{\mathfrak{p}_\infty}|_{\mathfrak{p}_\infty}\\ &= N\mathfrak{p}_1^{\,n_1}N\mathfrak{p}_2^{\,n_2}\,\cdots\,N\mathfrak{p}_k^{\,n_k}\cdot\prod\nolimits_{\mathfrak{p}\notin\mathrm{S}}N\mathfrak{p}^{n_{\mathfrak{p}}}\cdot\prod\nolimits_{\mathfrak{p}_\infty\in\mathrm{S}_\infty}t_{\mathfrak{p}_\infty}\end{aligned}$$

where $n_i\in\mathbb{Z}$, $n_{\mathfrak{p}}\leq 0$ for $\mathfrak{p}\notin\mathrm{S}$ and $t_{\mathfrak{p}_\infty}\in\mathbb{R}$. If $|r|\neq 0$ then we will see that $n_{\mathfrak{p}}=0$ for almost places $\mathfrak{p}\notin\mathrm{S}$. Let

$$\mathfrak{B} = \{\,f(r)\in L^2(\mathbb{A}_K)\,|\,(\mathcal{F}f)(\eta) = 0,\ |\eta|>\Omega\}.$$

Definition 4.2. For a given $\Omega>0$, let $\Omega = N\mathfrak{p}_1^{\,n_1}\cdots N\mathfrak{p}_k^{\,n_k}\cdot\prod_{\mathfrak{p}\notin\mathrm{S}}N\mathfrak{p}^{n_{\mathfrak{p}}}\cdot\prod_{\mathfrak{p}_\infty\in\mathrm{S}_\infty}t_{\mathfrak{p}_\infty}$. If $(\mathcal{F}f_{\mathfrak{p}_i})(\eta_{\mathfrak{p}_i}) = 0$ for $\eta_{\mathfrak{p}_i}$ $|\eta_{\mathfrak{p}_i}|_{\mathfrak{p}_i}>N\mathfrak{p}_i^{\,n_i}$, $(\mathcal{F}f_{\mathfrak{p}})(\eta_{\mathfrak{p}}) = 0$ for $\eta_{\mathfrak{p}}$ $|\eta_{\mathfrak{p}}|_{\mathfrak{p}}>N\mathfrak{p}^{n_{\mathfrak{p}}}$ $\mathfrak{p}\notin\mathrm{S}$ and $(\mathcal{F}f_{\mathfrak{p}_\infty})(\eta_{\mathfrak{p}_\infty}) = 0$ for $\eta_{\mathfrak{p}_\infty}$ $|\eta_{\mathfrak{p}_\infty}|_{\mathfrak{p}_\infty}>t_{\mathfrak{p}_\infty}$ then $f(r)\in\mathfrak{B}$.

Let

$$\mathfrak{D} = \{\,f(r)\in L^2(\mathbb{A}_K)\,|\,f(r) = 0,\ |r|>T\}.$$

Definition 4.3. For a given $T>0$, let $T = N\mathfrak{p}_1^{\,h_1}\cdots N\mathfrak{p}_k^{\,h_k}\cdot\prod_{\mathfrak{p}\notin\mathrm{S}}N\mathfrak{p}^{h_{\mathfrak{p}}}\cdot\prod_{\mathfrak{p}_\infty\in\mathrm{S}_\infty}s_{\mathfrak{p}_\infty}$. Put

$$Df_{\mathfrak{p}_i}(r_{\mathfrak{p}_i}) = \begin{cases} f_{\mathfrak{p}_i}(r_{\mathfrak{p}_i}) & \cdots\ |r_{\mathfrak{p}_i}|_{\mathfrak{p}_i}\leq N\mathfrak{p}_i^{\,h_i}\\ 0 & \cdots\ |r_{\mathfrak{p}_i}|_{\mathfrak{p}_i}>N\mathfrak{p}_i^{\,h_i}\end{cases},\qquad Df_{\mathfrak{p}}(r_{\mathfrak{p}}) = \begin{cases} f_{\mathfrak{p}}(r_{\mathfrak{p}}) & \cdots\ |r_{\mathfrak{p}}|_{\mathfrak{p}}\leq N\mathfrak{p}^{h_{\mathfrak{p}}}\\ 0 & \cdots\ |r_{\mathfrak{p}}|_{\mathfrak{p}}>N\mathfrak{p}^{h_{\mathfrak{p}}}\end{cases}$$

and

$$Df_{\mathfrak{p}_\infty}(r_{\mathfrak{p}_\infty}) = \begin{cases} f_{\mathfrak{p}_\infty}(r_{\mathfrak{p}_\infty}) & \cdots \ |r_{\mathfrak{p}_\infty}|_{\mathfrak{p}_\infty} \le s_{\mathfrak{p}_\infty} \\ 0 & \cdots \ |r_{\mathfrak{p}_\infty}|_{\mathfrak{p}_\infty} > s_{\mathfrak{p}_\infty} \end{cases} .$$

Then $\prod_{\mathfrak{p}} Df_{\mathfrak{p}}(r_{\mathfrak{p}}) \in \mathfrak{D}$.

Let $T = N\mathfrak{p}_1^{h_1} \cdots N\mathfrak{p}_k^{h_k} \cdot \prod_{\mathfrak{p} \notin S} N\mathfrak{p}^{h_{\mathfrak{p}}} \cdot \prod_{\mathfrak{p}_\infty \in S_\infty} s_{\mathfrak{p}_\infty}$ and let $Df(r) \in \mathfrak{D}$ for the given T.

(1) For the places of $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\} \subseteq S$,

$$Df_{\mathfrak{p}_i}(r_{\mathfrak{p}_i}) = \sum_{g=1}^{l_i} c_g \xi_{N\mathfrak{p}_i^{m_i}}(r_{\mathfrak{p}_i} - a_g)$$

and

$$(\mathcal{F}Df_{\mathfrak{p}_i})(\eta_{\mathfrak{p}_i}) = \sum_{g=1}^{l_i} c_g e^{-2\pi i \Lambda(a_g \eta_{\mathfrak{p}_i})} N\mathfrak{p}^{m_i} \xi_{N\mathfrak{p}_i^{-m_i}}(\eta_{\mathfrak{p}_i}).$$

$(\mathcal{F}Df_{\mathfrak{p}_i})(\eta_{\mathfrak{p}_i})$ vanishes for $|\eta_{\mathfrak{p}_i}| > N\mathfrak{p}_i^{-m_i}$.

(2) For the places $\mathfrak{p} \notin S$,

$$Df_{\mathfrak{p}}(r_{\mathfrak{p}}) = \begin{cases} \xi_{N\mathfrak{p}^{h_{\mathfrak{p}}}}(r_{\mathfrak{p}}) & \cdots \ h_{\mathfrak{p}} \le 0 \\ \xi_{N\mathfrak{p}^{0}}(r_{\mathfrak{p}}) & \cdots \ h_{\mathfrak{p}} > 0 \end{cases} .$$

Put $\{h_{\mathfrak{p}}\} = h_{\mathfrak{p}}$ if $h_{\mathfrak{p}} \le 0$ and $\{h_{\mathfrak{p}}\} = 0$ if $h_{\mathfrak{p}} > 0$. Then

$(\mathcal{F}Df_{\mathfrak{p}})(\eta_{\mathfrak{p}}) = N\mathfrak{p}^{\{h_{\mathfrak{p}}\}} \xi_{N\mathfrak{p}^{-\{h_{\mathfrak{p}}\}}}(\eta_{\mathfrak{p}})$ and it vanishes for $|\eta_{\mathfrak{p}}| > N\mathfrak{p}^{-\{h_{\mathfrak{p}}\}}$.

(3) For the places $\mathfrak{p}_\infty \in S_\infty$,

$Df_{\mathfrak{p}_\infty}(r_{\mathfrak{p}_\infty})$ can't be bandlimited. Only $0(r_{\mathfrak{p}_\infty})$ can be bandlimited.

Then

$(\mathcal{F}0)(\eta_{\mathfrak{p}_\infty})$ vanishes for $|\eta_{\mathfrak{p}_\infty}| > t_{\mathfrak{p}_\infty}$ where $t_{\mathfrak{p}_\infty}$ is an arbitrary positive real number.

Let

$$Df(t) = Df_{\mathfrak{p}_1}(r_{\mathfrak{p}_1}) \cdots Df_{\mathfrak{p}_k}(r_{\mathfrak{p}_k}) \cdot \prod_{\mathfrak{p} \notin S} Df_{\mathfrak{p}}(r_{\mathfrak{p}}) \cdot \prod_{\mathfrak{p}_\infty \in S_\infty} 0(r_{\mathfrak{p}_\infty}).$$

The Fourier transform of $Df(r)$ vanishes for $|\eta| > \Omega$ where $\Omega = N\mathfrak{p}_1^{-m_1} \cdots N\mathfrak{p}_k^{-m_k} \cdot \prod_{\mathfrak{p} \notin S} N\mathfrak{p}^{-\{h_{\mathfrak{p}}\}} \cdot \prod_{\mathfrak{p}_\infty \in S_\infty} t_{\mathfrak{p}_\infty}$.

Appendix

Here we define the Fourier transform of $f(t)$ as

$$(\mathcal{F}f)(\omega) = \mathrm{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\,.$$

The Fourier inverse transform is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathcal{F}f)(\omega)e^{i\omega t}\,d\omega\,.$$

cf. Define $(\mathcal{F}f)(\omega) = \mathrm{F}(2\pi\omega)$. Then

$$(\mathcal{F}f)(\omega) = \mathrm{F}(2\pi\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}\,dt\,.$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{F}(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{F}(2\pi\omega)e^{i2\pi\omega t}\,d2\pi\omega = \int_{-\infty}^{\infty} (\mathcal{F}f)(\omega)e^{2\pi i\omega t}\,d\omega\,.$$

The functions $\mathrm{S}_{0n}(c,\, t)$ are called "angular prolate spheroidal functions". They are real for real t, are continuous functions of c for $0 \le c$ and can be extended to be entire functions of the complex variable t. They are orthogonal in $(-1,\, 1)$ and are complete in $L^2(-1,\, 1)$. The functions $\mathrm{R}_{0n}{}^{(1)}(c,\, t)$ are called "radial prolate spheroidal functions". They differ from angular prolate spheroidal functions only by a real scale factor,

$$\mathrm{R}_{0n}{}^{(1)}(c,\, t) = k_n(c)\mathrm{S}_{0n}(c,\, t).$$

We have the following equations;

$$\frac{2c}{\pi}\mathrm{R}_{0n}{}^{(1)}(c,\, 1)^2\,\mathrm{S}_{0n}(c,\, t) = \int_{-1}^{1} \frac{\sin c(t-s)}{\pi(t-s)}\mathrm{S}_{0n}(c,\, s)\,ds\,, \qquad (1)$$

$$2i^n\,\mathrm{R}_{0n}{}^{(1)}(c,\, 1)\,\mathrm{S}_{0n}(c,\, t) = \int_{-1}^{1} e^{icts}\,\mathrm{S}_{0n}(c,\, s)\,ds \qquad n = 0,\, 1,\, 2,\, \cdots\,. \qquad (2)$$

Set $\lambda_n(c) = \frac{2c}{\pi}(\mathrm{R}_{0n}^{(1)}(c,\, 1))^2$ and set $u_n(c)^2 = \int_{-1}^{1} \mathrm{S}_{0n}(c,\, t)^2\,dt$. We define

$$\psi_n(c,\, t) = \frac{\sqrt{\lambda_n(c)}}{u_n(c)}\mathrm{S}_{0n}(c,\, \tfrac{2t}{T}).$$

Properties ii. follow from definitions and the orthogonality and completeness of $\mathrm{S}_{0n}(c,\, t)$ in $(-1,\, 1)$.

From the equation (1),

$$\frac{2c}{\pi}\mathrm{R}_{0n}{}^{(1)}(c,\, 1)^2\,\mathrm{S}_{0n}(c,\, \tfrac{2t}{T}) = \int_{-1}^{1} \frac{\sin c(\frac{2t}{T}-s)}{\pi(\frac{2t}{T}-s)}\mathrm{S}_{0n}(c,\, s)\,ds\,.$$

We have

$$\int_{-1}^{1} \frac{\sin c(\frac{2t}{T}-s)}{\pi(\frac{2t}{T}-s)}\mathrm{S}_{0n}(c,\, s)\,ds = \int_{-1}^{1} \frac{\sin c\cdot\frac{2}{T}(t-\frac{T}{2}\cdot s)}{\pi\frac{2}{T}(t-\frac{T}{2}\cdot s)}\mathrm{S}_{0n}(c,\, s)\,ds\,.$$

Put $\frac{T}{2}\cdot s=\sigma$. Then $ds=\frac{2}{T}d\sigma$. $-T/2\le\sigma\le T/2$ since $-1\le s\le 1$. So

$$\int_{-1}^{1}\frac{\sin c\cdot\frac{2}{T}(t-\frac{T}{2}\cdot s)}{\pi\frac{2}{T}(t-\frac{T}{2}\cdot s)}\mathrm{S}_{0n}(c,\ s)\,ds = \int_{-T/2}^{T/2}\frac{\sin c\cdot\frac{2}{T}(t-\sigma)}{\pi\cdot\frac{2}{T}(t-\sigma)}\mathrm{S}_{0n}(c,\frac{2\sigma}{T})\frac{2}{T}d\sigma$$
$$= \int_{-T/2}^{T/2}\frac{\sin\Omega(t-\sigma)}{\pi(t-\sigma)}\mathrm{S}_{0n}(c,\frac{2\sigma}{T})\ d\sigma \qquad c=\Omega\frac{T}{2}.$$

We obtain

$$\frac{2c}{\pi}\mathrm{R}_{0n}{}^{(1)}(c,\ 1)^2\,\mathrm{S}_{0n}(c,\ \frac{2t}{T}) = \int_{-T/2}^{T/2}\frac{\sin\Omega(t-\sigma)}{\pi(t-\sigma)}\mathrm{S}_{0n}(c,\frac{2\sigma}{T})\ d\sigma\,.$$

Multiplying both the sides by $\frac{\sqrt{\lambda_n(c)}}{u_n(c)}$,

$$\lambda_n(c)\psi_n(c,\ t) = \int_{-T/2}^{T/2}\frac{\sin\Omega(t-\sigma)}{\pi(t-\sigma)}\,\psi_n(c,\ \sigma)\ d\sigma\,.$$

The assertion of iii. is established.

From the equation (2),

$$2i^n\,\mathrm{R}_{0n}{}^{(1)}(c,\ 1)\,\mathrm{S}_{0n}(c,\ \frac{2t}{T}) = \int_{-1}^{1}e^{ic\cdot\frac{2t}{T}\cdot s}\mathrm{S}_{0n}(c,\ s)\,ds$$
$$= \int_{-1}^{1}e^{i\Omega ts}\,\mathrm{S}_{0n}(c,\ s)\,ds \quad c=\Omega\frac{T}{2}.$$

Put $s=\frac{\omega}{\Omega}$. Then $ds=\frac{1}{\Omega}d\omega$. $-\Omega\le\omega\le\Omega$ since $-1\le\frac{\omega}{\Omega}\le 1$. So

$$\int_{-1}^{1}e^{i\Omega ts}\,\mathrm{S}_{0n}(c,\ s)\,ds = \int_{-\Omega}^{\Omega}e^{i\Omega t\cdot\frac{\omega}{\Omega}}\,\mathrm{S}_{0n}(c,\ \frac{\omega}{\Omega})\,\frac{1}{\Omega}d\omega = \frac{1}{\Omega}\int_{-\Omega}^{\Omega}e^{i\omega t}\,\mathrm{S}_{0n}(c,\ \frac{\omega}{\Omega})d\omega\,.$$

Here

$$\mathrm{S}_{0n}(c,\ \frac{\omega}{\Omega}) = \mathrm{S}_{0n}(c,\ \frac{2\cdot\frac{\omega T}{2\Omega}}{T}).$$

We have

$$2i^n\,\mathrm{R}_{0n}{}^{(1)}(c,\ 1)\,\mathrm{S}_{0n}(c,\ \frac{2t}{T}) = \frac{1}{\Omega}\int_{-\Omega}^{\Omega}e^{i\omega t}\,\mathrm{S}_{0n}(c,\ \frac{2\cdot\frac{\omega T}{2\Omega}}{T})d\omega\,.$$

Thus

$$2i^n\Omega\mathrm{R}_{0n}{}^{(1)}(c,\ 1)\,\mathrm{S}_{0n}(c,\ \frac{2t}{T}) = \int_{-\Omega}^{\Omega}e^{i\omega t}\,\mathrm{S}_{0n}(c,\ \frac{2\cdot\frac{\omega T}{2\Omega}}{T})d\omega\,.$$

Multiplying both the sides by $\frac{1}{2\pi}\frac{\sqrt{\lambda_n(c)}}{u_n(c)}$,

$$\frac{i^n\Omega\mathrm{R}_{0n}^{(1)}(c,\ 1)}{\pi}\psi_n(c,\ t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}e^{i\omega t}\psi_n(c,\ \frac{\omega T}{2\Omega})d\omega\,.$$

Since $\mathrm{R}_{0n}{}^{(1)}(c,\ 1)=\sqrt{\frac{\lambda_n(c)\pi}{2c}}$,

$$\frac{i^n\Omega\mathrm{R}_{0n}^{(1)}(c,\ 1)}{\pi} = i^n\sqrt{\frac{\Omega^2\cdot\lambda_n(c)\pi}{\pi^2\cdot 2c}} = i^n\sqrt{\frac{\Omega\lambda_n(c)}{\pi T}} \qquad c=\Omega\frac{T}{2}.$$

Thus it turns out that

$$i^n\sqrt{\frac{\Omega}{\pi T}}\,\sqrt{\lambda_n(c)}\;\;\psi_n(c,\,t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\omega t}\psi_n(c,\;\frac{\omega T}{2\Omega})d\omega\,.$$

We have

$$\psi_n(c,\,t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\omega t}\left(i^{-n}\frac{1}{\sqrt{\lambda_n(c)}}\sqrt{\frac{\pi T}{\Omega}}\psi_n(c,\;\frac{\omega T}{2\Omega})\right)d\omega\,.$$

It means that

$$\mathcal{F}(\psi_n(c,\,t))(\omega) \;=\; \begin{cases} i^{-n}\frac{1}{\sqrt{\lambda_n(c)}}\sqrt{\frac{\pi T}{\Omega}}\psi_n(c,\;\frac{\omega T}{2\Omega}) & \cdots\;\; |\omega| \le \Omega \\ 0 & \cdots\;\; |\omega| > \Omega \end{cases}\,.$$

Namely $\psi_n(c,\,t)$ are bandlimited. The orthogonality and completeness of $\mathrm{S}_{0n}(c,\,t)$ in $(-1,\,1)$ leads the orthogonality and completeness of $\mathrm{S}_{0n}(c,\,\frac{\omega}{\Omega})$ in $(-\Omega,\,\Omega)$. Therefore $\psi_n(c,\,\frac{\omega T}{2\Omega})$ are orthogonal and complete in $(-\Omega,\,\Omega)$. Since $i^{-n}\frac{1}{\sqrt{\lambda_n(c)}}\sqrt{\frac{\pi T}{\Omega}}\psi_n(c,\;\frac{\omega T}{2\Omega})$ is the Fourier transform of $\psi_n(c,\,t)$, we can show the orthogonality and the completeness of $\psi_n(c,\,t)$ in $\mathfrak{B}$ by Parseval's theorem. The statement of i. is established.

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