THE ASYMPTOTIC SQUEEZE PRINCIPLE AND THE BINARY GOLDBACH CONJECTURE

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Abstract. In this paper, we prove the special squeeze principle for all sufficiently large $n \in 2\mathbb{N}$. This provides an alternative proof for the asymptotic version of the binary Goldbach conjecture in [3].

1. **Introduction and background**

In our seminal paper [2], we introduced and developed the method of circles of partition. This method is underpinned by a combinatorial structure that encodes certain additive properties of the subsets of the integers and invariably equipped with a certain geometric structure that allows to view the elements as points in the plane whose weights are just elements of the underlying subset. We call this combinatorial structure the circles of partition and is refereed to as the set of points

$$\mathcal{C}(n,\mathbb{M}) = \{[x] \mid x, n-x \in \mathbb{M}\}.$$

Each point in this set - except the center point - must have a uniquely distinct point that are join by a line which we refer to as an axis of the CoP. We denote an axis of a CoP with $\mathbb{L}_{[x],[y]}$ and an axis contained in the CoP as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) \text{ which means } [x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ with } x+y=n.$$

The method of circles of partition and their associated structures have been well advanced in [4], where the corresponding points have complex numbers as their weights and a line (axis) joining co-axis points. The following structure was considered as a complex circle of partition

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) = \{[z] \mid z, n-z \in \mathbb{C}_M, \Im(z)^2 = \Re(z)\left(n-\Re(z)\right)\}$$

where

$$\mathbb{C}_{\mathbb{M}} := \{z = x+iy \mid x \in \mathbb{M}, y \in \mathbb{R}\} \subset \mathbb{C}$$

with $\mathbb{M} \subseteq \mathbb{N}$. We abbreviate this complex additive structure as cCoP. The condition $\Im(z)^2 = \Re(z)(n-\Re(z)$ is referred to as the **circle condition** and it pretty much guarantees that all points on the cCoP lie on a circle in the complex. This circle is the embedding circle of the cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, denoted as $\mathfrak{C}_n$. The embedding circles of cCoPs have the property that they reside fully inside those embedding circle with a relatively larger generators, except the origin as a common point [4]. For each axis we have the following assignment

$$\mathbb{L}_{[z_1],[z_1]} \hat{\in} \mathcal{C}(n,\mathbb{C}_{\mathbb{M}}) \text{ which means } [z_1],[z_2] \in \mathcal{C}(n,\mathbb{C}_{\mathbb{M}}) \text{ with } z_1+z_2=n.$$

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The structure of the complex circles of partition is much more versatile and has extra structures that are not readily available in the structures of circles of partition. Most notably, for each axis $\mathbb{L}_{[z],[n-z]}$ of a cCoP there exists

$$\mathbb{L}_{\overline{[z]},\overline{[n-z]}}$$

a *conjugate axis*, where $\overline{[z]}, \overline{[n-z]}$ denotes the corresponding conjugate points. The space occupied by the embedding circles of partition and correspondingly outside the embedding circle had turned out to be very interesting, since this notion can be passed down to studying a certain ordering principle of the points of two interacting axes of distinct cCoPs. Much more striking is the fact which comes with ease by virtue of the circle condition that

$$|\mathbb{L}_{[z_1],[z_2]}| = n$$

for any axis $\mathbb{L}_{[z_1],[z_2]} \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}}) = \{[z] \mid z, n-z \in \mathbb{C}_M, \Im(z)^2 = \Re(z)\,(n - \Re(z))\}$. The squeeze principle [3] can be considered as a black box for studying the binary Goldbach conjecture. A slightly different version of this principle appears in [4]. For the sake of the reader, we provide a brief recap of this elegant principle as below

Lemma 1.1 (The squeeze principle). *Let $\mathbb{B} \subset \mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n + t, \mathbb{C}_{\mathbb{M}})$ with $t \geq 4$ be non–empty cCoPs with integers n, t, s of the same parity. If there exist an axis $\mathbb{L}_{[v_1],[w_1]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ with $w_1 \in \mathbb{C}_{\mathbb{B}}$ and an axis $\mathbb{L}_{[v_2],[w_2]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ with $v_2 \in \mathbb{C}_{\mathbb{B}}$ such that*

$$\Re(v_1) < \Re(v_2) \text{ and } \Re(w_1) < \Re(w_2) \tag{1.1}$$

then there exists an axis $\mathbb{L}_{[v_2],[w_1]} \hat{\in} \mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{B}})$ with $0 < s < t$. Hence also $\mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{M}})$ is not empty.

Proof. From the existence of an axis $\mathbb{L}_{[v_1],[w_1]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ follows $\Re(w_1) = n - \Re(v_1)$. With the requirement (1.1) we get

$$\Re(w_1) > n - \Re(v_2). \tag{1.2}$$

On the other hand from the existence of an axis $\mathbb{L}_{[v_2],[w_2]} \hat{\in} \mathcal{C}^o(n + t, \mathbb{C}_{\mathbb{M}})$ follows $\Re(w_2) = n + t - \Re(v_2)$ and with the requirement (1.1) and the result (1.2) we get

$$\begin{aligned} n - \Re(v_2) < \Re(w_1) &< n + t - \Re(v_2) \mid +\Re(v_2) \\ n < \Re(w_1) + \Re(v_2) &< n + t \\ n < n + s &< n + t. \end{aligned}$$

By virtue of the requirements $w_1, v_2 \in \mathbb{C}_{\mathbb{B}}$ and $n + s = \Re(w_1) + \Re(v_2)$ there is an axis $\mathbb{L}_{[v_2],[w_1]} \hat{\in} \mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{B}})$ and hence holds $\mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{B}}) \neq \emptyset$. And from $\mathbb{B} \subset \mathbb{M}$ follows immediately $\mathbb{C}_{\mathbb{B}} \subset \mathbb{C}_{\mathbb{M}}$ and therefore holds also $\mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{M}}) \neq \emptyset$. This completes the proof. □

Consequently, we obtain the special squeeze principle

Lemma 1.2 (Special squeeze principle). *Let $n, t, s \in 2\mathbb{N}$ and $\mathbb{P}$ be the set of all odd primes. If $t \geq 4$ and there exist an axis $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n)$ with $z_2 \in \mathbb{C}_{\mathbb{P}}$ and an axis $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(n + t)$ with $w_1 \in \mathbb{C}_{\mathbb{P}}$ such that*

$$\Re(z_1) < \Re(w_1) < \Re(z_1) + t$$

then there exists an axis $\mathbb{L}_{[w_1],[z_2]} \hat{\in} \mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{P}})$ with $0 < s < t$.

The Lemma 1.1 referred to as the squeeze principle may be regarded as a fundamental tool set for investigating the viability of dividing integers of a particular parity, utilizing constituent elements originating from a specific subset of the integers. The mechanism operates by discerning a pair of cCoPs that are both non-vacuous and share a common base set. Subsequently, supplementary cCoPs that are non-vacuous and have generators restrained within the interstice of these two generators are identified. This principle may be applied in a resourceful manner to investigate the overarching matter of the practicality of divvying up numbers such that each addend is a member of the identical subset of positive integers.

Remark 1.3. The CoP $\mathcal{C}(n,\mathbb{N}) := \mathcal{C}(n)$ is always non-empty and so is the cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{N}}) = \mathcal{C}^o(n)$.

2. **Application to the Binary Goldbach Conjecture**

In this section, we present an asymptotic proof for the binary Goldbach conjecture. The proof has been condensed into the language of cCoPs but can be reduced to the usual form of the conjecture.

Lemma 2.1 (juxtaposition principle)**.** *For all $n \geq 10$, there exist an axis $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{N}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{N}})$ for $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ such that $\Re(z_1) \neq \Re(w_1)$ and $\Re(z_2) \neq \Re(w_2)$ with $z_2 \in \mathbb{C}_{\mathbb{P}}$ and $w_1 \in \mathbb{C}_{\mathbb{P}}$ for $t \geq 4$.*

Proof. Let us choose a prime number $\Re(w_1) \leq \frac{n+t}{2}$ and choose a prime number $\Re(z_2) \in (\frac{n}{2}, n)$ for all $n \geq 10$ with $n \equiv 0 \pmod 2$, which is feasible by virtue of the prime number theorem. If $\Re(z_1) \neq \Re(w_1)$ and $\Re(z_2) \neq \Re(w_2)$ then there is nothing to do. Without loss of generality, suppose that $\Re(z_1) = \Re(w_1)$ then obviously $\Re(z_2) \neq \Re(w_2)$ since $n+t > n$. We note that $\pi(\frac{n+t}{2}) \geq 3$ for all $n \geq 10$ with $t \geq 4$ so that we can choose a prime number $\Re(w_1^{'}) \leq \frac{n+t}{2}$ such that $\Re(w_1^{'}) \neq \Re(w_1)$. Thus we replace $\Re(w_1)$ with $\Re(w_1^{'})$ and obtain the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{N}})$ and $\mathbb{L}_{[w_1^{'}],[w_2^{'}]} \hat{\in} \mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{N}})$ for $\Re(z_1) < \Re(z_2)$ and $\Re(w_1^{'}) < \Re(w_2^{'})$ such that $\Re(z_1) \neq \Re(w_1^{'})$. If $\Re(z_2) \neq \Re(w_2^{'})$ then we are done; otherwise, we choose another prime number $\Re(w_1^{''})$ such that $\Re(w_1^{''}) \neq \Re(w_1^{'})$ and $\Re(w_1^{''}) \neq \Re(w_1)$ since $\pi(\frac{n+t}{2}) \geq 3$ for all $n \geq 10$ and $t \geq 4$. By virtue of our construction, we obtain finally the axes of cCoPs $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{N}})$ and $\mathbb{L}_{[w_1^{''}],[w_2^{''}]} \hat{\in} \mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{N}})$ for $\Re(z_1) < \Re(z_2)$ and $\Re(w_1^{''}) < \Re(w_2^{''})$ such that $\Re(z_1) \neq \Re(w_1^{''})$ and $\Re(z_2) \neq \Re(w_2^{''})$ with $z_2 \in \mathbb{C}_{\mathbb{P}}$ and $w_1^{''} \in \mathbb{C}_{\mathbb{P}}$ for $t \geq 4$. □

Lemma 2.2. *(The prime number theorem) Let $\pi(n)$ denotes the number of prime numbers no more than n. Then we have*

$$\pi(n) = \frac{n}{\log n} + O(\frac{n}{\log^2 n}).$$

In particular, $\pi(n) \sim \frac{n}{\log n}$.

Lemma 2.3 (Bertrand's postulate)**.** *For $k \geq 89693$ there exists a prime number in the interval*

$$k < p \leq (1 + \frac{1}{\log^3 k})k.$$

Proof. The proof of this inequality appears in [1]. $\square$

Lemma 2.4. *Let p_n denotes the n^{th} prime number, then*

$$p_n = n\log n + O(n\log\log n).$$

In particular, $p_n \sim n\log n$.

Lemma 2.5. *There exist an $n_o \in \mathbb{N}$ such that for all even $n \geq n_o$ there exist an axis*
$\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{N}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{N}})$ for $\Re(z_1) \lesssim \Re(z_2)$ and $\Re(w_1) \lesssim \Re(w_2)$ such that $\Re(z_1) \lesssim \Re(w_1)$ and $\Re(z_2) \lesssim \Re(w_2)$ with $z_2 \in \mathbb{C}_{\mathbb{P}}$ and $w_1 \in \mathbb{C}_{\mathbb{P}}$ for $t \geq 4$.

Proof. Let us set $\Re(z_2)$ to be a prime number and choose $\Re(z_2)$ to be the $\pi(\frac{3n}{4})^{th}$ prime number. We note that via the prime number theorem holds

$$\begin{aligned}\pi(\frac{3n}{4}) &= \frac{\frac{3n}{4}}{\log(\frac{3n}{4})} + O(\frac{n}{\log^2 n})\\ &= \frac{3n}{4\log n} + O(\frac{n}{\log^2 n}).\end{aligned}$$

We also note via basic power series identities, we can write

$$\begin{aligned}-\log(1-\frac{1}{\log n}) &= \frac{1}{\log n} + \frac{1}{2(\log n)^2} + \frac{1}{3(\log n)^3} + \cdots\\ &= \frac{1}{\log n} + O(\frac{1}{(\log n)^2}).\end{aligned}$$

Then with Lemma 2.4 and Lemma 2.2 we obtain

$$\begin{aligned}\Re(z_2) = p_{\pi(\frac{3n}{4})} &= \pi(\frac{3n}{4})\log\pi(\frac{3n}{4}) + O(\pi(\frac{3n}{4})\log\log\pi(\frac{3n}{4}))\\ &= (\frac{3n}{4\log n} + O(\frac{n}{\log^2 n}))(\log(\frac{3n}{4\log n} + O(\frac{n}{\log^2 n})) + O(\pi(\frac{3n}{4})\log\log\pi(\frac{3n}{4}))\end{aligned}$$

We note that we can write

$$\begin{aligned}\log(\frac{3n}{4\log n} + O(\frac{n}{\log^2 n})) &= \log(\frac{3}{4}) + \log(\frac{n}{\log n}) + \log(1 + O(\frac{1}{\log n}))\\ &= \log n - \log\log n + O(1) \qquad (2.1)\end{aligned}$$

It follows from (2.1), we can write for the product

$$\begin{aligned}(\frac{3n}{4\log n} + O(\frac{n}{\log^2 n}))(\log(\frac{3n}{4\log n} + O(\frac{n}{\log^2 n})) &= (\frac{3n}{4\log n} + O(\frac{n}{\log^2 n}))(\log n - \log\log n + O(1))\\ &= \frac{3n}{4} + O(\frac{n\log\log n}{\log n}) \qquad (2.2)\end{aligned}$$

as the main term. Now we analyze the error term in a similar manner. By virtue of the prime number theorem, we can write

$$\pi(\frac{3n}{4})\log\log\pi(\frac{3n}{4}) = (\frac{3n}{4\log n} + O(\frac{n}{\log^2 n}))(\log\log(\frac{3n}{4\log n} + O(\frac{n}{\log^2 n}))) \qquad (2.3)$$

We observe that

$$\log\log(\frac{3n}{4\log n} + O(\frac{n}{\log^2 n})) = \log(\log(\frac{3n}{4\log n}) + \log(1 + O(\frac{1}{\log n}))) \ll \log\log n \qquad (2.4)$$

so that we obtain for the product

$$\pi(\frac{3n}{4})\log\log\pi(\frac{3n}{4}) = (\frac{3n}{4\log n}+O(\frac{n}{\log^2 n}))(\log\log(\frac{3n}{4\log n}+O(\frac{n}{\log^2 n})))$$
$$\ll \frac{n\log\log n}{\log n} \tag{2.5}$$

and by combining (2.1) and (2.5), we obtain

$$\Re(z_2) = \frac{3n}{4}+O(\frac{n\log\log n}{\log n})+O(\frac{n\log\log n}{\log n}) = \frac{3n}{4}+O(\frac{n\log\log n}{\log n}).$$

Consequently, we have for the real weight of the lower axis point

$$\begin{aligned}\Re(z_1) &= n-\Re(z_2)\\ &= n-\frac{3n}{4}+O(\frac{n\log\log n}{\log n})\\ &= \frac{n}{4}+O(\frac{n\log\log n}{\log n}).\end{aligned} \tag{2.6}$$

It is easy to see that

$$\Re(z_1)\sim\frac{n}{4}<\frac{n}{2}$$

and

$$\Re(z_2)\sim\frac{3n}{4}>\frac{n}{2}$$

Now, by virtue of Lemma 2.3, we set $\Re(w_1)$ to be a prime number and choose $\Re(w_1)$ so that

$$\frac{n}{4}<\Re(w_1)\leq(1+\frac{1}{\log^3\frac{n}{4}})(\frac{n}{4}) \tag{2.7}$$

for all $n\geq 358772$, then it implies that $\Re(z_1)\lesssim\Re(w_1)$. It is easy to see that

$$\Re(w_1)\lesssim\frac{n+t}{2}$$

for $t\geq 4$, since

$$(1+\frac{1}{\log^3\frac{n}{4}})(\frac{n}{4})\lesssim\frac{n}{2}$$

by virtue of the fact that

$$(1+\frac{1}{\log^3\frac{n}{4}})\sim 1.$$

It follows from (2.7) the lower bound

$$\begin{aligned}\Re(w_2) &= n+t-\Re(w_1)\\ &\geq n+t-(1+\frac{1}{\log^3\frac{n}{4}})(\frac{n}{4})\\ &\text{and since } (1+\frac{1}{\log^3\frac{n}{4}})\sim 1\\ &\sim n-\frac{n}{4}+t\\ &> n-\frac{n}{4}=\frac{3n}{4}\sim\Re(z_2)\end{aligned}$$

for all $t\geq 4$ and $n>n_o$ for some fixed $n_o\in\mathbb{N}$. This completes the proof. $\square$

We are now ready to prove the binary Goldbach conjecture for all even numbers greater than some $n_0 \in \mathbb{N}$. This result provides an alternative solution to our first result and in very few instances adopts the proof technique in [3]. The benefit of the strong version of Bertrand's postulate (Lemma 2.3) is good enough to verify the asymptotic version of the binary Goldbach conjecture using this version of the squeeze principle, which is a slight variation of the version that appears in the paper [3].

Theorem 2.6 (The asymptotic binary Goldbach theorem)**.** *There exist some $n_o \in \mathbb{N}$ such that every even number $n \geq n_o$ can be written as a sum of two prime numbers.*

Proof. We note that the above statement is equivalent to the statement that the cCoPs $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{P}})$ are non–empty for all even $n \geq n_o$.

By remark 1.3 all cCoPs basing on $\mathbb{C}_{\mathbb{N}}$ with generators ≥ 2 are non–empty. By virtue of Lemma 2.5 all cCoPs $\mathcal{C}^o(n)$ and $\mathcal{C}^o(n+4)$ with even generators $n \geq n_o$ fulfil the requirements of the special squeeze principle (Lemma 1.2). Hence for each such n there is always a non–empty cCoP $\mathcal{C}^o(n+2, \mathbb{C}_{\mathbb{P}})$. We start with $\mathcal{C}^o(n_o)$ and $\mathcal{C}^o(n_o+4)$ and continue this procedure with $\mathcal{C}^o(n_o+k)$ and $\mathcal{C}^o(n_o+k+4)$ for all even $k \geq 2$. We verify that all cCoPs $\mathcal{C}^o(n_o+k+2)$ for even $k \geq 2$ ad infinitum are non–empty. □

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