MULTINOMIAL DEVELOPMENT

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Abstract:
In this paper we obtain the multinomial theorem following the numbers A_n^p and C_n^p (Vandermonde's identity generalization). Using this notion we obtain generalization of products of numbers in arithmetic progression, arithmetic regression and their sum. From the generalisation we propose (define) the arithmetics sequences product.

1 INTRODUCTION

In combinatorial analysis we have numbers that allow us to calculate the cardinal of a set according to a well determined model among which we can quote: the numbers n^p, A_n^p and C_n^p.
Let n and p be two natural numbers. The drawing of p balls in an urn containing n balls, models many counting problems [2]

$$C_n^p = \binom{n}{p} = \frac{n!}{p!(n-p)!}$$

$$A_n^p = p!C_n^p$$

The number C_n^p plays an important role in enumerative combination and other discipline fields, see Chen and Kho [1] and Comtet [2] for more details on binomial coefficients. The term binomial coefficient comes from the fact that numbers C_n^p appear as coefficients in the development of $(x+y)^n$.

$$(x+y)^n = \sum_{i=0}^{n} C_n^i x^{n-i} y^i$$

The multinomial coefficients see [8] which are a generalization of the binomial coefficients allow to extend the development to more numbers. Let n and m be non-zero natural numbers, $x_1, x_2, ..., x_m$ real numbers:

$$(x_1 + x_2 + ... + x_m)^n = \sum_{k_1+k_2+...+k_m=n} \binom{n}{k_1, k_2, ..., k_m} x_1^{k_1} x_2^{k_2} ... x_m^{k_m}$$

$$\binom{n}{k_1, k_2, ..., k_m} = \frac{n!}{k_1! k_2! ... k_m!}$$

In the case of binomials, the sum of the powers of numbers in arithmetic progression using Bernoulli polynomials gives (see [5] and [7] for more details)

$$\sum_{k=0}^{n-1} (m + kr)^p = \frac{n^p}{p+1} (B_{p+1}(n + \frac{d}{r}) - B_{p+1}(\frac{d}{r}))$$

The sum of the first n products of p consecutive integers is given by the formula of M. LAISANT [6]. Let n and p be natural numbers:

$$S_p(n) = \sum_{k=1}^{n} k(k+1)(k+2)...(k+p) = \frac{1}{p+1} \prod_{i=0}^{p} (n+i)$$

Suppose m, r are non-zero positive reals and n is a non-zero natural number, the Euler gamma function [4] and [9] allows to generalize the product of n consecutive non-zero integers.

$$\prod_{k=0}^{n-1} (m + kr) = r^n \frac{\Gamma(n + 1 + \frac{m-r}{r})}{\Gamma(1 + \frac{m-r}{r})}$$

For every non-negative integer n we have :

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n!$$

The results of this paper are organized as follows: in section 2 the formula of the multinomial following the numbers A_n^p and C_n^p. In section 3 the formulas of product of p numbers in arithmetic progression or regression (divided by p!) are presented. In section 4 a generalization of the formula of M. LAISANT will be presented by proposing afterwards the arithmetic sequences produced.

2 MULTINOMIAL DEVELOPMENT

2.1 Multinomial development according to the number A_n^p

2.1.1 Theorem 1

Let m and n be two non-zero natural numbers and $x_1, x_2, ..., x_m$ natural numbers $n \leq x_i$. Then,

$$A^n_{(x_1+x_2+...+x_m)} = \sum_{k_1+k_2+...+k_m=n} \binom{n}{k_1, k_2, ..., k_m} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m}$$

Proof : we prove by recurrence on m

$$\begin{aligned} i)m = 1; A^n_{x_1} &= \sum_{k_1=n} \binom{n}{k_1} A^{k_1}_{x_1} \\ &= \binom{n}{n} A^n_{x_1} \\ A^n_{x_1} &= A^n_{x_1} \end{aligned}$$

ii) let's assume the equality is true for m > 1 we have :

$$A^n_{(x_1+x_2+...+x_m)} = \sum_{k_1+k_2+...+k_m=n} \binom{n}{k_1, k_2, ..., k_m} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m}$$

$$\begin{aligned} A^n_{(x_1+x_2+...+x_m+x_{m+1})} &= \sum_{i=0}^{n} \binom{n}{i} A^{n-i}_{(x_1+x_2+...+x_m)} A^i_{x_{m+1}} (Binomial formula) \\ &= \sum_{i=0}^{n} \binom{n}{i} [\sum_{k_1+k_2+...+k_m=n-i} \binom{n-i}{k_1, k_2, ..., k_m} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m}] A^i_{x_{m+1}} \\ &= \sum_{i=0}^{n} [\sum_{k_1+k_2+...+k_m=n-i} \binom{n-i}{k_1, k_2, ..., k_m} \binom{n}{i} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m} A^i_{x_{m+1}}] \\ &= \sum_{k_{m+1}=0}^{n} [\sum_{k_1+k_2+...+k_m=n-k_{m+1}} \binom{n-k_{m+1}}{k_1, k_2, ..., k_m} \binom{n}{k_{m+1}} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m} A^{k_{m+1}}_{x_{m+1}}] \\ &= \sum_{k_{m+1}=0}^{n} [\sum_{k_1+k_2+...+k_m+k_{m+1}=n} \binom{n}{k_1, k_2, ..., k_m, k_{m+1}} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m} A^{k_{m+1}}_{x_{m+1}}] \\ A^n_{(x_1+x_2+...+x_m+x_{m+1})} &= \sum_{k_1+k_2+...+k_m+k_{m+1}=n} \binom{n}{k_1, k_2, ..., k_m, k_{m+1}} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m} A^{k_{m+1}}_{x_{m+1}} \end{aligned}$$

Example

$$
\begin{aligned}
A^3_{(7+3+11)} &= \sum_{i+j+k=3} \binom{3}{i,j,k} A^i_7 A^j_3 A^k_{11} \\
&= A^3_7 A^0_3 A^0_{11} + 3A^2_7 A^1_3 A^0_{11} + 3A^2_7 A^0_3 A^1_{11} + 3A^1_7 A^2_3 A^0_{11} + 3A^1_7 A^0_3 A^2_{11} + 6A^1_7 A^1_3 A^1_{11} \\
&+ 3A^0_7 A^2_3 A^1_{11} + 3A^0_7 A^1_3 A^2_{11} + A^0_7 A^3_3 A^0_{11} + A^0_7 A^0_3 A^3_{11} \\
A^3_{(7+3+11)} &= 7980
\end{aligned}
$$

2.1.2 Formula of the binomial according to the number A^p_n

Let m, n be natural numbers and p a non-zero natural number, $p \leq m$, $p \leq n$ then :

$$
A^p_{(m+n)} = \sum_{i=0}^{p} C^i_p A^{p-i}_m A^i_n
$$

Proof (see 3.2)

2.2 Multinomial development according to the number C^p_n (multinomial vandermonde's identity)

2.2.1 Theorem 2

Let m and n be two non-zero natural numbers and $x_1, x_2, ..., x_m$ natural numbers $n \leq x_i$. Then,

$$
C^n_{(x_1+x_2+...+x_m)} = \sum_{k_1+k_2+...+k_m=n} C^{k_1}_{x_1} C^{k_2}_{x_2} ... C^{k_m}_{x_m}
$$

Proof : From theorem 1 we have:

$$
\begin{aligned}
A^n_{(x_1+x_2+...+x_m)} &= \sum_{k_1+k_2+...+k_m=n} \binom{n}{k_1,k_2,...,k_m} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m} \\
n! C^n_{(x_1+x_2+...+x_m)} &= \sum_{k_1+k_2+...+k_m=n} \frac{n!}{k_1!k_2!...k_m!} k_1! C^{k_1}_{x_1} k_2! C^{k_2}_{x_2} ... k_m! C^{k_m}_{x_m} \\
n! C^n_{(x_1+x_2+...+x_m)} &= \sum_{k_1+k_2+...+k_m=n} \frac{n!}{k_1!k_2!...k_m!} k_1!k_2!...k_m! C^{k_1}_{x_1} C^{k_2}_{x_2} ... C^{k_m}_{x_m} \\
C^n_{(x_1+x_2+...+x_m)} &= \sum_{k_1+k_2+...+k_m=n} C^{k_1}_{x_1} C^{k_2}_{x_2} ... C^{k_m}_{x_m}
\end{aligned}
$$

Example

$$
\begin{aligned}
C^3_{(7+3+11)} &= \sum_{i+j+k=3} C^i_7 C^j_3 C^k_{11} \\
&= C^3_7 C^0_3 C^0_{11} + C^2_7 C^1_3 C^0_{11} + C^2_7 C^0_3 C^1_{11} + C^1_7 C^2_3 C^0_{11} + C^1_7 C^0_3 C^2_{11} + C^1_7 C^1_3 C^1_{11} \\
&+ C^0_7 C^2_3 C^1_{11} + C^0_7 C^1_3 C^2_{11} + C^0_7 C^3_3 C^0_{11} + C^0_7 C^0_3 C^3_{11} \\
C^3_{(7+3+11)} &= 1330
\end{aligned}
$$

2.2.2 Formula of the binomial according to the number C^p_n

Let m, n be natural numbers and p a non-zero natural number, $p \leq m$, $p \leq n$ then :

$$
C^p_{(m+n)} = \sum_{i=0}^{p} C^{p-i}_m C^i_n (Vandermond'sidentity)
$$

Proof (see 3.3)

3 FORMULA OF DECOMPOSITION OF PRODUCT INTO SUM

3.1 Formula for decomposition of p product of numbers in arithmetic progression into sum

Let the product be defined by :

$$
\prod_{k=n}^{n+p-1} (m+kr) = (m+nr)(m+(n+1)r)...(m+(n+p-1)r)
$$

Let's set: $D_{m,r}^{k,p} = \prod_{i=1}^{k}(m+(p-i)r)$ with $D_{m,r}^{0,p} = 1$
Theorem 3 :
Let m be a non-zero real number, n be a natural number and p a non-zero natural number, then :

$$\prod_{k=n}^{n+p-1}(m+kr) = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i A_n^i$$

Proof :

$$i)p=1, \prod_{k=n}^{n+1-1}(m+kr) = \sum_{i=0}^{1} C_1^i D_{m,r}^{1-i,1} r^i A_n^i$$

$$m+nr = C_1^0 D_{m,r}^{1,1} r^0 A_n^0 + C_1^1 D_{m,r}^{0,1} r^1 A_n^1$$

$$m+nr = m+nr$$

ii) let's assume the equality is true for p > 1 we have :

$$\prod_{k=n}^{n+p-1}(m+kr) = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i A_n^i$$

$$\prod_{k=n}^{n+p}(m+kr) = (m+nr)(m+(n+1)r)...(m+(n+p-1)r)(m+(n+p)r)$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i A_n^i (m+(n+p)r)$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i A_n^i ((m+pr)+nr)$$

$$= \sum_{i=0}^{p} C_p^i (m+pr) D_{m,r}^{p-i,p} r^i A_n^i + \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^{i+1} n A_n^i$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i,p} r^{i+1} n A_n^i + C_p^p D_{m,r}^{0,p} r^{p+1} n A_n^p$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=0}^{p-1} C_p^i ((m+pr)-(p-i)r) D_{m,r}^{p-1-i,p} r^{i+1} n A_n^i$$

$$+ C_p^p D_{m,r}^{0,p} r^{p+1} n A_n^p$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=0}^{p-1} C_p^i (m+pr) D_{m,r}^{p-1-i,p} r^{i+1} n A_n^i$$

$$- \sum_{i=0}^{p-1} C_p^i (p-i) D_{m,r}^{p-1-i,p} r^{i+2} n A_n^i + C_p^p D_{m,r}^{0,p} r^{p+1} n A_n^p$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i,p+1} r^{i+1} n A_n^i$$

$$+ C_p^p D_{m,r}^{0,p} r^{p+1} n A_n^p - \sum_{i=0}^{p-1} C_p^i (p-i) D_{m,r}^{p-1-i,p} r^{i+2} n A_n^i$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p+1} r^{i+1} n A_n^i - \sum_{i=0}^{p-1} C_p^i (p-i) D_{m,r}^{p-1-i,p} r^{i+2} n A_n^i$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p+1} r^{i+1} ((n-i)+i) A_n^i$$

$$- \sum_{i=0}^{p-1} C_p^i (p-i) D_{m,r}^{p-1-i,p} r^{i+2} n A_n^i$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p+1} r^{i+1} (n-i) A_n^i$$

$$+ \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p+1} r^{i+1} i A_n^i - \sum_{i=0}^{p-1} C_p^i (p-i) D_{m,r}^{p-1-i,p} r^{i+2} n A_n^i$$

$$\prod_{k=n}^{n+p}(m+kr)=\sum_{i=0}^{p}C_p^iD_{m,r}^{p+1-i,p+1}r^iA_n^i+\sum_{i=0}^{p}C_p^iD_{m,r}^{p-i,p+1}r^{i+1}A_n^{i+1}$$
$$+\sum_{i=0}^{p}C_p^iD_{m,r}^{p-i,p+1}r^{i+1}iA_n^i-\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}nA_n^i$$
$$=\sum_{i=0}^{p}C_p^iD_{m,r}^{p+1-i,p+1}r^iA_n^i+\sum_{i=0}^{p}C_p^iD_{m,r}^{p-i,p+1}r^{i+1}A_n^{i+1}$$
$$+\sum_{i=0}^{p-1}C_p^iD_{m,r}^{p-i,p+1}r^{i+1}iA_n^i+C_p^pD_{m,r}^{0,p+1}r^{p+1}pA_n^p-\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}nA_n^i$$
$$=\sum_{i=0}^{p}C_p^iD_{m,r}^{p+1-i,p+1}r^iA_n^i+\sum_{i=0}^{p}C_p^iD_{m,r}^{p-i,p+1}r^{i+1}A_n^{i+1}$$
$$+\sum_{i=0}^{p-1}C_p^i((m+ir)+(p-i)r)D_{m,r}^{p-1-i,p}r^{i+1}iA_n^i+C_p^pD_{m,r}^{0,p+1}r^{p+1}pA_n^p$$
$$-\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}((n-i)+i)A_n^i$$
$$=\sum_{i=0}^{p}C_p^iD_{m,r}^{p+1-i,p+1}r^iA_n^i+\sum_{i=0}^{p}C_p^iD_{m,r}^{p-i,p+1}r^{i+1}A_n^{i+1}$$
$$+\sum_{i=0}^{p-1}C_p^i(m+ir)D_{m,r}^{p-1-i,p}r^{i+1}iA_n^i+\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}iA_n^i+C_p^pD_{m,r}^{0,p+1}r^{p+1}pA_n^p$$
$$-\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}(n-i)A_n^i-\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}iA_n^i$$
$$=\sum_{i=0}^{p}C_p^iD_{m,r}^{p+1-i,p+1}r^iA_n^i+\sum_{i=0}^{p}C_p^iD_{m,r}^{p-i,p+1}r^{i+1}A_n^{i+1}$$
$$+\sum_{i=0}^{p-1}C_p^iD_{m,r}^{p-i,p}r^{i+1}iA_n^i+C_p^pD_{m,r}^{0,p+1}r^{p+1}pA_n^p+\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}iA_n^i+$$
$$-\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}A_n^{i+1}-\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}iA_n^i$$
$$=\sum_{i=0}^{p}C_p^iD_{m,r}^{p+1-i,p+1}r^iA_n^i+\sum_{i=0}^{p}C_p^iD_{m,r}^{p-i,p+1}r^{i+1}A_n^{i+1}$$
$$+\sum_{i=0}^{p}C_p^iD_{m,r}^{p-i,p}r^{i+1}iA_n^i-\sum_{i=0}^{p-1}C_p^i(p-i)D_{m,r}^{p-1-i,p}r^{i+2}A_n^{i+1}$$

$$\prod_{k=n}^{n+p}(m+kr) = C_p^0 D_{m,r}^{p+1,p+1} r^0 A_n^0 + \sum_{i=1}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i,p+1} r^{i+1} A_n^{i+1}$$

$$+ C_p^p D_{m,r}^{p+1,p+1} r^{p+1} A_n^{p+1} + \sum_{i=1}^{p} C_p^i D_{m,r}^{p+1-i,p} r^i i A_n^i - \sum_{i=1}^{p} C_p^i (p-i+1) D_{m,r}^{p+1-i,p} r^{i+1} A_n^i$$

$$= C_p^0 D_{m,r}^{p+1,p+1} r^0 A_n^0 + \sum_{i=1}^{p} C_p^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + \sum_{i=1}^{p} C_p^{i-1} D_{m,r}^{p+1-i,p+1} r^i A_n^i$$

$$+ C_p^p D_{m,r}^{p+1,p+1} r^{p+1} A_n^{p+1} + \sum_{i=1}^{p} [iC_p^i - (p-i+1)C_p^{i-1}] D_{m,r}^{p+1-i,p} r^{i+1} A_n^i$$

$$= C_p^0 D_{m,r}^{p+1,p+1} r^0 A_n^0 + \sum_{i=1}^{p} (C_p^{i-1} + C_p^i) D_{m,r}^{p+1-i,p+1} r^i A_n^i + C_p^p D_{m,r}^{p+1,p+1} r^{p+1} A_n^{p+1}$$

$$= C_{p+1}^0 D_{m,r}^{p+1,p+1} r^0 A_n^0 + \sum_{i=1}^{p} C_{p+1}^i D_{m,r}^{p+1-i,p+1} r^i A_n^i + C_{p+1}^{p+1} D_{m,r}^{p+1,p+1} r^{p+1} A_n^{p+1}$$

$$\prod_{k=n}^{n+p}(m+kr) = \sum_{i=0}^{p+1} C_{p+1}^i D_{m,r}^{p+1-i,p+1} r^i A_n^i$$

Example

$$\prod_{k=7}^{7+3-1}(5+2k) = \sum_{i=0}^{3} C_3^i D_{5,2}^{3-i,3} 2^i A_7^i$$

$$= C_3^0 D_{5,2}^{3,3} 2^0 A_7^0 + C_3^1 D_{5,2}^{2,3} 2^1 A_7^1 + C_3^2 D_{5,2}^{1,3} 2^2 A_7^2 + C_3^3 D_{5,2}^{0,3} 2^3 A_7^3$$

$$= C_3^0 (9\times 7\times 5) 2^0 A_7^0 + C_3^1 (9\times 7) 2^1 A_7^1 + C_3^2 9\times 2^2 A_7^2 + C_3^3 2^3 A_7^3$$

$$\prod_{k=7}^{7+3-1}(5+2k) = 9177$$

3.2 Formula for decomposition of p product of numbers in arithmetic regression into sum

Let the product be defined by:

$$\prod_{k=0}^{p-1}(m+(n-k)r) = (m+nr)(m+(n-1)r)...(m+(n-p+1)r)$$

Let's set: $D_{m,r}^{k} = \prod_{i=1}^{k}(m-(i-1)r)$ with $D_{m,r}^{0} = 1$

Theorem 4 :

Let m be a non-zero real number, n be a natural number and p a non-zero natural number, then :

$$\prod_{k=0}^{p-1}(m+(n-k)r) = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^i A_n^i$$

Proof :

$$\begin{aligned} i)p=1, \prod_{k=0}^{1-1}(m+(n-k)r) &= \sum_{i=0}^{1} C_1^i D_{m,r}^{1-i} r^i A_n^i \\ m+nr &= C_1^0 D_{m,r}^{1} r^0 A_n^0 + C_1^1 D_{m,r}^{0} r^1 A_n^1 \\ m+nr &= m+nr \end{aligned}$$

ii) let's assume the equality is true for p > 1 we have :

$$\prod_{k=0}^{p-1}(m+(n-k)r) = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^i A_n^i$$

$$\prod_{k=0}^{p}(m+(n-k)r) = (m+nr)(m+(n-1)r)...(m+(n-p+1)r)(m+(n-p)r)$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^i A_n^i (m+(n-p)r)$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^i A_n^i ((m-(p-i)r)+(n-i)r)$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^i A_n^i (m-(p-i)r) + \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^{i+1} A_n^i (n-i)$$

$$= \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i} r^i A_n^i + \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^{i+1} A_n^{i+1}$$

$$= C_p^0 D_{m,r}^{p+1} r^0 A_n^0 + \sum_{i=1}^{p} C_p^i D_{m,r}^{p+1-i} r^i A_n^i + \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i} r^{i+1} A_n^{i+1} + C_p^p D_{m,r}^0 r^{p+1} A_n^{p+1}$$

$$= C_p^0 D_{m,r}^{p+1} r^0 A_n^0 + \sum_{i=1}^{p} C_p^i D_{m,r}^{p+1-i} r^i A_n^i + \sum_{i=1}^{p} C_p^{i-1} D_{m,r}^{p+1-i} r^i A_n^i + C_p^p D_{m,r}^0 r^{p+1} A_n^{p+1}$$

$$= C_p^0 D_{m,r}^{p+1} r^0 A_n^0 + \sum_{i=1}^{p} (C_p^{i-1} + C_p^i) D_{m,r}^{p+1-i} r^i A_n^i + C_p^p D_{m,r}^0 r^{p+1} A_n^{p+1}$$

$$= C_{p+1}^0 D_{m,r}^{p+1} r^0 A_n^0 + \sum_{i=1}^{p} C_{p+1}^i D_{m,r}^{p+1-i} r^i A_n^i + C_{p+1}^{p+1} D_{m,r}^0 r^{p+1} A_n^{p+1}$$

$$\prod_{k=0}^{p}(m+(n-k)r) = \sum_{i=0}^{p+1} C_{p+1}^i D_{m,r}^{p+1-i} r^i A_n^i$$

Remark : For r = 1, $D_{m,1}^{p-i} = A_m^{p-i}$ we find the binomial formula according to the number A_n^p

3.3 Formula for decomposition of p product of numbers in arithmetic progression divided by p! into sum

Let the product be defined by : $\dfrac{\prod_{k=n}^{n+p-1}(m+kr)}{p!}$

Let's set : $D_{m,r}^{k,p} = \prod_{i=1}^{k}(m+(p-i)r)$ with $D_{m,r}^{0,p} = 1$

Theorem 5 : (This theorem is a deduction from Theorem 3)

Let m be a non-zero real number, n be a natural number and p a non-zero natural number, then :

$$\frac{\prod_{k=n}^{n+p-1}(m+kr)}{p!} = \sum_{i=0}^{p} \frac{D_{m,r}^{p-i,p}}{(p-i)!} r^i C_n^i$$

Proof :

We have : $\displaystyle\prod_{k=n}^{n+p-1}(m+kr) = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i A_n^i$

$$\begin{aligned}
\frac{\prod_{k=n}^{n+p-1}(m+kr)}{p!} &= \frac{1}{p!}\sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i A_n^i \\
&= \sum_{i=0}^{p} C_p^i \frac{D_{m,r}^{p-i,p}}{p!} r^i A_n^i \\
&= \sum_{i=0}^{p} \frac{p!}{i!(p-i)!} \frac{D_{m,r}^{p-i,p}}{p!} r^i i! C_n^i \\
\frac{\prod_{k=n}^{n+p-1}(m+kr)}{p!} &= \sum_{i=0}^{p} \frac{D_{m,r}^{p-i,p}}{(p-i)!} r^i C_n^i
\end{aligned}$$

3.4 Formula for decomposition of p product of numbers in arithmetic regression divided by p! into sum

Let the product be defined by : $\dfrac{\prod_{k=0}^{p-1}(m+(n-k)r)}{p!}$

Let's set : $D_{m,r}^{k} = \displaystyle\prod_{i=1}^{k}(m-(i-1)r)$ with $D_{m,r}^{0} = 1$

Theorem 6 : (This theorem is a deduction from Theorem 4)

Let m be a non-zero real number, n be a natural number and p a non-zero natural number, then :

$$\frac{\prod_{k=0}^{p-1}(m+(n-k)r)}{p!} = \sum_{i=0}^{p} \frac{D_{m,r}^{p-i}}{(p-i)!} r^i C_n^i$$

Proof :

We have : $\displaystyle\prod_{k=0}^{p-1}(m+(n-k)r) = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^i A_n^i$

$$\begin{aligned}
\frac{\prod_{k=0}^{p-1}(m+(n-k)r)}{p!} &= \frac{1}{p!}\sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} r^i A_n^i \\
&= \sum_{i=0}^{p} C_p^i \frac{D_{m,r}^{p-i}}{p!} r^i A_n^i \\
&= \sum_{i=0}^{p} \frac{p!}{i!(p-i)!}\frac{D_{m,r}^{p-i}}{p!} r^i i! C_n^i \\
\frac{\prod_{k=0}^{p-1}(m+(n-k)r)}{p!} &= \sum_{i=0}^{p} \frac{D_{m,r}^{p-i}}{(p-i)!} r^i C_n^i
\end{aligned}$$

Remark : For r = 1, $D_{m,1}^{p-i} = A_m^{p-i}$ we find the binomial formula according to the number C_n^p

4 SUMS COMPUTATIONS

4.1 Sum of p products of numbers in arithmetic progression

Consider the sum define by:
$S_p = m(m+r)...(m+(p-1)r) + (m+r)(m+2r)...(m+pr) + (m+(n-1)r)...(m+(n+p-2)r)$

Let's set : $D_{m,r}^{k,p} = \displaystyle\prod_{i=1}^{k}(m+(p-i)r)$ with $D_{m,r}^{0,p} = 1$

Theorem 7 : (This theorem is a deduction from Theorem 3)
Let m be a non-zero real number, n be a natural number and p a non-zero natural number, then :

$$\sum_{j=0}^{n-1}\prod_{k=j}^{j+p-1}(m+kr) = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i \frac{A_n^{i+1}}{i+1}$$

Proof
$m(m+r)...(m+(p-1)r) = C_p^0 (m+(p-1)r)...(m+r)mr^0 A_0^0$
$(m+r)(m+2r)...(m+pr) = C_p^0 (m+(p-1)r)...(m+r)mr^0 A_1^0 + C_p^1 (m+(p-1)r)...(m+r)r^1 A_1^1$
$(m+2r)(m+3r)..(m+(p+1)r) = C_p^0 (m+(p-1)r)...(m+r)mr^0 A_2^0 + C_p^1 (m+(p-1)r)...(m+r)r^1 A_2^1 + C_p^2 (m+(p-1)r)...(m+2r)r^2 A_2^2$

:: :: :: :: ::

$(m+pr)(m+(p+1)r)..(m+(2p+1)r) = C_p^0(m+(p-1)r)...(m+r)mr^0A_p^0 + C_p^1(m+(p-1)r)...(m+r)r^1A_p^1 + C_p^2(m+(p-1)r)...(m+2r)r^2A_p^2 + ... + C_p^{P-1}(m+(p-1)r)r^{P-1}A_p^{P-1} + C_p^P r^P A_p^P$

:: :: :: :: ::

$(m+(n-1)r)(m+nr)..(m+(n+p-2)r) = C_p^0(m+(p-1)r)...(m+r)mr^0A_{n-1}^0 + C_p^1(m+(p-1)r)...(m+r)r^1A_{n-1}^1 + C_p^2(m+(p-1)r)...(m+2r)r^2A_{n-1}^2 + ... + C_p^{P-1}(m+(p-1)r)r^{P-1}A_{n-1}^{P-1} + C_p^P r^P A_{n-1}^P$

Summing member to member we get:
$S_p = C_p^0(m+(p-1)r)...(m+r)mr^0[A_0^0+A_1^0+...+A_{n-1}^0] + C_p^1(m+(p-1)r)...(m+r)r^1[A_1^1+A_2^1+...+A_{n-1}^1] + C_p^2(m+(p-1)r)...(m+2r)r^2[A_2^2+A_3^2+...+A_{n-1}^2] + ... + C_p^{p-1}(m+(p-1)r)r^{p-1}[A_{p-1}^{p-1}+A_p^{p-1}+...+A_{n-1}^{p-1}] + C_p^p r^p[A_p^p+A_{p+1}^p+...+A_{n-1}^p]$
From [6] we have:
$S_p = C_p^0(m+(p-1)r)...(m+r)mr^0\frac{A_n^1}{1} + C_p^1(m+(p-1)r)...(m+r)r^1\frac{A_n^2}{2} + C_p^2(m+(p-1)r)...(m+2r)r^2\frac{A_n^3}{3} + ... + C_p^{p-1}(m+(p-1)r)r^{p-1}\frac{A_n^p}{p} + C_p^p r^p\frac{A_n^{p+1}}{p+1}$
$S_p = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i \frac{A_n^{i+1}}{i+1}$

4.2 Sum of p products of numbers in arithmetic progression divided by p!

Consider the sum defined by :
$S'_p = \frac{m}{1}\frac{m+r}{2}...\frac{m+(p-1)r}{p} + \frac{m+r}{1}\frac{m+2r}{2}...\frac{m+pr}{p} + ... + \frac{m+(n-1)r}{1}\frac{m+nr}{2}...\frac{m+(n+p-2)r}{p}$
Let's set : $D_{m,r}^{k,p} = \prod_{i=1}^{k}(m+(p-i)r)$ with $D_{m,r}^{0,p} = 1$
Theorem 8 : (This theorem is a deduction from Theorem 5)
Let m be a non-zero real number, n be a natural number and p a non-zero natural number, then :

$$\sum_{j=0}^{n-1}\frac{\prod_{k=j}^{j+p-1}(m+kr)}{p!} = \sum_{i=0}^{p}\frac{D_{m,r}^{p-i,p}}{(p-i)!}r^iC_n^{i+1}$$

Proof :
$S'_p = \frac{m}{1}\frac{m+r}{2}...\frac{m+(p-1)r}{p} + \frac{m+r}{1}\frac{m+2r}{2}...\frac{m+pr}{p} + ... +$
$\frac{m+(n-1)r}{1}\frac{m+nr}{2}...\frac{m+(n+p-2)r}{p}$
$S'_p = \frac{1}{p!}S_p$

$$\begin{aligned} S'_p &= \frac{1}{p!}\sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} r^i \frac{A_n^{i+1}}{i+1} \\ &= \sum_{i=0}^{p} C_p^i \frac{D_{m,r}^{p-i,p}}{p!} r^i \frac{A_n^{i+1}}{i+1} \\ &= \sum_{i=0}^{p} \frac{p!}{i!(p-i)!}\frac{D_{m,r}^{p-i,p}}{p!} r^i (i+1)! \frac{C_n^{i+1}}{i+1} \\ S'_p &= \sum_{i=0}^{p} \frac{D_{m,r}^{p-i,p}}{(p-i)!} r^i C_n^{i+1} \end{aligned}$$

5 ARITHMETIC SEQUENCES PRODUCTS

5.1 Arithmetic sequence of type A_n^p

5.1.1 Definition

$(U_{n,p})$ is an arithmetic sequence of type A_n^p if there exists a real r called reason such that, for all natural numbers n and p :
$\frac{U_{n+1,p} - U_{n,p}}{pU_{n+1,p-1}} =$ r or $U_{n+1,p} - U_{n,p} = prU_{n+1,p-1}$

5.1.2 Expression of $U_{n,p}$ in terms of n

If the sequence $U_{n,p}$ is arithmetic of type A_n^p of first term, $U_{k,p} = \prod_{k=1}^{p}(m+(i-1)r)$ and reason r then : $U_{n,p} = \sum_{i=0}^{p} C_p^i U_{k,p-i} r^i A_{n-k}^i$; $U_{k,p-i} = \prod_{j=i+1}^{p}(m+(j-1)r)$ with $U_{k,0} = 1$

5.1.3 Sum of consecutive terms

$S_{n,p} = \sum_{i=0}^{p} C_p^i U_{k,p-i} r^i \frac{A_{numberofterms}^{i+1}}{i+1}$
Remark : p = 1, $S_{n,1}$ is the sum of a classical arithmetic sequence

5.2 Example

Let be the sequence $(V_{n,3})$ defined by : $V_{n,3} = (\frac{2}{3} + \frac{1}{4}n)^3 - \frac{1}{4^2}(\frac{2}{3} + \frac{1}{4}n)$, n ∈ N*
1a) Express $V_{n,3}$ as a product of consecutive factors in terms of n.
b)Prove that $V_{n,3}$ is an arithmetic sequence of type A_n^p whose reason will be determined
2a) Calculate $V_{1,3}$ and give a new expression of $V_{n,3}$ as a function of n.
b) Express the sum $S_{n,3}$ of the first n terms of $V_{n,3}$ as a function of n.
3) Let $U_{n,3} = (\frac{2}{3} + \frac{1}{4}n)^3$, n ∈ N* and $S'_{n,3}$ sum of the n first terms of $U_{n,3}$.
Express $S'_{n,3}$ as a function of $S_{n,3}$, $S_{n,1}$ and n.

5.2.1 Resolution:

1a) Expression of $V_{n,3}$ as a product of factor

$$\begin{aligned} V_{n,3} &= (\frac{2}{3} + \frac{1}{4}n)^3 - \frac{1}{4^2}(\frac{2}{3} + \frac{1}{4}n) \\ &= (\frac{2}{3} + \frac{1}{4}n)[(\frac{2}{3} + \frac{1}{4}n)^2 - \frac{1}{4^2}] \\ &= (\frac{2}{3} + \frac{1}{4}n)(\frac{2}{3} + \frac{1}{4}n - \frac{1}{4})(\frac{2}{3} + \frac{1}{4}n + \frac{1}{4}) \\ V_{n,3} &= (\frac{2}{3} + \frac{1}{4}(n-1))(\frac{2}{3} + \frac{1}{4}n)(\frac{2}{3} + \frac{1}{4}(n+1)) \end{aligned}$$

b) Let us prove $V_{n,3}$ is an arithmetic sequence of type A_n^p whose reason will be determined

$$\begin{aligned} \frac{V_{n+1,3-V_{n,3}}}{3V_{n+1,2}} &= \frac{(\frac{2}{3} + \frac{1}{4}n)(\frac{2}{3} + \frac{1}{4}(n+1))(\frac{2}{3} + \frac{1}{4}(n+2)) - (\frac{2}{3} + \frac{1}{4}(n-1))(\frac{2}{3} + \frac{1}{4}n)(\frac{2}{3} + \frac{1}{4}(n+1))}{3(\frac{2}{3} + \frac{1}{4}n)(\frac{2}{3} + \frac{1}{4}(n+1))} \\ &= \frac{(\frac{2}{3} + \frac{1}{4}n)(\frac{2}{3} + \frac{1}{4}(n+1))[(\frac{2}{3} + \frac{1}{4}(n+2)) - (\frac{2}{3} + \frac{1}{4}(n-1))]}{3(\frac{2}{3} + \frac{1}{4}n)(\frac{2}{3} + \frac{1}{4}(n+1))} \\ \frac{V_{n+1,3-V_{n,3}}}{3V_{n+1,2}} &= \frac{1}{4}, \end{aligned}$$

Hence $V_{n,3}$ is an arithmetic sequence of type A_n^p of reason $r = \frac{1}{4}$
2)a) The value of $V_{1,3}$

$$V_{n,3} = (\frac{2}{3} + \frac{1}{4}(n-1))(\frac{2}{3} + \frac{1}{4}n)(\frac{2}{3} + \frac{1}{4}(n+1))$$
$$V_{1,3} = \frac{2}{3} \times \frac{11}{12} \times \frac{7}{6}$$
$$V_{1,3} = \frac{77}{108}$$

New expression of $V_{n,3}$ as a function of n

$$V_{n,3} = \sum_{i=0}^{3} C_3^i V_{1,3-i}(\frac{1}{4})^i A_{n-1}^i$$
$$= C_3^0 V_{1,3}(\frac{1}{4})^0 A_{n-1}^0 + C_3^1 V_{1,2}(\frac{1}{4})^1 A_{n-1}^1 + C_3^2 V_{1,1}(\frac{1}{4})^2 A_{n-1}^2 + C_3^3 V_{1,0}(\frac{1}{4})^3 A_{n-1}^3$$
$$= \frac{2}{3} \times \frac{11}{12} \times \frac{7}{6} + 3(\frac{11}{12} \times \frac{7}{6})\frac{1}{4}(n-1) + 3(\frac{7}{6})(\frac{1}{4})^2(n-1)(n-2) + (\frac{1}{4})^3(n-1)(n-2)(n-3)$$
$$V_{n,3} = \frac{77}{108} + \frac{77}{96}(n-1) + \frac{7}{32}(n-1)(n-2) + \frac{1}{64}(n-1)(n-2)(n-3)$$

b) Let's express the $S_{n,3}$ sum of the first n terms of $V_{n,3}$ as a function of n

$$S_{n,3} = \sum_{i=0}^{3} C_3^i V_{1,3-i}(\frac{1}{4})^i \frac{A_n^{i+1}}{i+1}$$
$$= C_3^0 V_{1,3}(\frac{1}{4})^0 \frac{A_n^1}{1} + C_3^1 V_{1,2}(\frac{1}{4})^1 \frac{A_n^2}{2} + C_3^2 V_{1,1}(\frac{1}{4})^2 \frac{A_n^3}{3} + C_3^3 V_{1,0}(\frac{1}{4})^3 \frac{A_n^4}{4}$$
$$S_{n,3} = \frac{77}{108}n + \frac{77}{192}n(n-1) + \frac{7}{96}n(n-1)(n-2) + \frac{1}{256}n(n-1)(n-2)(n-3)$$

3) Let' express $S'_{n,3}$ as a function of $S_{n,3}$ and $S_{n,1}$
$U_{n,3} = (\frac{2}{3} + \frac{1}{4}n)^3$, n ∈ N*

$$V_{n,3} = (\frac{2}{3} + \frac{1}{4}n)^3 - \frac{1}{4^2}(\frac{2}{3} + \frac{1}{4}n)$$
$$(\frac{2}{3} + \frac{1}{4}n)^3 = V_{n,3} + \frac{1}{4^2}(\frac{2}{3} + \frac{1}{4}n)$$
$$U_{n,3} = V_{n,3} + \frac{1}{4^2}(\frac{2}{3} + \frac{1}{4}(n-1+1))$$
$$= V_{n,3} + \frac{1}{4^2}(\frac{2}{3} + \frac{1}{4}(n-1)) + \frac{1}{4^3}$$
$$U_{n,3} = V_{n,3} + \frac{1}{4^2}V_{n,1} + \frac{1}{4^3}$$

$$S'_{n,3} = U_{1,3} + U_{2,3} + U_{3,3} + ... + U_{n,3}$$
$$= (V_{1,3} + \frac{1}{4^2}V_{1,1} + \frac{1}{4^3}) + (V_{2,3} + \frac{1}{4^2}V_{2,1} + \frac{1}{4^3}) + (V_{3,3} + \frac{1}{4^2}V_{3,1} + \frac{1}{4^3}) + ... + (V_{n,3} + \frac{1}{4^2}V_{n,1} + \frac{1}{4^3})$$
$$= (V_{1,3} + V_{2,3} + V_{3,3} + ... + V_{n,3}) + \frac{1}{4^2}(V_{1,1} + V_{2,1} + V_{3,1} + ... + V_{n,1}) + (\frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} + ... + \frac{1}{4^3})$$
$$S'_{n,3} = S_{n,3} + \frac{1}{4^2}S_{n,1} + \frac{n}{4^3}$$

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