

COMPLEX CIRCLES OF PARTITION AND THE SQUEEZE PRINCIPLE

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ABSTRACT. In this paper we continue the development of the circles of partition by introducing the notion of complex circles of partition. This is an enhancement of such structures from subsets of the natural numbers as base sets to the complex area as base and bearing set. The squeeze principle as a basic tool for studying the possibilities of partitioning of numbers is demonstrated.

1. INTRODUCTION

In our paper [2], motivated in part by the binary Goldbach conjecture and its variants, we developed a method which we feel might be a valuable resource and a recipe for studying problems concerning partition of numbers in specified subsets of \mathbb{N} . The method is very elementary in nature and has parallels with configurations of points on the geometric circle.

Let us suppose that for any $n \in \mathbb{N}$ we can write $n = u + v$ where $u, v \in \mathbb{M} \subset \mathbb{N}$ then our method associate each of this summands to points on the circle generated in a certain manner by $n > 2$ and a line joining any such associated points on the circle. This geometric correspondence turned out to be useful in our development. Due to the geometric correspondence we call the combinatorial structure in this development as the **circle of partition**, abbreviated as **CoP**.

Now we repeat the base results of our special method.

Definition 1.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subseteq \mathbb{N}$. We denote with

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, n - x \in \mathbb{M}\}$$

the **Circle of Partition** generated by n with respect to the subset \mathbb{M} . We will abbreviate this in the further text as **CoP**. We call members of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$. We denote with $\|[x]\| := x$ the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n, \mathbb{M})$ as $\|\mathcal{C}(n, \mathbb{M})\|$. Obviously holds

$$\|\mathcal{C}(n)\| = \{1, 2, \dots, n - 1\}. \tag{1.1}$$

Definition 1.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then we call it as a **degenerated**

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axis $\mathbb{L}_{[x]}$ in comparison to the **real axes** $\mathbb{L}_{[x],[y]}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n, \mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}) \text{ which means } [x], [y] \in \mathcal{C}(n, \mathbb{M}) \text{ with } x + y = n.$$

In the following we consider only real axes. Therefore we abstain from the attribute *real* in this section.

Proposition 1.3. *Each axis is uniquely determined by points $[x] \in \mathcal{C}(n, \mathbb{M})$.*

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n, \mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 1.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. \square

Lemma 1.4. *Each point of a CoP $\mathcal{C}(n, \mathbb{M})$ except its center has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n, \mathbb{M})$ be a point without an axis partner being not the center of the CoP. Then holds for every point $[y] \neq [x]$ except the center

$$x + y \neq n.$$

This is a contradiction to the Definition 1.1. Due to Proposition 1.3 the case of more than one axis partners is impossible. This completes the proof. \square

Notation. We denote by

$$\nu(n, \mathbb{M}) := |\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M})\}| \quad (1.2)$$

the number of real axes of the CoP $\mathcal{C}(n, \mathbb{M})$. It is evident that holds

$$\nu(n, \mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor$$

if $\mathcal{C}(n, \mathbb{M})$ has k members.

2. Complex Circles of Partition

At first we define a special subset of the set of the complex numbers to use it as base set of CoPs.

Definition 2.1. Let $\mathbb{M} \subseteq \mathbb{N}$ and

$$\mathbb{C}_{\mathbb{M}} := \{z = x + iy \mid x \in \mathbb{M}, y \in \mathbb{R}\} \subset \mathbb{C}$$

be a subset of the complex numbers where the real part is from $\mathbb{M} \subseteq \mathbb{N}$. Then a CoP with a special requirement

$$\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}}) = \{[z] \mid z, n - z \in \mathbb{C}_{\mathbb{M}}, \Im(z)^2 = \Re(z)(n - \Re(z))\}$$

will be denoted as a **complex Circle of Partition**, abbreviated as **cCoP**. The special requirement will be called as the *circle condition*.

The components x and y we will call as *real weight* resp. *imaginary weight*. The CoP $\mathcal{C}(n, \mathbb{M})$ will be called as the *source CoP*. Since in the case $\mathbb{M} = \mathbb{N}$ the source CoP is shortened as $\mathcal{C}(n)$ therefore we set

$$\mathcal{C}^o(n) := \mathcal{C}^o(n, \mathbb{C}_{\mathbb{N}}). \quad (2.1)$$

In order to distinguish between points $[z]$ of cCoPs and points z in the complex plane \mathbb{C} we denote the latter as *complex points*.

Definition 2.2. Let $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be a cCoP and $[z] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ a point of it with $z = x + iy$. Then $[n - z]$ with the weight $(n - x) - iy$ denotes the axis partner of $[z]$.

With it the first requirement of a CoP is fulfilled

$$||[z]|| + |[n - z]| = x + iy + n - x - iy = n.$$

Important: For axis partners $[z_1]$ and $[z_2] = [n - z_1]$ holds always

$$\Im(z_1) = -\Im(z_2). \quad (2.2)$$

Definition 2.3. Let $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be a cCoP and $[z] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ a point of it with $z = x + iy$. Then $[\bar{z}]$ with the weight $x - iy$ denotes the *conjugate point* of $[z]$.

Definition 2.4. Let $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be a cCoP and $\mathbb{L}_{[z], [n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ an axis of it. Then

$$\mathbb{L}_{[z], [\overline{n-z}]}$$

denotes the *conjugate axis* of $\mathbb{L}_{[z], [n-z]}$. We do not distinguish between axes $\mathbb{L}_{[z], [n-z]}$ and $\mathbb{L}_{[n-z], [z]}$, since we do not consider axes as different up to rearrangement of resident points.

Definition 2.5. Corresponding with Definition 1.2 we define

$$\nu^o(n, \mathbb{C}_{\mathbb{M}}) := |\{\mathbb{L}_{[z], [n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})\}|$$

as the number of axes of the cCoP $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$. Evidently holds

$$\nu^o(n, \mathbb{C}_{\mathbb{M}}) = \begin{cases} 2\nu(n, \mathbb{M}) \text{ resp.} \\ 2\nu(n, \mathbb{M}) + 1 \end{cases} \quad (2.3)$$

if the CoP $\mathcal{C}(n, \mathbb{M})$ contains a degenerated axis.

We will see that the circle condition

$$\Im(z)^2 = \Re(z)(n - \Re(z)) \quad (2.4)$$

guarantees that all points of a cCoP lie on a circle in the complex plane \mathbb{C} .

Theorem 2.6. *Let $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be a non-empty cCoP. The weights of all its points are located on a circle in the complex plane \mathbb{C} with its center on the real axis at $\frac{n}{2}$ and a diameter n .*

Proof. We consider an arbitrary point $[z] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and its axis partner $[n - z]$. We set $x := \Re(z)$ and $y := \Im(z)$ ¹. With the circle condition (2.4) holds

$$y^2 = x(n - x). \quad (2.5)$$

By virtue of Definition 2.1 is $x \in \mathbb{M} \subseteq \mathbb{N}$. Hence holds $x > 0$. The second requirement for $[z] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ is $n - x \in \mathbb{M}$. Therefore must be $0 < x < n$.

¹We use this setting also in the sequel.

Now we search the greatest imaginary part of the complex point z_o such that holds $[z_o] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$. That means that we search the root of the derivation of (2.5)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \sqrt{x(n-x)} \\ &= \frac{1}{2} \frac{n-2x}{\sqrt{x(n-x)}} \stackrel{!}{=} 0. \end{aligned}$$

Thus we get as root $x_o = \frac{n}{2}$ under the condition that the denominator not becomes zero. We set for x in (2.5) $\frac{n}{2}$ and get

$$y_o^2 = \frac{n}{2} \left(n - \frac{n}{2} \right) = \left(\frac{n}{2} \right)^2$$

and hence $|y_o| = |\Im(z_o)| = \frac{n}{2}$.

Obviously holds $\Im(n-z) = \Im(\bar{z}) = -\Im(z)$. Therefore the points $[z], [\bar{z}], [n-z]$ form a right-angled triangle with the diagonal $\mathcal{L}_{[z],[n-z]}$ and the legs $2y$ and $n-2x$. With the Theorem of *Pythagoras* we get

$$\begin{aligned} |\mathcal{L}_{[z],[n-z]}|^2 &= (2y)^2 + (n-2x)^2 \\ &\text{and with (2.5)} \\ &= 4nx - 4x^2 + n^2 - 4nx + 4x^2 \\ &= n^2 \text{ and hence} \\ |\mathcal{L}_{[z],[n-z]}| &= n. \end{aligned}$$

Since the sum of z and $n-z$ equals n both points $[z], [n-z]$ are end points of an axis $\mathbb{L}_{[z],[n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and at once of a diameter of a circle containing the complex points $z, \bar{z}, n-z$ because their imaginary parts fulfil the circle condition. This is a circle with center on the real axis at $\frac{n}{2}$ and a diameter n . \square

Definition 2.7. The circle in the complex plane \mathbb{C} with center on the real axis at $\frac{n}{2}$ and a diameter n will be denoted as the **embedding circle** \mathfrak{C}_n of the cCoP $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$. It holds

$$\mathfrak{C}_n = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq n, \Im(z)^2 = \Re(z)(n - \Re(z))\}.$$

Additionally let

$$\begin{aligned} \mathfrak{I}_n &:= \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq n, \Im(z)^2 < \Re(z)(n - \Re(z))\} \text{ and} \\ \mathfrak{X}_n &:= \mathbb{C} \setminus (\mathfrak{I}_n \cup \mathfrak{C}_n) \end{aligned}$$

be the sets of all complex points $z \in \mathbb{C}$ inside resp. outside of the embedding circle \mathfrak{C}_n .

Obviously holds for a non-empty cCoP $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ with $m < n$

$$\begin{aligned} |\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})| &\subset \mathfrak{C}_m \subset \mathfrak{I}_n \\ &\text{and} \\ \mathfrak{I}_m &\subset \mathfrak{I}_n \text{ and } \mathfrak{X}_n \subset \mathfrak{X}_m. \end{aligned} \tag{2.6}$$

Corollary 2.8. For all subsets $\mathbb{M} \subseteq \mathbb{N}$ holds that their cCoPs $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ for a fixed generator n have the same embedding circle \mathfrak{C}_n .

Lemma 2.9. *Let \mathfrak{C}_m and \mathfrak{C}_n be two embedding circles with $m \neq n$. Then both circles have the origin as only common point*

$$\mathfrak{C}_m \cap \mathfrak{C}_n = \{(0, 0)\}.$$

Proof. Let $z_m \in \mathfrak{C}_m$ and $z_n \in \mathfrak{C}_n$. We assume that $z_m = z_n$ as a common complex point of both circles. Then holds $\Re(z_m) = \Re(z_n)$. For the imaginary parts we get by virtue of the circle condition (2.4)

$$\begin{aligned} \Im(z_m)^2 &= \Re(z_m)(m - \Re(z_m)) \quad \text{and} \\ \Im(z_n)^2 &= \Re(z_n)(n - \Re(z_n)), \quad \text{which is} \\ &= \Re(z_m)(n - \Re(z_m)) \quad \text{and as difference} \\ \Im(z_m)^2 - \Im(z_n)^2 &= \Re(z_m)(m - n) = \Re(z_n)(m - n). \end{aligned}$$

Since $m \neq n$ this is only equal zero if $\Re(z_m) = \Re(z_n) = 0$. Then for the imaginary part by virtue of the circle condition, we get

$$\Im(z_m)^2 = 0(m - 0) = 0(n - 0) = \Im(z_n)^2.$$

Hence the origin is the only common point of \mathfrak{C}_m and \mathfrak{C}_n . \square

Corollary 2.10 (Big Bang). *If $m < n$ then the circle \mathfrak{C}_m resides fully inside of the circle \mathfrak{C}_n except the common origin. Hence the origin is the only common complex point of **all** embedding circles with growing diameters, is the "Big Bang" of all embedding circles.*

Theorem 2.11. *Let $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be two non-empty cCoPs with $m \neq n$. Then both cCoPs have no common point*

$$\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}}) \cap \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}}) = \emptyset.$$

Proof. In virtue of (2.6) and Lemma 2.10 the origin could be the only common point of both cCoPs. But since $\mathbb{M} \subseteq \mathbb{N}$ the real weight of a point of any cCoP cannot be 0. Hence both cCoPs have no common point. \square

Proposition 2.12. *Let $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be two non-empty cCoPs with $n \neq m$. They have points $[z_m] \in \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ and $[z_n] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ with a common real weight $\Re(z_m) = \Re(z_n) = x \in \mathbb{M}$ if and only if their source CoPs $\mathcal{C}(m, \mathbb{M})$ and $\mathcal{C}(n, \mathbb{M})$ have a common point $[x]$.*

Proof. Let $[x]$ be a common point of $\mathcal{C}(m, \mathbb{M})$ and $\mathcal{C}(n, \mathbb{M})$. Then $m - x$ and $n - x$ are members of \mathbb{M} and $m - x - iy_m$ and $n - x - iy_n$ are members of $\mathbb{C}_{\mathbb{M}}$. And then also their axis partners $x + iy_m$ and $x + iy_n$ are members of $\mathbb{C}_{\mathbb{M}}$. This means that with $z_m = x + iy_m$ and $z_n = x + iy_n$ holds

$$[z_m] \in \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}}) \quad \text{and} \quad [z_n] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$$

with $x = \Re(z_m) = \Re(z_n)$.

This conclusion chain can be inverted and it holds that from $x = \Re(z_m) = \Re(z_n)$ follows $[x] \in \mathcal{C}(m, \mathbb{M}) \cap \mathcal{C}(n, \mathbb{M})$. \square

Corollary 2.13. *From Proposition 2.13 follows that a cCoP $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ is non-empty if and only if its source CoP $\mathcal{C}(n, \mathbb{M})$ is non-empty.*

Proposition 2.14. *In the special case $\mathbb{M} = \mathbb{N}$ all cCoPs $\mathcal{C}^o(n)$ for integers $n \geq 2$ are non-empty.*

Proof. The source CoPs of such cCoPs are $\mathcal{C}(n)$ by virtue of Definition 1.1. And these are non-empty for all integers $n \geq 2$ by virtue of (1.1). Due to Corollary 2.14 also their cCoPs are non-empty. \square

Corollary 2.15. *Since the considered points $[z]$ in Theorem 2.6 were arbitrary holds that all axes of a cCoP $\mathcal{C}^o(n, \mathbb{C}_M)$ have equal lengths*

$$|\mathbb{L}_{[z], [n-z]}| = n \text{ for all points } [z] \in \mathcal{C}^o(n, \mathbb{C}_M).$$

Now we specify the calculation of the length of a chord joining arbitrary points of a cCoP under the circle condition.

Theorem 2.16. *Let $\mathcal{C}^o(n, \mathbb{C}_M)$ be a non-empty cCoP and $[z_1], [z_2] \in \mathcal{C}^o(n, \mathbb{C}_M)$ be two arbitrary points of it. Then we get for the length $\Gamma([z_1], [z_2])$ ² of the chord $\mathcal{L}_{[z_1], [z_2]}$*

$$|\mathcal{L}_{[z_1], [z_2]}| = \Gamma([z_1], [z_2]) = |\sqrt{x_1(n-x_2)} \pm \sqrt{x_2(n-x_1)}|,$$

whereby "−" will be taken if $\text{sign}(y_1) = \text{sign}(y_2)$ and "+" else.

Proof.

$$\begin{aligned} |\mathcal{L}_{[z_1], [z_2]}|^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 \pm 2|y_1y_2| \\ &\text{and with (2.5)} \\ &= x_1^2 + x_2^2 - 2x_1x_2 \pm 2|y_1y_2| + nx_1 - x_1^2 + nx_2 - x_2^2 \\ &= nx_1 - x_1x_2 + nx_2 - x_1x_2 \pm 2|y_1y_2| \\ &= x_1(n-x_2) + x_2(n-x_1) \pm 2|y_1y_2| \\ &= x_1(n-x_2) + x_2(n-x_1) \pm 2\sqrt{x_1(n-x_1)} \cdot \sqrt{x_2(n-x_2)} \\ &= x_1(n-x_2) + x_2(n-x_1) \pm 2\sqrt{x_2(n-x_1)} \cdot \sqrt{x_1(n-x_2)} \\ &= \left(\sqrt{x_1(n-x_2)} \pm \sqrt{x_2(n-x_1)} \right)^2. \end{aligned}$$

Hence the function $\Gamma([z_1], [z_2])$ for the chord length simplifies to

$$\Gamma([z_1], [z_2]) = |\sqrt{x_1(n-x_2)} \pm \sqrt{x_2(n-x_1)}|. \quad (2.7)$$

\square

In the case that $[z_2]$ becomes $[n-z_1]$ then the chord $\mathcal{L}_{[z_1], [z_2]}$ becomes a diameter. Then holds $y_2 = -y_1$ and $x_2 = n-x_1$ and therefore

$$\begin{aligned} \Gamma([z_1], [n-z_1]) &= |\sqrt{x_1(n-x_2)} + \sqrt{x_2(n-x_1)}| \\ &= |\sqrt{x_1x_1} + \sqrt{(n-x_1)(n-x_1)}| \\ &= |x_1 + n - x_1| = n. \end{aligned}$$

If $[z_2]$ is the axis partner of the conjugate point of $[z_1]$ then holds that $x_2 = n-x_1$ and $y_2 = y_1$. Since the signs of both y are equal we get in this case

$$\begin{aligned} \Gamma([z_1], [n-z_1]) &= |\sqrt{x_1x_1} - \sqrt{x_2x_2}| \\ &= |x_1 - x_2|. \end{aligned}$$

²See [2, p. 2] Definition 2.2

This result coincides with the chord length in a CoP in virtue of its definition in [2].

A degenerated axis of a cCoP coincides with the diameter that is parallel to the imaginary axis. It is a real diameter but with the property that it equals to its conjugate axis. In this case, we get from (2.7) with $x_2 = x_1 = \frac{n}{2}$ and $y_2 = -y_1$

$$\Gamma([z_1], [n - z_1]) = \left| \sqrt{\left(\frac{n}{2}\right)^2} + \sqrt{\left(\frac{n}{2}\right)^2} \right| = n.$$

3. Interior and Exterior Points of Complex Circles of Partition

Theorem 3.1. *Let $\mathcal{C}^o(n, \mathbb{C}_M)$ be a non-empty cCoP. Then the distance from every complex point of $|\mathcal{C}^o(n, \mathbb{C}_M)|$ to every complex point in \mathfrak{I}_n is less than n and from some complex point in $|\mathcal{C}^o(n, \mathbb{C}_M)|$ to every complex point in \mathfrak{X}_n greater than n .*

Proof. The diameter of \mathfrak{C}_n is the longest line from any complex point on this circle to any complex point inside or on the circle. Hence all complex points of \mathfrak{I}_n have a smaller distance to any complex point on \mathfrak{C}_n than the diameter. Since (2.6) this relation is also valid between any complex points of $|\mathcal{C}^o(n, \mathbb{C}_M)|$ and \mathfrak{I}_n . Therefore their distances are less than the diameter of \mathfrak{C}_n , which is n .

And vice versa the distances between some complex point of $\mathcal{C}^o(n, \mathbb{C}_M)$ and every complex point in \mathfrak{X}_n is greater than n since \mathfrak{X}_n are the complex points outside of the embedding circle \mathfrak{C}_n . This completes the proof. \square

Corollary 3.2. *For two non-empty cCoPs $\mathcal{C}^o(m, \mathbb{C}_M)$ and $\mathcal{C}^o(n, \mathbb{C}_M)$ with $m < n$ holds that all distances between points of them are less than n and some are greater than m .*

Definition 3.3. Since $\mathfrak{I}_n, \mathfrak{X}_n$ are defined in Definition 2.8 as all complex points **inside** resp. **outside** of the embedding circle \mathfrak{C}_n , we call the points $z \in \mathfrak{I}_n \cap \mathbb{C}_M$ as **interior** points with respect to \mathfrak{C}_n and denote the set of all such points as $\text{Int}[\mathfrak{C}_n]$.

Correspondingly, we call the complex points $z \in \mathfrak{X}_n \cap \mathbb{C}_M$ as **exterior** points with respect to \mathfrak{C}_n and denote the set of all these points as $\text{Ext}[\mathfrak{C}_n]$.

Obviously holds

$$\text{Int}[\mathfrak{C}_n] = \mathfrak{I}_n \cap \mathbb{C}_M \text{ and } \text{Ext}[\mathfrak{C}_n] = \mathfrak{X}_n \cap \mathbb{C}_M.$$

Definition 3.4. Let $\mathcal{C}^o(n, \mathbb{C}_M)$ be a non-empty cCoP and \mathfrak{C}_n its embedding circle. Then we call the complex point $z \in \text{Int}[\mathfrak{C}_n]$ as an **interior** point with respect to the cCoP $\mathcal{C}^o(n, \mathbb{C}_M)$ if and only if for **all** points $[w] \in \mathcal{C}^o(n, \mathbb{C}_M)$ holds $|z - w| < n$. We denote the set of all such points as $\text{Int}[\mathcal{C}^o(n, \mathbb{C}_M)]$.

Correspondingly, we call the complex point $z \in \text{Ext}[\mathfrak{C}_n]$ as **exterior** point with respect to $\mathcal{C}^o(n, \mathbb{C}_M)$ if and only if for **some** points $[w] \in \mathcal{C}^o(n, \mathbb{C}_M)$ holds $|z - w| > n$ and denote the set of all such points as $\text{Ext}[\mathcal{C}^o(n, \mathbb{C}_M)]$.

Let $n_o \in \mathbb{N}$ be the least generator for all cCoPs. If $n > n_o$ and $\mathcal{C}^o(n, \mathbb{C}_M)$ is an empty cCoP, then $\text{Int}[\mathcal{C}^o(n, \mathbb{C}_M)]$ and $\text{Ext}[\mathcal{C}^o(n, \mathbb{C}_M)]$ are empty by definition.

Corollary 3.5. *If $\mathcal{C}^\circ(n, \mathbb{C}_M)$ is a non-empty cCoP then by virtue of Theorem 3.1 holds*

$$\begin{aligned} \text{Int}[\mathcal{C}^\circ(n, \mathbb{C}_M)] &= \text{Int}[\mathfrak{C}_n] = \mathfrak{J}_n \cap \mathbb{C}_M \\ &\text{and} \\ \text{Ext}[\mathcal{C}^\circ(n, \mathbb{C}_M)] &= \text{Ext}[\mathfrak{C}_n] = \mathfrak{X}_n \cap \mathbb{C}_M. \end{aligned} \tag{3.1}$$

Proposition 3.6. *Let $\mathcal{C}^\circ(m, \mathbb{C}_M)$ and $\mathcal{C}^\circ(n, \mathbb{C}_M)$ be two non-empty cCoPs. If and only if $m < n$ holds*

$$\text{Int}[\mathcal{C}^\circ(m, \mathbb{C}_M)] \subset \text{Int}[\mathcal{C}^\circ(n, \mathbb{C}_M)] \text{ and } \text{Ext}[\mathcal{C}^\circ(n, \mathbb{C}_M)] \subset \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)].$$

Proof. Let $m < n$, then since (3.1) holds

$$\begin{aligned} \text{Int}[\mathcal{C}^\circ(m, \mathbb{C}_M)] &= \mathfrak{J}_m \cap \mathbb{C}_M \text{ and since (2.6)} \\ &\subset \mathfrak{J}_n \cap \mathbb{C}_M = \text{Int}[\mathcal{C}^\circ(n, \mathbb{C}_M)]. \end{aligned}$$

Vice versa holds

$$\begin{aligned} \text{Ext}[\mathcal{C}^\circ(n, \mathbb{C}_M)] &= \mathfrak{X}_n \cap \mathbb{C}_M \text{ and since (2.6)} \\ &\subset \mathfrak{X}_m \cap \mathbb{C}_M = \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)]. \end{aligned}$$

On the other hand from $\text{Int}[\mathcal{C}^\circ(m, \mathbb{C}_M)] \subset \text{Int}[\mathcal{C}^\circ(n, \mathbb{C}_M)]$ follows $\mathfrak{J}_m \cap \mathbb{C}_M \subset \mathfrak{J}_n \cap \mathbb{C}_M$, which is only with $m < n$ solvable. Analogously follows from $\text{Ext}[\mathcal{C}^\circ(n, \mathbb{C}_M)] \subset \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)]$ also $m < n$. \square

Proposition 3.7. *Let $\mathcal{C}^\circ(m, \mathbb{C}_M)$ and $\mathcal{C}^\circ(n, \mathbb{C}_M)$ be two non-empty cCoPs. If and only if $m < n$ holds*

$$\|\mathcal{C}^\circ(m, \mathbb{C}_M)\| \subset \text{Int}[\mathcal{C}^\circ(n, \mathbb{C}_M)] \text{ and } \|\mathcal{C}^\circ(n, \mathbb{C}_M)\| \subset \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)].$$

Proof. Let $m < n$, then since (2.6) and $\|\mathcal{C}^\circ(m, \mathbb{C}_M)\| \subset \mathbb{C}_M$ holds

$$\begin{aligned} \|\mathcal{C}^\circ(m, \mathbb{C}_M)\| &\subset \mathfrak{C}_m \cap \mathbb{C}_M \\ &\subset (\mathfrak{C}_m \cap \mathbb{C}_M) \cup \mathfrak{J}_m \\ &\subset (\mathfrak{C}_m \cup \mathfrak{J}_n) \cap \mathbb{C}_M \text{ and since } \mathfrak{C}_m \subset \mathfrak{J}_n \\ &= \mathfrak{J}_n \cap \mathbb{C}_M \text{ and because of (3.1)} \\ &= \text{Int}[\mathcal{C}^\circ(n, \mathbb{C}_M)]. \end{aligned}$$

In a similar manner $\|\mathcal{C}^\circ(n, \mathbb{C}_M)\| \subset \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)]$ can be proved.

On the other hand, the embedding $\|\mathcal{C}^\circ(m, \mathbb{C}_M)\| \subset \text{Int}[\mathcal{C}^\circ(n, \mathbb{C}_M)]$ implies $\mathfrak{J}_m \cap \mathbb{C}_M \subset \mathfrak{J}_n \cap \mathbb{C}_M$, which is only with $m < n$ solvable. Analogously follows from $\text{Ext}[\mathcal{C}^\circ(n, \mathbb{C}_M)] \subset \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)]$ also $m < n$. \square

Proposition 3.8. *Let $\mathcal{C}^\circ(m, \mathbb{C}_M) \neq \emptyset$. If $[z_1], [z_2]$ are axis partners of the cCoP $\mathcal{C}^\circ(n, \mathbb{C}_M)$ and $|\mathbb{L}_{[z_1], [z_2]}| = n > m$, then $z_1, z_2 \in \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)]$.*

Proof. From the requirement $\mathbb{L}_{[z_1], [z_2]} \hat{\in} \mathcal{C}^\circ(n, \mathbb{C}_M)$ with $n > m$ and Proposition 3.6, it follows that

$$\begin{aligned} \|\mathcal{C}^\circ(n, \mathbb{C}_M)\| &\subset \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)] \text{ and therefore} \\ z_1, z_2 &\in \text{Ext}[\mathcal{C}^\circ(m, \mathbb{C}_M)]. \end{aligned}$$

\square

Proposition 3.9. *Let $\mathcal{C}^o(m, \mathbb{C}_M) \neq \emptyset$. If $\text{Int}[\mathcal{C}^o(m, \mathbb{C}_M)] \subset \text{Int}[\mathcal{C}^o(n, \mathbb{C}_M)]$, then $\mathcal{C}^o(n, \mathbb{C}_M) \neq \emptyset$.*

Proof. The conditions above with Definition 3.3 implies that $\text{Int}[\mathcal{C}^o(m, \mathbb{C}_M)] \neq \emptyset$ and $\text{Int}[\mathcal{C}^o(n, \mathbb{C}_M)] \supset \emptyset$, and hence $\mathcal{C}^o(n, \mathbb{C}_M) \neq \emptyset$. \square

We state a sort of converse of the above result in the following theorem.

Theorem 3.10. *Let $\mathcal{C}^o(m, \mathbb{C}_M), \mathcal{C}^o(n, \mathbb{C}_M) \neq \emptyset$. If $m < n$, then holds for each point $[z] \in \mathcal{C}^o(n, \mathbb{C}_M)$ that its value z is not a member of $\text{Int}[\mathcal{C}^o(m, \mathbb{C}_M)]$.*

Proof. By virtue of Definition 2.8 holds $\mathfrak{C}_n \cap \mathfrak{I}_n = \emptyset$ and $|\mathcal{C}^o(n, \mathbb{C}_M)| \subset \mathfrak{C}_n$, it follows easily that $\mathfrak{I}_n \cap |\mathcal{C}^o(n, \mathbb{C}_M)| = \emptyset$. Since for each point $[z] \in \mathcal{C}^o(n, \mathbb{C}_M)$ holds $z \notin \mathfrak{I}_n$ and because of $m < n$ holds additionally $\mathfrak{I}_m \subset \mathfrak{I}_n$ and hence

$$z \notin \mathfrak{I}_n \supset \mathfrak{I}_m \supset \mathfrak{I}_m \cap \mathbb{C}_M = \text{Int}[\mathcal{C}^o(m, \mathbb{C}_M)].$$

\square

4. The Squeeze Principle

We do not distinguish between axes $\mathbb{L}_{[z],[n-z]}$ and $\mathbb{L}_{[n-z],[z]}$ (see Definition 2.4) in this section, since axes are not considered as different up to rearrangement of resident points. We consider only axis $\mathbb{L}_{[z_1],[z_2]}$ with

$$\Re(z_1) < \Re(z_2)$$

in this section.

Lemma 4.1 (Axial Points Ordering Principle). *Let $M \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_M)$ and $\mathcal{C}^o(n+t, \mathbb{C}_M)$ with $t > 0$ be non-empty cCoPs with integers n, t of the same parity and axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_M)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_M)$. It holds*

$$\Re(z_1) < \Re(w_1) \text{ and } \Re(z_2) < \Re(w_2) \quad (4.1)$$

if and only if holds

$$\Re(z_1) < \Re(w_1) < \Re(z_1) + t. \quad (4.2)$$

Proof. We note that the left inequalities are equal. Hence we have to show that the right ones are equivalent. At first, we assume (4.1). From the right inequality and the existence of $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_M)$ we get

$$\Re(z_2) < \Re(w_2) = n+t - \Re(w_1) \longrightarrow \Re(w_1) < n+t - \Re(z_2) = \Re(z_1) + t.$$

This is the right side of (4.2).

If on the other hand the right side of (4.2) holds, then we get together with $\Re(w_1) = n+t - \Re(w_2)$

$$\Re(w_1) = n+t - \Re(w_2) < \Re(z_1) + t = n - \Re(z_2) + t \longrightarrow \Re(z_2) < \Re(w_2).$$

This is the right inequality of (4.1). \square

Corollary 4.2. *Obviously holds for the size of the limitation interval (4.3)*

$$(\Re(z_1) + t) - \Re(z_1) = t.$$

This is independent from the choice of the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$.

Lemma 4.3 (The squeeze principle). *Let $\mathbb{B} \subset \mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ with $t \geq 4$ be non-empty cCoPs with integers n, t, s of the same parity. If there exist an axis $\mathbb{L}_{[z_1], [z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ with $z_2 \in \mathbb{C}_{\mathbb{B}}$ and an axis $\mathbb{L}_{[w_1], [w_2]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ with $w_1 \in \mathbb{C}_{\mathbb{B}}$ such that*

$$\mathfrak{R}(z_1) < \mathfrak{R}(w_1) < \mathfrak{R}(z_1) + t \quad (4.3)$$

then there exists an axis $\mathbb{L}_{[w_1], [z_2]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{B}})$ with $0 < s < t$. Hence also $\mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{M}})$ is not empty.

Proof. From the existence of the axis $\mathbb{L}_{[z_1], [z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ follows $\mathfrak{R}(z_2) = n - \mathfrak{R}(z_1)$. With the left side of the requirement (4.3), we get

$$\mathfrak{R}(z_2) > n - \mathfrak{R}(w_1). \quad (4.4)$$

From the right inequality of (4.3) follows

$$\mathfrak{R}(w_1) < \mathfrak{R}(z_1) + t = n - \mathfrak{R}(z_2) + t \longrightarrow \mathfrak{R}(z_2) < n + t - \mathfrak{R}(w_1)$$

and we get

$$\begin{aligned} n - \mathfrak{R}(w_1) &< \mathfrak{R}(z_2) < n + t - \mathfrak{R}(w_1) \quad | \quad +\mathfrak{R}(w_1) \\ n &< \mathfrak{R}(z_2) + \mathfrak{R}(w_1) < n + t \\ n &< n + s < n + t. \end{aligned}$$

By virtue of the requirements $z_2, w_1 \in \mathbb{C}_{\mathbb{B}}$ and $n + s = \mathfrak{R}(z_2) + \mathfrak{R}(w_1)$ there is an axis $\mathbb{L}_{[w_1], [z_2]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{B}})$ and hence holds $\mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{B}}) \neq \emptyset$. And from $\mathbb{B} \subset \mathbb{M}$ follows immediately $\mathbb{C}_{\mathbb{B}} \subset \mathbb{C}_{\mathbb{M}}$ and therefore holds also $\mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{M}}) \neq \emptyset$. This completes the proof. \square

Lemma 4.3 can be viewed as a basic tool-box for studying the possibility of partitioning numbers of a particular parity with components belonging to a special subset of the integers. It works by choosing two non-empty cCoPs with the same base set and finding further non-empty cCoPs with generators trapped in between these two generators. This principle can be used in an ingenious manner to study the broader question concerning the feasibility of partitioning numbers with each summand belonging to the same subset of the positive integers. We formulate the following lemma as a special case of Lemma 4.3.

Lemma 4.4 (Special squeeze principle). *Let $n, t, s \in 2\mathbb{N}$ and \mathbb{P} be the set of all odd primes. If $t \geq 4$ and there exist an axis $\mathbb{L}_{[z_1], [z_2]} \hat{\in} \mathcal{C}^o(n)$ with $z_2 \in \mathbb{C}_{\mathbb{P}}$ and an axis $\mathbb{L}_{[w_1], [w_2]} \hat{\in} \mathcal{C}^o(n+t)$ with $w_1 \in \mathbb{C}_{\mathbb{P}}$ such that*

$$\mathfrak{R}(z_1) < \mathfrak{R}(w_1) < \mathfrak{R}(z_1) + t$$

then there exists an axis $\mathbb{L}_{[w_1], [z_2]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{P}})$ with $0 < s < t$.

Proof. Since holds $\mathbb{P} \subset \mathbb{N}$ and by virtue of Proposition 2.15 all cCoPs $\mathcal{C}^o(n)$ with even $n \geq 2$ are non-empty the requirements of Lemma 4.3 are fulfilled. \square

Since $n, t, s \in 2\mathbb{N}$ for $t = 4$ due to $0 < s < t$ only $s = 2$ is possible. If there exists an axis $\mathbb{L}_{[z_1], [z_2]} \hat{\in} \mathcal{C}^o(n)$ with $z_2 \in \mathbb{C}_{\mathbb{P}}$ such that $z_1 + 2 \in \mathbb{C}_{\mathbb{P}}$ then exists an axis $\mathbb{L}_{[z_1+2], [z_2+2]} \hat{\in} \mathcal{C}^o(n+4)$ with $z_1 + 2 \in \mathbb{C}_{\mathbb{P}}$ and it holds $\mathfrak{R}(z_1) < \mathfrak{R}(z_1 + 2)$ and $\mathfrak{R}(z_2) < \mathfrak{R}(z_2 + 2)$. Hence the special squeeze principle reduces to the following Theorem.

Theorem 4.5. *Let m be an integer ≥ 3 and \mathbb{P} like in Lemma 4.4. If there exist an axis*

$$\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(2m) \tag{4.5}$$

with $z_2 \in \mathbb{C}_{\mathbb{P}}$ such that holds

$$z_1 + 2 \in \mathbb{C}_{\mathbb{P}} \tag{4.6}$$

then there exists an axis $\mathbb{L}_{[z_1+2],[z_2]} \hat{\in} \mathcal{C}^o(2m + 2, \mathbb{C}_{\mathbb{P}}) \neq \emptyset$.

Proof. From (4.5) we have $\Re(z_1) + \Re(z_2) = 2m$. Hence holds

$$\Re(z_1 + 2) + \Re(z_2) = \Re(z_1) + \Re(z_2) + 2 = 2m + 2.$$

Due to (4.6) and the requirement $z_2 \in \mathbb{C}_{\mathbb{P}}$ there is an axis

$$\mathbb{L}_{[z_1+2],[z_2]} \hat{\in} \mathcal{C}^o(2m + 2, \mathbb{C}_{\mathbb{P}}).$$

□

It is well known that for all odd primes $p \geq 5$ holds

$$p \equiv \pm 1 \pmod{6}.$$

On the other hand all cCoPs with base set $\mathbb{C}_{\mathbb{P}}$ are empty for odd generators $2m - 1$. Therefore there remain for generators of non-empty cCoPs $\mathcal{C}^o(2m, \mathbb{C}_{\mathbb{P}})$ only the residue classes $2m \equiv -2 \pmod{6}$, $2m \equiv 0 \pmod{6}$ and $2m \equiv +2 \pmod{6}$.

Lemma 4.6. *Let the requirements of Theorem 4.5 be fulfilled. If holds $2m \equiv +2 \pmod{6}$ then must be $\Re(z_2) \equiv -1 \pmod{6}$ and if holds $2m \equiv 0 \pmod{6}$ then must be $\Re(z_2) \equiv +1 \pmod{6}$.*

Proof. At first, we consider the case $2m \equiv +2 \pmod{6}$. Then holds if $\Re(z_2) \equiv +1 \pmod{6}$

$$\Re(z_1) = 2m - \Re(z_2) \equiv 2 - 1 \pmod{6} \equiv 1 \pmod{6} \longrightarrow \Re(z_1 + 2) \equiv 3 \pmod{6},$$

which means that $z_1 + 2$ cannot be a member of $\mathbb{C}_{\mathbb{P}}$.

If holds $2m \equiv 0 \pmod{6}$ and $\Re(z_2) \equiv -1 \pmod{6}$ we get

$$\Re(z_1) = 2m - \Re(z_2) \equiv 0 + 1 \pmod{6} \equiv 1 \pmod{6} \longrightarrow \Re(z_1 + 2) \equiv 3 \pmod{6},$$

which also means that $z_1 + 2$ cannot be a member of $\mathbb{C}_{\mathbb{P}}$. The case $\Re(z_2) = 3$ can be excluded, since $\Re(z_1 + 2)$ must be less than $\Re(z_2)$. It is easy to check that in all other cases $z_1 + 2$ can be a member of $\mathbb{C}_{\mathbb{P}}$. □

REFERENCES

1. Estermann, Theodor *On Goldbach's problem: Proof that almost all even positive integers are sums of two primes*, Proceedings of the London Mathematical Society, vol. 2:1, Wiley Online Library, 1938, pp. 307–314.
2. Agama, Theophilus and Gensel, Berndt *Studies in Additive Number Theory by Circles of Partition*, arXiv:2012.01329, 2020.
3. Helfgott, Harald A *The ternary Goldbach conjecture is true*, arXiv preprint arXiv:1312.7748, 2013.
4. Green, Ben and Tao, Terence *The primes contain arbitrarily long arithmetic progressions*, Annals of Mathematics, JSTOR, 2008, pp. 481–547.
5. Chen, Jing-run *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, The Goldbach Conjecture, World Scientific, 2002, pp. 275–294.

6. Heath-Brown, D Roger and Puchta, J-C *Integers represented as a sum of primes and powers of two*, Asian J. Math, vol: 6(3), 2002, 535–566.
7. Shnirel'man, Lev Genrikhovich, *On the additive properties of numbers*, Uspekhi Matematicheskikh Nauk, vol. 2:6, Russian Academy of Sciences, Steklov Mathematical Institute of Russian , 1939, pp. 9–25.
8. Chudakov, Nikolai Grigor'evich, *The Goldbach's problem*, Uspekhi Matematicheskikh Nauk, vol. 4, Russian Academy of Sciences, Steklov Mathematical Institute of Russian , 1938, 14–33.
9. Hardy, Godfrey H and Littlewood, John E *Some problems of Partitio Numerorum(V): A further contribution to the study of Goldbach's problem*, Proceedings of the London Mathematical Society, vol. 2(1), Wiley Online Library, 1924, pp. 46–56.

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