

The randomness in the prime numbers

Ihsan Raja Muda Nasution

August 24, 2023

Abstract

The prime numbers have very irregular pattern. The problem of finding pattern in the prime numbers is the long-standing open problem in mathematics. In this paper, we try to solve the problem axiomatically. And we propose some natural properties of prime numbers.

MSC: 11A41

Keywords: prime numbers, distribution of primes, Riemann zeta-function, structure of primes

1 Introduction

In 1859, Riemann [Rie59] showed a deep connection between non-trivial zeros of the Riemann zeta-function and the prime numbers. Our goal is to axiomatize the structure of primes.

2 The pattern of numbers

These below are some patterns of number.

Let t_n denote the n th triangular number. Then

$$t_n = \binom{n+1}{2} \quad n \geq 1,$$

where $\binom{n+1}{2}$ is the binomial coefficients [Bur02, p. 15].

Let F_n be the n th Fibonacci number. Then

$$F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}},$$

where n is a positive integer [Wei03, p. 1042].

Let B_n be the n th Bernoulli number. Then

$$B_n = (-1)^{n+1} n \zeta(1 - n),$$

where $\zeta(1 - n)$ is the Riemann zeta-function [Wei03, p. 189].

If $p(n)$ denotes the total number of partitions of n , then

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}},$$

where n is a positive integer [Har99, p. 116].

Postulate 2.1 (Peano Postulates). Given the number 0, the set \mathbf{N} , and the function σ . Then:

1. $0 \in \mathbf{N}$.
2. $\sigma : \mathbf{N} \rightarrow \mathbf{N}$ is a function from \mathbf{N} to \mathbf{N} .
3. $0 \notin \text{range}(\sigma)$.
4. The function σ is one-to-one.
5. If $I \subset \mathbf{N}$ such that $0 \in I$ and $\sigma(n) \in I$ whenever $n \in I$, then $I = \mathbf{N}$.

We define $1 = \sigma(0)$, $2 = \sigma(1)$, $3 = \sigma(2)$, etc. [Eis96, p. A64].

3 The axiomatic approach

In this section, we try to solve the problem of prime number axiomatically. We propose the basic characteristics of prime numbers. The prime numbers stands on several basic assumptions. Given the prime numbers p, q and the set \mathbf{N} . Then

Postulate 3.1. $2 \leq p$.

Postulate 3.1 says that 2 is the least prime number. 2 is the only one even prime number. 2 is also assumed as a generator of primes. For a composite number k , $4 \leq k$.

Postulate 3.2. $4 \nmid p$.

Postulate 3.2 holds generally for any prime number. Postulate 3.2 expresses the indivisibility of primes over the number 4. The number 4 is the least composite number which cannot divide any prime number. For a composite number k , $4 \mid k$ or $4 \nmid k$.

Definition 3.3. For an integer $n > 1$, where $\tau(n)$ denote the number of positive divisors of n . The function $\chi(n)$ is defined by

$$\chi(n) = \begin{cases} 0 & \text{if } \tau(n) = 2 \\ 1 & \text{if } \tau(n) > 2. \end{cases}$$

Postulate 3.4. $(-1)^{\chi(p)} = 1$.

Postulate 3.4 shows the connection between -1 and the prime number p . For a composite number k , $(-1)^{\chi(k)} = -1$. Hence, $(-1)^{\chi(n)} = \pm 1$ for $n \geq 2$. $\sigma(n)$ denotes the sum of positive divisors of n . Then

Postulate 3.5. $3 \leq \sigma(p)$.

Postulate 3.5 expresses that 3 is the lower bound of $\sigma(p)$. Postulate 3.1 can deduce Postulate 3.5. Given $2 \leq p$. Then $2 + 1 \leq p + 1$ implies $3 \leq \sigma(p)$. Postulate 3.5 is the one of basic patterns of the function $\sigma(n)$. For a composite number k , $7 \leq \sigma(k)$.

Definition 3.6. Given an integer $n > 1$, let $\Delta(n)$ denote the number of positive divisors of n besides 1 and n .

Postulate 3.7. $\Delta(p) = 0$.

Postulate 3.7 shows that the zeros of the function Δ are any prime number. If $\Delta(p) = 0$, then $\Delta^{-1}(0) = p$. For a composite number k , $\Delta(k) \neq 0$.

Postulate 3.8. If $p, q \in \mathbf{N}$, then $pq \in \mathbf{N}$.

Postulate 3.8 explains the closure property of multiplication over the prime numbers p and q . By using induction, Postulate 3.8 can imply the prime factorization of a positive integer. And Postulate 3.8 shows the existence of semiprimes.

References

[Bur02] D. M. Burton. *Elementary Number Theory*. McGraw-Hill, New York, 5th edition, 2002.

- [Eis96] M. Eisenberg. *The Mathematical Method*. Prentice-Hall, Inc., Upper Saddle River, NJ, 1996.
- [Har99] G. H. Hardy. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*. Chelsea, New York, 3rd edition, 1999.
- [Rie59] B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsber. Akad. Berlin*, pages 671–680, 1859.
- [Wei03] E. W. Weisstein. *CRC Concise Encyclopedia of Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2nd edition, 2003.