Rutherford Cross Section in the laboratory frame-PartII

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(Dated: April 22, 2023)

In this pedagogical article, we extend the direct derivation of the classical non-relativistic Rutherford scattering cross section, differential, in the laboratory frame of two equal mass particles, ala relativistic quantum mechanics as presented in the book of Bjorken and Drell up to $(v/c)^2$ order.

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I. INTRODUCTION

The classical non-relativistic Rutherford scattering cross section, differential, in the laboratory frame is computed in the classical mechanics in two steps. First the Rutherford scattering cross section, differential, in the center of mass frame is calculated, [1], as $\frac{k^2}{16E^2_{c.o.m}}\frac{1}{\sin^4\frac{\Theta}{2}}$, where, Θ is the c.o.m scattering angle. Then using transformation from the the center of mass to the laboratory frame, [1], the differential scattering cross section is calculated in the laboratory frame, for two particles of equal mass and elastic scattering, as $\frac{k^2}{E^2_{lab}}\frac{cos\theta}{\sin^4\theta}$, where, θ is the laboratory frame scattering angle. On the other hand, in the relativistic quantum mechanics of Dirac, the differential scattering cross section can be calculated in the laboratory frame directly, [2]. Taking non-relativistic limit has yielded, [3] us the classical non-relativistic differential cross-section, $\frac{k^2}{E^2_{lab}}\frac{cos\theta}{\sin^4\theta}$. In this paper, we extend the analysis up to $(v/c)^2$ order.

II.

Starting from the Dirac equation of a spin half particle, of rest mass m_0 and charge q, in the presence of electromagnetic field, A^μ, is given by $(i\hbar\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m_0c)\psi(x) = 0$ and considering a scattering between a a hypothetical proton and an electron, assuming the hypothetical proton's mass being equal to that of the electron i.e. seting $m_0 = M_0$ we obtained the differential scattering cross section ala [2], in our previous paper, [3], as

$$\frac{d\sigma}{d\Omega_f}|_{lab} = 4(\frac{e^2}{4\pi\epsilon_0})^2\frac{1}{m_0c^2E_iv_i}\int_{m_0c^2}^{E_i+m_0c^2}\frac{|\vec{p_f}|dE_f}{(p_f-p_i)^4}\delta(2m_0^2c^2 - 2\frac{E_i}{c}\frac{E_f}{c} + 2\vec{p_i}.\vec{p_f} + 2m_0(E_i - E_f))$$
$$[2m_0^2E_fE_i + m_0(p_i.p_f)(E_i - E_f - m_0c^2) + m_0^3c^2[2(E_f - E_i) + m_0c^2]].$$

Resorting to non-relativistic limit keeping terms upto $(\frac{v}{c})^2$ order, tentamounts to,

$$E_i = m_0c^2 + m_0\frac{v_i^2}{2} + \frac{3}{8}m_0v_i^2\beta_i^2,$$
$$\frac{E_i}{c}\frac{E_f}{c} = m_0^2c^2 + \frac{1}{2}m_0^2(v_i^2+v_f^2) + \frac{3}{8}m_0^2(v_i^2+v_f^2)(\beta_i^2+\beta_f^2),$$
$$E_i - E_f = \frac{1}{2}m_0(v_i^2 - v_f^2) + \frac{3}{8}m_0(v_i^2\beta_i^2 - v_f^2\beta_f^2),$$
$$|\vec{p_f}| = m_0v_f(1+\frac{1}{2}\beta_f^2),$$
$$\vec{p_i}.\vec{p_f} = m_0^2v_iv_fcos\theta(1+\frac{\beta_i^2+\beta_f^2}{2}),$$
$$dE_f = m_0v_fdv_f(1+\frac{3}{4}\beta_f^2)$$

one achieves,

$$\delta(2m_0^2c^2 - 2\frac{E_i}{c}\frac{E_f}{c} + 2\vec{p_i}.\vec{p_f} + 2m_0(E_i - E_f))$$
$$= \delta(-m_0^2(v_i^2+v_f^2) + 2m_0^2v_iv_fcos\theta(1+\frac{\beta_i^2+\beta_f^2}{2}) - \frac{3}{4}m_0^2(v_i^2+v_f^2)(\beta_i^2+\beta_f^2) + m_0^2(v_i^2-v_f^2) + \frac{3}{4}m_0^2(v_i^2\beta_i^2 - v_f^2\beta_f^2))$$
$$= \frac{1}{2m_0^2v_f}\delta(v_f - v_icos\theta + \frac{3}{4}(\beta_i^2+\beta_f^2)(v_f - \frac{2}{3}v_icos\theta)) = \frac{1}{2m_0^2v_f}\delta(v_f - v_icos\theta + \frac{3}{4}(\beta_i^2+\beta_i^2cos^2\theta)(v_icos\theta - \frac{2}{3}v_icos\theta))$$
$$= \frac{1}{2m_0^2v_f}\delta(v_f - v_icos\theta(1-\frac{1}{4}\beta_i^2(1+cos^2\theta)))$$
$$(p_f-p_i)^2 = 2m_0^2c^2 - 2p_i.p_f = 2[m_0^2c^2 - (\frac{E_i}{c}\frac{E_f}{c} - \vec{p_i}.\vec{p_f})] = m_0^2[-v_i^2 - v_f^2 + 2v_iv_fcos\theta(1+\frac{1}{2}(\beta_i^2+\beta_f^2)) - \frac{3}{4}(v_i^2+v_f^2)(\beta_i^2+\beta_f^2)]$$
$$(p_f-p_i)^2|_{v_f - v_icos\theta(1-\frac{1}{4}\beta_i^2(1+cos^2\theta))=0} = -m_0^2v_i^2sin^2\theta[1 - \frac{\beta_i^2}{4}\frac{(1+cos^2\theta)(cos^2\theta-3)}{sin^2\theta}]$$
$$\frac{1}{(p_f-p_i)^4}|_{v_f - v_icos\theta(1-\frac{1}{4}\beta_i^2(1+cos^2\theta))=0} = \frac{1}{m_0^4v_i^4sin^4\theta}[1 + \frac{\beta_i^2}{2}\frac{(1+cos^2\theta)(cos^2\theta-3)}{sin^2\theta}]$$
$$[2m_0^2E_fE_i + m_0(p_i.p_f)(E_i - E_f - m_0c^2) + m_0^3c^2[2(E_f - E_i) + m_0c^2]]$$
$$= m_0^4c^4 + m_0^2c^2(\frac{E_i}{c}\frac{E_f}{c} + \vec{p_i}.\vec{p_f}) + m_0(E_f - E_i)(2m_0^2c^2 - \frac{E_i}{c}\frac{E_f}{c} + \vec{p_i}.\vec{p_f}) = 2m_0^4c^4[1 + \frac{1}{2}\beta_f(\beta_f + \beta_icos\theta)]$$
$$[2m_0^2E_fE_i + m_0(p_i.p_f)(E_i - E_f - m_0c^2) + m_0^3c^2[2(E_f - E_i) + m_0c^2]]|_{v_f - v_icos\theta(1-\frac{1}{4}\beta_i^2(1+cos^2\theta))=0} = 2m_0^4c^4[1 + \beta_i^2cos^2\theta]$$

Hence, the differential scattering cross-section, in the NR, for two non-identical particles of equal mass and equal but opposite charges, in the laboratory frame i.e. when one particle is at rest initially is given by, up to $(\frac{v}{c})^2$ or, β^2, order

$$\frac{d\sigma}{d\Omega_f}|_{lab} = 4(\frac{e^2}{4\pi\epsilon_0})^2\frac{1}{m_0c^2E_iv_i}\int_{m_0c^2}^{E_i+m_0c^2}\frac{|\vec{p_f}|dE_f}{(p_f-p_i)^4}\delta(2m_0^2c^2-2\frac{E_i}{c}\frac{E_f}{c}+2\vec{p_i}.\vec{p_f}+2m_0(E_i-E_f))$$
$$[2m_0^2E_fE_i+m_0(p_i.p_f)(E_i-E_f-m_0c^2)+m_0^3c^2[2(E_f-E_i)+m_0c^2]]$$
$$=4(\frac{e^2}{4\pi\epsilon_0})^2\frac{1}{m_0c^2m_0c^2v_i}(1-\frac{1}{2}\beta_i^2)\int_{m_0c}^{2m_0c}\frac{m_0v_f(1+\frac{1}{2}\beta_f^2)m_0v_fdv_f(1+\frac{3}{4}\beta_f^2)}{m_0^4v_i^4\sin^4\theta}$$
$$[1+\frac{\beta_i^2}{2}\frac{(1+cos^2\theta)(cos^2\theta-3)}{sin^2\theta}]\frac{1}{2m_0^2v_f}\delta(v_f-v_icos\theta(1-\frac{1}{4}\beta_i^2(1+cos^2\theta)))2m_0^4c^4[1+\beta_i^2cos^2\theta]$$
$$=4(\frac{e^2}{4\pi\epsilon_0})^2\frac{cos\theta}{m_0^2v_i^4\sin^4\theta}(1-\frac{1}{4}\beta_i^2(1+cos^2\theta))(1-\frac{1}{2}\beta_i^2)(1+\beta_i^2cos^2\theta)[1+\frac{\beta_i^2}{2}\frac{(1+cos^2\theta)(cos^2\theta-3)}{sin^2\theta}](1+\frac{5}{4}\beta_i^2cos^2\theta)$$
$$=4(\frac{e^2}{4\pi\epsilon_0})^2\frac{cos\theta}{(2E_{lab})^2\sin^4\theta}(1-\frac{1}{4}\beta_i^2(1+cos^2\theta)+\beta_i^2[-\frac{1}{2}+cos^2\theta+\frac{5}{4}cos^2\theta+\frac{(1+cos^2\theta)(cos^2\theta-3)}{2sin^2\theta}])$$
$$=(\frac{e^2}{4\pi\epsilon_0})^2\frac{cos\theta}{E_{lab}^2\sin^4\theta}[1-\beta_i^2(\frac{2}{sin^2\theta}+\frac{1}{4}-\frac{3}{2}cos^2\theta)]$$
$$=\frac{k^2}{E_{lab}^2}\frac{cos\theta}{\sin^4\theta}[1-\beta_i^2(\frac{2}{sin^2\theta}+\frac{1}{4}-\frac{3}{2}cos^2\theta)]$$

III. ACKNOWLEDGEMENT

The reference where this is done, has not reached the author. Hopefully, nothing new has been presented in the paper.

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[1] Herbert Goldstein, Charles Poole, John Safko, *Classical Mechanics* third edition, Pearson, Inc., New Delhi 110017, India. ISBN 978-81-7758-283-3.

[2] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*(McGraw-Hill Book Compnay, Inc., USA,1964).

[3] Anindya Kumar Biswas, "Rutherford Cross Section in the laboratory frame", viXra:2304.0163[Classical Physics].