Integrability of Continuous Functions in 2 Dimensions

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Abstract. In this paper it is shown that the Banach space of continuous, $\mathbb{R}^2$- or $\mathbb{C}$-valued functions on a simply connected either 2-dimensional real or 1-dimensional complex compact region can be decomposed into the topological direct sum of two subspaces, a subspace of integrable (and conformal) functions, and another one of unintegrable (and anti-conformal) functions. It is shown that complex integrability is equivalent to complex analyticity. This can be extended to real functions. The existence of a conjugation on that Banach space will be proven, which maps unintegrable functions onto integrable functions.
The boundary of a 2-dimensional simply connected compact region is defined by a Jordan curve, from which it is known to topologically divide the domain into two disconnected regions. The choice of which of the two regions is to be the inside, defines the orientation. The conjugation above will be seen to be the inversion of orientation. Analyticity, integrability, and orientation on $\mathbb{R}^2$ (or $\mathbb{C}$) therefore are intimately related.

1. Introduction: Prelminaries and problem statement

Let $\mathbb{K}$ stand for either $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{C}$. A function f from V to either $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{C}$ is is called continuous on V, if is well-defined and continuous in an open environment $U \subset \mathbb{K}$ of V. The set of continuous $\mathbb{K}$-valued functions on V then is a Banach space $\mathbb{C}(V, \mathbb{K})$ with the supremum norm $\|\dot{\|} : f \mapsto sup_{x\in V} \|f(x)\|_{\mathbb{K}}$, where $\|\cdot\|_{\mathbb{K}}$ stands for the absolute value for $\mathbb{K} = \mathbb{R}$, the Euclidean norm for $\mathbb{K} = \mathbb{R}^2$, and the absolute value for $\mathbb{K} = \mathbb{C}$. In the following, we'll briefly write $\mathbb{C}(V)$ for $\mathbb{C}(V, \mathbb{K})$, when it is clear what the target space $\mathbb{K}$ is.

A path γ in V is a continuous mapping $\gamma : [0, 1] \to V$, where $[0, 1]$ denotes the closed real interval from 0 to 1. V s called connected, if for each $x, y \in V$ there is a path γ in V with $\gamma(0) = x$ and $\gamma(1) = y$. A compact set $V \subset \mathbb{K}$ is a closed and bounded subset of $\mathbb{K}$. V will be called closed region, if it is the closure of a non-void open and connected set. The path γ is called closed if $\gamma(0) = \gamma(1)$, and a connected V is called *simply connected*, if all closed paths in V are point homotopic in V, i.e.: if V has no holes. Let V be a simply connected, closed region and $f \in \mathcal{C}(V, \mathbb{K})$. Then for every piecewise continuously differentiable path $\gamma : [0,1] \to V$, the path integral $\int_\gamma f(s)ds := \int_0^1 f(\gamma(t))\frac{d\gamma(t)}{dt}dt$ is a well-defined, continuous linear functional on $\mathcal{C}(V, \mathbb{K})$. A function $f \in \mathcal{C}(V, \mathbb{K})$ is called *integrable*, if and only if $\int_\gamma f(s)ds = 0$ for every closed path γ in V. In all cases, if f is integrable, then the path integrals from a fixed startpoint in V to the variable endpoint in V define a function If, which is commonly called *primitive* of f. (Since two primitives of the same function f differ utmost by the choice of the startpoint, which adds an additive constant, the primitives are naturally defined as equivalence classes.) While this is trivial for one real dimension, i.e: for $V \subset \mathbb{R}$, and it is simple in the complex (also 1-dimensional) case, with two real dimensions $V \subset \mathbb{R}^2$, both $f \in \mathcal{C}(V, \mathbb{R})$ and $f \in \mathcal{C}(V, \mathbb{R}^2)$ there is a twist: primitives of integrable $f \in \mathcal{C}(V, \mathbb{R})$ are functions $If \in \mathcal{C}(V, \mathbb{R}^2)$, while the primitives of $f \in \mathcal{C}(V, \mathbb{R}^2)$ are functions $If \in \mathcal{C}(V, \mathbb{R})$. So, if If itself is integrable again to $I^2 f$, then $I^2 f$ will be in the same space of continuous functions on V as f, and the m^{th} order primitive $I^m f$ of f will be element of $\mathcal{C}(V, \mathbb{R})$ or $\mathcal{C}(V, \mathbb{R}^2)$, depending on whether m is even or odd.

For now, let us restrict to the unproblematic complex case $\mathcal{C}(V, \mathbb{C})$ with $V \subset \mathbb{C}$:
If $f \in \mathcal{C}(V, \mathbb{C})$ is integrable, then f can be uniquely path integrated from a fixed $z_0 = x_0 + iy_0$ in the interior of V to any other $z = x + iy \in V$, which – up to an additive constant of integration – defines a function $If \in \mathcal{C}(V, \mathbb{C})$, which is complex differentiable and for which $\frac{dIf(z)}{dz}$ holds. If is therefore called *anti-derivative* or *primitive* of f. Clearly, if f is integrable, then it is integrable to all orders, i.e. the n^{th} primitive $I^n f$ exists for all $n \in \mathbb{N}$. For the next, a definition of complex analyticity is needed: $If \in \mathcal{C}(V, \mathbb{C})$ is called *(complex) analytic*, if for all $z_0 \in V$ there is an environment $U_\epsilon(z_0)$, such that for all $z \in U_\epsilon(z_0)$: $f(z) = \sum_{k \geq 0} c_k(z - z_0)^k$ is on $U_\epsilon(z_0)$ the uniform limit of the power series $\sum_{k \geq 0} c_k(z - z_0)^k$, where $c_k \in \mathbb{C}$ for all k.

We'll refer to *Cauchy theory* as the contents of his original article [3]. The following shows that it can be based on the integrability only:

Proposition 1.1 (Corollary of Cauchy theory). *If $V \subset \mathbb{C}$ is a compact and simply connected region and $f : V \ni z \mapsto f(z)$ is continuous and integrable (w.r.t. dz), then f is analytic on V.*

Its proof uses the following

Lemma 1.2. *Let $V \subset \mathbb{C}$ be a compact and simply connected region.*

(i) If $f \in \mathcal{C}(V, \mathbb{C})$ is integrable and If is its primitive, then the square $If^2 := If \cdot If$ is integrable.

(ii) If $f, g \in \mathcal{C}(V, \mathbb{C})$ are integrable, the product $If \cdot Ig$ of their primitives If and Ig is integrable.

Proof. (i) follows from $\frac{d}{dz}If^2(z) = 2If(z) \cdot f(z)$, because then $2If(z) \cdot f(z)$ has a primitive, namely If^2, so If^2 itself is integrable either. To prove (ii), we consider the square $(If+Ig)^2$ of the primitives If and Ig for two integrable functions f and g. Since then $If+Ig$ is integrable, the square is integrable, so $If \cdot Ig = \frac{1}{2}((If+Ig)^2 - If^2 - Ig^2)$ is integrable. □

Proof of proposition 1.1. First, we may assume that f is already the primitive of a continuous function on V: Because if we prove that the primitive If of f is analytic and therefore differentiable to any order, then f itself will be analytic. By Cauchy theorem, $g(z) := \frac{1}{z-z_0}$ is integrable in all convex regions of $\mathbb{C} \setminus \{z_0\}$, and by the integration formula, $\int_\gamma g(z)dz = 2\pi i$ for all closed paths that wind in positive orientation around z_0 once (see: [1][Theorem 6]). Let $D_\epsilon(z_0)$ be the disk of radius ϵ around z_0, and let f be continuous and integrable on the closure V of $D_\epsilon(z_0)$. Then the product fg is continuous and integrable an all convex subsets of $V \setminus \{z_0\}$ (by 1.2), $f(z) - f(z_0)$ converges to zero as $z \to z_0$, and therefore, with γ_r being the circular path of radius $r > 0$ around z_0 with $r < \epsilon$:

$$\left| \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq 2\pi sup_{|z-z_0| \leq r} \left| f(z) - f(z_0) \right|,$$

which (by continuity of f in z_0) converges to zero as $r \to 0$. So, by integrability of fg outside of z_0, $\int_\gamma \frac{f(z)}{z-z_0}dz = \frac{f(z_0)}{z-z_0}dz = 2\pi i f(z_0)$ for every closed curve in $V \setminus \{z_0\}$, which with positive orientation winds exactly once around z_0. The rest is standard: We have $f(\zeta) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z-\zeta}dz$ for all $\zeta \in D_r(z_0)$ with $r < \epsilon$, which is a locally analytic and bounded function within the interior of the punctured disk $D_r(z_0) \setminus \{z_0\}$, and therefore is analytic in the interior of $D_r(z_0)$. □

The above offers numerous open topics to explore:

(1) The characteristic properties of integrable functions as a subspace of $\mathcal{C}(V, \mathbb{C})$ should be examined:
   - is it closed?
   - is it open?
   - does if have a topological compement, and if so: what is this complement?

(2) For $V \subset \mathbb{R}^2$ the *complex isomorphism* $\iota : V \ni (x, y) \mapsto x + iy \in \mathbb{C}$ isomorphically transforms $f \in \mathbb{C}(V, \mathbb{R}^2)$ to $\iota f \iota^{-1} \in \mathcal{C}(\iota V, \mathbb{C})$. So

$$T_\iota : \mathbb{C}(V, \mathbb{R}^2) \ni f \mapsto \iota f \iota^{-1} \in \mathcal{C}(\iota V, \mathbb{C})$$

is an isomorphism of Banach spaces, which will be called *complex isomorphism*, either. Then it is to expect that every relation for the complex functions can be mapped via T_ι^{-1} from $\mathcal{C}(\iota V, \mathbb{C})$ to $\mathbb{C}(V, \mathbb{R}^2)$, and this includes integrability and analyticity along with it. By the Weierstraß convergence theorem ([1][Ch. 8 1.1]) this pulled-back space of analytic functions should be closed in $\mathbb{C}(V, \mathbb{R}^2)$, and therefore the complex analytic functions would be closed in $\mathcal{C}(\iota V, \mathbb{C})$.

2. Integrability decomposition

Let V be a simply connected, closed and compact region of $\mathbb{R}^2$ or $\mathbb{C}$ and $f \in \mathbb{C}(V, \mathbb{K})$, where $\mathbb{K}$ stands for either $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{C}$. f will be called *integrable at* the point $z \in V$, if and only if there are some $h_0 > 0$ such that the path integrals $\int_{\gamma_h} f(s)ds$ of positively (i.e. counter-clockwise) orientated, closed paths once around the boundaries of circles of radius $h < h_0$ around z are defined, and such that $\int_{\gamma_h} f(s)ds = o(h^m)$ holds for any $m \in \mathbb{N} \cup \{0\}$ as $h \to 0$, which means that $\frac{1}{h^m}\left|\int_{\gamma_h} f(s)ds\right| \to 0$ as $h \to 0$. Because the value of the path integration gets inverted in sign, $\frac{1}{h^m}\left|\int_{\gamma_h} f(s)ds\right| \to 0$ for $h \to 0$ likewise holds if the paths γ_h go the opposite way with negative orientation. A function $f \in \mathbb{C}(V, \mathbb{K})$, which is not integrable at $z \in V$ will be called *unintegable at z*. As such f is unintegrable at $z \in V$, if and only if there is some $h_0 > 0$, some $C_0 > 0$, and $m \in \mathbb{N}$, such that for any $\delta > 0$ with $\delta < h_0$ there is a positive $h < \delta$: $\left|\int_{\gamma_h} f(s)ds\right| \geq C_0 h^m$, where again $(\gamma_h)_{h>0}$ is the familiy of positively orientated paths with (winding) index 1 along circles of radius h around z.

Proposition 2.1 (Integrability decomposition). *Let $V \subset \mathbb{R}^2$ be be a simply connected compact region and $\mathbb{K}$ stand for either $\mathbb{R}$ or $\mathbb{R}^2$.*

(i) $\mathbb{C}(V, \mathbb{K})$ *is the topological direct sum of two subspaces: the space of integrable functions $\mathcal{Y}_+(V, \mathbb{K})$ and a complementary space $\mathcal{Y}_-(V, \mathbb{K})$ of unintegrable functions.*

(ii) $\mathbb{C}(\iota V, \mathbb{C})$ *is the topological direct sum of two subspaces: the space of integrable functions $\mathcal{Y}_+(\iota V, \mathbb{C})$ and a space $\mathcal{Y}_-(\iota V, \mathbb{C})$ of (strictly) unintegrable functions.*

Proof. The asserted decomposition of $\mathbb{C}(\iota V, \mathbb{C})$ follows from the decomposition of $\mathbb{C}(V, \mathbb{K})$ through the complex isomorphism T_ι. So it suffices to prove the first statement.

So, let $f \in \mathbb{C}(V, \mathbb{K})$. Then f is to be continuous on an open superset U of V, and we define $\mathcal{Q}$ as set of all squares $Q(d, x, y) = \{(x', y') \in \mathbb{R}^2 \mid |x' - x|, |y' - y| \leq d/2\}$ for $(x, y) \in V$ and some $d > 0$. Let $\Gamma(\mathcal{Q})$ be the set of all positively (i.e.: anti-clockwise) orientated paths $\gamma(d, x, y)$ around the boundaries of the $Q(h, x, y)$ with $d > 0$ and $(x, y) \in V$. Then $p_\gamma : f \mapsto p_\gamma(f) := \|\int_\gamma f(s)ds\| \geq 0$, $(\gamma \in \Gamma(\mathcal{Q}))$, defines a family of continuous seminorms on $\mathbb{C}(V, \mathbb{K})$. The set of all $f \in \mathbb{C}(V, \mathbb{K})$, for which $p_\gamma(f) = 0$ for all $\gamma \in \Gamma(\mathcal{Q})$ then is closed in $\mathbb{C}(V, \mathbb{K})$, since it is the intersection of the closed sets. It contains all integrable(, continuous) functions on V.

Let $\mathcal{Y}_+(V, \mathbb{K})$ denote this closed space of $\mathbb{C}(V, \mathbb{K})$. Then its complement is an algebraic subspace, which is open in $\mathcal{C}(V, \mathbb{K})$. We call it space of *non-integrable* functions and denote it by $\mathcal{Y}_-(V, \mathbb{K})$.

To finish up, it remains to be shown that $\mathcal{Y}_+(V, \mathbb{K})$ is also open, or equivalently to prove that $\mathcal{Y}_-(V, \mathbb{K})$ is closed. We need to refine this family of seminorms, in order to make further progress:

For each $f \in \mathcal{C}(V, \mathbb{K})$ the function

$$F : [0, d] \times V \ni (h, x, y) \mapsto \int_{\gamma(h,x,y)} f(s)ds \in \mathbb{K}$$

is uniformly continuous on $[0, d] \times V$, but also: $\left|F(h, x, y) - F(h', x, y)\right| = o(h - h')$ (for $h, h' < d$). So, F is (right) differentiable (at $h = 0$) in its first argument for $h \to 0$, and F is continuously differentiable in h for each $(x, y) \in V$ for $0 < h < d$. And because every $f \in \mathcal{C}(V, \mathbb{K})$ can be isometrically extended as a continuous function onto the closed d-environment of V, the mapping

$$p : \mathcal{C}(V, \mathbb{K}) \ni f \mapsto sup_{h\in[0,d],(x-y)\in V} \frac{1}{4h}\left|F(h, x, y)\right| \geq 0$$

is a well-defined semi-norm on $\mathcal{C}(V, \mathbb{K})$, and it is a norm on its (open) subspace $\mathcal{Y}_-(V, \mathbb{K})$ of unintegrable functions. Let's inspect the last statement in detail: For $f \in \mathcal{Y}_-(V, \mathbb{K})$, there is some $(x, y) \in V$, such that for any $\delta > 0$ there is an $h > 0$ with $h < \delta$ and $\left|\int_{\gamma(h,x,y)} f(s)ds\right| > 0$, where $\gamma_{(h,x,y)}$ is the path once around the boundary of the h-square centered at (x, y). So, $\gamma_{(h,x,y)}$ is the sum of two paths, $\gamma_{(h,x,y)} = \gamma_{(h,x,y),R} - \gamma_{(h,x,y),L}$, where $\gamma_{(h,x,y),L}$ starts from the lower left corner along the y-axis to the upper left corner, then along the upper upper side along the x-axis from top left to upper right corner, and $\gamma_{(h,x,y),R}$ is the path from the lower left corner to upper right corner across the lower right corner. Unintegrability of f at (x, y) then mandates $\int_{\gamma(h,x,y)} f(s)ds = 2\int_{\gamma(h,x,y),R} f(s)ds$. So, by continuity of f:

$$lim_{h\to0} \sup_{(x,y)\in V} \left|\frac{1}{4h}\int_{\gamma(h,x,y)} f(s)ds\right| \geq \left|f(x, y)\int_{\gamma(h,x,y),R} \frac{1}{4h}ds\right|,$$

and therefore $p(f) \geq \frac{1}{2}\sup_{(x,y)\in V}\left|f(x, y)\right|$. So, p is stronger than the supremum norm, so p itself is a norm on $\mathcal{Y}_-(V, \mathbb{K})$. On the other hand, clearly: $p(f) \leq \sup_{(x,y)\in V}\left|f(x, y)\right|$, so on $\mathcal{Y}_-(V, \mathbb{K})$, p is equivalent to the supremum norm. Hence $\mathcal{Y}_-(V, \mathbb{K})$ is closed, its algebraic complement $\mathcal{Y}_+(V, \mathbb{K})$ is open, the canonical projections to the quotient spaces $\pi_\pm : \mathcal{C}(V, \mathbb{K}) \ni f \mapsto [f]_\pm \in \mathcal{C}(V, \mathbb{K})/\mathcal{Y}_\pm(V, \mathbb{K})$ are (bi-)continuous, and $\mathcal{C}(V, \mathbb{K})$ is the topological direct sum of its closed and open subspaces $\mathcal{Y}_\pm(V, \mathbb{K})$ – as was asserted. □

The decomposition into the spaces $\mathcal{Y}_\pm(V, \mathbb{K})$ and $\mathcal{Y}_\pm(\iota V, \mathbb{C})$ resp. is a provisional result and not the final decomposition: One would want the integrable and unintegrable subspaces to be isomorphic. We'll see next, that there are conjugations on $\mathcal{C}(V, \mathbb{R}^2)$ and $\mathcal{C}(\iota V, \mathbb{C})$, which map the $\mathcal{Y}_-$-spaces into their complementary $\mathcal{Y}_+$-spaces, but leave a subspace of the $\mathcal{Y}_+$-spaces invariant. The goal then will be to extract that subspace and to decompose $\mathcal{Y}_+$ further.

3. Conjugation, Jacobians, and $\mathcal{C}_0$-spaces

Again, let $V \subset \mathbb{R}^2$ be a simply connected compact region.
For $f = (f_1, f_2) \in \mathcal{C}(V, \mathbb{R}^2)$ and $f = Re(f) + iIm(f) \in \mathcal{C}(\iota V, \mathbb{C})$ the functions

$$f^c := (f_1 - f_2) \text{ and } f^c := \bar{(}f) := Re(f) - iIm(f)$$

will be called *conjugates* of f, where in particular $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. So, in the complex case, $f^c(z) := \overline{f(z)}$.

Then the *conjugation* is a an isometric isomorphism on $\mathcal{C}(V, \mathbb{R}^2)$ and an isometric antilinear bijection on $\mathcal{C}(\iota V, \mathbb{C})$, such that $(f^c)^c = f$ for all f, i.e.: the conjugation is an idempotent mapping in all cases.

We now examine the spaces of integrable and unintegrable functions, in order to identify the conjugation-invariant subspaces. We may restrict mainly to $\mathcal{C}(V, \mathbb{R}^2)$, as the results will carry over to the complex case via the complex isomorphism.

Both, $\mathcal{C}(V, \mathbb{R}^2)$ and $\mathcal{C}(\iota V, \mathbb{C})$, have the infinitely differentiable functions $\mathcal{C}^\infty(V, \mathbb{R}^2)$ and $\mathcal{C}^\infty(\iota V, \mathbb{C})$ as dense subspaces (see: [4]). Restricting to these has the advantage that the structure of the subspaces can be classified by the types of the Jacobi matrices (i.e.: the derivatives) of its elements. With this we have: The derivative of every continuously differentiable $f \in \mathcal{C}(V, \mathbb{R}^2)$ can be represented by matrix-valued function Df, called the *Jacobian*, given by

$$Df(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}, \quad \text{with } a, b, c, d \in \mathcal{C}(V, \mathbb{R})$$

By Green's theorem (see e.g.: [1][Ch. 5 5.2]), a continuously differentiable function $f \in \mathcal{C}(V, \mathbb{R}^2)$ is integrable if and only if its Jacobian Df is a symmetric matrix

$$Df(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}, \quad \text{where } a, b, c \in \mathcal{C}(V, \mathbb{R}).$$

These then comprise all continuously differentiable elements from $\mathcal{Y}_+(V, \mathbb{R}^2)$. And the unintegrable, continuously differentiable $f \mathcal{Y}_-(V, \mathbb{R}^2)$ then have the Jacobian Df

$$Df(x,y) = \begin{pmatrix} 0 & -b(x,y) \\ b(x,y) & 0 \end{pmatrix}, \quad \text{where } b \in \mathcal{C}(V, \mathbb{R}) \setminus \{0\}.$$

The conjugation on $\mathcal{C}(V, \mathbb{R}^2)$ now maps the Jacobian

$$Df(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}$$

for an abitrary continuously differentiable $f \in \mathcal{C}(V, \mathbb{R}^2)$ to:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} = \begin{pmatrix} a(x,y) & b(x,y) \\ -c(x,y) & -d(x,y) \end{pmatrix},$$

hence it inverts the integrability: It maps $\mathcal{Y}_-(V,\mathbb{R}^2)$ into $\mathcal{Y}_+(V,\mathbb{R}^2)$, but it is not onto, because its image does not contain the diagonal elements

$$\begin{pmatrix} a(x,y) & 0 \\ 0 & d(x,y) \end{pmatrix}.$$

This determines its invariant subspace w.r.t. integrability inversion, which will be denoted by $\mathcal{C}_0(V,\mathbb{R}^2)$. Because these diagonal matrix functions are globally diagonal on V, a must not change in the y-direction, and d must be constant in the x-direction. So,

$$Df(x,y) = \begin{pmatrix} a(x,y) & 0 \\ 0 & d(x,y) \end{pmatrix}$$

mandate $a(x,y) = a(x)$ and $d(x,y) = d(y)$ for $(x,y) \in V$, so a and d are functions on the x- and y-coordinate projections on V, namely $V_x := \{x \in \mathbb{R} \mid (x,y) \in V\}$ and $V_y := \{y \in \mathbb{R} \mid (x,y) \in V\}$, and both are bounded, closed intervals, since V is to be a simply connected compact region. And a and b have primitives given by $Ia(x) := \int_{-\infty}^{x} a(t)dt$ and $Ib(y) := \int_{-\infty}^{y} b(t)dt$, so that the primitive of Df is given by the pair of functions $f : V \ni (x,y) \mapsto (Ia(x), Ib(y))$. And because the set of continuously differentiable functions is dense in $\mathcal{C}(V,\mathbb{R}^2)$, it follows that $\mathcal{C}_0(V,\mathbb{R}^2)$ is the closed subspace of all $f : V \ni (x,y) \mapsto (f_1(x), f_2(y)) \in \mathbb{R}^2$ with $f_1 \in \mathcal{C}(V_x,\mathbb{R})$ and $f_2 \in \mathcal{C}(V_y,\mathbb{R})$. Next, $\mathcal{C}_0(V,\mathbb{R}^2)$ is open too, because for every $f = (f_1,f_2) \neq 0$, either $f_1 \neq 0$ or $f_2 \neq 0$, where both are continuous, real-valued functions on intervals. If $f_1 \neq 0$, then $\left|f_1(x)\right| > \epsilon$ for some $(x,y) \in V$ an some $\epsilon > 0$. Then there is a function $g \in \mathcal{C}(V_x,\mathbb{R})$ contained in the ϵ-environment of f_1, and likewise there is for f_2, in case $f_2 \neq 0$. That proves the openedness of $\mathcal{C}_0(V,\mathbb{R}^2)$.

As announced above, we then define $\mathcal{C}_+(V,\mathbb{R}^2) := \mathcal{Y}_+(V,\mathbb{R}^2)/\mathcal{C}_0(V,\mathbb{R}^2)$, rename $\mathcal{C}_-(V,\mathbb{R}^2) := \mathcal{Y}_-(V,\mathbb{R}^2)$, and get the desired decomposition

$$\mathcal{C}(V,\mathbb{R}^2) = \mathcal{C}_+(V,\mathbb{R}^2) \oplus \mathcal{C}_0(V,\mathbb{R}^2) \oplus \mathcal{C}_-(V,\mathbb{R}^2)$$

into the the topological direct sum of its constituents $\mathcal{C}_\pm(V,\mathbb{R}^2)$ and $\mathcal{C}_0(V,\mathbb{R}^2)$.

We consider $\mathcal{C}(V,\mathbb{R})$: There is no conjugation defined on it, yet the $\mathcal{Y}_\pm(V,\mathbb{R})$ are both non-trvial, and they have an integrability inversion with a nontrivial $\mathcal{C}_0(V,\mathbb{R})$ as invariant subspace of $\mathcal{C}(V,\mathbb{R})$:
If $f \in \mathcal{C}(V,\mathbb{R})$ is continuously differentiable, then its derivative is given by its gradient $\nabla f := (\partial_x f, \partial_y f)$. It exists irrespective of whether ∇f is integrable again to its primitive, or not. Suppose, that ∇f was not integrable. What we know from the above is that ∇f is the sum of three components $\nabla f = g_+ + g_- + g_0$ with $g_\pm \in \mathcal{C}_\pm(V,\mathbb{R}^2)$ and $g_0 \in \mathcal{C}_0(V,\mathbb{R}^2)$, where $g_- \neq 0$. To enforce the integration of ∇f back to f, g_- must be transformed to its integrable counterpart via conjugation: $(Ig_0^c)^c$; this would allow to retain f from ∇f, even when non-integrable.

It is well-known that $\mathcal{C}_\pm(V,\mathbb{R})$ are both non-trivial:
$f(x = r\cos(t), y = r\sin(t)) := r^2 \sin(t/r)$ for $(x,y) \neq 0$ and $f(0,0) := 0$ with (x,y) in $V := \{(x,y) \mid -1 \leq x,y \leq 1\}$, is an example of an unintegrable function at the origin, so represents a non-zero element $f \in \mathcal{C}_-(V,\mathbb{R})$, and

hence its conjugate represents a member of $\mathcal{C}_+(V,\mathbb{R})$.
To show that $\mathcal{C}_0(V,\mathbb{R})$ is non-trivial either, it suffices to integrate $f \in \mathcal{C}_0(V,\mathbb{R}^2)$: $f(x,y) = (f_1(x), f_2(y))$ is integrable and has $If(x,y) = If_1(x) + If_2(y)$ as primitives, where again If_1, If_2 are the primitives $If_1(x) := \int_{-\infty}^{x} f_1(t)dt$ and $If_2(y) := \int_{-\infty}^{y} f_2(t)dt$. By differentiating the continuously differentiable $f \in \mathcal{C}_0(V,\mathbb{R}^2)$, we even get the general result directly for all $g \in \mathcal{C}_0(V,\mathbb{R})$: it consists of all functions $g = g_1 + g_2$ with $g_1 \in \mathcal{C}_0(V_x,\mathbb{R})$ and $g_2 \in \mathcal{C}_0(V_y,\mathbb{R})$. And again, this is an open and closed subspace of $\mathcal{C}(V,\mathbb{R})$.

An immediate consequence of the above is that primitives of (integrable) functions of $\mathcal{C}(V,\mathbb{K})$ are integrable again to any order. (The special case $\mathbb{K} = \mathbb{C}$ is analogous to $\mathbb{K} = \mathbb{R}$.)
As to the differentiation, the situation then is similar: if $f \in \mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_0(V,\mathbb{K})$ is n times continuously differentiables, then all its n derivatives are in $\mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_0(V,\mathbb{K})$ for some $\mathbb{K} = \mathbb{R}, \mathbb{R}^2, \mathbb{C}, \mathbb{C}^2$. However: if $f \in \mathcal{C}_-(V,\mathbb{K})$, then latest at the 2^{nd} derivative, the anti-symmetry of the Jacobian

$$Dg(x,y) = \begin{pmatrix} 0 & -b(x,y) \\ b(x,y) & 0 \end{pmatrix}, \quad \text{where } b \neq 0$$

impedes further differentiability, because of $\partial_y \partial_x g(x,y) = \partial_x b(x,y) = -\partial_x \partial_y g(x,y)$.
That said, $f \in \mathcal{C}(V,\mathbb{K})$ is continuously differentiable to an order of 2 or more, only if $f \in \mathcal{C}(V,\mathbb{K}) \in \mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_0(V,\mathbb{K})$.

Summarizing, it was shown:

Proposition 3.1.
1. *The subspaces $\mathcal{Y}_+(V,\mathbb{K})$ and $\mathcal{Y}_+(\iota V,\mathbb{C})$ contain open and closed invariant subspaces $\mathcal{C}_0(V,\mathbb{K})$ and $\mathcal{C}_0(\iota V,\mathbb{C})$ consisting of continuous functions f, for which $\partial_x \partial_y f = \partial_y \partial_x f \equiv 0$ holds.*
2. *$\mathcal{Y}_-(V,\mathbb{K})$ and $\mathcal{Y}_-(\iota V,\mathbb{C})$ are isomorphic to the quotient spaces $\mathcal{Y}_+(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$ and $\mathcal{Y}_+(\iota V,\mathbb{C})/\mathcal{C}_0(\iota V,\mathbb{C})$, resp.*
3. *For $\mathcal{C}(V,\mathbb{R}^2)$ and $\mathcal{C}(\iota V,\mathbb{C})$, the conjugation $f \to f^c$ maps $\mathcal{Y}_-(V,\mathbb{K})$ onto $\mathcal{Y}_+(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$, and $\mathcal{Y}_-(\iota V,\mathbb{C})$ onto $\mathcal{Y}_+(\iota V,\mathbb{C})/\mathcal{C}_0(\iota V,\mathbb{C})$.*
4. *Primitives of integrable functions are integrable.*

We define $\mathcal{C}_+(\iota V,\mathbb{C}) := \mathcal{Y}_+(\iota V,\mathbb{C})/\mathcal{C}_0(\iota V,\mathbb{C})$ and $\mathcal{C}_-(\iota V,\mathbb{C}) := \mathcal{Y}_-(\iota V,\mathbb{C})$ in line with $\mathcal{C}_+(V,\mathbb{K}) := \mathcal{Y}_+(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$ and $\mathcal{C}_-(V,\mathbb{K}) := \mathcal{Y}_-(V,\mathbb{K})$, the corresponding canonical projections will be denoted by

$\Pi_0 : \mathcal{C}(V,\mathbb{K}) \to \mathcal{C}_0(V,\mathbb{K})$,
$\Pi_0 : \mathcal{C}(\iota V,\mathbb{C}) \to \mathcal{C}_0(\iota V,\mathbb{C})$,
$\Pi_\pm : \mathcal{C}(V,\mathbb{K}) \to \mathcal{C}_\pm(V,\mathbb{K})$ as well as
$\Pi_\pm : \mathcal{C}(\iota V,\mathbb{C}) \to \mathcal{C}_\pm(\iota V,\mathbb{C})$.

Since integrable functions have been defined as elements from the $\mathcal{Y}_+$-spaces, which include the $\mathcal{C}_0$-spaces as a subspace, the functions from $\mathcal{C}_+(V,\mathbb{K})$ and $\mathcal{C}_+(\iota V,\mathbb{C})$ will be called *strictly integrable.*

Then we can state:

Corollary 3.2. *The following holds as a topological direct sum:*
1. $\mathcal{C}(V,\mathbb{K}) = \mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_0(V,\mathbb{K}) \oplus \mathcal{C}_-(V,\mathbb{K})$
2. $\mathcal{C}(\iota V,\mathbb{C}) = \mathcal{C}_+(\iota V,\mathbb{C}) \oplus \mathcal{C}_0(\iota V,\mathbb{C}) \oplus \mathcal{C}_-(\iota V,\mathbb{C})$

From inspection of the Jacobians, note that the product of two integrable functions from $\mathcal{C}(V,\mathbb{R})$ or $\mathcal{C}_+(V,\mathbb{R}^2)$ is integrable again (where for $\mathcal{C}_+(V,\mathbb{R}^2)$ the product is a function in $\mathcal{C}(V,\mathbb{R})$).

4. Conformality, holomorphic and anti-holomorphic functions

As was shown above, $\mathcal{C}(\iota V,\mathbb{C})$ splits into the topological sum of a strictly integrable, a strictly unintegrable, and an invariant subspace. From Proposition 1.1 we know that all integrable functions are analytic, and then it will be straightforward to derive the analyticity of the unintegrable ones (see below).

The a-priori concern however is, how the vector space of holomorphic functions will fit into this, especially regarding the closedness of the space $\mathcal{C}_+(\iota V,\mathbb{C})$ in $\mathcal{C}(\iota V,\mathbb{C})$. So, let's look into this:

The Jacobian for a continuously differentiable $f \in \mathcal{C}_+(V,\mathbb{R}^2)\oplus\mathcal{C}_0(V,\mathbb{R}^2)$ is given by

$$Df(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}, \quad \text{where } a,b,c \in \mathcal{C}(V,\mathbb{R}).$$

Under the complex isomorphism T_ι it transforms to

$$D(T_\iota f)(x,y) = D(\iota f \iota^{-1})(x,y) = \begin{pmatrix} a(x,y) & -ib(x,y) \\ ib(x,y) & c(x,y) \end{pmatrix}, \quad \text{where } a,b,c \in \mathcal{C}(\iota V,\mathbb{R}).$$

But: The definition of an holomorphic function demands $c \equiv a$ (see: e.g. [1]). This is solved by splitting the diagonal matrix up into the sum of a symmetric and an anti-symmetric part:

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+c & 0 \\ 0 & a+c \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a-c & 0 \\ 0 & -(a-c) \end{pmatrix},$$

which defines continuous projections on $\mathcal{C}_0(V,\mathbb{R}^2)$ and $\mathcal{C}_0(\iota V,\mathbb{C})$, respectively. The spaces $\mathcal{C}_0(V,\mathbb{R}^2)$ and $\mathcal{C}_0(\iota V,\mathbb{C})$ therefore decompose into toplogical direct sums of symmetric subspaces $\mathcal{C}_{0,sym}(V,\mathbb{R}^2)$ and $\mathcal{C}_{0,sym}(\iota V,\mathbb{C})$, as well as anti-symmetric subspaces $\mathcal{C}_{0,asym}(V,\mathbb{R}^2)$ and $\mathcal{C}_{0,asym}(\iota V,\mathbb{C})$. So,

$$\mathcal{C}(V,\mathbb{R}^2) = \mathcal{C}_{conf}(V,\mathbb{R}^2) \oplus \mathcal{C}_{aconf}(V,\mathbb{R}^2), \text{ where}$$

$$\mathcal{C}_{conf}(V,\mathbb{R}^2) := \mathcal{C}_+(V,\mathbb{R}^2) \oplus \mathcal{C}_{0,sym}(V,\mathbb{R}^2),$$

$$\mathcal{C}_{aconf}(V,\mathbb{R}^2) := \mathcal{C}_-(V,\mathbb{R}^2) \oplus \mathcal{C}_{0,asym}(V,\mathbb{R}^2) \text{ and likewise}$$

$$\mathcal{C}(\iota V,\mathbb{C}) = \mathcal{C}_{conf}(\iota V,\mathbb{C}) \oplus \mathcal{C}_{aconf}(\iota V,\mathbb{C}), \text{ where}$$

$$\mathcal{C}_{conf}(\iota V,\mathbb{C}) := \mathcal{C}_+(\iota V,\mathbb{C}) \oplus \mathcal{C}_{0,sym}(\iota V,\mathbb{C}), \text{ and}$$

$$\mathcal{C}_{aconf}(\iota V,\mathbb{C}) := \mathcal{C}_-(\iota V,\mathbb{C}) \oplus \mathcal{C}_{0,sym}(\iota V,\mathbb{C}).$$

The functions of $\mathcal{C}_{conf}(V,\mathbb{R}^2)$ and $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ are called *conformal*, and the functions of $\mathcal{C}_{aconf}(V,\mathbb{R}^2)$ and $\mathcal{C}_{aconf}(\iota V,\mathbb{C})$ are defined as *anti-conformal* functions. With this, a real-valued function $f \in \mathcal{C}(V,\mathbb{R})$ will be called conformal, if and only if it is integrable and its primitive (which then is an $\mathbb{R}^2$-valued function) is conformal.

Remark 4.1. $\mathcal{C}_{conf}(V,\mathbb{R}^2)$ is the closure of the subspace of all differentiable $f=(f_1,f_2)$ of $\mathcal{C}(V,\mathbb{R}^2)$, for which $\partial_x f_1=\partial_y f_2$ holds, $\mathcal{C}_{aconf}(V,\mathbb{R}^2)$ the closure of differentiable $f\in\mathcal{C}(V,\mathbb{R}^2)$, for which $\partial_x f_1=-\partial_y f_2$. Analogously, $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ is the closure of all $f\in\mathcal{C}(\iota V,\mathbb{C})$, for which the partial derivatives exist and $\partial_x Re(f)=\frac{\partial Im(f)(x+iy)}{i\partial_y}$ holds, and $\mathcal{C}_{aconf}(\iota V,\mathbb{C})$ the closure of f with existing partial derivatives, such that $\partial_x Re(f)=-\frac{\partial Im(f)(x+iy)}{i\partial_y}$.

The decomposition of $\mathcal{C}(V,\mathbb{R}^2)$ and $\mathcal{C}(V,\mathbb{C})$ into the topological direct sum of their conformal and anti-conformal subspaces will be called *conformal split.*
Then we get:

Proposition 4.2. *Let $V\subset\mathbb{R}^2$ be a simply connected compact region. The functions of $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ are exactly those, which obey the Cauchy-Riemann equations (see:* [1]*), which – by the definition – are holomorphic functions on V. Its (complex) conjugated space $\mathcal{C}_{aconf}(\iota V,\mathbb{C})$ therefore consists of all* anti-holomorphic functions *on V.*

Proof. The functions in $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ are integrable. By Proposition 1.1 these functions then are analytic on V, so continuously differentiable on V in its x- and y-coordinates. Because all elements of $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ are conformal, they are holomorphic (which by definition means that they satisfy the functions are continuously differentiable in x- and y-coordinate and satisfy the Cauchy-Riemann equations). All non-zero elements in its topological complement are either not integrable or anti-conformal, conflicting the Cauchy-Riemann equations. So, no other holomorphic functions exist on V. □

Remark 4.3. The conformal split allows a pragmatic access to integrability: $f_{conf}=(f_1,f_2)\in\mathcal{C}_{conf}(V,\mathbb{R}^2)$ if and only if $f_1\equiv f_2$. Likewise, $f_{aconf}=(f_1,f_2)$ is in $\mathcal{C}_{aconf}(V,\mathbb{R}^2)$ if and only if $f_1\equiv -f_2$. So, $f_{conf}=(g,g)$ and $f_{aconf}=(h^c,-h^c)$ for some conformal functions $g,h\in\mathcal{C}(V,\mathbb{R})$. As a conformal function, g is integrable to a function (Ig,Ig), so the primitive If_{conf} of f_{conf} is Ig, which we can write as $If_{conf}=Ig(1,1)$; the second order primitive of f_{conf} then writes to $I^2f_{conf}=(I^2g,I^2g)$, and so forth. Analogously, we can assign $If_{aconf}:=(Ih)^c(1,-1)$ as the primitive of f_{aconf}, $I^2f_{aconf}:=((I^2h)^c,(I^2h)^c)$ as 2^{nd}-order primitive, and so forth.

Lemma 4.4. *Let $V\subset\mathbb{R}^2$ be a simply connected compact region. If $f\in\mathcal{C}(\iota V,\mathbb{C})$ is analytic on ιV, then its conjugate f^c is analytic on $(-i)\overline{\iota V}$.*

Proof. If $f(z)=\sum_k c_k(z-z_0)^k$ is analytic (on V), then $\bar f(z):=\sum_k \bar c_k(z-z_0)^k$ is analytic (on V). The conjugate f^c is defined by $f^c:z\mapsto\overline{f(z)}$, so we have $f^c(z)=\bar f(\bar z)$. Now, $g:(-i)\overline{\iota V}\ni(ix+y)\mapsto\sum_k\bar c_k(-i)^k((ix+y)-(ix_0+y_0))^k$ is analytic on $(-i)\overline{\iota V}$, and $f^c=g$, since $(-i)((ix+y)-(ix_0+y_0))=(x-iy)-(x_0-iy_0)$. □

Because every $f\in\mathcal{C}(\iota V,\mathbb{C})$ can be extended to a continuous function $\tilde f$ on a square area $Q(h)\supset\iota V$ with the origin as center and of sufficiently large side length $h > 0$, such that $\sup_{z\in Q(h)}\left|\tilde{f}(z)\right| \le 2\sup_{z\in\iota V}\left|f(z)\right|$, $\mathcal{C}(\iota V,\mathbb{C})$ is continuously embedded into $\mathcal{C}(Q(h),\mathbb{C})$, and we can ensure ιV to contain $-iz$ and $\bar{z}$ with every $z \in \iota V$. So, there appears to be no substantial reason, to exclude conjugates of analytic functions from being analytic functions.

The results can be summarized for $\mathcal{C}(\iota V,\mathbb{C})$ as:

Corollary 4.5. *Let $\iota V \subset \mathbb{C}$ be a simply connected compact region. $\mathcal{C}(\iota V,\mathbb{C})$ is the topological direct sum of the subspace $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ of analytic and holomorphic functions $f(x+iy) = g(x) + ih(iy)$, and its conjugated subspace $\mathcal{C}_{aconf}(\iota V,\mathbb{C})$ of anti-holomorphic functions.*

That solves the integrability and analyticity posed as to the complex space, but still we have no analogous results for the spaces $\mathcal{C}(V,\mathbb{R}^2)$ (and $\mathcal{C}(V,\mathbb{R})$. This asks for some explanation:
Complex analysis is essentially built upon the 2-dimensional Laplace equation

$$\Delta f(x,y) := (\partial_x^2 + \partial_y^2)f(x,y) \equiv 0.$$

Within $\mathbb{C}$, Δ factors into the commuting product $\Delta = (\partial_x - i\partial_y)(\partial_x + i\partial_y)$. Hence, in there, $\Delta f \equiv 0$ reduces to first order differential equations, and the solutions are the sums of functions that solve $(\partial_x - i\partial_y)f = 0$ or $(\partial_x + i\partial_y)f = 0$. So, the idea was to pick any differentiable function $f(x+iy)$, for which then $(\partial_x - i\partial_y)f(x+iy) \equiv 0$, so $\Delta f \equiv 0$. The hindsight: these functions are analytic (by Cauchy theory). The problem: By the Weierstraß convergence theorem, these functions proved not to be dense in the space of continuous functions $f : V \ni z \mapsto f(z) \in \mathbb{C}$, where $V \neq \emptyset$ is a simply connected open region in $\mathbb{C}$. What was proved in here was, that the conjugated differentiable functions $f^c : z \mapsto f(\bar{z})$ are needed either, in order to get $\Delta f \equiv 0$ fulfilled for a dense set of continuous functions $f : U \to \mathbb{C}$.

What one would then obviously would want to do, is to pull the results in the complex via the complex isomorphism T_ι^{-1} to the $\mathcal{C}(V,\mathbb{R}^2)$. The concern is, that for a well-behaved, integrable complex function $f(re^{i\phi})$, the preimage $T_\iota^{-1}f$ is a function $g(r,\phi)$ with a polar symmetry, which generally will be strictly unintegrable at the origin: For example, if $g(r,\phi) = r^2\sin(4\phi)$, the path integral along ϕ from 0 to 2π will not vanish. And as discussed above, this means that $\partial_x\partial_y g = -\partial_y\partial_x g$ (at the origin), which in turn suggests to look for a (possibly compact) Lie group to apply. However, there is apparently no suitable one. To get at results for $\mathcal{C}(V,\mathbb{R}^2)$ at all, it will be necessary to build from ground up.

5. Algebraic extension of $\mathbb{R}^2$ and $\mathcal{C}(V,\mathbb{R}^2)$

An *orientation* on the vector field $\mathbb{R}^n$ is an embedding

$$\varphi : \mathbb{K}^n \ni (x_1,\ldots,x_n) \mapsto \sum_{1\le k\le n} a_k x_k \in \mathcal{A}$$

into an associative algebra $\mathcal{A}$ over the field $\mathbb{R}$ with unit element 1, such that $a_k a_j = -a_j a_k$ for all $1 \le k < j \le n$ and $a_k^2 = 1$ for all $k = 1,\ldots,n$. (For

$\mathbb{K} = \mathbb{C}$ and $n = 1$ the orientation is implicitly interpreted to be "in line with" or as "given by" the direction of the real part.)

In the 2-dimensional case, $n = 2$, we define two numbers $\mathfrak{e}_1$ and $\mathfrak{e}_2$ (not contained in $\mathbb{C}$), for which

(i) $\mathfrak{e}_1\mathfrak{e}_1 = \mathfrak{e}_2\mathfrak{e}_2 \equiv 1$,
(ii) $\mathfrak{e}_1\mathfrak{e}_2 = -\mathfrak{e}_2\mathfrak{e}_1$, and
(iii) $\mathfrak{e}_1\mathfrak{e}_2 \equiv +i$.

(From conditions (i) and (ii) follows that $\mathfrak{e}_1\mathfrak{e}_2 = \pm i$, and in order to determine the sign of that value, (iii) is needed.)

Then $\varphi_+ : \mathbb{R}^2 \ni (x, y) \mapsto \zeta := \mathfrak{e}_1 x + \mathfrak{e}_2 y$ and $\varphi_- : \mathbb{R}^2 \ni (x, y) \mapsto \tilde{\zeta} := \mathfrak{e}_1 x - \mathfrak{e}_2 y$ are a vector space isomorphisms of $\mathbb{R}^2$ onto the target spaces $\varphi_\pm \mathbb{R}^2$, which we denote by $\mathbb{R}^2_\pm$.

By defining on $\mathbb{R}^2_\pm$ the metrics, induced by the quadratic form

$$Q : \varphi_\pm \mathbb{R}^2 \ni \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \mapsto (\mathfrak{e}_1 x \pm \mathfrak{e}_2 y)^2 = x^2 + y^2 = \|\mathfrak{e}_1 x \pm \mathfrak{e}_2 y\|^2,$$

$\varphi_\pm$ become isometries.

Along with $\zeta = \mathfrak{e}_1 x + \mathfrak{e}_2 y$ also $\zeta' = \mathfrak{e}_2 x + \mathfrak{e}_1 y$ solves the algebraic equation $(a + b)^2 = a^2 + b^2$. Because of $\mathfrak{e}_1\mathfrak{e}_2 = i$, $i(\mathfrak{e}_2 x + \mathfrak{e}_1 y) = \mathfrak{e}_1 x - \mathfrak{e}_2 y$, and $i\zeta' = (\mathfrak{e}_1 x - \mathfrak{e}_2 y)$ follows. To be in line with the complex functions, $i\zeta'$ will be called *conjugate* of ζ and denoted with either ζ^c or $\tilde{\zeta}$.

$$\varphi_\pm : \mathbb{R}^2 \ni (x, y) \mapsto \zeta = \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \in \mathbb{R}^2_\pm$$

then define two global coordinate charts over the manifold $(\mathbb{R}^2, \varphi_\pm)$ of positive and negative orientation.

Remark 5.1.
1. $\mathfrak{e}_1$ and $\mathfrak{e}_1$ are numbers, not just symbols: they are defined solely based on the imaginary i, which is not a symbol, but a number.
2. Sofar, $\mathbb{R}^2_\pm$ are vector spaces, which are equivalent to $\mathbb{R}^2$, but they readily extend to a non-commutative, associative algebra, which will be denoted by $\mathbb{A}$, in which the product is defined as algebra extension of:
$$\cdot : \mathbb{R}^2_\pm \times \mathbb{R}^2_\pm \ni (\mathfrak{e}_1 x \pm \mathfrak{e}_2 y, \mathfrak{e}_1 x' \pm \mathfrak{e}_2 y') \mapsto xx' + yy' \pm (ixy' - iyx') \in \mathbb{A}.$$
3. Due to $\mathfrak{e}_1\mathfrak{e}_2 = i$, the algebra $\mathbb{A}$ is inevitably complex. However it is not an algebra over the field $\mathbb{C}$: As an algebra over $\mathbb{C}$, i would commute with all elements, which is not the case for $\mathbb{A}$.
4. Anti-commutativity of $\mathfrak{e}_1$ and $\mathfrak{e}_2$ with $\mathfrak{e}_1\mathfrak{e}_2 \equiv +i$ implies: $i\mathfrak{e}_k = -\mathfrak{e}_k i$, $(k = 1, 2)$, so $(\mathfrak{e}_k x + iy)^2 = x^2 - y^2$ follows for $k = 1, 2$.

Next, $\zeta^2 > 0$ for all non-zero $\zeta \in \mathbb{R}^2_\pm$. Therefore the the Euclidean topology of $\mathbb{R}^2$ (and its isometric space $\mathbb{R}^2_\pm$) extends onto $\mathbb{A}$, so $\mathbb{R}^2$ and $\mathbb{R}^2_\pm$ are isometrically embedded into $\mathbb{A}$.

With this we define $\mathcal{C}_+(\varphi_+ V, \mathbb{A})$ as vector space of all functions $T_{\varphi_+} := \varphi_+ f_{conf} \varphi_+^{-1}$, where $f_{conf} \in \mathcal{C}_{conf}(V, \mathbb{R}^2)$, and likewise $\mathcal{C}_-(\varphi_- V, \mathbb{A})$ is defined as vector space of all $T_{\varphi_-} := \varphi_- f_{aconf} \varphi_+^{-1}$ with $f_{aconf} \in \mathcal{C}_{aconf}(V, \mathbb{R}^2)$. For $\zeta = \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \in \mathbb{R}^2_\pm \neq 0$ the multiplicative inverse $\frac{1}{\zeta} = \frac{\zeta}{x^2+y^2}$ is well-defined, and likewise $\zeta^m = (\mathfrak{e}_1 x \pm \mathfrak{e}_2 y)^m$ exists for $m \in \mathbb{N}$.

Since $\mathbb{R}^2_\pm$ and $\mathbb{A}$ are finite dimensional normed spaces, the vector spaces $\mathcal{C}(\varphi_\pm V, \mathbb{A})$ of $\mathbb{A}$-valued continuous functions on $\varphi_\pm$ are well-defined, and are Banach spaces with the supremum norm, which isometrically embed $\mathcal{C}_\pm(\varphi_\pm V, \mathbb{A})$ as closed subspaces.

For $\zeta_0 \in \varphi_\pm V$ a function $f \in \mathcal{C}_\pm(\varphi_\pm V, \mathbb{A})$ will be called *differentiable* in ζ_0 if and only if $\frac{df(\zeta=\zeta_0)}{d\zeta} := \lim_{\zeta\to\zeta_0}(f(\zeta) - f(\zeta_0))\frac{1}{\zeta-\zeta_0}$ exists (as an $\mathbb{A}$-valued function). $\frac{df(\zeta)}{d\zeta}$ will be called *derivative* of f.

Remark 5.2. (i) Note that the divisional term $\frac{1}{\zeta-\zeta_0}$ is factored to the right side of f: This is to ensure uniqueness of the limit in the case that the target values $f(\zeta)$ do not commute with the variable ζ. As long as $f(\zeta)$ is real-valued, however, the ordering of the product is irrelevant: "left" and "right" derivative coincide.

(ii) In particular, we then have: $(\frac{df(\zeta)}{d\zeta})^c = \frac{df^c(\tilde{\zeta})}{d\tilde{\zeta}}$, where $f^c : \tilde{\zeta} \mapsto (f(\tilde{\zeta}))^c$.

Since $\mathbb{A}$ is a finite-dimensional algebra, the Euclidean metrics defines a natural topology on $\mathcal{A}$, through which differentiability of functions $f : U \to \mathbb{A}$ for open $U \subset \mathbb{A}$ get well-defined.
The chain rule also holds for differentiable functions $g : \varphi_\pm V \to \mathbb{A}$ and $f : g(\varphi_\pm V) \to \mathbb{A}$, where $\frac{df(u=g(\zeta))}{du}$ now denotes the derivative $Df(u)$ of f at $u \in g(\varphi_\pm V) \subset \mathbb{A}$.
Also, the product rule holds for two commuting, differentiable functions $f, g : \varphi_\pm V \to \mathbb{A}$: if $f(\zeta)g(\zeta) = g(\zeta)f(\zeta)$ for all $\zeta \in \varphi_\pm V$, then $\frac{d(f(\zeta)g(\zeta))}{d\zeta} = \frac{df(\zeta)}{d\zeta}g(\zeta) + f(\zeta)\frac{dg(\zeta)}{d\zeta}$.

In view of the isometry of $\varphi_\pm : \mathbb{R}^2 \to \mathbb{R}^2_\pm$, a real-valued function $f \in \mathcal{C}(V, \mathbb{R})$ is differentiable in some point (x_0, y_0), if and only if $\varphi_\pm V \ni \zeta \mapsto f(\zeta = \varphi_\pm(x, y))$ is differentiable in $\zeta_0 = \mathfrak{e}_1 x_0 \pm \mathfrak{e}_2 y_0$.

Because $\mathcal{C}_\pm(\varphi_\pm V, \mathbb{A})$ are the images of conformal and anti-conformal subspaces of $\mathcal{C}(V, \mathbb{R}^2)$, every $f \in \mathcal{C}_+(\varphi_+ V, \mathbb{A})$ writes as

$$f(\zeta) = \mathfrak{e}_1 g(\zeta) + \mathfrak{e}_2 g(\zeta), \text{ where } g : V \ni (x, y) \mapsto g(\zeta := \mathfrak{e}_1 x + \mathfrak{e}_1 x) \in \mathbb{R}$$

is conformal. Then path integration of f along a path $\gamma \subset \varphi_+ V$ in $\mathfrak{e}_1 x$- and $\mathfrak{e}_2 y$-coordinates from $\zeta_0 = \mathfrak{e}_1 x_0 + \mathfrak{e}_2 y_0$ to $\zeta = \mathfrak{e}_1 x + \mathfrak{e}_2 y$ equals the path-invariant integral of $G : (x, y) \mapsto (g(x, y), g(x, y))$ from (x_0, y_0) to (x, y), so f is integrable, and $If = (Ig)_1 \equiv (Ig)_2$, where $(Ig)_1$ and $(Ig)_2$ denote the projections of $Ig = ((Ig_1, (Ig)_2)$ onto its x- and y-coordinates. The 2^{nd} primitive $I^2 f_{conf}$ of f_{conf} then results into

$$I^2 f : \zeta \mapsto \mathfrak{e}_1 I^2 g(x, y) + \mathfrak{e}_2 I^2 g(x, y).$$

By induction, f_{conf} is integrable to all orders, If is differentiable, and $\frac{dIf_{conf}}{d\zeta} = f_{conf}$.

Likewise, for $f \in \mathcal{C}(\varphi_- V, \mathbb{A})$, $f(\tilde{\zeta}) = \mathfrak{e}_1 g^c(x, y) - \mathfrak{e}_2 g^c(x, y)$, where $g \in \mathcal{C}(V, \mathbb{R})$ is conformal again, and $If(\tilde{\zeta}) = (Ig^c(x, y))^c$ defines the primitive of f. Then $I^2 f = (I^2 g^c(x, y))^c$ is its second primitive, f has primitives of all orders, If is differentiable on $(\varphi_- V)$ w.r.t. $\tilde{\zeta}$, and $\frac{dIf(\tilde{\zeta})}{d\tilde{\zeta}} = f(\tilde{\zeta})$.

Next, we define analyticity:
A function $f \in \mathcal{C}(\varphi_+ V, \mathbb{A})$ is called *analytic* on $\varphi_+ V$, if for each ζ_0 in $\varphi_+ V$ there is an open neighbourhood $U \subset \mathbb{R}^2_+$, of ζ_0, such that $f(\zeta) = \sum_{k\geq 0} c_k(\zeta - \zeta_0)^k$, where the power series is to converge uniformly on U. Analogously, $f \in \mathcal{C}(\varphi_- V, \mathbb{A})$ is called analytic, if every $\tilde{\zeta}_0 \in \varphi_- V$ has an open neighbourhood $U \subset \mathbb{R}^2_-$ of $\tilde{\zeta}_0$, on which f the uniformly converging limit $f(\tilde{\zeta}) = \sum_{k\geq 0} c_k(\tilde{\zeta} - \tilde{\zeta}_0)^k$.

On the positive/negative orientated $\varphi_\pm \mathbb{R}^2$ let

$$\Psi_\pm : \zeta = \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \mapsto \frac{1}{\zeta}$$

be the *Cauchy function.* Then $\Psi_\pm(\zeta_0 - \zeta) = \frac{1}{\zeta_0} \sum_{k\geq 0} \left(\zeta_0^{-1}\zeta\right)^k$ exists for $\left|\zeta_0^{-1}\zeta\right| < 1$, and the series uniformly converges in ζ on all compact simply connected regions not containing the pole ζ_0. So, it is *analytic* on these regions.

Remark 5.3. Ψ_+ is conformal on simply connected regions not containing the origin, because

(1) the constant function and the identity $id : \mathbb{R}^2_+ \ni \zeta \mapsto \zeta \in \mathbb{A}$ are conformal,
(2) the addition $f + g$ of two conformal functions f and g is conformal,
(3) if f is conformal, then $\frac{1}{f}$ is conformal on all simply connected regions, on which f has no zeros.

Since also the product of two conformal functions is conformal again, the path integral $\int_\gamma f(\zeta)\Psi_+(\zeta_0 - \zeta)d\zeta$ along $f \in \mathcal{C}_+(\varphi_+ V, \mathbb{A})$ along a (piecewise smooth) path $\gamma \subset \varphi_+ V \setminus \{\zeta_0\}$ is a conformal function of ζ_0.

6. Analyticity of $\mathcal{C}(V, \mathbb{R}^2)$

For $r > 0$ the paths $\gamma_\pm : [0, 2\pi] \ni t \mapsto r(\mathfrak{e}_1 \cos(t) \pm \mathfrak{e}_2 \sin(t)) \in \mathbb{R}^2_\pm$ are circular paths around the origin with positive and negative orientation from and to $\mathfrak{e}_1 r$. The path integrals $\int_{\gamma_+} \Psi_+(\zeta)d\zeta$ and $\int_{\gamma_-} \Psi_-(\tilde{\zeta})d\tilde{\zeta}$ along these paths then calculate to

$$\begin{aligned}\int_{\gamma_+} \Psi_+(\zeta)d\zeta &= \int_0^{2\pi} (\mathfrak{e}_1 \cos(t) + \mathfrak{e}_2 \sin(t))(-\mathfrak{e}_1 \sin(t) + \mathfrak{e}_2 \cos(t))dt \\ &= \int_0^{2\pi} (\mathfrak{e}_1\mathfrak{e}_2(\cos^2(t) + \sin^2(t)))dt = \int_0^{2\pi} i dt = 2\pi i, \text{ and}\end{aligned} \tag{6.1}$$

$$\begin{aligned}\int_{\gamma_-} \Psi_-(\tilde{\zeta})d\tilde{\zeta} &= \int_0^{2\pi} (\mathfrak{e}_1 \cos(t) - \mathfrak{e}_2 \sin(t))(-\mathfrak{e}_1 \sin(t) - \mathfrak{e}_2 \cos(t))dt \\ &= \int_0^{2\pi} -(\mathfrak{e}_1\mathfrak{e}_2(\cos^2(t) + \sin^2(t)))dt = \int_0^{2\pi} -i dt = -2\pi i.\end{aligned} \tag{6.2}$$

This gives

Proposition 6.1. *1. Every conformal $f_{conf} \in \mathcal{C}(V, \mathbb{R}^2)$ extends as an analytic function $f_+ : \varphi_+ V \to \mathbb{A}$, where $\varphi_+ : \mathbb{R}^2 \to \mathbb{R}^2_+$ is the chart with positive orientation. The Cauchy-formula holds for f_+: $\int_\gamma f_+(\zeta) \frac{1}{\zeta - \zeta_0} d\zeta = 2\pi i f_+(\zeta_0)$, where $\gamma \subset \varphi_+ V$ is a positively orientated Jordan curve around ζ_0 (i.e: a piecewise continuously differentiable closed curve looping once around ζ_0 at some distance $\epsilon > 0$ from ζ_0 with positive orientation).*

2. Every anti-conformal $f_{aconf} \in \mathcal{C}(V, \mathbb{R}^2)$ extends as analytic function $f_- : \varphi_- V \to \mathbb{A}$, where $\varphi_- : \mathbb{R}^2 \to \mathbb{R}^2_-$ is the chart with negative orientation. The Cauchy-formula holds for f_-: $\int_\gamma f_-(\tilde{\zeta}) \frac{1}{\tilde{\zeta} - \tilde{\zeta}_0} d\tilde{\zeta} = -2\pi i f_-(\tilde{\zeta}_0)$, where $\gamma \subset \varphi_- V$ is a negatively orientated Jordan curve around $\tilde{\zeta}_0$.

Proof. Since $f \in \mathcal{C}_+(\varphi_+ V, \mathbb{R}^2)$ is integrable on V, the path integrals (within $\varphi_+ V$) from startpoint $a \in \varphi_+ V$ to endpoint $b \in \varphi_+ V$ are path independent. By the above, the Cauchy function $\Psi_+(\zeta) = \frac{1}{\zeta}$ is analytic on convex sets not containg the origin, hence integrable on there. The Cauchy-formula $f(\zeta_0) = \frac{1}{2\pi i} \int_\gamma f(\zeta) \frac{1}{\zeta - \zeta_0} dz$ then follows from equation 6.1 togther with the continuity of f for all closed, positively orientated Jordan curves $\gamma \subset \iota V$ around ζ_0. Then, as in Proposition 1.1, $f(\zeta_0)$ is within the encircled open region the uniform limit of a power series on ϵ-neighbourhoods of ζ_0, so analytic in there.
For $f \in \mathcal{C}_-(\varphi_- V, \mathbb{R}^2)$ the the proof is analogous with equation 6.2. □

Corollary 6.2. *The complex isomorphism T_ι maps $\mathcal{C}_+(\varphi_+ V, \mathbb{R}^2)$ onto the complex subspace $\mathcal{C}_{conf}(\iota V, \mathbb{C})$ of holomorphic functions, and is given by: $T_\iota : f \mapsto \mathfrak{e}_1 f \mathfrak{e}_1$. Hence, the power series expansion $f(z) = \sum_k c_k (z - z_0)^k$ of any holomorphic function $f \in \mathcal{C}(\iota V, \mathbb{C})$, determines the power series expansion for $T_\iota^{-1} f \in \mathcal{C}_+(\varphi_+ V, \mathbb{R}^2)$ to be $(T_\iota^{-1} f)(\zeta) = \sum_k (\mathfrak{e}_1 c_k)(\zeta - \zeta_0)^k$.*

7. Summary and outlook

As its essence, it was shown that analyticity is driven by integrability, rather than by differentiability: while differentiability has a strictly local, pointwise definition, integrability relies on simply connected compact regions. Path integration always comes with a right-handed, positive and a left-handed, negative orientation. Parity mandates the symmetry of both, and that brings in the perhaps unexpected anti-conformal "fermionic-like" algebraic structure besides the expected conformal "bosonic-like" one:

$$\mathcal{C}(V, \mathbb{K}) = \mathcal{C}_{conf}(V, \mathbb{K}) \oplus \mathcal{C}_{aconf}(V, \mathbb{K}),$$

where conformal and anti-conformal subspaces contain a parity-invariant subspace $\mathcal{C}_0(V, \mathbb{K})$, such that $\mathcal{C}(V, \mathbb{K})$ allows a decomposition into $\mathcal{C}_0(V, \mathbb{K})$ and its symmetric and anti-symmetric complements $\mathcal{C}_\pm(V, \mathbb{K})$, which are bosonic and fermionic transversal harmonic functions.

For real dimensions of $n > 2$ (or complex dimension $n \geq 2$), similar results follow by replacement of the numbers $\mathfrak{e}_1$ and $\mathfrak{e}_2$ with anti-commuting Hermititian $n \times n$-matrices $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1^2 = \cdots = \alpha_n^2 \equiv 1$ holds.

The implications to physics are clear: The symmetric $\mathcal{C}_+$-subspaces of continuous functions describe a bosonic behaviour, while their conjugated $\mathcal{C}_-$-subspaces are fermionic.

Another direct consequence is to the stability of mechanical systems: It is currently held that such dynamical systems may evolve into chaotic systems, even by small perturbations of the system. Due to the above shown analyticity, that should not happen in conserved mechanical systems for small perturbations.

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