

Proof of the triple and twin prime conjectures using the Sindaram sieve method

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Abstract

Since Dr. Yitang Zhang proved in 2013 that there are infinitely many pairs of prime numbers differing by 70 million, it has been proved now that there are infinitely many pairs of prime numbers differing by 246. In this paper, we use the sieve method invented by Sindaram in 1934 to find out the solution of triple prime numbers and twin prime numbers, and find the general solution formula of the subset, i.e. $an_1 + b$ which is result of each subset, such as $3n + 1, 5n + 2, 7n + 3, 9n + 4, 11n + 5, 13n + 6, 15n + 7, 17n + 8, \dots$ in $2mn + n + m$, modulo x respectively ($x \leq 3$ takes prime). This general solution formula is used to prove the triple prime conjecture and the twin prime conjecture.

1. Introduction

In 1934, Sindaram invented a new sieve method, centered on the use of the general term of the number array $a_{mn} = (2m + 1)n + m$ to construct the following number array - Sindaram's table (see Literature [1]).

$$\begin{array}{cccccccc} 4 & 7 & 10 & 13 & 16 & 19 & 22 & \dots \\ 7 & 12 & 17 & 22 & 27 & 32 & 37 & \dots \\ 10 & 17 & 24 & 31 & 38 & 45 & 52 & \dots \\ 13 & 22 & 31 & 40 & 49 & 58 & 67 & \dots \end{array} \quad (1)$$

Sindaram found that $2N + 1$ is not prime if the natural number N appears in the above number array; if N does not appear in the above number array, then $2N + 1$ must be prime (see Literature [1]).

In this paper, note that $K = \{2mn + n + m | m, n \in N\}$, $L = \{2mn + n + m - 1 | m, n \in N\}$, $S = \{2mn + n + m - 3 | m, n \in N\}$, where $m \leq 1, n \leq 1$ are natural numbers, $N^+ : 0, 1, 2, 3, \dots$ are 0 and natural numbers, $N : 1, 2, 3, \dots$ are natural numbers.

Since $(2m + 1)n + m = 2mn + n + m$, note $K = \{2mn + n + m | m, n \in N\}$, so we have the following: let q be positive integer, then $2q + 1$ is prime ($q \notin K$).

2. Basic Knowledge

Proposition 2.1 All prime numbers p greater than 2 can be expressed as (see Literature [2])

$$p = 2q + 1 \quad (2)$$

where $q \notin K$ takes positive integer.

Proposition 2.2 All twin prime numbers can be expressed as (see Literature [3])

$$\begin{cases} 2q + 1 \\ 2(q + 1) + 1 \end{cases} \quad (3)$$

where $q \notin K \cup L$ takes positive integer.

Proposition 2.3 All triple primes can be expressed as (see Literature [4])

$$\begin{cases} 2q + 1 \\ 2(q + 1) + 1 \\ 2(q + 3) + 1 \end{cases} \quad (4)$$

where $q \notin K \cup L \cup S$ takes positive integer. Obviously, the following definitions are available from Proposition 2.1, Proposition 2.2, and Proposition 2.3.

Definition 2.1 Positive integers not belonging to K are the root of odd prime numbers.

Definition 2.2 Positive integers not belonging to K nor L are the root of twin prime.

Definition 2.3 Positive integers not belonging to K , L or S are the root of triple prime.

3. Applications

3.1. Finding odd prime numbers by the Sindaram sieve method

The values of $2mn + n + m$ are arranged in descending order as follows if m, n are the natural number, which are obtained from the Sindaram table (See Literature [1]).

$$4 \quad 7 \quad 10 \quad 13 \quad 16 \quad 19 \quad 22 \quad \dots \quad (5)$$

The remaining positive integer corresponding to the above is the value of the odd prime root q in equation (2.1), i.e.,

$$q = 1, 2, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, \dots \quad (6)$$

Substituting the above equation into (2.1) yields

$$3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \dots \quad (7)$$

which are all odd prime numbers.

3.2. Finding twin primes by the Sindaram sieve method

The values of $2mn + n + m$ are arranged in descending order as follows if m, n are the natural number, which are obtained from Sindaram's table.

$$4 \quad 7 \quad 10 \quad 13 \quad 16 \quad 19 \quad 22 \quad \dots \quad (8)$$

Then the values of $2mn + n + m - 1$ are arranged in order from smallest to largest as follows.

$$3 \quad 6 \quad 9 \quad 11 \quad 12 \quad 15 \quad 16 \quad 18 \quad 21 \quad \dots \quad (9)$$

Then the values in the equations $2mn + n + m$ and $2mn + n + m - 1$ are arranged in the following order from smallest to largest.

$$3 \quad 4 \quad 6 \quad 7 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19 \quad 21 \quad 22 \quad \dots \quad (10)$$

The remaining positive integer corresponding to the above is the value of the twin prime root q in equation (2.2), i.e.,

$$q = 1, 2, 5, 8, 14, 20, \dots \quad (11)$$

Substituting $q = 1, 2, 5, 8, 14, 20, \dots$ into equation (2.2) yields:

$$3, 5; 5, 7; 11, 13; 17, 19; 29, 31; 41, 43; \dots \quad (12)$$

all of which are twin prime numbers.

3.3. Finding the triple prime by the Sindaram sieve method

The values of $2mn + n + m$ are arranged in descending order as follows if m, n are natural numbers, which are obtained from Sindaram's table.

$$4 \ 7 \ 10 \ 12 \ 13 \ 16 \ 17 \ 19 \ 22 \ \dots \quad (13)$$

Then the values of $2mn + n + m - 1$ are arranged in order from smallest to largest as follows.

$$3 \ 6 \ 9 \ 11 \ 12 \ 15 \ 16 \ 18 \ 21 \ \dots \quad (14)$$

Then the values of $2mn + n + m - 3$ are arranged in order from smallest to largest as follows.

$$1 \ 4 \ 7 \ 9 \ 10 \ 13 \ 14 \ 16 \ 19 \ \dots \quad (15)$$

The values in the equations $2mn + n + m$, $2mn + n + m - 1$, and $2mn + n + m - 3$ are then arranged in the following order from smallest to largest.

$$1, 3, 4, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, \dots \quad (16)$$

The remaining positive integer corresponding to the above is the value of the triple prime root q in equation (2.3), i.e.,

$$q = 2, 5, 8, 20, \dots \quad (17)$$

Substituting $q = 2, 5, 8, 20, \dots$ into equation (2.3) yields:

$$5, 7, 11; 11, 13, 17; 17, 19, 23; 41, 43, 47; \dots \quad (18)$$

all of which are triple prime numbers.

4. Introduction of the equivalence proposition of infinite number of triple prime numbers

We firstly identify the pattern from the following example and then introduce the equivalent proposition of infinite number of triple primes.

Example 4.1 In the five sets $5n, 5n + 1, 5n + 2, 5n + 3, 5n + 4$.

- 1). Substituting $n = 1$ into $5n$ yields 5, and substituting $q = 5$ into equation (2.3) yields: 11, 13, 17.
- 2). Substituting $n = 1$ into $5n + 3$ yields 8. Substituting $q = 8$ into equation (2.3) yields: 17, 19, 23.

Checking the prime number table shows that:

$$11, 13, 17; 17, 19, 23 \quad (19)$$

are two groups of triple prime numbers, then according to Proposition 2.3: 5 and 8 are both positive integers that do not belong to $K \cup L \cup S$. Then according to Definition 2.3: 5 and 8 are both triple prime roots, so when $n \geq 0$ takes positive integer, among the five sets $5n, 5n + 1, 5n + 2, 5n + 3, 5n + 4$, at least 2 sets are the ones containing triple prime roots.

Example 4.2 In the seven sets $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$.

- 1). Substituting $n = 1$ into $7n + 1$ yields 8, substituting $q = 8$ into equation (2.3) yields: 17, 19, 23.
- 2). Substituting $n = 7$ into $7n + 4$ yields 53, and substituting $q = 53$ into equation (2.3) yields: 107, 109, 113.
- 3). Substituting $n = 24$ into $7n + 5$ yields 173, substituting $q = 173$ into equation (2.3) yields: 347, 349, 353.
- 4). Substituting $n = 2$ into $7n + 6$ yields 20, substituting $q = 20$ into equation (2.3) yields: 41, 43, 47.

Checking the prime number table shows that:

$$17, 19, 23; 107, 109, 113; 347, 349, 353; 41, 43, 47 \quad (20)$$

are four groups of triple prime numbers, then according to Proposition 2.3: 8, 53, 173, 20 are all positive integers that do not belong to $K \cup L \cup S$. Then according to Definition 2.3: 8, 53, 173, 20 are all triple prime roots, so when $n \geq 0$ takes positive integer, among the seven sets $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$, at least 4 sets are the ones containing triple prime roots.

Example 4.3 In the eleven sets $11n, 11n+1, 11n+2, 11n+3, 11n+4, 11n+5, 11n+6, 11n+7, 11n+8, 11n+9, 11n+10$.

- 1). Substituting $n = 40$ into $11n$ yields 440, and substituting $q = 440$ into equation (2.3) yields: 881, 883, 887.
- 2). Substituting $n = 14$ into $11n + 1$ yields 155, and substituting $q = 155$ into equation (2.3) yields: 311, 313, 317.
- 3). Substituting $n = 10$ into $11n + 3$ yields 113, and substituting $q = 113$ into equation (2.3) yields: 227, 229, 233.
- 4). Substituting $n = 4$ into $11n + 6$ yields 50, and substituting $q = 50$ into equation (2.3) yields: 101, 103, 107.
- 5). Substituting $n = 8$ into $11n + 7$ yields 95, and substituting $q = 95$ into equation (2.3) yields: 191, 193, 197.
- 6). Substituting $n = 15$ into $11n + 8$ yields 173, and substituting $q = 173$ into equation (2.3) yields: 347, 349, 353.
- 7). Substituting $n = 1$ into $11n + 9$ yields 20, and substituting $q = 20$ into equation (2.3) yields: 41, 43, 47.
- 8). Substituting $n = 20$ into $11n + 10$ yields 230, and substituting $q = 230$ into equation (2.3) yields: 461, 463, 467.

Checking the prime number table shows that:

$$881, 883, 887; 311, 313, 317; 227, 229, 233; 101, 103, 107; 191, 193, 197; 347, 349, 353; 41, 43, 47; 461, 463, 467 \quad (21)$$

are 8 groups of triple prime numbers, then according to Proposition 2.3, we know that: 440, 155, 113, 50, 95, 173, 20, 230 are all positive integers not belonging to $K \cup L \cup S$. Then according to Definition 2.3: 440, 155, 113, 50, 95, 173, 20, 230 are triple prime roots, so when $n \geq 0$ takes positive integer, among the eleven sets $11n, 11n + 1, 11n + 2, 11n + 3, 11n + 4, 11n + 5, 11n + 6, 11n + 7, 11n + 8, 11n + 9, 11n + 10$, at least 8 sets are the ones containing triple prime roots.

Example 4.4 In the thirteen sets $13n, 13n + 1, 13n + 2, 13n + 3, 13n + 4, 13n + 5, 13n + 6, 13n + 7, 13n + 8, 13n + 9, 13n + 10, 13n + 11, 13n + 12$ thirteen.

- 1). Substituting $n = 50$ into $13n$ yields 650, then substituting $q = 650$ into equation (2.3) yields: 1301, 1303, 1307.
- 2). Substituting $n = 4$ into $13n + 1$ yields 53, then substituting $q = 53$ into equation (2.3) yields: 107, 109, 113.
- 3). Substituting $n = 57$ into $13n + 2$ yields 743, then substituting $q = 743$ into equation (2.3) yields: 1487, 1489, 1493.
- 4). Substituting $n = 7$ into $13n + 4$ yields 95, then substituting $q = 95$ into equation (2.3) yields: 191, 193, 197.
- 5). Substituting $n = 1$ into $13n + 7$ yields 20, then substituting $q = 20$ into equation (2.3) yields: 41, 43, 47.
- 6). Substituting $n = 24$ into $13n + 8$ yields 320, then substituting $q = 320$ into equation (2.3) yields: 641, 643, 647.
- 7). Substituting $n = 8$ into $13n + 9$ yields 113, then substituting $q = 113$ into equation (2.3) yields: 227, 229, 233.
- 8). Substituting $n = 61$ into $13n + 10$ yields 803, then substituting $q = 803$ into equation (2.3) yields: 1607, 1609, 1613.
- 9). Substituting $n = 3$ into $13n + 11$ yields 50, then substituting $q = 50$ into equation (2.3) yields: 101, 103, 107.
- 10). Substituting $n = 11$ into $13n + 12$ yields 155, then substituting $q = 155$ into equation (2.3) yields: 311, 313, 317.

Checking the prime number table shows that:

$$\begin{aligned} &1301, 1303, 1307; 107, 109, 113; 1487, 1489, 1493; 191, 193, 197; 41, 43, 47; \\ &641, 643, 647; 227, 229, 233; 1607; 1609, 1613; 101, 103, 17; 311, 313, 317 \end{aligned} \quad (22)$$

are 10 groups of triple primes, then according to Proposition 2.3: 650, 53, 743, 95, 20, 320, 113, 803, 50, 155 are all positive integers that do not belong to $K \cup L \cup S$. Then according to Definition 2.3: 650, 53, 743, 95, 20, 320, 113, 803, 50, 155 are all triple prime roots. So when $n \geq 0$ takes positive integer, among the thirteen sets at least 10 of the 13 sets $13n, 13n + 1, 13n + 2, 13n + 3, 13n + 4, 13n + 5, 13n + 6, 13n + 7, 13n + 8, 13n + 9, 13n + 10, 13n + 11, 13n + 12$, at least 10 sets are the ones containing triple prime roots.

Clearly from Example 4.1, Example 4.2, Example 4.3, Example 4.4, there is a common feature found that when p takes prime number 5, 7, 11, 13, and $n \geq 1$ is natural number, among the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$, there are all $p - 3$ sets containing the roots of the triple prime. From this, we introduce the equivalent proposition of infinite number of triple primes as follows.

Proposition 4.1 When $p \geq 5$ is prime and $n \geq 0$ is natural number, the p sets $pn, pn + 1, pn + 2, \dots, pn + p - 1$, at least $p - 3$ sets are the ones that containing triple prime roots.

Clearly when $p \geq 5$ is prime and $n \geq 0$ is natural number, there is no repeated values among the p sets $pn, pn + 1, pn + 2, \dots, pn + p - 1$.

Also because there are infinite prime numbers, and only prime 2 is even, then there are infinite odd prime numbers, and then odd prime number $p - 3$ is still infinite.

Therefore, from infinite numbers of sets containing the roots of triple prime numbers, there are infinite numbers of groups of triple prime numbers, namely.

If Proposition 4.1 holds, then there are infinite many triple prime numbers.

So Proposition 4.1 is an equivalent proposition of the infinity of triple prime .

Clearly, there is a group of twin prime in every group of triple primes, so if there are infinite many triple primes, then there are infinite many twin primes.

5. Three general solution formulas

From the above discussion it follows that: by the Sindaram sieve method, prime, twin prime, and triple prime numbers are related to the integer in $2mn + n + m$, so the simplest subset of integer in $2mn + n + m$ is studied firstly.

5.1. The $3n + 1; 5n + 2; 7n + 3; 9n + 4; 11n + 5; 13n + 6; 15n + 7; 17n + 8; \dots$ subsets of $2mn + n + m$, each modulo x ($x \geq 3$ takes prime) with all the corresponding $an_1 + b$ subsets of the general solution formulas and the $3m + 1; 5m + 2; 7m + 3; 9m + 4; 11m + 5; 13m + 6; 15m + 7; 17m + 8; \dots$ subsets of $2mn + n + m$, each modulo x ($x \geq 3$ takes prime), with all the corresponding $an_1 + b$ subsets of general solution formulas.

According to the unique decomposition theorem, every integer greater than 1 could be divided into the concatenated product of prime factors, which is $a = p_1 p_2 \cdots p_k, k \geq 1$. Here, $p_1 p_2 \cdots p_k$ are odd prime numbers, of which there may be same values, e.g., $63 = 3 \times 3 \times 7, 75 = 3 \times 5 \times 5$ (see Literature [5]).

Then odd number $3, 5, 7, 9, 11, 13, 15, 17, \dots$ greater than 1 could also be denoted by $p_1 p_2 \cdots p_k$ (here, $p_1 p_2 \cdots p_k$ are odd prime, of which there may be same values, $k = 1, 2, 3, \dots$ takes natural number, e.g., $63 = 3 \times 3 \times 7, 75 = 3 \times 5 \times 5$).

Since $2mn + n + m = (2m + 1)n + m$, substituting $m = 1, 2, 3, 4, 5, 6, \dots$ natural number into $(2m + 1)n + m$ yields respectively: $3n + 1; 5n + 2; 7n + 3; 9n + 4; 11n + 5; 13n + 6; 15n + 7; 17n + 8; \dots$.

Then $2mn + n + m: 3n + 1; 5n + 2; 7n + 3; 9n + 4; 11n + 5; 13n + 6; 15n + 7; 17n + 8; \dots$ n with coefficient $3, 5, 7, 9, 11, 13, 15, \dots$ odd $an + b$ (where a takes odd number, n takes natural number, $1 \leq b \leq \frac{a-1}{2}$) subset could be expressed as the following general solution formula:

$$p_1 p_2 \cdots p_k n + \frac{p_1 p_2 \cdots p_k - 1}{2} \quad (23)$$

of which p_1, p_2, \dots, p_k take odd prime number, $n \geq 1$ takes natural number, and $k = 1, 2, 3, \dots$ takes natural number.

Clearly, when $p_1 p_2 \cdots p_k n + \frac{p_1 p_2 \cdots p_k - 1}{2}$ modulo x ($x \geq 3$ takes prime), according to the residual theory we have:

$$\begin{aligned} & \{p_1 p_2 \cdots p_k n + \frac{p_1 p_2 \cdots p_k - 1}{2} \mid n \in N^+\} \\ &= \{x p_1 p_2 \cdots p_k n_1 + \frac{p_1 p_2 \cdots p_k - 1}{2} \mid n_1 \in N^+\} \\ & \cup \{x p_1 p_2 \cdots p_k n_1 + p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} \mid n_1 \in N^+\} \\ & \cup \{x p_1 p_2 \cdots p_k n_1 + 2 p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} \mid n_1 \in N^+\} \\ & \quad \cup \dots \\ & \cup \{x p_1 p_2 \cdots p_k n_1 + (x - 1) p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} \mid n_1 \in N^+\} \end{aligned} \quad (24)$$

Therefore, the following general solution could be derived from the set of residual system on the right side of the above equation.

That is, the coefficient $3n + 1; 5n + 2; 7n + 3; 9n + 4; 11n + 5; 13n + 6; 15n + 7; 17n + 8; \dots$ n of $2mn + n + m$ are $an + b$ of $3, 5, 7, 9, 11, 13, 15, \dots$ odd number (where a takes odd number, n takes natural number, $1 \leq b \leq \frac{a-1}{2}$) After the subsets are each modulo x ($x \geq 3$ takes prime), the corresponding general solution of the subset $an_1 + b$ is

$$\begin{cases} x p_1 p_2 \cdots p_k n_1 + \frac{p_1 p_2 \cdots p_k - 1}{2} \\ x p_1 p_2 \cdots p_k n_1 + p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} \\ x p_1 p_2 \cdots p_k n_1 + 2 p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} \\ \dots \\ x p_1 p_2 \cdots p_k n_1 + (x - 1) p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} \end{cases} \quad (25)$$

where m, n, k, n_1 takes natural number, p_1, p_2, \dots, p_k may be identical, but none is equal to x , p_1, p_2, \dots, p_k are all odd prime. $\frac{p_1 p_2 \cdots p_k - 1}{2} \leq b \leq (x - 1) p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2}$.

Similarly, substituting $n = 1, 2, 3, 4, 5, 6, \dots$ natural number into $(2m + 1)n + m$ yields: $3m + 1; 5m + 2; 7m + 3; 9m + 4; 11m + 5; 13m + 6; 15m + 7; 17m + 8; \dots; (2m + 1)n + m$, respectively.

Since $2mn + n + m$ is symmetric, it shall find the general solution of subset $an_1 + b$ of $2mn + n + m$ in terms of $3m + 1; 5m + 2; 7m + 3; 9m + 4; 11m + 5; 13m + 6; \dots$ (where a takes odd number, m, n both take natural number,

$1 \leq b \geq \frac{a-1}{2}$) when the set is modulo x ($x \geq 3$ takes prime) respectively. It is the same as finding general solution of subset $an_1 + b$ (5.1) of $2mn + n + m$ in terms of $3n + 1; 5n + 2; 7n + 3; 9n + 4; 11n + 5; 13n + 6; \dots$ (where a takes odd number, m, n take natural number, $1 \leq b \geq \frac{a-1}{2}$) when the set is modulo x ($x \geq 3$ takes prime) respectively. An example is given as follows.

Example 5.1 (1) Substituting $m = 1$ into $2mn + n + m$ yields $3n + 1$, then $\{3n + 1 \mid n \in N\} \in \{2mn + m + n \mid n \in N\}$, i.e., $3n + 1$ is a subset of $2mn + n + m$ when m, n take natural number. Substituting $n = 5n_1$ into $3n + 1$ yields $15n_1 + 1$. Substituting $n = 5n_1 + 1$ into $3n + 1$ yields $15n_1 + 4$. Substituting $n = 5n_1 + 2$ into $3n + 1$ yields $15n_1 + 7$. Then $\{15n_1 + 1 \mid n_1 \in N\} \in \{3n + 1 \mid n \in N\}$, $\{15n_1 + 4 \mid n_1 \in N\} \in \{3n + 1 \mid n \in N\}$, $\{15n_1 + 7 \mid n_1 \in N\} \in \{3n + 1 \mid n \in N\}$. That is, when n take natural number, $15n_1 + 1, 15n_1 + 4, 15n_1 + 7$ are all subsets of $3n + 1$. Since $3n + 1$ is a subset of $2mn + n + m$ when m, n take natural number. Then it could be drawn from the set transitivity that when n_1, m, n take natural number, $15n_1 + 1, 15n_1 + 4, 15n_1 + 7$ are all subsets of $2mn + m + n$.

(2) Substituting $n = 1$ into $2mn + n + m$ yields $3m + 1$, then $\{3m + 1 \mid m \in N\} \in \{2mn + m + n \mid m, n \in N\}$, i.e., $3m + 1$ is a subset of $2mn + n + m$ when m, n take natural number. Substituting $m = 5n_1$ into $3m + 1$ yields $15n_1 + 1$. Substituting $m = 5n_1 + 1$ into $3m + 1$ yields $15n_1 + 4$. Substituting $m = 5n_1 + 2$ into $3m + 1$ yields $15n_1 + 7$. Then $\{15n_1 + 1 \mid n_1 \in N\} \in \{3m + 1 \mid m \in N\}$, $\{15n_1 + 4 \mid n_1 \in N\} \in \{3m + 1 \mid m \in N\}$, $\{15n_1 + 7 \mid n_1 \in N\} \in \{3m + 1 \mid m \in N\}$. That is, when n_1, m take natural number, $15n_1 + 1, 15n_1 + 4, 15n_1 + 7$ are all subsets of $3m + 1$. Since $3m + 1$ is a subset of $2mn + n + m$ when m, n take natural number. Then it could be drawn from the set transitivity that when n_1, m, n take natural number, $15n_1 + 1, 15n_1 + 4, 15n_1 + 7$ are all subsets of $2mn + m + n$.

Clearly the same result in (1) of Example 5.1 and (2) of Example 5.1 is obtained when n_1, m, n take natural number, $15n_1 + 1, 15n_1 + 4, 15n_1 + 7$ are all subsets of $2mn + m + n$, etc, so one of them could be omitted. In this paper, we use the $3n + 1, 5n + 2, 7n + 3, 7n + 4, 11n + 5, 13n + 6, \dots$ subsets of $2mn + n + m$ to solve Eq. (5.1), and no longer use the $3m + 1; 5m + 2; 7m + 3; 9m + 4; 11m + 5; 13m + 6; \dots$ subsets of $2mn + n + m$ to solve Eq. (5.1).

That is, equation (5.1) not only contains $3n + 1; 5n + 2; 7n + 3; 9n + 4; 11n + 5; 13n + 6; 15n + 7; 17n + 8; \dots$ subsets of $2mn + n + m$ modulo x ($x \geq 3$ takes prime) corresponding to all $an_1 + b$ subsets, but also contains $3m + 1, 5m + 2, 7m + 3$ of $2mn + n + m, 9m + 4, 11m + 5, 13m + 6, 15m + 7, 17m + 8, \dots$ subsets modulo x ($x \geq 3$ takes prime) respectively, which is corresponding to all $an_1 + b$ subsets.

5.2. The subsets $3n; 5n + 1; 7n + 2; 9n + 3; 11n + 4; 13n + 5; 15n + 6; 17n + 7; \dots$ of $2mn + n + m - 1$ modulo x ($x \geq 3$ takes prime) and all the corresponding $an_1 + b$ subset general solution formula and the subsets $3m; 5m + 1; 7m + 2; 9m + 3; 11m + 4; 13m + 5; 15m + 6; 17m + 7; \dots$ of $2mn + n + m - 1$ modulo x ($x \geq 3$ takes prime) respectively, which is corresponding to all $an_1 + b$ subset general solution formula.

Since $2mn + n + m - 1 = (2m + 1)n + m - 1$, substituting $m = 1, 2, 3, 4, 5, 6, \dots$ natural number into $(2m + 1)n + m - 1$ yields: $3n; 5n + 1; 7n + 2; 9n + 3; 11n + 4; 13n + 5; 15n + 6; 17n + 7; \dots$ respectively. The same method as in 5.1 yields $3n; 5n + 1; 7n + 2; 9n + 3; 11n + 4; 13n + 5; 15n + 6; 17n + 7; \dots$ subsets of $2mn + n + m - 1$ modulo x ($x \geq 3$ taken as prime), respectively, which is corresponding to all $an_1 + b$ subset general solution:

$$\begin{cases} xp_1p_2 \cdots p_k n_1 + \frac{p_1p_2 \cdots p_k - 1}{2} - 1 \\ xp_1p_2 \cdots p_k n_1 + p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 1 \\ xp_1p_2 \cdots p_k n_1 + 2p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 1 \\ \dots \\ xp_1p_2 \cdots p_k n_1 + (x - 1)p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 1 \end{cases} \quad (26)$$

where m, n, k, n_1 take natural number, p_1, p_2, \dots, p_k may be identical, but none is equal to x . p_1, p_2, \dots, p_k , are all odd prime, $\frac{p_1p_2 \cdots p_k - 1}{2} \leq b \leq (x - 1)p_1, p_2, \dots, p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 1$. As in equation (5.1), equation (5.2) not only contains all $an_1 + b$ subsets of $2mn + n + m - 1$ of $3n; 5n + 1; 7n + 2; 9n + 3; 11n + 4; 13n + 5; 15n + 6; 17n + 7; \dots$ modulo x ($x \geq 3$ takes prime) respectively, corresponding to all $a + b$ subsets, but also contains $3m$ of $2mn + n + m - 1; 5m + 1; 7m + 2; 9m + 3; 11m + 4; 13m + 5; 15m + 6; 17m + 7; \dots$ modulo x ($x \geq 3$ takes prime), respectively, corresponding to all $an_1 + b$ subsets.

5.3. The subsets $3n - 2; 5n - 1; 7n; 9n + 1; 11n + 2; 13n + 3; 15n + 4; 17n + 5; \dots$ of $2mn + n + m - 3$ modulo x ($x \geq 3$ takes prime) and all the corresponding $an_1 + b$ subset general solution formula and the subsets $3m - 2; 5m - 1; 7m; 9m + 1; 11m + 2; 13m + 3; 15m + 4; 17m + 5; \dots$ of $2mn + n + m - 1$ modulo x ($x \geq 3$ takes prime) respectively, which is corresponding to all $an_1 + b$ subset general solution formula.

Since $2mn + n + m - 3 = (2m + 1)n + m - 3$, substituting $m = 1, 2, 3, 4, 5, 6, \dots$ natural number into $(2m + 1)n + m - 3$ yields: $3n - 2; 5n - 1; 7n; 9n + 1; 11n + 2; 13n + 3; 15n + 4; 17n + 5; \dots$ respectively. The same method as in 5.1 yields $3m - 2; 5m - 1; 7m; 9m + 1; 11m + 2; 13m + 3; 15m + 4; 17m + 5; \dots$ subsets of $2mn + n + m - 3$ modulo x ($x \geq 3$ takes prime) respectively, which is corresponding to all $an_1 + b$ subset general solution:

$$\begin{cases} xp_1p_2 \cdots p_k n_1 + \frac{p_1p_2 \cdots p_k - 1}{2} - 3 \\ xp_1p_2 \cdots p_k n_1 + p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 3 \\ xp_1p_2 \cdots p_k n_1 + 2p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 3 \\ \dots \\ xp_1p_2 \cdots p_k n_1 + (x - 1)p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 3 \end{cases} \quad (27)$$

where m, n, k, n_1 take natural number, p_1, p_2, \dots, p_k may be identical, but none is equal to x . p_1, p_2, \dots, p_k are all odd prime, $\frac{p_1p_2 \cdots p_k - 1}{2} - 3 \leq b \leq (x - 1)p_1, p_2, \dots, p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 3$. As in equation (5.1), equation (5.3) not only contains all $an_1 + b$ subsets corresponding to the subsets $3n - 2; 5n - 1; 7n; 9n + 1; 11n + 2; 13n + 3; 15n + 4; 17n + 5; \dots$ of $2mn + n + m - 3$ modulo x ($x \geq 3$ takes prime) respectively, but also contains $3m - 2; 5m - 1; 7m; 9m + 1; 11m + 2; 13m + 3; 15m + 4; 17m + 5; \dots$ of $2mn + n + m - 3$ modulo x ($x \geq 3$ takes prime) respectively, corresponding to all $an_1 + b$ subsets. Also, since this paper is written: $K = \{2mn + n + m \mid m, n \in N\}$, $L = \{2mn + n + m - 1 \mid m, n \in N\}$, $S = \{2mn + n + m - 3 \mid m, n \in N\}$, where $m, n \geq 1$ take natural number, $N : 1, 2, 3, \dots$ takes natural number.

The equation (5.1) contains all $an_1 + b$ subsets of $K \cup L \cup S$ modulo x ($x \geq 3$ taken as prime) corresponding to $3n + 1; 5n + 2; 7n + 3; 9n + 4; 11n + 5; 13n + 6; 15n + 7; 17n + 8; \dots$ subsets modulo x ($x \geq 3$ taken as prime), as well as $3m + 1; 5m + 2; 7m + 3; 9m + 4$ of $K \cup L \cup S$; $11m + 5; 13m + 6; 15m + 7; 17m + 8; \dots$ subsets modulo x ($x \geq 3$ taken as prime), respectively, which is corresponding to all an_1 .

The equation (5.2) contains all $an_1 + b$ subsets modulo x ($x \geq 3$ takes prime) corresponding to $3n; 5n + 1; 7n + 2; 9n + 3; 11n + 4; 13n + 5; 15n + 6; 17n + 7; \dots$ subsets of $K \cup L \cup S$. It also contains all $an_1 + b$ subsets modulo x ($x \geq 3$ takes prime) respectively, corresponding to $3m; 5m + 1; 7m + 2; 9m + 3; 11m + 4; 13m + 5; 15m + 6; 17m + 7; \dots$ subsets of $K \cup L \cup S$.

The equation (5.3) contains all $an_1 + b$ subsets modulo x ($x \geq 3$ takes prime) corresponding to $3n - 2; 5n - 1; 7n; 9n + 1; 11n + 2; 13n + 3; 15n + 4; 17n + 5; \dots$ subsets of $K \cup L \cup S$ as well as all $an_1 + b$ subsets modulo x ($x \geq 3$ takes prime) respectively, corresponding to $3m - 2; 5m - 1; 7m; 9m + 1; 11m + 2; 13m + 3; 15m + 4; 17m + 5; \dots$ subsets of $K \cup L \cup S$.

6. Preliminary propositions

Proposition 6.1 In $\frac{p_1p_2 \cdots p_k - 1}{2} - 3 \leq b \leq (x - 1)p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2}$ closed interval, the corresponding $an_1 + b$ subset general solution formula of $K \cup L \cup S$ are (5.1), (5.2), (5.3) only. (where m, n, k, n_1 take natural number, $p_1p_2 \cdots p_k$ may be identical, but none is equal to x , $p_1p_2 \cdots p_k$ take odd prime, $a \leq 3$ takes odd).

In the 1st $an_1 + b$ set in equation (5.1) $b = \frac{p_1p_2 \cdots p_k - 1}{2}$

In the last $an_1 + b$ set in equation (5.1) $b = (x - 1)p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2}$

In the 1st $an_1 + b$ set in equation (5.2) $b = \frac{p_1p_2 \cdots p_k - 1}{2} - 1$

In the last $an_1 + b$ set in equation (5.2) $b = (x - 1)p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 1$

In the 1st $an_1 + b$ set in equation (5.3) $b = \frac{p_1p_2 \cdots p_k - 1}{2} - 3$

In the last $an_1 + b$ set in equation (5.3) $b = (x - 1)p_1p_2 \cdots p_k + \frac{p_1p_2 \cdots p_k - 1}{2} - 3$ closed interval and since it is clear that when k takes natural number, x , the $p_1p_2 \cdots p_k$ are odd prime,

$$\frac{p_1p_2 \cdots p_k - 1}{2} - 3 < \frac{p_1p_2 \cdots p_k - 1}{2} - 1 < \frac{p_1p_2 \cdots p_k - 1}{2} \quad (28)$$

$$(x - 1)p_1 \cdots p_k + \frac{p_1 \cdots p_k - 1}{2} - 3 < (x - 1)p_1 \cdots p_k + \frac{p_1 \cdots p_k - 1}{2} - 1 < (x - 1)p_1 \cdots p_k + \frac{p_1 \cdots p_k - 1}{2} - 3 \quad (29)$$

Then b in the set $an_1 + b$ is in $\frac{p_1 p_2 \cdots p_k - 1}{2} - 3$ to $(x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2}$ contains b in $\frac{p_1 p_2 \cdots p_k - 1}{2}$ to $(x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2}$ closed interval and contains b in $\frac{p_1 p_2 \cdots p_k - 1}{2} - 1$ to $(x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} - 1$ closed interval and also contains b in $\frac{p_1 p_2 \cdots p_k - 1}{2} - 3$ to $(x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} - 3$ closed interval.

Suppose that $K \cup L \cup S$ has $an_1 \pm b$ subset of general solution in addition to equations (5.1), (5.2), (5.3) (where a is odd number and m, n, k, n_1 are natural number, $p_1 p_2 \cdots p_k$ may be identical, but none is equal to x . $p_1 p_2 \cdots p_k$ are all odd prime, $\frac{p_1 p_2 \cdots p_k - 1}{2} - j \leq b \leq (x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} - j$).

According to the unique decomposition theorem, the coefficient a of n_1 in $an_1 \pm b$, every integer greater than 1, the result is unique, if it decomposes prime factors regardless of their decomposition order (see Literature [5]).

Obviously, if $K \cup L \cup S$ has another $an_1 \pm b$ subset general solutions in addition to Eqs. (5.1), (5.2), (5.3), then the coefficient n_1 must be same with coefficient n_1 in (5.1), (5.2), (5.3) and only the remainder is different, then it could be written in the following form.

$$\begin{cases} xp_1 p_2 \cdots p_k n_1 + \frac{p_1 p_2 \cdots p_k - 1}{2} - j \\ xp_1 p_2 \cdots p_k n_1 + p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} - j \\ xp_1 p_2 \cdots p_k n_1 + 2p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} - j \\ \dots \\ xp_1 p_2 \cdots p_k n_1 + (x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2} - j \end{cases} \quad (30)$$

$p_1, p_2, \dots, p_k, n_1, m, n, x, k$ are consistent with the requirements in equations (5.1), (5.2), (5.3).

Obviously, according to equations (5.1), (5.2) and (5.3), equation (6.1) is subsets of $2mn + n + m - j : 3n + 1 - j; 5n + 2 - j; 7n + 3 - j; 9n + 4 - j; 11n + 5 - j; 13n + 6 - j; 15n + 7 - j; 17n + 8 - j; \dots$ modulo x ($x \geq 3$ takes prime) respectively. Which is corresponding to all $a + b$ subset general solution formula, also contains subsets of $2mn + n + m - j : 3m + 1 - j; 5m + 2 - j; 7m + 3 - j; 9m + 4 - j; 11m + 5 - j; 13m + 6 - j; 15m + 7 - j; 17m + 8 - j; \dots$ modulo x ($x \geq 3$ takes prime), which is corresponding to all $a + b$ subset.

(a) When $j = 0$, equation (6.1) is in line with equation (5.1).

(b) When $j = 1$, equation (6.1) is in line with equation (5.2).

(c) When $j = 3$, equation (6.1) is in line with equation (5.3).

Then $j \neq 0, j \neq 1, j \neq 3$. Also, since when $j \neq 0, j \neq 1, j \neq 3$, it could be drawn from Proposition 2.1 that $2Q + 1; 2(Q + 1) + 1; 2(Q + 3) + 1; 2(Q + j) + 1$ are four odd prime numbers when $Q \neq 2mn + n + m, Q + 1 \neq 2mn + n + m, Q + 3 \neq 2mn + n + m$, and $Q + j \neq 2mn + n + m$, and $2Q + 1; 2(Q + 3) + 1; 2(Q + j) + 1$ are beyond the triple prime. A contradiction appears and the hypothesis is not valid. Then Proposition 6.1 holds.

7. Proofs of the Samson prime conjecture and the twin prime conjecture

From what is stated in Paper 4, Proposition 4.1 is an equivalent proposition of the infinity of triple prime. If Proposition 4.1 holds is proved, then the infinity of triple prime holds. Proposition 4.1 is proved as follows.

7.1. $p = 5$

Substituting $p = 5$ into $pn, pn + 1, pn + 2, \dots, pn + p - 1$ yields: $5n, 5n + 1, 5n + 2, 5n + 3, 5n + 4$. In the fives sets of $5n, 5n + 1, 5n + 2, 5n + 3, 5n + 4$.

1) Substituting $n = 1$ into $5n$ yields 5, and substituting $q = 5$ into equation (2.3) yields: 11, 13, 17.

2) Substituting $n = 1$ into $5n + 3$ yields 8, and substituting $q = 8$ into equation (2.3) yields: 17, 19, 23.

Checking the prime number table shows that: 11, 13, 17; 17, 19, 23 are two groups of triple prime number, then according to Proposition 2.3 we have: 5 and 8 are both positive integers not belonging to $K \cup L \cup S$. Then according to Definition 2.3 we have: 5 and 8 are both triple prime roots. So when $n \geq 0$ takes positive integer, among the five sets $5n, 5n + 1, 5n + 2, 5n + 3, 5n + 4$, at least 2 sets are the ones containing triple prime roots. So, when $p = 5$, Proposition 4.1 holds.

7.2. $p = 7$

Substituting $p = 7$ into $pn, pn + 1, pn + 2, \dots, pn + p - 1$ yields: $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$.

The first proof method 1) Substituting $n = 1$ into $7n + 1$ yields 8, substituting $q = 8$ into equation (2.3) yields: 17, 19, 23.

2) Substituting $n = 7$ for $7n + 4$ yields 53, and substituting $q = 53$ into equation (2.3) yields: 107, 109, 113.

3) Substituting $n = 24$ into $7n + 5$ yields 173, substituting $q = 173$ into equation (2.3) yields: 347, 349, 353.

4) Substituting $n = 2$ into $7n + 6$ yields 20, substituting $q = 20$ into equation (2.3) yields: 41, 43, 47.

Checking the prime number table shows that: 17, 19, 23; 107, 109, 113; 347, 349, 353; 42, 43, 47 are four groups of triple prime numbers, then according to Proposition 2.3: 8, 53, 173, 20 are all positive integers not belonging to $K \cup L \cup S$. Then according to Definition 2.3: 8, 53, 173, 20 are all triple prime roots, so when $n \neq 0$ takes positive integer, among the seven sets $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$, at least 4 sets are the ones containing triple prime roots. So, when $p = 7$, Proposition 4.1 holds. Since the first proof requires looking up the prime number table, it is obvious that this method does not work when analyze the relationship between $p, pn + 1, pn + 2, \dots, pn + p - 1$ and $K \cup L \cup S$ when p in $pn, pn + 1, pn + 2, \dots, pn + p - 1$ takes infinite prime.

The second proof method Step I: $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ modulo 11 respectively, by the theory of complete system of residues yields 7 groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11, i.e.,

$$\begin{aligned} \{7n \mid n \in N^+\} &= \{77n_1 \mid n_1 \in N^+\} \cup \{77n_1 + 7 \mid n_1 \in N^+\} \cup \{77n_1 + 14 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 21 \mid n_1 \in N^+\} \cup \{77n_1 + 28 \mid n_1 \in N^+\} \cup \{77n_1 + 35 \mid n_1 \in N^+\} \cup \{77n_1 + 42 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 49 \mid n_1 \in N^+\} \cup \{77n_1 + 56 \mid n_1 \in N^+\} \cup \{77n_1 + 63 \mid n_1 \in N^+\} \cup \{77n_1 + 70 \mid n_1 \in N^+\} \end{aligned} \quad (31)$$

$$\begin{aligned} \{7n + 1 \mid n \in N^+\} &= \{77n_1 + 1 \mid n_1 \in N^+\} \cup \{77n_1 + 8 \mid n_1 \in N^+\} \cup \{77n_1 + 15 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 22 \mid n_1 \in N^+\} \cup \{77n_1 + 29 \mid n_1 \in N^+\} \cup \{77n_1 + 36 \mid n_1 \in N^+\} \cup \{77n_1 + 43 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 50 \mid n_1 \in N^+\} \cup \{77n_1 + 57 \mid n_1 \in N^+\} \cup \{77n_1 + 64 \mid n_1 \in N^+\} \cup \{77n_1 + 71 \mid n_1 \in N^+\} \end{aligned} \quad (32)$$

$$\begin{aligned} \{7n + 2 \mid n \in N^+\} &= \{77n_1 + 2 \mid n_1 \in N^+\} \cup \{77n_1 + 9 \mid n_1 \in N^+\} \cup \{77n_1 + 16 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 23 \mid n_1 \in N^+\} \cup \{77n_1 + 30 \mid n_1 \in N^+\} \cup \{77n_1 + 37 \mid n_1 \in N^+\} \cup \{77n_1 + 44 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 51 \mid n_1 \in N^+\} \cup \{77n_1 + 58 \mid n_1 \in N^+\} \cup \{77n_1 + 65 \mid n_1 \in N^+\} \cup \{77n_1 + 72 \mid n_1 \in N^+\} \end{aligned} \quad (33)$$

$$\begin{aligned} \{7n + 3 \mid n \in N^+\} &= \{77n_1 + 3 \mid n_1 \in N^+\} \cup \{77n_1 + 10 \mid n_1 \in N^+\} \cup \{77n_1 + 17 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 24 \mid n_1 \in N^+\} \cup \{77n_1 + 31 \mid n_1 \in N^+\} \cup \{77n_1 + 38 \mid n_1 \in N^+\} \cup \{77n_1 + 45 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 52 \mid n_1 \in N^+\} \cup \{77n_1 + 59 \mid n_1 \in N^+\} \cup \{77n_1 + 66 \mid n_1 \in N^+\} \cup \{77n_1 + 73 \mid n_1 \in N^+\} \end{aligned} \quad (34)$$

$$\begin{aligned} \{7n + 4 \mid n \in N^+\} &= \{77n_1 + 4 \mid n_1 \in N^+\} \cup \{77n_1 + 11 \mid n_1 \in N^+\} \cup \{77n_1 + 18 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 25 \mid n_1 \in N^+\} \cup \{77n_1 + 32 \mid n_1 \in N^+\} \cup \{77n_1 + 39 \mid n_1 \in N^+\} \cup \{77n_1 + 46 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 53 \mid n_1 \in N^+\} \cup \{77n_1 + 60 \mid n_1 \in N^+\} \cup \{77n_1 + 67 \mid n_1 \in N^+\} \cup \{77n_1 + 74 \mid n_1 \in N^+\} \end{aligned} \quad (35)$$

$$\begin{aligned} \{7n + 5 \mid n \in N^+\} &= \{77n_1 + 5 \mid n_1 \in N^+\} \cup \{77n_1 + 12 \mid n_1 \in N^+\} \cup \{77n_1 + 19 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 26 \mid n_1 \in N^+\} \cup \{77n_1 + 33 \mid n_1 \in N^+\} \cup \{77n_1 + 40 \mid n_1 \in N^+\} \cup \{77n_1 + 47 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 54 \mid n_1 \in N^+\} \cup \{77n_1 + 61 \mid n_1 \in N^+\} \cup \{77n_1 + 68 \mid n_1 \in N^+\} \cup \{77n_1 + 75 \mid n_1 \in N^+\} \end{aligned} \quad (36)$$

$$\begin{aligned} \{7n + 6 \mid n \in N^+\} &= \{77n_1 + 6 \mid n_1 \in N^+\} \cup \{77n_1 + 13 \mid n_1 \in N^+\} \cup \{77n_1 + 20 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 27 \mid n_1 \in N^+\} \cup \{77n_1 + 34 \mid n_1 \in N^+\} \cup \{77n_1 + 41 \mid n_1 \in N^+\} \cup \{77n_1 + 48 \mid n_1 \in N^+\} \\ &\cup \{77n_1 + 55 \mid n_1 \in N^+\} \cup \{77n_1 + 62 \mid n_1 \in N^+\} \cup \{77n_1 + 69 \mid n_1 \in N^+\} \cup \{77n_1 + 76 \mid n_1 \in N^+\} \end{aligned} \quad (37)$$

The total gives 77 sets of $an_1 + b$ with n_1 coefficients all be 77.

Feature 1: In the above 77 sets,

- There are only 7 sets divided by 11 with remainder 0.
- There are only 7 sets divided by 11 with remainder 1.
- There are only 7 sets divided by 11 with remainder 2.
- There are only 7 sets divided by 11 with remainder 3.
- There are only 7 sets divided by 11 with remainder 4.
- There are only 7 sets divided by 11 with remainder 5.
- There are only 7 sets divided by 11 with remainder 6.
- There are only 7 sets divided by 11 with remainder 7.
- There are only 7 sets divided by 11 with remainder 8.
- There are only 7 sets divided by 11 with remainder 9.

There are only 7 sets divided by 11 with remainder 10.

A total of 7 sets of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11.

Feature 2: The coefficients of n_1 in sets $an_1 + b$ are all 77.

Feature 3: The value of the remainder b of set $an_1 + b$ is in the closed interval 0 to 76.

The total gives 77 sets of $an_1 + b$ with n_1 coefficients of all be 77. From these three characteristics and the transitivity of set, the following result could be derived.

If the seven sets $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ are all subsets of $K \cup L \cup S$. Then in $K \cup L \cup S$, there must be 7 groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11, and these 7 groups of complete system of residues with remainder 0,1,2,3,4,5,6,7,8,9,10 divided by 11 are the ones with remainder b in $K \cup L \cup S$ in the closed interval 0 to 76 and the coefficients are all 77 of $a + b$ subsets, which otherwise do not satisfy the transitivity of set.

So we want to analyze $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ with $K \cup L \cup S$ using the characteristics of $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ modulo 11 respectively, to find $an_1 + b$ in $K \cup L \cup S$ whose remainder b has values in the closed interval from 0 to 76 and whose n_1 coefficients are all 77 sets.

Step 2: Find the set of $an_1 + b$ with remainder b of $K \cup L \cup S$ is in the closed interval 0 to 76 and the coefficient of n_1 is 77.

Since it could be drawn from Section 5: Eq. (5.1) is the general solution formula of the subset corresponding to $2mn + n + m$, i.e., $an_1 + b$, which is result of each subset, such as $3n + 1, 5n + 2, 7n + 3, 9n + 4, \dots$ of $K \cup L \cup S$. Actually, it is also the general solution formula of the subset corresponding to $2mn + n + m$, i.e., $an_1 + b$, which is result of each subset, such as $3m + 1, 5m + 2, 7m + 3, 9m + 4, \dots$ of $K \cup L \cup S$.

Eq. (5.2) is the general solution formula of the subset corresponding to $2mn + n + m - 1$, i.e., $an_1 + b$, which is result of each subset, such as $3n, 5n + 1, 7n + 2, 9n + 3, \dots$ of $K \cup L \cup S$. Actually, it is also the general solution formula of the subset corresponding to $2mn + n + m - 1$, i.e., $an_1 + b$, which is result of each subset, such as $3m, 5m + 1, 7m + 2, 9m + 3, \dots$ of $K \cup L \cup S$.

Eq. (5.3) is the general solution formula of the subset corresponding to $2mn + n + m - 3$, i.e., $an_1 + b$, which is result of each subset, such as $3n - 2, 5n - 1, 7n, 9n + 1, \dots$ of $K \cup L \cup S$. Actually, it is also the general solution formula of the subset corresponding to $2mn + n + m - 3$, i.e., $an_1 + b$, which is result of each subset, such as $3m - 2, 5m - 1, 7m, 9m + 1, \dots$ of $K \cup L \cup S$.

From equations (5.1), (5.2), (5.3), it is obvious that when the coefficient of n_1 is 77, there are only two cases, i.e., $p_1 p_2 \cdots p_k = 7$ with $x = 11$, and $x = 7$ with $p_1 p_2 \cdots p_k = 11$.

7.2.1 The case of $p_1 p_2 \cdots p_k = 7, x = 11$

Substituting $p_1 p_2 \cdots p_k = 7, x = 11$ into (5.1) yields: $77n_1 + 3, 77n_1 + 10, 77n_1 + 17, 77n_1 + 24, 77n_1 + 31, 77n_1 + 38, 77n_1 + 45, 77n_1 + 52, 77n_1 + 59, 77n_1 + 66, 77n_1 + 73$. There are eleven sets and one set each with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11.

Substituting $p_1 p_2 \cdots p_k = 7, x = 11$ into (5.2) yields: $77n_1 + 2, 77n_1 + 9, 77n_1 + 16, 77n_1 + 23, 77n_1 + 30, 77n_1 + 37, 77n_1 + 44, 77n_1 + 51, 77n_1 + 58, 77n_1 + 65, 77n_1 + 72$. There are eleven sets and one set each with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11.

Substituting $p_1 p_2 \cdots p_k = 7, x = 11$ into (5.3) yields: $77n_1, 77n_1 + 7, 77n_1 + 14, 77n_1 + 21, 77n_1 + 28, 77n_1 + 35, 77n_1 + 42, 77n_1 + 49, 77n_1 + 56, 77n_1 + 63, 77n_1 + 70$. There are eleven sets and one set each with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11.

7.2.2 The case of $p_1 p_2 \cdots p_k = 11, x = 7$

Substituting $p_1 p_2 \cdots p_k = 11, x = 7$ into (5.1) yields: $77n_1 + 5, 77n_1 + 16, 77n_1 + 27, 77n_1 + 38, 77n_1 + 49, 77n_1 + 60, 77n_1 + 71$. There are seven sets and all are divided by 11 with remainder 5.

Substituting $p_1 p_2 \cdots p_k = 11, x = 7$ into (5.2) yields: $77n_1 + 4, 77n_1 + 15, 77n_1 + 26, 77n_1 + 37, 77n_1 + 48, 77n_1 + 59, 77n_1 + 70$. There are seven sets and all are divided by 11 with remainder 4.

Substituting $p_1 p_2 \cdots p_k = 11, x = 7$ into (5.3) yields: $77n_1 + 2, 77n_1 + 13, 77n_1 + 24, 77n_1 + 35, 77n_1 + 46, 77n_1 + 57, 77n_1 + 68$. There are seven sets and all are divided by 11 with remainder 2.

In summary, the following are as described in 7.2.1 and 7.2.2.

1) We get $77n_1 + 3, 77n_1 + 10, 77n_1 + 17, 77n_1 + 24, 77n_1 + 31, 77n_1 + 38, 77n_1 + 45, 77n_1 + 52, 77n_1 + 59, 77n_1 + 66, 77n_1 + 73$ for a total of eleven sets and one set each of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ divided by 11, while $77n_1 + 3$ is the smallest and $77n_1 + 73$ is the largest.

2) We get $77n_1 + 2, 77n_1 + 9, 77n_1 + 16, 77n_1 + 23, 77n_1 + 30, 77n_1 + 37, 77n_1 + 44, 77n_1 + 51, 77n_1 + 58, 77n_1 + 65, 77n_1 + 72$ for a total of eleven sets and one set each of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ divided by 11, while $77n_1 + 2$ is the smallest and $77n_1 + 72$ is the largest.

3) We get $77n_1, 77n_1 + 7, 77n_1 + 14, 77n_1 + 21, 77n_1 + 28, 77n_1 + 35, 77n_1 + 42, 77n_1 + 49, 77n_1 + 56, 77n_1 + 63, 77n_1 + 70$ for a total of eleven sets and one set each of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ divided by 11, while $77n_1$ is the smallest and $77n_1 + 70$ is the largest.

4) We get $77n_1 + 5, 77n_1 + 16, 77n_1 + 27, 77n_1 + 38, 77n_1 + 49, 77n_1 + 60, 77n_1 + 71$ seven sets which are all divided by 11 with remainder 5, while $77n_1 + 5$ is the smallest and $77n_1 + 71$ is the largest.

5) We get $77n_1 + 4, 77n_1 + 15, 77n_1 + 26, 77n_1 + 37, 77n_1 + 48, 77n_1 + 59, 77n_1 + 70$ seven sets which are all divided by 11 with remainder 4, while $77n_1 + 4$ is the smallest and $77n_1 + 70$ is the largest.

6) We get $77n_1 + 2, 77n_1 + 13, 77n_1 + 24, 77n_1 + 35, 77n_1 + 46, 77n_1 + 57, 77n_1 + 68$ seven sets which are all divided by 11 with remainder 2, while $77n_1 + 2$ is the smallest and $77n_1 + 68$ is the largest.

From 1), 2), 3), 4), 5), 6), they totally give $11 + 11 + 11 + 7 + 7 + 7 = 54$ subsets. The smallest of these sets is $77n_1$, and the largest set is $77n_1 + 73$.

The above 54 subsets are further classified which is divided by 11 with the remainder $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ as follows.

There are only 3 sets with the remainder 0 divided by 11.

There are only 3 sets with the remainder 1 divided by 11.

There are only 3 + 7 sets with the remainder 2 divided by 11.

There are only 3 sets with the remainder 3 divided by 11.

There are only 3 + 7 sets with the remainder 4 divided by 11.

There are only 3 + 7 sets with the remainder 5 divided by 11.

There are only 3 sets with the remainder 6 divided by 11.

There are only 3 sets with the remainder 7 divided by 11.

There are only 3 sets with the remainder 8 divided by 11.

There are only 3 sets with the remainder 9 divided by 11.

There are only 3 sets with the remainder 10 divided by 11.

Totally there are $3 \times 11 + 3 \times 7 = 54$ sets of $an_1 + b$.

Feature 1: Among the 54 $an_1 + b$ sets obtained above, since there are only 3 sets each with remainder $0, 1, 3, 6, 7, 8, 9, 10$ divided by 11, at most three groups of complete system of residues with remainder $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ divided by 11 could be formed.

Feature 2: The coefficient n_1 of the 54 $an_1 + b$ sets are all 77.

Feature 3: The remainder b of the 54 $an_1 + b$ sets are in the closed interval 0 to 73.

It could be drawn from Proposition 6.1 that in the $\frac{p_1 p_2 \cdots p_k - 1}{2} - 3 \leq b \leq (x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2}$ closed interval, there are only equations (5.1), (5.2), (5.3) corresponding to all $an_1 + b$ subsets of $K \cup L \cup S$ (m, n, k, n_1 all take natural number, $p_1 p_2 \cdots p_k$ may be identical, but none of them is equal to x . $p_1 p_2 \cdots p_k$ all take odd prime number).

1) Substituting $p_1 p_2 \cdots p_k = 7, x = 11$ into $\frac{p_1 p_2 \cdots p_k - 1}{2} - 3 \leq b \leq (x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2}$ yields $0 \leq b \leq 73$.

2) Substituting $p_1 p_2 \cdots p_k = 11, x = 7$ into $\frac{p_1 p_2 \cdots p_k - 1}{2} - 3 \leq b \leq (x - 1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k - 1}{2}$ yields $2 \leq b \leq 71$.

Clearly $0 \leq b \leq 73$ contains $2 \leq b \leq 71$.

Then the remainder b in the set $an_1 + b$ obtained by equations (5.1), (5.2), (5.3) is in the closed interval 0 to 73.

Therefore, there are only 54 sets of $an_1 + b$ in $K \cup L \cup S$ of which all the remainder b are in the closed interval 0 to 73 and the coefficient n_1 is 77. Since among the 54 sets, there are only 3 sets each with remainder $0, 1, 3, 6, 7, 8, 9, 10$ divided by 11, at most three groups of complete system of residues with remainder $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ divided by 11 could be formed.

Also, because the positive integer values of the remainder b in the closed interval 0 to 76 have three more values 74, 75, 76 than the positive integer values of the remainder b in the closed interval 0 to 73.

So the remainder b of $K \cup L \cup S$ is in the closed interval from 0 to 73 and the coefficients n_1 is all 77, plus the sets $77n_1 + 74, 77n_1 + 75, 77n_1 + 76$, it could ensure b is in the closed interval 0 to 76.

Then considering the case of adding three sets $77n_1 + 74, 77n_1 + 75, 77n_1 + 76$.

Since among the 54 sets of $K \cup L \cup S$ above, except for the sets with remainder $2, 4, 5$ divided by 11, there are only 3 sets each with remainder $0, 1, 3, 6, 7, 8, 9, 10$ divided by 11, at most three groups of complete system of residues with remainder $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ divided by 11 could be formed. If we add another group of complete system of residues with

remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11, we must add at least each set with remainder 0, 1, 3, 6, 7, 8, 9, 10 divided by 11, then we must add at least 8 sets. Obviously ,adding the three sets $77n_1 + 74, 77n_1 + 75, 77n_1 + 76$,it still could not reach 8 sets at least.Then it could not add the complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11.

It means that adding $77n_1 + 74, 77n_1 + 75, 77n_1 + 76$ to the above 54 sets of $K \cup L \cup S$ still could forms at most 3 groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11.

So all remainder b in $K \cup L \cup S$ is in the closed interval from 0 to 73 and coefficient n_1 is all 77, which could form at most 3 groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11.

Step 3: Analyze the 7 sets of $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ with $K \cup L \cup S$.

From the result of the first step of 7.2 that, if the 7 sets of $7n, 7n+1, 7n+2, 7n+3, 7n+4, 7n+5, 7n+6$ are all $K \cup L \cup S$ subsets, then there must be 7 groups of complete system of residues in $K \cup L \cup S$ with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11, and these 7 groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11 are the ones with remainder b in $K \cup L \cup S$ is in the closed interval 0 to 76 and the coefficient n_1 is all 77 of $an_1 + b$ subsets .

And, since it could be drawn from the second step of 7.2 that the set of all $an_1 + b$ sets in $K \cup L \cup S$ with remainder b in the closed interval 0 to 76 and coefficient n_1 is all 77, which could form at most three groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by 11.

Therefore, at most 3 of the 7 sets of $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ satisfy the set transitivity by transferring the corresponding subsets to $K \cup L \cup S$.

So, at most 3 of the 7 sets of $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ are subsets of $K \cup L \cup S$.

Then, when $n \geq 0$ takes natural number, there are at least 4 sets out of the 7 sets $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ that are not subsets of $K \cup L \cup S$.

According to Proposition 2.3, the set of positive integer not belonging to $K \cup L \cup S$, must be the set containing triple prime roots.

Then, when $n \geq 0$ takes natural number, at least four of the seven sets $7n, 7n + 1, 7n + 2, 7n + 3, 7n + 4, 7n + 5, 7n + 6$ are sets containing triple prime roots.

So, when $p = 7$, Proposition 4.1 holds.

7.3. $p \geq 11$ takes prime

Step 1: Analyze the results of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ each modulo 7. Let $pn, pn + 1, pn + 2, \dots, pn + p - 1$ modulo 7 respectively, the theory of complete system of residues yields p groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7, i.e.,

$$\begin{aligned} \{pn \mid n \in N^+\} &= \{7pn_1 \mid n_1 \in N^+\} \cup \{7pn_1 + p \mid n_1 \in N^+\} \cup \{7pn_1 + 2p \mid n_1 \in N^+\} \\ &\cup \{7pn_1 + 3p \mid n_1 \in N^+\} \cup \{7pn_1 + 4p \mid n_1 \in N^+\} \\ &\cup \{7pn_1 + 5p \mid n_1 \in N^+\} \cup \{7pn_1 + 6p \mid n_1 \in N^+\} \end{aligned} \quad (38)$$

$$\begin{aligned} \{pn + 1 \mid n \in N^+\} &= \{7pn_1 + 1 \mid n_1 \in N^+\} \cup \{7pn_1 + p + 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 2p + 1 \mid n_1 \in N^+\} \\ &\cup \{7pn_1 + 3p + 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 4p + 1 \mid n_1 \in N^+\} \\ &\cup \{7pn_1 + 5p + 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 6p + 1 \mid n_1 \in N^+\} \end{aligned} \quad (39)$$

$$\begin{aligned} \{pn + 2 \mid n \in N^+\} &= \{7pn_1 + 2 \mid n_1 \in N^+\} \cup \{7pn_1 + p + 2 \mid n_1 \in N^+\} \cup \{7pn_1 + 2p + 2 \mid n_1 \in N^+\} \\ &\cup \{7pn_1 + 3p + 2 \mid n_1 \in N^+\} \cup \{7pn_1 + 4p + 2 \mid n_1 \in N^+\} \\ &\cup \{7pn_1 + 5p + 2 \mid n_1 \in N^+\} \cup \{7pn_1 + 6p + 2 \mid n_1 \in N^+\} \end{aligned} \quad (40)$$

...

$$\begin{aligned} \{pn + p - 1 \mid n \in N^+\} &= \{7pn_1 + p - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 2p - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 3p - 1 \mid n_1 \in N^+\} \\ &\cup \{7pn_1 + 4p - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 5p - 1 \mid n_1 \in N^+\} \\ &\cup \{7pn_1 + 6p - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 7p - 1 \mid n_1 \in N^+\} \end{aligned} \quad (41)$$

pn is divided into each set with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7 of the union of seven subsets of $7pn_1, 7pn_1 + p, 7pn_1 + 2p, 7pn_1 + 3p, 7pn_1 + 4p, 7pn_1 + 5p, 7pn_1 + 6p$.

$pn + 1$ is divided into each set with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7 of the union of seven subsets of $7pn_1 + 1, 7pn_1 + p + 1, 7pn_1 + 2p + 1, 7pn_1 + 3p + 1, 7pn_1 + 4p + 1, 7pn_1 + 5p + 1, 7pn_1 + 6p + 1$.

$pn + 2$ is divided into each set with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7 of the union of seven subsets of $7pn_1 + 2, 7pn_1 + p + 2, 7pn_1 + 2p + 2, 7pn_1 + 3p + 2, 7pn_1 + 4p + 2, 7pn_1 + 5p + 2, 7pn_1 + 6p + 2$.

...

$pn + p - 1$ is divided into each set with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7 of the union of seven subsets of $7pn_1 + p - 1, 7pn_1 + 2p - 1, 7pn_1 + 3p - 1, 7pn_1 + 4p - 1, 7pn_1 + 5p - 1, 7pn_1 + 6p - 1, 7pn_1 + 7p - 1$.

Feature 1: It gives a total of $7p$ sets of $an_1 + b$, coefficient n_1 is all $7p$, and forms p groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7.

There are only p sets with the remainder 0 divided by 7;

There are only p sets with the remainder 1 divided by 7;

There are only p sets with the remainder 2 divided by 7;

There are only p sets with the remainder 3 divided by 7;

There are only p sets with the remainder 4 divided by 7;

There are only p sets with the remainder 5 divided by 7;

There are only p sets with the remainder 6 divided by 7;

Total $7p$ collections.

Feature 2: The coefficient n_1 in $7p an_1 + b$ sets is all $7p$.

Feature 3: The resulting $7p an_1 + b$ set with remainder b has values in the closed interval from 0 to $7p - 1$.

From the above three features and set transitivity, it could be drawn that if the p sets $pn, pn + 1, pn + 2, \dots, pn + p - 1$ are all subsets of $K \cup L \cup S$, then when the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ are each modulo 7, the corresponding p groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7 necessarily in the subsets of $an_1 + b$ of the ones with remainder b in $K \cup L \cup S$ is in the closed interval 0 to 76 and the coefficient n_1 is all $7p - 1$.

So we want to use the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ each modulo 7 to analyze the relationship between the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ and $K \cup L \cup S$ by finding the values of the remainder b in $K \cup L \cup S$ is in the closed interval from 0 to $7p - 1$ and coefficients n_1 is all $7p$ of the $an_1 + b$ sets.

Step 2: Find the remainder b of $K \cup L \cup S$ in the closed interval from 0 to $7p - 1$ and the coefficient n_1 is $7p$ of the $an_1 + b$ set. It is clear from equations (5.1), (5.2), (5.3) that when the coefficient n_1 is $7p$, there are and only two cases, i.e., $p_1 p_2 \dots p_k = 7, x = p$ and $x = 7, p_1 p_2 \dots p_k = p$.

7.3.1 The case of $p_1 p_2 \dots p_k = 7, x = p$

Substituting $p_1 p_2 \dots p_k = 7, x = p$ into equation (5.1) yields: $7pn_1 + 3, 7pn_1 + 10, 7pn_1 + 17, \dots, 7pn_1 + 7(p - 1) + 3$ total of p sets, and all of the sets are divided by 7 with remainder 3.

Substituting $p_1 p_2 \dots p_k = 7, x = p$ into equation (5.2) yields: $7pn_1 + 2, 7pn_1 + 9, 7pn_1 + 16, \dots, 7pn_1 + 7(p - 1) + 2$ total of p sets, and all of the sets are divided by 7 with remainder 2.

Substituting $p_1 p_2 \dots p_k = 7, x = p$ into equation (5.3) yields: $7pn_1, 7pn_1 + 7, 7pn_1 + 14, \dots, 7pn_1 + 7(p - 1)$ total of p sets, and all of the sets are divided by 7 with remainder 0.

A total of $3p$ sets obtained.

7.3.2 The case of $p_1 p_2 \dots p_k = p, x = 7$

Substituting $p_1 p_2 \dots p_k = p, x = 7$ into equation (5.1) yields $7pn_1 + \frac{p-1}{2}, 7pn_1 + p + \frac{p-1}{2}, 7pn_1 + 2p + \frac{p-1}{2}, 7pn_1 + 3p + \frac{p-1}{2}, 7pn_1 + 4p + \frac{p-1}{2}, 7pn_1 + 5p + \frac{p-1}{2}, 7pn_1 + 6p + \frac{p-1}{2}$, which are total 7 sets.

For $\{7pn_1 + \frac{p-1}{2} \mid n_1 \in N^+\} \cup \{7pn_1 + p + \frac{p-1}{2} \mid \in N^+\} \cup \{7pn_1 + 2p + \frac{p-1}{2} \mid \in N^+\} \cup \{7pn_1 + 3p + \frac{p-1}{2} \mid \in N^+\} \cup \{7pn_1 + 4p + \frac{p-1}{2} \mid \in N^+\} \cup \{7pn_1 + 5p + \frac{p-1}{2} \mid \in N^+\} \cup \{7pn_1 + 6p + \frac{p-1}{2} \mid \in N^+\} = \{pn + \frac{p-1}{2} \mid n \in N^+\}$, i.e., $7pn_1 + \frac{p-1}{2}, 7pn_1 + p + \frac{p-1}{2}, 7pn_1 + 2p + \frac{p-1}{2}, 7pn_1 + 3p + \frac{p-1}{2}, 7pn_1 + 4p + \frac{p-1}{2}, 7pn_1 + 5p + \frac{p-1}{2}, 7pn_1 + 6p + \frac{p-1}{2}$. A decomposed group of complete system of residues each with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7, based on $pn + \frac{p-1}{2}$ modulo 7.

Then in $7pn_1 + \frac{p-1}{2}, 7pn_1 + p + \frac{p-1}{2}, 7pn_1 + 2p + \frac{p-1}{2}, 7pn_1 + 3p + \frac{p-1}{2}, 7pn_1 + 4p + \frac{p-1}{2}, 7pn_1 + 5p + \frac{p-1}{2}, 7pn_1 + 6p + \frac{p-1}{2}$: with remainder 0 divided by 7 occupies a set, with remainder 1 divided by 7 occupies a set, with remainder 2 divided by 7 occupies a set, with remainder 3 divided by 7 occupies a set; with remainder 4 divided by 7 occupies a set, with remainder 5 divided by 7 occupies a set, with remainder 6 divided by 7 occupies a set, for a total of 7 sets.

Substituting $p_1 p_2 \cdots p_k = p$, $x = 7$ into equation (5.2) yields $7pn_1 + \frac{p-1}{2} - 1, 7pn_1 + p + \frac{p-1}{2} - 1, 7pn_1 + 2p + \frac{p-1}{2} + 1, 7pn_1 + 3p + \frac{p-1}{2} + 1, 7pn_1 + 4p + \frac{p-1}{2} + 1, 7pn_1 + 5p + \frac{p-1}{2} + 1, 7pn_1 + 6p + \frac{p-1}{2} - 1$ for a total of 7 sets.

For $\{7pn_1 + \frac{p-1}{2} - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + p + \frac{p-1}{2} - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 2p + \frac{p-1}{2} - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 3p + \frac{p-1}{2} - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 4p + \frac{p-1}{2} - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 5p + \frac{p-1}{2} - 1 \mid n_1 \in N^+\} \cup \{7pn_1 + 6p + \frac{p-1}{2} - 1 \mid n_1 \in N^+\} = \{pn + \frac{p-1}{2} - 1 \mid n \in N^+\}$, i.e., $7pn_1 + \frac{p-1}{2} - 1, 7pn_1 + p + \frac{p-1}{2} - 1, 7pn_1 + 2p + \frac{p-1}{2} + 1, 7pn_1 + 3p + \frac{p-1}{2} + 1, 7pn_1 + 4p + \frac{p-1}{2} + 1, 7pn_1 + 5p + \frac{p-1}{2} + 1, 7pn_1 + 6p + \frac{p-1}{2} - 1$ is $pn + \frac{p-1}{2} - 1$. A decomposed group of complete system of residues each with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7, based on $pn + \frac{p-1}{2} - 1$ modulo 7.

Then in $7pn_1 + \frac{p-1}{2} - 1, 7pn_1 + p + \frac{p-1}{2} - 1, 7pn_1 + 2p + \frac{p-1}{2} + 1, 7pn_1 + 3p + \frac{p-1}{2} - 1, 7pn_1 + 4p + \frac{p-1}{2} - 1, 7pn_1 + 5p + \frac{p-1}{2} - 1, 7pn_1 + 6p + \frac{p-1}{2} - 1$: with remainder 0 divided by 7 occupies a set, with remainder 1 divided by 7 occupies a set, with remainder 2 divided by 7 occupies a set, with remainder 3 divided by 7 occupies a set; with remainder 4 divided by 7 occupies a set, with remainder 5 divided by 7 occupies a set, with remainder 6 divided by 7 occupies a set, for a total of 7 sets.

Substituting $p_1 p_2 \cdots p_k = p$, $x = 7$ into equation (5.3) yields $7pn_1 + \frac{p-1}{2} - 3, 7pn_1 + p + \frac{p-1}{2} - 3, 7pn_1 + 2p + \frac{p-1}{2} - 3, 7pn_1 + 3p + \frac{p-1}{2} - 3, 7pn_1 + 4p + \frac{p-1}{2} - 3, 7pn_1 + 5p + \frac{p-1}{2} - 3, 7pn_1 + 6p + \frac{p-1}{2} - 3$ for a total of 7 sets.

For $\{7pn_1 + \frac{p-1}{2} - 3 \mid n_1 \in N^+\} \cup \{7pn_1 + p + \frac{p-1}{2} - 3 \mid n_1 \in N^+\} \cup \{7pn_1 + 2p + \frac{p-1}{2} - 3 \mid n_1 \in N^+\} \cup \{7pn_1 + 3p + \frac{p-1}{2} - 3 \mid n_1 \in N^+\} \cup \{7pn_1 + 4p + \frac{p-1}{2} - 3 \mid n_1 \in N^+\} \cup \{7pn_1 + 5p + \frac{p-1}{2} - 3 \mid n_1 \in N^+\} \cup \{7pn_1 + 6p + \frac{p-1}{2} - 3 \mid n_1 \in N^+\} = \{pn + \frac{p-1}{2} - 3 \mid n \in N^+\}$, i.e., $7pn_1 + \frac{p-1}{2} - 3, 7pn_1 + p + \frac{p-1}{2} - 3, 7pn_1 + 2p + \frac{p-1}{2} - 3, 7pn_1 + 3p + \frac{p-1}{2} - 3, 7pn_1 + 4p + \frac{p-1}{2} - 3, 7pn_1 + 5p + \frac{p-1}{2} - 3, 7pn_1 + 6p + \frac{p-1}{2} - 3$. A decomposed group of complete system of residues each with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7, based on $pn + \frac{p-1}{2} - 3$ modulo 7.

Then in $7pn_1 + \frac{p-1}{2} - 3, 7pn_1 + p + \frac{p-1}{2} - 3, 7pn_1 + 2p + \frac{p-1}{2} - 3, 7pn_1 + 3p + \frac{p-1}{2} - 3, 7pn_1 + 4p + \frac{p-1}{2} - 3, 7pn_1 + 5p + \frac{p-1}{2} - 3, 7pn_1 + 6p + \frac{p-1}{2} - 3$: with remainder 0 divided by 7 occupies a set, with remainder 1 divided by 7 occupies a set, with remainder 2 divided by 7 occupies a set, with remainder 3 divided by 7 occupies a set; with remainder 4 divided by 7 occupies a set, with remainder 5 divided by 7 occupies a set, with remainder 6 divided by 7 occupies a set, for a total of 7 sets.

A total of $p + p + p + 7 + 7 + 7 + 7 = 3p + 21$ sets of $an_1 + b$ is obtained as described in 7.3.1 and 7.3.2.

The above set of $3p + 21$ is then classified which is divided by 7 with the remainder 0, 1, 2, 3, 4, 5, 6 as follows.

There are only $3 + p$ sets divided by 7 with the remainder 0.

There are only 3 sets divided by 7 with the remainder 1.

There are only $3 + p$ sets divided by 7 with the remainder 2.

There are only $3 + p$ sets divided by 7 with the remainder 3.

There are only 3 sets divided by 7 with the remainder 4.

There are only 3 sets divided by 7 with the remainder 5.

There are only 3 sets divided by 7 with the remainder 6.

Features: A total of $3p + 21$ sets of $an_1 + b$ is obtained, the most important of which are only 3 sets each divided by 7 with remainder 1, 4, 5, 6, and $3 + p$ sets each divided by 7 with remainder 0, 2, 3, then three groups of complete system of residues divided by 7 with remainder 0, 1, 2, 3, 4, 5, 6 could be formed.

It could be drawn from Proposition 6.1 that in $\frac{p_1 p_2 \cdots p_k}{2} - 3 \leq b \leq (x-1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k}{2}$ closed interval, there are only equations (5.1), (5.2), (5.3) corresponding to all $an_1 + b$ subsets of $K \cup L \cup S$ (m, n, k, n_1 all take natural number, $p_1 p_2 \cdots p_k$ may be identical, but none of them is equal to x . $p_1 p_2 \cdots p_k$ all take odd prime number).

1) Substituting $p_1 p_2 \cdots p_k = 7$, $x = p$ into $\frac{p_1 p_2 \cdots p_k}{2} - 3 \leq b \leq (x-1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k}{2}$ yields: $0 \leq b \leq 7p - 4$

2) Substituting $p_1 p_2 \cdots p_k = p$, $x = 7$ into $\frac{p_1 p_2 \cdots p_k}{2} - 3 \leq b \leq (x-1)p_1 p_2 \cdots p_k + \frac{p_1 p_2 \cdots p_k}{2}$ yields: $-3 \leq b \leq 6p + \frac{p-1}{2}$ and because when $p \geq 11$ takes prime, $0 < \frac{p-1}{2} - 3, 7p - 4 > 6p + \frac{p-1}{2}$

Then when $p \geq 11$ takes prime, $0 \leq b \leq 7p - 4$ contains $\frac{p-1}{2} - 3 \leq b \leq 6p + \frac{p-1}{2}$.

That is, the remainder b in the set $an_1 + b$ from equations (5.1), (5.2), (5.3) is all in the closed interval from 0 to $7p - 4$.

So when $p_1 p_2 \cdots p_k = 7$, $x = p$ and $p_1 p_2 \cdots p_k = p$, $x = 7$, $p \geq 11$ takes prime, the remainder b of $K \cup L \cup S$ is in the closed interval from 0 to $7p - 4$ and the coefficient n_1 is $7p$ of the $an_1 + b$ set, which has only $3p + 21$.

Clearly the remainder b in the closed interval from 0 to $7p - 1$ has three more values $7p - 3, 7p - 2, 7p - 1$ than the remainder b in the closed interval from 0 to $7p - 4$.

So, the remainder b of $K \cup L \cup S$ is in the closed interval from 0 to $7p - 4$ and the coefficients n_1 is $7p$ of $an_1 + b$ set plus $7pn_1 + 7p - 3, 7pn_1 + 7p - 2, 7pn_1 + 7p - 1$ three sets, then it could ensure b is in the closed interval from 0 to $7p - 1$.

Then considering the case of adding three sets $7pn_1 + 7p - 3, 7pn_1 + 7p - 2, 7pn_1 + 7p - 1$ to $K \cup L \cup S$ beyond the $3p + 21$ sets.

Since there are only 3 sets in $3p + 21$ sets of $K \cup L \cup S$ with remainder 1, 4, 5, 6, at most 3 sets of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7 could be formed. If one adds another group of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7, we must add at least each set with remainder 1, 4, 5, 6 divided by 7, i.e., at least 4 more sets must be added. Obviously, adding $7pn_1 + 7p - 3, 7pn_1 + 7p - 2, 7pn_1 + 7p - 1$ three sets, it still could not reach 4 sets at least. Then it could not add the complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7. That is,

It means that adding $7pn_1 + 7p - 3, 7pn_1 + 7p - 2, 7pn_1 + 7p - 1$ to the above $3p + 21$ sets of $K \cup L \cup S$ still could forms at most 3 groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7.

So, all remainder b in $K \cup L \cup S$ is in the closed interval from 0 to $7p - 1$ and the coefficient n_1 is $7p$ of $an_1 + b$ set, which could form at most 3 groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7.

Step 3: Analyze the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ with $K \cup L \cup S$. From the result of the first step of 7.3 that, if the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ are all $K \cup L \cup S$ subsets, then there must be p groups of complete residue in $K \cup L \cup S$ with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7, and the p groups of complete residue in $K \cup L \cup S$ with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7 are the ones with remainder b in $K \cup L \cup S$ is in the closed interval 0 to $7p - 1$ and the coefficients n_1 is all $7p$ of $an_1 + b$ subsets. And, since it could be drawn from the second step of 7.3 that the set of all $an_1 + b$ sets in $K \cup L \cup S$ with remainder b in the closed interval 0 to $7p - 1$ and coefficient n_1 is $7p$, which could form at most three groups of complete system of residues with remainder 0, 1, 2, 3, 4, 5, 6 divided by 7.

So, at most 3 of the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ satisfy the set transitivity by transferring the corresponding subsets to $K \cup L \cup S$.

So, at most 3 of the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ are subsets of $K \cup L \cup S$.

Then, when $n \geq 0$ takes natural number, at least $p - 3$ out of the p sets of $pn, pn + 1, pn + 2, \dots, pn + p - 1$ are not subsets of $K \cup L \cup S$.

According to Proposition 2.3, the set of positive integer not belonging to $K \cup L \cup S$ must be the set containing the triple prime roots.

Then, when $n \geq 0$ takes natural number, at least $p - 3$ of the p -sets in $pn, pn + 1, pn + 2, \dots, pn + p - 1$ are sets containing the triple prime roots.

Then Proposition 4.1 holds when $p \geq 11$ takes prime.

Since Proposition 4.1 holds when $p = 5$ and $p = 7$ has been proved in 7.1 and 7.2.

Then Proposition 4.1 holds when $p \geq 5$ takes prime.

Since Proposition 4.1 was proved above to be an equivalent proposition of the infinity of triple prime, then the infinity of triple prime holds.

Clearly there is a group of twin prime numbers in every group of triple prime numbers, then there are also infinite many twin prime numbers, i.e., both the triple and twin prime numbers are infinite.

Also, since the triple prime conjecture is the infinity of triple prime and the twin prime conjecture is the infinity of twin prime, then the triple prime conjecture and the twin prime conjecture hold.

Certificate completed.

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