Hydrodynamical limit of a homogeneous electron gas

Matthew Stephenson*
Stanford University

I. INTRODUCTION

IR limit vs. phenomenological hydrodynamics
interactions generated by non-linear coordinate transformation

II. NON-LINEAR COORDINATE TRANSFORMATIONS AND INTERACTIONS

Normal modes are the non-interacting degrees of freedom of a harmonic system. However, the simple physical picture in terms of normal modes changes drastically when we consider a description of the dynamics of a non-linear combination of coordinates. For instance, the Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2, \tag{1}$$

of a harmonic oscillator assumes the form

$$L = \frac{m}{4Q}\dot{Q}^2 - m\omega^2 Q, \tag{2}$$

in terms of the non-linear coordinate $Q = q^2/2$. The role of the position-dependent effective mass, $m(Q) = m/2Q$, is to speed up the oscillations and prevent the new coordinate from passing through the point $Q = 0$, when the system starts the some initial value $Q_{in} > 0$.

It is instructive to consider a system with several harmonic degrees of freedom, x_n, with $n = 1, \dots, N$, governed by a quadratic action $S_0[x]$. The effective dynamics of a collective variable $y = F(x_1, \dots, x_N)$, can be defined by the action $S[y, x] = S_0[x] + S_i[y, x]$, where the part of the action S_i is a Lagrange multiplier term

$$S_i[y, x] = -K \int dt \left\{ y(t) - F\left[x_1(t), \dots, x_N(t)\right]\right\}^2, \tag{3}$$

containing an arbitrary constant K. The effective action, $S_{eff}[y] = S\left[y, x[y]\right]$, is obtained from $S[y, x]$ by eliminating the original set of variables $\{x_1, \dots, x_N\}$, by their equations of motion for an arbitrary, fixed trajectory $y(t)$. This algorithm produces a quadratic, harmonic effective action for y only if the function $F[x_1, \dots, x_N]$ is linear in x.

III. CTP

$$\begin{aligned}
\hat{D} &= \begin{pmatrix} D_n + iD_i & D_f - iD_i \\ D_f + iD_i & D_n - iD_i \end{pmatrix}, \\
\hat{D}^{-1} &= \hat{\sigma} \begin{pmatrix} \Delta_n + i\Delta_i & \Delta_f - i\Delta_i \\ \Delta_f + i\Delta_i & \Delta_n - i\Delta_i \end{pmatrix} \hat{\sigma}
\end{aligned} \tag{4}$$

$D_{\substack{r\\a}} = D_n \pm D_f$, $D^{-1}_{\substack{r\\a}} = \Delta_n \pm \Delta_f$

---

*Electronic address: `matthewjstephenson@icloud.com`

IV. CURRENT DYNAMICS IN A HOMOGENEOUS ELECTRON GAS

We shall work below with electrons at finite density and vanishing temperature. The Green's functions are generated by the functional

$$e^{\frac{i}{\hbar}W[\hat{a},\hat{j}]} = \int D[\hat{\psi}]D[\hat{\bar{\psi}}]D[\hat{A}]e^{\frac{i}{\hbar}\hat{\bar{\psi}}[\hat{F}^{-1}+\not{\hat{a}}-\frac{e}{c}\hat{\sigma}\not{\hat{A}}]\hat{\psi}+\frac{i}{2\hbar}\hat{A}\hat{D}_0^{-1}\hat{A}+\frac{i}{\hbar}\hat{j}\hat{A}} \tag{5}$$

where the scalar products denotes space-time integrations,

$$fg = \frac{1}{c}\int d^4x f(x)g(x) = c\int \frac{d^4q}{(2\pi)^4}f(-q)g(q) \tag{6}$$

with

$$f(q) = \frac{1}{c}\int d^4x e^{ixq}f(x), \tag{7}$$

fields with hat stand for the CTP doublets, eg.

$$\hat{\psi} = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \qquad \hat{A} = \begin{pmatrix}A^+\\ A^-\end{pmatrix}, \tag{8}$$

and

$$\hat{\sigma} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{9}$$

The free photon and electron propagators are given by

$$\hat{D}_0^{\mu\nu}(p) = -\hat{D}_+(p;0)T^{\mu\nu}, \tag{10}$$

with

$$T^{\mu\nu} = g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}, \tag{11}$$

and

$$\hat{F}(p) = (\not{p}+m)\left[\hat{D}_-(p;m) + 2\pi i\delta(p^2-m^2)n(p)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right], \tag{12}$$

where

$$\hat{D}_\pm(p;m^2) = \begin{pmatrix}\frac{1}{p^2-m^2+i\epsilon} & \pm 2\pi i\delta(p^2-m^2)\Theta(-p^0)\\ \pm 2\pi i\delta(p^2-m^2)\Theta(p^0) & -\frac{1}{p^2-m^2-i\epsilon}\end{pmatrix}. \tag{13}$$

The occupation number density,

$$n(p) = \frac{\Theta(p^0)}{e^{\beta(\epsilon_{\boldsymbol{p}}-\mu)}+1} + \frac{\Theta(-p^0)}{e^{\beta(\epsilon_{\boldsymbol{p}}+\mu)}+1} \tag{14}$$

is given in terms of the single particle energy $\epsilon_{\boldsymbol{p}} = c\sqrt{m^2c^2+\hbar^2\boldsymbol{p}^2}$ and the chemical potential μ. The removal of the UV divergences is achieved as in the usual, single time axes formalism, [15], and the counterterms will be suppressed in what follows.

The generator functional $W[\hat{a},\hat{j}]$ is used to construct the effective action by functional Legendre transformation,

$$W[\hat{a},\hat{j}] = \Gamma[\hat{J},\hat{A}] + \hat{a}\hat{J} + \hat{j}\hat{A}, \tag{15}$$

where the new variables

$$\hat{J} = \frac{\delta W[\hat{a},\hat{j}]}{\delta\hat{a}}, \quad \hat{A} = \frac{\delta W[\hat{a},\hat{j}]}{\delta\hat{j}} \tag{16}$$

are the expectation values, $J^\mu(x) = \mathrm{Tr}[\rho\bar\psi(x)\gamma^\mu\psi(x)] = (c\rho, \boldsymbol{j})$, $A_\mu(x) = \mathrm{Tr}[\rho A_\mu(x)]$ for $\hat a = \hat j = 0$. The Legendre transformation of a real, convex function can be defined either geometrically or algebraically. We follow the latter route here and use eqs. (16)-(16) to define the effective action, a complex functional in an algebraic manner. The inverse transformation is based on the variable

$$\frac{\delta\Gamma[\hat J, \hat A]}{\delta \hat J} = -\hat a, \quad \frac{\delta\Gamma[\hat J, \hat A]}{\delta \hat A} = -\hat j, \tag{17}$$

therefore the variational equation of the effective action is satisfied by the expectation values, obtained for vanishing external source. We are interested in the effective current dynamics and the corresponding effective action, $\Gamma[\hat J]$ will be obtained by eliminating $\hat A$ from $\Gamma[\hat J, \hat A]$ by the help of the second equation in (17). The effective action is real in the physical case, $j_+ = -j_-$, $a_+ = -a_-$, therefore it is sufficient to retain the real part, $\Re\Gamma$, in the equation of motion.

The quadratic part of the generator functional W and the effective action Γ will be calculated by expanding in $\hbar$ and retaining the first two orders. We start with W by integrating out the electron field,

$$e^{\frac{i}{\hbar}W[\hat a, \hat j]} = \int D[\hat A] e^{\mathrm{Tr}[\hat F^{-1} + \not{\hat a} - \frac{e}{c}\hat\sigma\not{\hat A}] + \frac{i}{2\hbar}\hat A\hat D_0^{-1}\hat A + \frac{i}{\hbar}\hat j\hat A}. \tag{18}$$

We keep the $\mathcal{O}\left(\hbar^0\hat a^2\right)$ part of the exponent,

$$e^{\frac{i}{\hbar}W[\hat a, \hat j]} = \int D[\hat A] e^{-\frac{i}{2}\hat a\hat G\hat a + \frac{i}{\hbar}\hat k\hat A + \frac{i}{2\hbar}\hat A\hat D^{-1}\hat A}, \tag{19}$$

where

$$\hat D^{-1} = \hat D_0^{-1} - \frac{e^2}{c^2}\hat\sigma\hat G\sigma, \tag{20}$$

and the source

$$\hat k = \hat j + \frac{e}{c}\hat\sigma\hat G\hat a, \tag{21}$$

are given in terms of the one-loop current-current Green function

$$G_{(\sigma_1\mu_1)(\sigma_2\mu_2)}(x_1, x_2) = -i\hbar\,\mathrm{tr}[\hat F_{\sigma_1\sigma_2}(x_1, x_2)\gamma_{\mu_2}\hat F_{\sigma_2\sigma_1}(x_2, x_1)\gamma_{\mu_1}]. \tag{22}$$

The presence of a neutralizing, homogeneous background charge was assumed in Eq. (21). The Gaussian integral, (19) yields

$$W[\hat a, \hat j] = -\frac{\hbar}{2}\hat a\hat G\hat a - \frac{e}{c}\hat j\hat D_0\sigma\hat G\hat a - \frac{1}{2}\hat j\left(\hat D_0 + \frac{e^2}{c^2}\hbar\hat D_0\hat\sigma\hat G\hat\sigma\hat D_0\right)\hat j \tag{23}$$

in the desired order.

Though the orders of the expansion of $W[\hat a, \hat j]$ in $\hbar$ and in the number of loops correspond to each other in the usual manner, this is not the case anymore when the effective action is considered. The reason is that the variables of the effective action have different orders in $\hbar$, $\hat J = \mathcal{O}(\hbar)$ and $\hat A = \mathcal{O}\left(\hbar^0\right)$, as opposed to the variables of W, $\hat a = \mathcal{O}\left(\hbar^0\right)$, $\hat j = \mathcal{O}\left(\hbar^0\right)$. The Legendre transformation (15) gives

$$\Gamma[\hat J, \hat A] = \frac{1}{2\hbar}\hat J\hat G^{-1}\hat J + \frac{1}{2}\hat A\hat D_0^{-1}\hat A - e\hat A\hat\sigma\hat J \tag{24}$$

for the $\mathcal{O}(\hbar)$ effective action. After the effective action has been derived in the desired accuracy we set $\hbar = 1$ for the rest of this work.

The Maxwell equation,

$$\hat A = e\hat D_0\sigma\hat J \tag{25}$$

can be used to eliminate the photon field and arrive at the effective action,

$$\Gamma[\hat J] = \frac{1}{2}\hat J\left(\hat G_0^{-1} - \frac{e^2}{c^2}\hat\sigma\hat D_0\hat\sigma\right)\hat J. \tag{26}$$

As any bosonic two-point function, $\hat G$, too, has the CTP block structure of Eq. (4), allowing us to define the retarded and advanced parts,

$$\left(\hat G^{-1} - \frac{e^2}{c^2}\hat\sigma\hat D_0\hat\sigma\right)_{\substack{r\\a}} = (\hat G_{\substack{r\\a}})^{-1} - \frac{e^2}{c^2}\hat D_{0\substack{r\\a}}. \tag{27}$$

V. EQUATIONS OF MOTION

It is advantegous to use the parametrization $a^{\pm} = \bar{a}/2 \pm a$ for the source, where a stands for a physical source and $\bar{a}$ is a book-keeping device. The effective action for the current is then defined by

$$W[\bar{a}, a] = \Gamma[J, J_d] + \bar{a}J + aJ_d, \tag{28}$$

where

$$J = \frac{\delta W[\bar{a}, a]}{\delta \bar{a}}, \quad J_d = \frac{\delta W[\bar{a}, a]}{\delta a}. \tag{29}$$

When the book-keeping variable is set to zero, $\bar{a} = 0$ then

$$J = \frac{\delta W[\hat{a}]}{\delta a^+}_{|a^+ = -a_- = a} = \frac{\delta W[\hat{a}]}{\delta a^-}_{|a^+ = -a_- = a} \tag{30}$$

becomes the expectation value of the current in the presence of a physical external source a and the auxiliary field J_d is vanishing.

The effective action, defined by Eq.s (28)-(29),

$$\Gamma[\hat{J}] = -\frac{i}{2}J_d G^{r-1} G^{a-1} J_d + \frac{1}{2}J_d G^{r-1} J + \frac{1}{2}J G^{a-1} J_d, \tag{31}$$

yielding the equation of motion,

$$a = -G^{r-1} J, \tag{32}$$

for the true expectation value, obtained for $\bar{a} = 0$ when $J_d = 0$.

Appendix A: Current two-point function at finite density in the non-relativistic limit

The result of the calculation of the current-current Green function, given by Eq. (22),

$$G^{\mu\nu}_{\sigma\tau}(q) = -i \int \frac{d^4 p}{(2\pi)^4} \mathrm{tr}[\gamma^\mu F_{\sigma\tau}(q+p)\gamma^\nu F_{\tau\sigma}(p)] \tag{A1}$$

at finite density and vanishing temperature in the non-relativistic limit, $c \to \infty$, is briefly summarized in this Appendix.

1. Lorentz structure

The two-point function is symmetric, $G_{(\sigma\mu)(\sigma'\nu)}(p) = G_{(\sigma'\nu)(\sigma\mu)}(-p)$ and transverse, $p^\mu G_{(\sigma\mu)(\sigma'\nu)}(p) - 0$. Furthermore it is covariant and depends on two four vectors, p^μ and β^μ, defined as $\beta^\mu = (1, \mathbf{0})$ in the inertial frame where the electron gas is at rest which give two independent kinematical, scalar combinations, $\boldsymbol{q}^2 = -[q - u(uq)]^2$ and $\xi = uq/|\boldsymbol{q}| = \omega/c|\boldsymbol{q}|$ where the notation $q^\mu = (\omega/c, \boldsymbol{q})$ is used. Such a tensor can be parametrized by two Lorentz scalars as

$$G^{\mu\nu} = G_\ell P^{\mu\nu}_\ell + D_t P^{\mu\nu}_t \tag{A2}$$

where P_t and P_ℓ are projectors onto the three dimensional transverse and longintudinal subspaces,

$$\begin{aligned} P^{\mu\nu}_t &= -\begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{T} \end{pmatrix}, \\ P^{\mu\nu}_\ell &= \frac{1}{1-\xi^2}\begin{pmatrix} 1 & \boldsymbol{n}\xi \\ \boldsymbol{n}\xi & \xi^2 \boldsymbol{L} \end{pmatrix}, \end{aligned} \tag{A3}$$

respectively with $\boldsymbol{n} = \boldsymbol{k}/|\boldsymbol{k}|$, $\boldsymbol{L} = \boldsymbol{n} \otimes \boldsymbol{n}$ and $\boldsymbol{T} = 1\!\!1 - \boldsymbol{L}$. The inverse, defined by $G\mu\ _\rho G^{-1\rho\nu} = T^{\mu\nu}$ is given by

$$G^{-1} = \frac{1}{G_\ell}P_\ell + \frac{1}{G_t}P_t. \tag{A4}$$

2. Vacuum contribution

The two-point function is the sum of a vacuum and a finite density contributions and we have

$$\hat{G} = \hat{G}_{vac} + \hat{G}_{gas}, \tag{A5}$$

and both the vacuum and the finite density contributions are of the form (4). The vacuum contributions, $\hat{G}^{\mu\nu}_{vac} = \hat{G}_{vac}T^{\mu\nu}$ are easy to find. The CTP diagonal block, $G^{++}_{\ell\ vac} = G^{++}_{t\ vac} = G^{++}_{vac}$, is the standard one,

$$\begin{aligned} G^{++}_{vac}(q) &= \frac{1}{3\pi}q^2\left\{\frac{1}{3} + 2\left(1+\frac{2m^2c^2}{q^2}\right)\left[\sqrt{\frac{4m^2c^2}{q^2}-1}\ \mathrm{arccot}\sqrt{\frac{4m^2c^2}{q^2}-1}-1\right]\right\} \\ &= \frac{q^2}{15\pi}\left[\frac{q^2}{m^2c^2} + \mathcal{O}\left(\left(\frac{q^2}{m^2c^2}\right)^2\right)\right]. \end{aligned} \tag{A6}$$

The CTP non-diagonal block, calculated by means of the free propagator (12) reads as $G^{+-}_{\ell\ vac} = G^{+-}_{t\ vac} = G^{+-}_{vac}$, with

$$G^{+-}(q) = \frac{i}{3}\int\frac{d^4p}{(2\pi)^4}2\pi\delta((p+q)^2-m^2)\Theta(-p^0-q^0)2\pi\delta(p^2-m^2)\Theta(p^0)\mathrm{tr}N(p+q,q), \tag{A7}$$

where the trace is for the Lorentz indices of the trace formule,

$$\begin{aligned} N^{\mu\nu}(p,q) &= \mathrm{tr}\gamma^\mu[(p_\alpha\gamma^\alpha+mc)\gamma^\nu(q_\beta\gamma^\beta+mc)] \\ &= 4(m^2c^2-pq)g^{\mu\nu}+4p^\mu q^\nu+4p^\nu q^\mu. \end{aligned} \tag{A8}$$

Simple steps lead to

$$G^{+-}_{vac} = \frac{i(c^2m^2+\frac{q^2}{2})}{3\pi c|\boldsymbol{q}|}\Theta(-q^0-m)\int_0^{\sqrt{q^{02}-m^2c^2}}\frac{dpp}{\omega_p}\Theta(2p|\boldsymbol{q}|-|q^2+2\omega_p q^0|), \tag{A9}$$

an expression to be neglected in the non-relativistic limit.

3. Fermi-sphere contribution

To find $\hat{G}_\ell$ and $\hat{G}_t$ for the electron gas we have need $\hat{T} = \hat{G}^{00}_{gas}$ and the spatial trace, $\hat{S} = \hat{G}^{jj}_{gas}$. For this end we use the trace factors

$$\begin{aligned} t &= N^{00}(p+q,p)_{|p^2=m^2c^2} = t' - 2(q^2+2pq), \\ s &= N^{jj}(p+q,p)_{|p^2=m^2c^2} = s' + 2(q^2+2pq), \end{aligned} \tag{A10}$$

where $t' = 8(p^{02}+p^0q^0)+2q^2$ and $s' = 8(p^0q^0+\boldsymbol{p}^2)-2q^2$ in the loop integrals

$$\begin{aligned} \binom{T}{S}_{++}(q) &= i\int_p\binom{t}{s}\Big[2\pi\delta((q+p)^2-m^2c^2)n_{q+p}2\pi\delta(p^2-m^2c^2)n_p \\ &\quad -i\frac{2\pi\delta(p^2-m^2c^2)n_p}{(p+q)^2-m^2c^2+i\epsilon} - i\frac{2\pi\delta((p+q)^2-m^2c^2)n_{p+q}}{p^2-m^2c^2+i\epsilon}\Big], \\ \binom{T}{S}_{+-}(q) &= -i\int_p\binom{t'}{s'}2\pi\delta((q+p)^2-m^2c^2)2\pi\delta(p^2-m^2c^2)[\Theta(-p^0-q^0)n_p+\Theta(p^0)n_{p+q}-n_{q+p}n_p]. \end{aligned} \tag{A11}$$

It is advantegous to introduce at this point the integrals

$$\begin{aligned} I^{(1)}[q;f] &= \int\frac{d^4p}{(2\pi)^4}f(p,q)2\pi\delta(p^2-m^2c^2), \\ I^{(2)}[q;f] &= \int\frac{d^4p}{(2\pi)^4}f(p,q)2\pi\delta(p^2-m^2c^2)2\pi\delta(q^2+2pq), \end{aligned} \tag{A12}$$

and write

$$
\begin{aligned}
\binom{T}{S}_{++}(q) &= I^{(1)}\left[q;\frac{\binom{t}{s}n_p}{q^2+2pq+i\epsilon}\right]+\frac{i}{2}I^{(2)}[q;\binom{t'}{s'}n_pn_{q+p}]+(q\to -q),\\
\binom{T}{S}_{+-}(q) &= -iI^{(2)}[q;\binom{t'}{s'}(\Theta(-p^0-q^0)n_p+\Theta(p^0)n_{p+q}-n_{q+p}n_p)]. \qquad (A13)
\end{aligned}
$$

The integrals $I^{(1)}$ and $I^{(2)}$ in these expressions are evaluated by assuming that the integrands are spherically symmetric and nonvanishing for $p^0>0$ and $|\boldsymbol{p}|\ll mc$. One finds

$$
\begin{aligned}
I^{(1)}[q;f(p^0,\boldsymbol{p})] &= \frac{1}{4\pi^2mc}\int dpp^2f(mc^2,\boldsymbol{p}),\\
I^{(1)}\left[q;\frac{g(p^0,\boldsymbol{p})}{q^2+2pq+i\epsilon}\right] &= \frac{1}{16\pi^2|\boldsymbol{q}|mc}\int_0^\infty dppg(mc^2,\boldsymbol{p})\ln\frac{k+p+i\epsilon}{k-p+i\epsilon},\\
I^{(2)}\left[q;h\left((p^0,\boldsymbol{p}),(\frac{\omega}{c},\boldsymbol{q})\right)\right] &= \frac{1}{16\pi^2|\boldsymbol{q}|mc}\int d^3ph\left((mc^2,\boldsymbol{p}),(\frac{\omega}{c},\boldsymbol{q})\right)\delta(k-p_z), \qquad (A14)
\end{aligned}
$$

where $k=\frac{q^2+2m\omega}{2|\boldsymbol{q}|}$.

We need the Fourier transform of the real part of the CTP diagonal block in the leading order in $1/c$,

$$
\begin{aligned}
\Re\binom{T}{S}_{++}(q) &= \frac{1}{2}[\binom{T}{S}_{++}(q)+\binom{T}{S}^*_{++}(-q)]\\
&= \Re I^{(1)}\left[\frac{\binom{t}{s}n_p}{q^2+2pq+i\epsilon}\right]+(\omega\to-\omega)\\
&= \frac{1}{16\pi^2|\boldsymbol{q}|mc}\int_0^{k_F}dppn_p\left[\binom{8m^2c^2}{8(m\omega+p^2)+2\boldsymbol{q}^2}\ln\left|\frac{k+p}{k-p}\right|+8|\boldsymbol{q}|p\binom{-1}{1}\right]+(\omega\to-\omega) \qquad (A15)
\end{aligned}
$$

The momentum integral can easily be carried out at vanishing temperature, $n_p=\Theta(p^0)\Theta(k_F-|\boldsymbol{p}|)$ with the result

$$
\begin{aligned}
\Re T_{++}(q) &= \frac{k_F^2mc}{2\pi^2|\boldsymbol{q}|}L^{(1)}(r)+(\omega\to-\omega)\\
\Re S_{++}(q) &= \frac{k_F^2}{2\pi^2mc|\boldsymbol{q}|}\left[k_F^2L^{(3)}(r)+\left(m\omega+\frac{\boldsymbol{q}^2}{4}\right)L^{(1)}(r)\right]+\frac{k_F^3c}{6\pi^2m}+(\omega\to-\omega), \qquad (A16)
\end{aligned}
$$

where $r=\frac{q^2+2m\omega}{2|\boldsymbol{q}|k_F}$ and

$$
\begin{aligned}
L^{(1)}(r) &= \int_0^1dkk\ln\left|\frac{a+k}{a-k}\right|=r+\frac{1}{2}(1-r^2)\ln\left|\frac{r+1}{r-1}\right|,\\
L^{(3)}(r) &= \int_0^1dkk^3\ln\left|\frac{r+k}{r-k}\right|=\frac{r}{6}+\frac{r^3}{2}+\frac{1}{4}(1-r^4)\ln\left|\frac{r+1}{r-1}\right|. \qquad (A17)
\end{aligned}
$$

The leading order $1/c$ contribution to the off-diagonal CTP block is

$$
\begin{aligned}
\binom{T}{S}_{+-}(q) &= -iI^{(2)}[\binom{t'}{s'}[\Theta(p^0)-n_p]n_{q+p}]\\
&= -\frac{imc}{2\pi^2|\boldsymbol{q}|}\int d^3p\binom{1}{\frac{1}{m^2c^2}(m\omega+p^2+\frac{\boldsymbol{q}^2}{4})}\Theta(1-|\boldsymbol{p}+\boldsymbol{q}|)\Theta(|\boldsymbol{p}|-k_F)\delta(p_z-rk_F). \qquad (A18)
\end{aligned}
$$

This is an integral on the part of a surface, orthogonal to $\boldsymbol{q}$ outside of the Fermi sphere centered at the origin and inside in another Fermi sphere which is centered at $-\boldsymbol{q}$. It is advantageous to parametrize these integrals by the dimensionless variables $x=\frac{\omega m}{|\boldsymbol{q}|k_F}$ and $y=\frac{|\boldsymbol{q}|}{k_F}$. There are three different functional forms for these integrals, according to the cases, shown in Figs. **??** and **??**: (a): $y>2$, $-y-1<r_+<-y+1$, (b): $y<2$, $-1-y<r_-<-1$, and (c): $y<2$, $1<r_+<-\frac{y}{2}$, where $r_\pm=\frac{q^2\pm2m\omega}{2|\boldsymbol{q}|k_F}=\pm x-\frac{y}{2}$ in the non-relativistic limit. A straightforward integration over the regions, shown in the figures yields

$$
\begin{aligned}
T_{+-}(q) &= -\frac{imck_F^2}{2\pi|\boldsymbol{q}|}M^{(1)}(x,y),\\
S_{+-}(q) &= -\frac{ik_F^2}{2\pi mc|\boldsymbol{q}|}\left[k_F^2M^{(1)}(x,y)+\left(\omega m+\frac{\boldsymbol{q}^2}{4}\right)M^{(1)}(x,y)\right], \qquad (A19)
\end{aligned}
$$

where

$$M_1^{(1)} = 2\int_{p_1}^{p_2} dpp = \begin{cases} 1-r_-^2 & (a) \\ 1-r_-^2 & (b) \\ -2xy & (c) \end{cases},$$

$$M_3^{(3)} = 2\int_{p_1}^{p_2} dpp^3 = \frac{1}{2}\begin{cases} (1-r_-^2)^2 & (a) \\ (1-r_-^2)^2 & (b) \\ xy(y^2+4x^2-4) & (c) \end{cases}, \tag{A20}$$

and the integrals over the interval given by $p_2 = \sqrt{1-r_-^2}$ and (a): $p_1 = 0$, (b): $p_1 = 0$, (c): $p_1 = \sqrt{1-r_+^2}$. The real part of the off-diagonal CTP block is therefore

$$\Re \left(\begin{smallmatrix} T \\ S \end{smallmatrix}\right)_{+-}(q) = \frac{1}{2}\left[\left(\begin{smallmatrix} T \\ S \end{smallmatrix}\right)_{+-}(q) - \left(\begin{smallmatrix} T \\ S \end{smallmatrix}\right)_{+-}(-q)\right]. \tag{A21}$$

4. Retarded Green function:

The retarded Green function, $G_r = G_n + G_f = G_{++} - G_{+-}$, has a longitudinal and transverse component according to the parametrization (A2). These components are defined in the non-relativistic limit by

$$G_{r\ \ell} = \frac{1}{c}\Re[T_{++} - T_{+-}],$$
$$G_{r\ t} = \frac{c}{2}\Re[(T_{++} - T_{+-})\xi^2 - S_{++} + S_{+-})]. \tag{A22}$$

The expressions (A16) and (A19) yield

$$G_{r\ \ell}(q) = \frac{k_F^2 m}{4\pi^2|\boldsymbol{q}|}[2L^{(1)}(r_+) + i\pi M^{(1)}(x,y) + 2L^{(1)}(r_-) - i\pi M^{(1)}(-x,y)]$$

$$G_{r\ t}(q) = -\frac{k_F^2}{32\pi^2 m|\boldsymbol{q}|}\{(\boldsymbol{q}^2 + 4m\omega)[2L^{(1)}(r_+) + i\pi M^{(1)}(x,y)] + (\boldsymbol{q}^2 - 4m\omega)[2L^{(1)}(r_-) - i\pi M^{(1)}(-x,y)]$$
$$+4k_F^2[2L^{(3)}(r_+) + i\pi M^{(3)}(x,y) + 2L^{(3)}(r_-) - i\pi M^{(3)}(-x,y)]\} + \frac{k_F^3}{6\pi^2 m} - \frac{x^2 k_F^2}{2m^2} G_{r\ \ell}(q). \tag{A23}$$

---

[1] H. D. Zeh, Found. Phys. **1**, 69 (1970).
[2] O. Kübler, H. D. Zeh, Ann. Phys. (NY) **76**, 405 (1973).
[3] W. H. Zurek, Phys. Rev. D **24**, 1516 (1981).
[4] W. H. Zurek, Phys. Rev. D **26**, 1862 (1982).
[5] E. Joos, H. D. Zeh, Z. Phys. B. **59**, 223 (1985).
[6] E. M. Lifsitz, L. P. Pitaevskii, *Course of Theoretical Physics, Vol 10., Physical Kinetics* Pergamon Press, Oxford, 1981.
[7] J. Schwinger, J. Math. Phys. **2**, 407 (1961); *Particles and Sources*, vol. I., II., and III.,Addison-Wesley, Cambridge, Mass. 1970-73.
[8] A. O. Caldeira, A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981).
[9] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) (Sov. Phys. JETP **20**, 1018 (1965)).
[10] R. P. Feynman, F. L. Vernon, Ann. Phys. **24**, 118 (1963).
[11] H. F. Dowker, J. J. Halliwell, Phys. Rev. **D46**, 1580 (1992).
[12] C. Anastopoulos, Phys. Rev. **D63**, 125024 (2001).
[13] J. J. Halliwell, Phys. Rev. **D58**, 105015 (1998).
[14] Y. Aharonov, P. G. Bergmann, J. L. Lebowitz, Phys. Rev. **B134**, 1410 (1964).
[15] J. Polonyi, Phys. Rev. **D74**, 065014 (2006).
[16] U. Weiss, *Quantum Dissipative Systems*, World Scientific, 1993, Singapore.
[17] H. D. Zeh, *The Direction of Time*, Springer-Verlag, Berlin, 1989.
[18] P. Ehrenfest, T. Ehrenfest, *Conceptual Foundations of the Statistical Approach in Mechanics*, Cornell University Press, Ithaca, 1959.

[19] E. Jaynes, Phys. Rev. **106**, 620 (1957).
[20] J. Gemmer, M. Michel, G. Mahler, *Quantum Thermodyanmics*, Springer Verlag, Berlin, 2004.
[21] P. Bergmann, J. Lebowitz, Phys. Rev. **99**, 578 (1955).
[22] R. B. Griffith, J. Stat. Phys. **36**, 219 (1984).
[23] R. Omnès, J. Stat. Phys. **53**, 893 (1988).
[24] M. Gell-Mann, J. B. Hartle, Phys. Rev. **D47**, 3345 (1993).
[25] J. Polonyi, Phys. Rev. D **77**, 125018 (2008).
[26] P. C. Clemmow, J. P. Dougherty, *Electrodynamics of Particles and Plasmas* Addison-Wesley, 1969.
[27] J. A. Wheeler, in *Quantum Theory and Measurement*, ed. J. A. Wheeler, W. H. Zuker, Princeton University Press, Princeton, 1983.
[28] J. A. Wheeler, R. Feynman, Rev. Mod. Phys. **17**, 157 (1945).