Identification of universal features in the conductivity of classes of two-dimensional QFTs using the AdS/CFT correspondence

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April 18, 2023

Abstract: We study the electrical conductivity of strongly disordered, strongly coupled quantum field theories, holographically dual to non-perturbatively disordered uncharged black holes. The computation reduces to solving a diffusive hydrostatic equation for an emergent horizon fluid. We demonstrate that a large class of theories in two spatial dimensions have a universal conductivity independent of disorder strength, and rigorously rule out disorder-driven conductor-insulator transitions in many theories. We present a (fine-tuned) axion-dilaton bulk theory which realizes the conductor-insulator transition, interpreted as a classical percolation transition in the horizon fluid. We address aspects of strongly disordered holography that can and cannot be addressed via mean-field modeling, such as massive gravity.

1 Introduction

We examined electrical transport in strongly coupled holographic quantum field theories at zero charge density, constructing perfect metals amidst disorder. Our findings have implications for realistic models of disordered strange metals.

2 Conductivity

Consider a static, asymptotically anti-de Sitter space with a black hole horizon sourced entirely by uncharged bulk matter and a dynamical metric. We can choose the bulk metric using diffeomorphism invariance.

$$\mathrm{d}s^2 = L^2\left[P\mathrm{d}r^2 - Q\mathrm{d}t^2 + G_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]. \tag{1}$$

i, j indices represent the spatial boundary directions, while M, N represent all dimensions, and L is AdS radius. All functions in the metric are functions of r and $\mathbf{x}$. We further choose bulk coordinate $0 < r < \infty$, with $r = 0$ black hole horizon, and $r = \infty$ AdS boundary. Uncharged matter not required, energy conditions obeyed.

We add a U(1) gauge field to the bulk, so the action of our theory is

$$S = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(\mathcal{L}_{\text{uncharged}} - \frac{Z}{4}F^2\right). \tag{2}$$

Function Z is a parameter of (uncharged) scalar matter, but for our purposes it's an arbitrary function of r and $\mathbf{x}$. Gauge field's two-point functions correspond to current-current correlation functions in the boundary theory, including electrical conductivity matrix σ^{ij}. The conductivity may be related, via membrane paradigm [1], to data on the horizon of the black hole alone. The expected value of the boundary current is given by

$$J^i = \sigma^{ij}E_j = \mathbb{E}\left[Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right], \tag{3}$$

where E_j is the applied electric field, $\mathbb{E}[\cdots]$ denotes a uniform spatial average, $\gamma_{ij} = G_{ij}(r=0)$ is the induced metric on the horizon, and α is the unique function which obeys equation

$$0 = \partial_i\left(Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right) \tag{4}$$

with appropriate boundary conditions (for example, periodicity in compact boundary spatial directions). The membrane paradigm was used in holographic systems in [2], and similar computations appear in [3, 4, 5] for black holes with translational symmetry broken only in one direction. These results are special cases of this general formula. This formula may break down if black hole horizon fragments and becomes disconnected, as was considered in [6, 7].

We can interpret (4) as a hydrostatic equation enforcing local charge conservation in an emergent horizon fluid. This is subtle – the local "electric current" in (4) is *not the same* as the expected value of the local current in the dual theory; only their spatial averages are equal. A powerful set of techniques have been developed to understand the qualitative behavior of transport in such fluids [8]; for example, it immediately follows from (4) that $\sigma_{ij} = \sigma_{ji}$.

In particular, if $\sigma[Z;\gamma_{ij}]$ is the conductivity matrix with given Z and γ_{ij}:

$$\det\left(\sigma[Z;\gamma_{ij}]\right)\det\left(\sigma\left[\frac{1}{Z};\gamma_{ij}\right]\right) = \frac{1}{e^8}. \tag{5}$$

If we set $Z=1$, (5) gives

$$\det(\sigma) = \frac{1}{e^4}. \tag{6}$$

If we expect that on average for a disordered sample, the conductivity matrix is isotropic ($\sigma^{ij} = \sigma\delta^{ij}$), that fixes conductivity to be $\sigma = 1/e^2$, exactly the clean result!

A simple way to understand this result: suppose that in local coordinates, the metric is given by

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \approx a_x^2\mathrm{d}x^2 + a_y^2\mathrm{d}y^2. \tag{7}$$

Then we expect "locally" $\sigma_{xx} \sim a_y/a_x$ and $\sigma_{yy} \sim a_x/a_y$ [9]. On average a_y and a_x should have identical distributions, so we expect that σ_{xx} and $1/\sigma_{xx}$ have the same distributions. This implies $\sigma = 1/e^2$; analogous statements are known for random resistor lattices in $d=2$ with analogous (e.g., log-normal) resistance distributions [**?**]. And more generally, if $\log Z$ is symmetrically distributed about 0, then in an isotropic theory, $\sigma = 1/e^2$ follows from (5) in the thermodynamic limit.

The robustness of σ in these strongly disordered $d=2$ models is remarkable, and deserves further comments. In models where momentum dissipation is introduced through massive gravity [10] or "Q-lattice" axions [11], one finds the hydrodynamic result [12]

$$\sigma = \sigma_{\mathrm{Q}} + \frac{\mathcal{Q}^2\tau}{\epsilon + P}, \tag{8}$$

where $\mathcal{Q}$ is charge density, ϵ energy density, P pressure, σ_{Q} dissipative "quantum critical" conductivity without disorder, and τ a "momentum relaxation time", inversely related to graviton mass. Before now, it was unclear whether the fact that (8) holds beyond the hydrodynamic limit was an unrealistic feature of massive gravity or similar theories. Our work confirms this is a sensible prediction of massive gravity for many systems at $\mathcal{Q} = 0$. (8) further implies another mechanism, $\tau \to 0$, by which the conductivity can reach its lower bound, σ_{Q}. The conductivity saturating this lower bound, at least qualitatively, is likely to occur at strong disorder [8]. Confirmation that strongly-disordered charged holographic models (with $Z = 1$) have a conductivity no smaller than $1/e^2$ in $d = 2$ would be a further non-trivial test of predictions of simple mean-field physics.

In $d \neq 2$, and/or if Z is distributed more generically, it's valuable to employ insight gained from equivalence between Markov chains on lattices and resistance of a resistor lattice [13]. For arbitrary Z, this analogy can be leveraged to find lower and upper bounds to σ, for a self-averaging disordered sample: [8]

$$\frac{L^{d-2}}{e^2}\mathbb{E}\left[\frac{\gamma_{ii}}{dZ\sqrt{\gamma}}\right]^{-1} \leq \sigma \leq \frac{L^{d-2}}{e^2}\frac{\mathbb{E}[Z\sqrt{\gamma}\gamma^{ii}]}{d}. \tag{9}$$

It is straightforward to test these results and bounds by numerically solving (4) for various disorder realizations. Good agreement with our exact analytic results and consistency with our bounds is obtained.

3 Conductor-Insulator Transition

(9) constrains σ to deviate from the clean result by the strength of fluctuations in Z and γ_{ij}. It's evident from (9) that if γ_{ij} and Z are finite at all points on the horizon, then the black hole necessarily conducts electrical current, no matter how strong the disorder. This is a remarkable result. In contrast, in non-interacting quantum field theory, a conductor-insulator transition occurs at a finite disorder strength [14] in $d > 2$, and at arbitrarily small disorder in $d \leq 2$ [15]. This transition relates to the destructive interference of matter waves scattering off of the disorder. Apparently, bulk fluctuations of the gauge field in holographic theories do not suffer from such interference. While it's known [16, 17] that metal-insulator transitions occur at a finite disorder strength in an interacting quantum system, even such systems ultimately succumb to (many-body) localization at strong disorder. Perhaps holographic models have taken the "coupling $\to \infty$" limit first, rendering such a transition impossible.

Realizing a holographic conductor-insulator transition takes more care. A "helical lattice" approach has generated such a transition in [18, 19], but there is no satisfying physical interpretation. However, even in these papers, the conductivity in the insulating phase only decays as algebraically in T as $T \to 0$, in contrast to canonical insulators.

Assuming $d = 2$ and a probe limit with AdS-Schwarzschild geometry, we need a large $\mathbb{E}[1/Z]$ for $\mathcal{C} = 0$, requiring percolating $Z \to 0$ bubbles across the horizon. When these finite-Z regions disconnect, charge transport is halted, causing a disorder-driven holographic metal-insulator transition, similar to random resistor lattices [20].

Numerically compute conductivity for Z ansatz with "bubbles" where $Z \to 0$ percolate across horizon to test proposal. Numerics support this; see Figures 1 and 2.

3.1 Holographic Realizations

We now ask whether the percolation mechanism proposed above for a disorder-driven metal-insulator transition can occur in a "realistic" holographic model: a bottom-up Einstein-Maxwell-dilaton (Φ)-axion

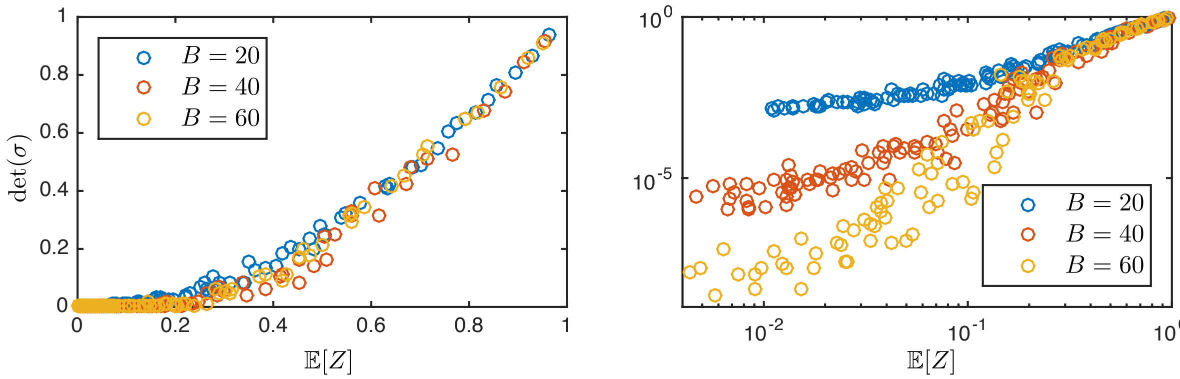

Figure 1: $\det(\sigma)$ from a black hole horizon for a theory in $d = 2$; we set $e = 1$, and use periodic boundary conditions with $|x|, |y| \leq \pi$, with a discretized spatial grid of 701^2 points. We take $\gamma_{ij} = \delta_{ij}$ and $Z = \exp[-B\mathcal{Z}/(1+2\mathcal{Z})]$, where $\mathcal{Z} = \sum_{j=1}^{N} \exp(-(\sin^2(\phi_{jx}+x/2)+\sin^2(\phi_{jy}+y/2))/2\xi^2)$, with ϕ_{jx} and ϕ_{jy} independent random phases, and $B > 0$ is a random constant. We took various values of B and fixed $\xi = 20\pi/701$. When $\mathbb{E}[Z] \gtrsim 0.28 \equiv Z^*$, curves at different B approximately collapse, implying that current avoids the non-conducting bubbles; when $\mathbb{E}[Z] \lesssim Z^*$, the value of conductivity is sensitive to B. In the limit $B \to \infty$ and $\xi \to 0$, a metal-insulator transition appears at Z^*.

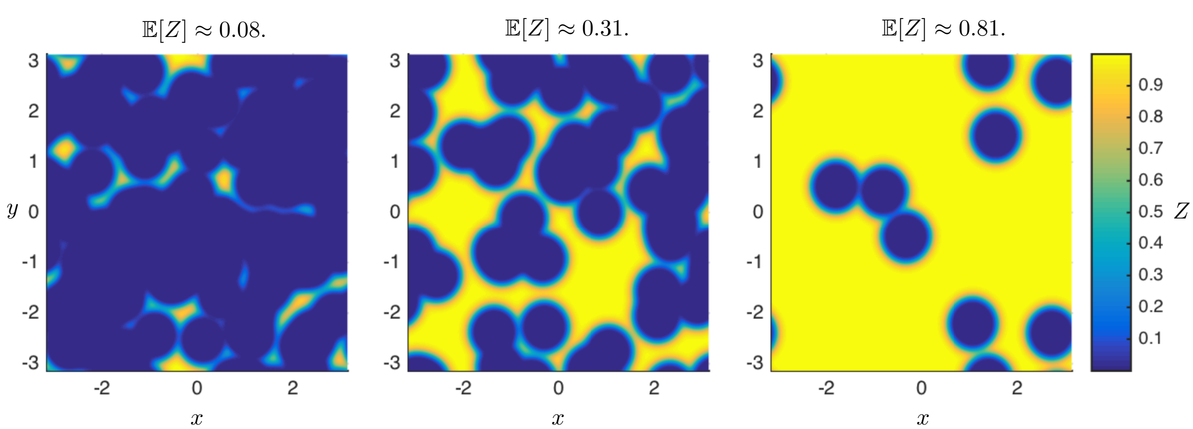

Figure 2: Surface plots of $Z(x, y)$ for various bubble densities. Depending on whether regions of high or low Z percolate across the horizon determines whether we're in the metallic or insulating phase, as is clear upon comparing with Figure 1.

(α) theory with action

$$S = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(\frac{R-2\Lambda}{16\pi G} - \mathcal{M}^d\left[\frac{1}{2}(\partial\Phi)^2 + \frac{\mathrm{e}^{-\Phi}}{2}(\partial\alpha)^2 - \frac{V(\alpha)+U(\Phi)}{L^2}\right] - \frac{Z(\alpha)}{4e^2}F^2\right). \tag{10}$$

Here $\mathcal{M}$ is a mass scale, whose precise value is unimportant – we choose it so that Φ is strictly dimensionless, for simplicity, and

$$\Lambda = -\frac{d(d+1)}{2L^2}. \tag{11}$$

At $G \to 0$, generalizing choices yields similar results, but (10) with axio-dilaton scalar kinetic terms is essential. $Z(\alpha)$'s cosine potentials may suit our needs, and arise due to instanton effects in effective actions (as in QCD). In our holographic model, $Z(\alpha)$ isn't suppressed by G/L^d (the scale of bulk's quantum corrections).

For conductor-insulator transitions, $V(\alpha)$ must have at least two minima, α_{c} and α_{i}, with $Z(\alpha_{\mathrm{c}}) > 0$ and $Z(\alpha_{\mathrm{i}}) = 0$. α drives the transition and Φ stabilizes it, although theories with finite Lifshitz or hyperscaling-violating exponents may also work [21]. Insulators form when bubbles of $\alpha = \alpha_{\mathrm{i}}$ percolate across the horizon; we aim to demonstrate how to create and maintain these bubbles at low temperatures. [21] is a citation. For this purpose, a simple choice of potentials, though certainly not the only, is

$$U(\Phi) = \frac{7\lambda^2}{2} - 3\lambda\Phi - 4\lambda^2\mathrm{e}^{-\Phi/\lambda} + \frac{\lambda^2}{2}\mathrm{e}^{-2\Phi/\lambda}, \tag{12a}$$

$$V(\alpha) = -\alpha^2 + \frac{\alpha^4}{2\alpha_0^2}, \tag{12b}$$

$$Z(\alpha) = \left(1 - \frac{\alpha}{\alpha_0}\right)^2. \tag{12c}$$

Using Z in [22], we set $\alpha_0 \to 0$, $\lambda > 2$, and $V(\alpha)$ marginal to avoid axion backreaction on the dilaton. The Harris criterion [21] implies inability to source disordered modes of all wavelengths without UV geometry backreaction.

Let us begin by sourcing the dilaton with (positive) δ-like sources on the AdS boundary – analogous to point-like impurities in the dual theory. Each impurity produces an expanding bubble which becomes insulating; width of the "bubbles" of α is $\sim 1/T$. If density of the impurities is n, then the bubbles percolate across the horizon when $T \lesssim \sqrt{n}$. Within each bubble, $\alpha \to \alpha_0$, and thus at low temperatures we obtain an insulator

A second mechanism for obtaining the transition is as follows: suppose α.

As $T \to 0$ in the insulating phase, we predict:

$$\sigma(T) \sim \exp\left[-\frac{8}{\lambda}\left(\frac{\zeta}{T}\right)^{\lambda/2}\right] \tag{13}$$

4 Outlook

Recent models [23, 24, 25, 26] propose momentum non-conservation in (quasi-2d) strange metals. We constructed perfect conductors in strong disorder and predict finite charge density will not decrease conductivity. We encourage extending holographic approach to charged black holes and finding non-holographic field theories with disorder-resistant σ_{Q}.

Acknowledgements

We thank Ed Witten for discussions. We especially thank Veronica Toro Arana and Anna Maria Wojtyra for providing some code for solving elliptic partial differential equations. This research was funded through Nvidia.

References

[1] K. S. Thorne, R. H. Price, and D. A. MacDonald (eds.). *Black Holes: The Membrane Paradigm* (Yale University Press, 1986).

[2] N. Iqbal and H. Liu. "Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm", *Physical Review* **D79** 025023 (2009), arXiv:0809.3808.

[3] S. Ryu, T. Takayanagi, and T. Ugajin. "Holographic conductivity in disordered systems", *Journal of High Energy Physics* **04** 115 (**2011**), arXiv:1103.6068.

[4] A. Donos and J. P. Gauntlett. "The thermoelectric properties of inhomogeneous holographic lattices", *Journal of High Energy Physics* **01** 035 (**2015**), arXiv:1409.6875.

[5] M. Rangamani, M. Rozali, and D. Smyth. "Spatial modulation and conductivities in effective holographic theories", arXiv:1505.05171.

[6] D. Anninos, T. Anous, F. Denef, and L. Peeters. "Holographic vitrification", arXiv:1309.0146.

[7] G. T. Horowitz, N. Iqbal, J. E. Santos, and B. Way. "Hovering black holes from charged defects", *Classical & Quantum Gravity* **32** 105001 (2015), arXiv:1412.1830.

[8] A. Lucas. "Hydrostatic transport in strongly coupled disordered quantum field theories", to appear.

[9] P. Chesler, A. Lucas, and S. Sachdev. "Conformal field theories in a periodic potential: results from holography and field theory", *Physical Review* **D89** 026005 (2014), arXiv:1308.0329.

[10] M. Blake and D. Tong. "Universal resistivity from holographic massive gravity", *Physical Review* **D88** 106004 (2013), arXiv:1308.4970.

[11] B. Goutéraux. "Charge transport in holography with momentum dissipation", *Journal of High Energy Physics* **04** 181 (**2014**), arXiv:1401.5436.

[12] S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev. "Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes", *Physical Review* **B76** 144502 (2007), arXiv:0706.3215.

[13] D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov Chains and Mixing Times* (AMS, 2009).

[14] P. W. Anderson. "Absence of diffusion in certain random lattices", *Physical Review* **109** 1492 (1958).

[15] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan. "Scaling theory of localization: absence of quantum diffusion in two dimensions", *Physical Review Letters* **42** 673 (1979).

[16] T. Giamarchi and H. J. Schulz. "Localization and interaction in one-dimensional quantum fluids", *Europhysics Letters* **3** 1287 (1987).

[17] D. M. Basko, I. L. Aleiner, and B. L. Altshuler. "Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states", *Annals of Physics* **321** 1126 (2006), arXiv:cond-mat/0506617.

[18] A. Donos and S. A. Hartnoll. "Interaction-driven localization in holography", *Nature Physics* **9** 649 (2013), arXiv:1212.2998.

[19] A. Donos, B. Goutéraux, and E. Kiritsis. "Holographic metals and insulators with helical symmetry", *Journal of High Energy Physics* **09** 038 (**2014**), arXiv:1406.6351.

[20] B. Derrida and J. Vannimenus. "A transfer-matrix approach to random resistor networks", *Journal of Physics* **A15** L557 (1982).

[21] A. Lucas, S. Sachdev, and K. Schalm. "Scale-invariant hyperscaling-violating holographic theories and the resistivity of strange metals with random-field disorder", *Physical Review* **D89** 066018 (2014), arXiv:1401.7993.

[22] E. Mefford and G. T. Horowitz. "Simple holographic insulator", *Physical Review* **D90** 084042 (2014), arXiv:1406.4188.

[23] S. A. Hartnoll. "Theory of universal incoherent metallic transport", *Nature Physics* **11** 54 (2015), arXiv:1405.3651.

[24] P. Kovtun. "Fluctuation bounds on charge and heat diffusion", arXiv:1407.0690.

[25] S. A. Hartnoll and A. Karch. "Scaling theory of the cuprate strange metals", *Physical Review* **B91** 155126 (2015), arXiv:1501.03165.

[26] D. V. Khveshchenko. "Constructing (un)successful phenomenologies of the normal state of cuprates", arXiv:1502.03375.