The Navier-Stokes equations from a "minimal" effective field theory

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We use an effective Schwinger-Keldysh field theory of long-range massless modes to derive the Navier-Stokes equations as an energy-momentum balance equation. The fluid will be invariant under the linear subgroup of the volume-preserving diffeomorphisms, which are the non-linear, time-independent spatial translations.

A. Introduction

In this work, we extend the systematic approach to dissipative hydrodynamics, developed in [1], to derive the full Navier-Stokes equations with the bulk and the shear viscosity terms.

Our goal in this paper will be to find the "minimal" dissipative extension of [1–5], sufficient to give us the full Navier-Stokes equations, which are known to be the universal description of fluids at low energy in Nature.

and at second order by [6]

Quantum fluctuations of the effective action [7] on a lattice [8, 9]. Non-perturbative vacuum state? Quantised ideal fluid backreaction give viscosity [10]

the linear response theory and dissipation [11]

hydrodynamic correlation functions from an effective action was recently proposed in [12, 13].

Ideal fluid action from holography [14, 15], as an extension of [16] in the context of Wilsonian renormalisation group in gauge/gravity duality [17–23].

Schwinger-Keldysh, closed-time-path (CTP) [24–27]

Kinetic theory in CTP [28, 29]

the CTP formalism in classical physics [30, 31]

B. The Schwinger-Keldysh effective field theory

Let us consider a microscopic unitary QFT, such as QED or QCD, in which ψ represents all the fundamental fields in the theory. The generating functional for the correlation functions of fields or composite operators, given an arbitrary initial density matrix of the system, can be written as

$$\begin{aligned} \exp\left\{iW_{CTP}[\hat{h}_{\mu\nu}]\right\} &= \int \mathcal{D}\hat{\psi}\,\rho_i[\hat{\psi}(t_i,\boldsymbol{x})] \\ &\times \exp\left\{iS_{CTP}[\hat{\psi}] + i\int d^4x\,\hat{T}^{\mu\nu}[\hat{\psi}]\,\hat{h}_{\mu\nu}\right\}, \end{aligned} \tag{1}$$

where we for concreteness consider the W_{CTP} for the stress-energy tensor $T^{\mu\nu}$. The metric perturbation $\eta_{\mu\nu} + h_{\mu\nu}$ of the flat space thus acts as its source. In terms of the operator language, Eq. (1) enables us to compute time-dependent expectation values, such as

$$\mathrm{Tr}\left[\hat{T}^{\mu\nu}\rho_i\right]_{(t,\boldsymbol{x})} = -i\frac{\delta}{\delta\hat{h}_{\mu\nu}}\exp\left\{iW_{CTP}[\hat{h}_{\mu\nu}]\right\}\Big|_{\hat{h}=0}, \tag{2}$$

on each of the time axes. After equations of motion are imposed in classical field theory (on-shell), $T^+_{\mu\nu} = T^-_{\mu\nu}$. Hence, in the Keldysh basis, $T_{\mu\nu} = \frac{1}{2}\left(T^+_{\mu\nu} + T^-_{\mu\nu}\right)$ is often considered as the *classical* operator, while $T^+_{\mu\nu} - T^-_{\mu\nu} = 0$, on-shell. In this paper, we will use an equivalent basis in which we will only vary fields on the + axis.

The microscopic CTP action in (1) has the form

$$S_{CTP}[\psi^+,\psi^-] = S_s[\psi^+] - S^*_s[\psi^-], \tag{3}$$

where S_s stands for the unitary *single-axis* action of ψ. S_{CTP} is thus invariant under the CTP symmetry,[1]

$$S_{CTP}[\psi^+,\psi^-] = -S^*_{CTP}[\psi^-,\psi^+]. \tag{4}$$

Let us now consider the Wilsonian effective theory of the massless excitations, which we collectively denote by φ. Such a theory follows from integrate out all the massive modes by using the CTP formalism and keeping (1) invariant. As an example of a non-Wilsonian effective field theory calculation from a microscopic QFT, [34] calculated the classical equation of motion for a conserved Noether current in QED.

An effective CTP action, S_{eff}, has two important structural feature due to the off-diagonal, on-*mass*-shell Wightman propagators that connect ψ^+ and ψ^-. Firstly, an effective action can directly couple φ^+ with φ^- as in the influence functional [36], and secondly, S_{eff} can be complex, i.e. $S_{eff} = \Re S_{eff} + i\Im S_{eff}$. It is important to stress that the CTP symmetry (3) is preserved in the $\varphi^\pm$ variable.

Can see decoherence, as for example recently discussed in [34, 37–39].

C. Hydrodynamics

From the point of view of Wilsonian field theory, hydrodynamics can be formulated as an effective theory of

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[1] For a detailed discussion on the background material regarding the Schwinger-Keldysh CTP formalism and effective actions, see [1, 32–35].

the gapless IR modes that remain in the theory after all the massive modes have been integrated out above some mass gap scale. Such effective theories are most naturally expressed in terms of the gradient expansion of the massless fields. Within this framework, dissipationless hydrodynamics was developed in [2–5].

In this formalism, the low energy effective dynamics of a *charged* fluid in $d+1$ space-time dimensions is parametrised in terms of $d+1$ scalar fields. We will work in $d=3$ and only consider fluids in flat space, where it is sufficient to introduce the scalar modes ϕ^I, with $I=\{0,1,2,3\}$.[2] Since the field indices I directly correspond to the space-times indices μ, we will also use μ as field labels.

The fields ϕ^μ should be permitted to describe large extended background configurations of the fluid. It is therefore convenient to split them as

$$\phi^\mu(x) = \Phi^\mu(x) + \pi^\mu(x). \tag{5}$$

The equilibrium background expectation values are [2, 3]

$$\langle \Phi^\mu \rangle = x^\mu. \tag{6}$$

The theory describing a fluid should be invariant under constant translations, typical for Goldstone modes, and spatial rotations,

$$\phi^\mu \to \phi^\mu + a^\mu, \qquad a^\mu = const., \tag{7}$$

$$\phi^i \to R^i{}_j\,\phi^j, \qquad R^i{}_j \in SO(3). \tag{8}$$

For simplicity, we will only consider the *uncharged fluids*. Thus, we henceforth set the fluctuation $\pi^0=0$. It is important to note that even though $\Phi^0 \neq 0$, there is no additional degree of freedom associated with the ϕ^0 field, only with ϕ^i. The system is therefore uncharged and no non-trivial Lagrangian consistent with Eq. (7) can be written down, as $\partial_\mu\phi^0 = \delta_\mu{}^0$. Nevertheless, the introduction of π^0 is important to allow to construct a manifestly relativistic theory.

It is clear from Eq. (6) that ϕ^μ cannot be used directly in the gradient expansion. Instead, since we are treating π^i as small fluctuations around Φ^μ, we will perform a gradient expansion in π^i.

In [1–6], the group of volume-preserving diffeomorphisms, SDiff$\,(\mathbb{R}^{1,3})$,

$$\phi^i \to \xi^i(\phi), \qquad \det\left(\frac{\partial \xi^i}{\partial \phi^j}\right) = 1. \tag{9}$$

played a central role in constraining the effective action. However, while it has been established that ideal hydrodynamics is invariant under SDiff$\,(\mathbb{R}^{1,3})$ [40, 41], [2, 3, 5, 14, 15], dissipative viscous hydrodynamics with that symmetry in [1] only incorporated non-zero bulk viscosity ζ, but no shear viscosity η. Similarly, dissipationless second-order hydrodynamics of [6] resulted in a reduced number of transport coefficients in comparison with those obtained from the gradient expansion [42–44]. Furthermore, it was shown in [7] that the linear-response based approach that gave the right form of the π^i-field coupling to some UV composite operators, which could formally permit for a shear viscosity term, explicitly broke SDiff$\,(\mathbb{R}^{1,3})$.

In this work, we propose a resolution to this situation. We take the view that the SDiff$\,(\mathbb{R}^{1,3})$ group should be understood as an accidental emergent symmetry of ideal hydrodynamics, but not of the full theory of the non-linear ϕ^i fields. Instead, we will impose the linearised SDiff$(\mathbb{R}^{1,3})$ invariance, denoted as SDiff$_\pi(\mathbb{R}^{1,3})$, only on the *fluctuation fields*,

$$\pi^i(t,\boldsymbol{x}) \to \pi^i(t,\boldsymbol{x}) + \alpha^i(\boldsymbol{x}), \qquad \partial_i\alpha^i = 0, \tag{10}$$

leaving Φ^μ completely unconstrained. This is consistent with [5] where these transformations were discussed in the context of the coset construction that resulted in the invariance of ideal hydrodynamics under the full SDiff$\,(\mathbb{R}^{1,3})$. Note that these transformations are nothing but the non-linear time-independent spatial translations — a local extension of Eq. (7) for the fluctuation fields.

In order to construct a theory invariant under SDiff$_\pi(\mathbb{R}^{1,3})$, it is convenient to define the expectation values of all the manifestly invariant derivatives of ϕ^μ as

$$\left\langle \partial_0\phi^0 \right\rangle = 1 \equiv -v^0, \tag{11}$$

$$\left\langle \partial_0\phi^i \right\rangle = \left\langle \partial_0\pi^i \right\rangle \equiv -v^i, \tag{12}$$

$$\left\langle \partial_\mu\phi^\mu \right\rangle = 4 + \left\langle \partial_i\pi^i \right\rangle \equiv \omega. \tag{13}$$

On top of the three SDiff$_\pi(\mathbb{R}^{1,3})$ invariants, we can also use $\partial_\mu\Phi^\nu$ to construct the effective theory, which will explicitly break (9).

In a CTP effective field theory, we double the fields $\phi^i \to \{\phi^{0i},\phi^{1i}\}$ and impose

It is clear from Eq. (6) that ϕ^I cannot be used directly in the gradient expansion, as $\partial_\mu\phi^I = b_0^{1/3}\left(\delta_\mu{}^I + \partial_\mu\pi^I\right)$ may be any order of magnitude. Although one could form a gradient expansion in derivatives of $\partial_\mu\phi^I$, the fact that any space-time contracted "powers" of $\partial_\mu\phi^I\partial_\nu\phi_I$ or $\epsilon_{IJK}\partial_\mu\phi^I\partial_\nu\phi^J\partial_\rho\phi^K$ are still at zeroth order makes such a theory very complicated. In an effective theory that is invariant under the volume-preserving diffeomorphisms, SDiff$\left(\mathbb{R}^{1,3}\right)$, the possible tensor structures are vastly restricted and only

$$K^\mu = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJK}\partial_\nu\phi^I\partial_\rho\phi^J\partial_\sigma\phi^K \tag{14}$$

can be used to form the gradient expansion.

[2] Throughout this work, we use the metric tensor signature $\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)$. Lower case Greek letters will be used as Lorentzian space-time indices and lower case Latin letters from the middle of the alphabet (i, j, ...) as spatial indices. We will use lower case Latin letters from the beginning of the alphabet as the CTP indices, $a \in \{0,1\}$, with 0 and 1 representing $-$ and $+$, respectively.

In the CTP effective action, the doubled rotational invariance $SO(3)^+ \times SO(3)^-$ gets broken to a diagonal subgroup after the equations of motion are imposed, i.e. *on shell*,

$$SO(3)^+ \times SO(3)^- \to SO(3). \tag{15}$$

The off-shell variations are

$$\delta^+ \partial_\mu \Phi^{a\nu} = a\partial_\mu \left(\xi^\lambda \partial_\lambda \Phi^{a\nu}\right), \tag{16}$$
$$\delta^+ \partial_\mu \pi^{ai} = a\partial_\mu \left(\xi^\lambda \partial_\lambda \pi^{ai}\right). \tag{17}$$

become

$$\left\langle \delta^+ \partial_\mu \Phi^{ai} \right\rangle = a\partial_\mu \xi^i, \tag{18}$$
$$\left\langle \delta^+ \partial_\mu \pi^{ai} \right\rangle = a\partial_\mu \left(\xi_\lambda \omega^{\lambda i}\right). \tag{19}$$

D. The dissipative action and the Navier-Stokes equations

By employing the above discussed ingredients, we can now systematically write down an effective relativistic CTP action for a dissipative fluid. In the gradient expansion to second order in derivatives of π^i,

$$S = \int d^4x \, [\ldots], \tag{20}$$

with the Lagrangian that is symmetric under the off-shell doubled CTP $\mathrm{SDiff}_\pi(\mathbb{R}^{1,3}) \times \mathrm{SDiff}_\pi(\mathbb{R}^{1,3})$ symmetry of Eq. (10). The coefficients $\mathcal{A}$, ..., are dimensionful constants.

The energy-momentum balance equation, i.e. the equation of motion for the space-time variation $x^\mu \to x^\mu + \xi^\mu(x)$, can be derived by using the variations in Eqs. (16) and (17)

E. Discussion

Particularly interesting is the quantisation, and understanding what would be the maximal possible subgroup of $\mathrm{SDiff}(\mathbb{R}^{3,1})$ that could describe the behaviour of viscous fluids, with non-linear extensions beyond the Navier-Stokes equations.

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