The Exact Renormalisation Group and Quantum Mechanics

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Abstract: The exact renormalization group equation is rewritten as a Schrödinger type equation and analyzed.

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Contents

1 Introduction

We begin by considering the exact Wilsonian renormalisation group equation

$$\partial_t S = \int_p (c+2p^2)\left(\frac{\delta^2 S}{\delta\phi_p \delta\phi_{-p}} - \frac{\delta S}{\delta\phi_p}\frac{\delta S}{\delta\phi_{-p}} + \phi_p \frac{\delta S}{\delta\phi_p}\right) , \tag{1.1}$$

where we denote

$$\int_p = \int \frac{d^d p}{(2\pi)^d}$$

for a more compact notation. As an equation for the action S, this is a non-linear equation, containing terms quadratic in the action. Note however that if we instead consider the functional

$$\psi = e^{-S} ,$$

then this equation can be rewritten as

$$\partial_t \psi = \int_p (c+2p^2)\left(\frac{\delta^2 \psi}{\delta\phi_p \delta\phi_{-p}} + \phi_p \frac{\delta \psi}{\delta\phi_p}\right) , \tag{1.2}$$

or

$$\partial_t \psi = \mathcal{H}\psi , \tag{1.3}$$

where

$$\mathcal{H} = \int_p (c+2p^2)\left(\frac{\delta^2}{\delta\phi_p \delta\phi_{-p}} + \phi_p \frac{\delta}{\delta\phi_p}\right) . \tag{1.4}$$

This then turns exact RG theory into solving a linear differential equation.

Note that in the Wilsonian case, the "Hamiltonian" $\mathcal{H}$ is independent of the RG-time t. This is not the case for generic functional RG equations, like the Polchinski equation. If however the RG-time dependence of $\mathcal{H}$ can be pulled out in a function, so

$$\mathcal{H}(t) = f(t)\,\mathcal{H}_0 ,$$

where $\mathcal{H}_0$ is RG-time independent. Then the RG-equation can be written as

$$\partial_{\tilde{t}}\psi = \mathcal{H}_0\psi ,$$

where the new variable $\tilde{t}$ is defined by

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} = f(t) .$$

The point is that since (1.3) is a linear equation with a t-independent right hand side, it can be solved using methods similar to that of quantum mechanics. In particular, (1.3) looks very much like a Schrödinger type equation.

2 Functional Quantum Mechanics

Inspired by the look of (1.3), we proceed to "solve" this equation, employing similar methods to that of Quantum Mechanics. First, let $|\psi_i\rangle$ be an eigenvector of $\mathcal{H}$, i.e.

$$\mathcal{H}|\psi_i\rangle = \lambda_i|\psi_i\rangle , \tag{2.1}$$

and we have gone to bra-ket notation. We assume that the set of eigenvectors $\{|\psi_i\rangle\}$ is complete and that the functions can be labeled by an index i. Note that i is not necessarily discrete.

By completeness, any functional ψ may then be written as

$$\psi = \sum_i \alpha_i\psi_i . \tag{2.2}$$

We further assume that we have an inner product (,) on the space of functionals, making it into a Hilbert space. E.g. we can take the inner product to be the path integral

$$(\psi_1, \psi_2) = \langle\psi_1|\psi_2\rangle = \int \mathcal{D}\phi\, \psi_1[\phi]\overline{\psi}_2[\phi] ,$$

where we have allowed for complex valued functionals as well. We assume the eigenfunctionals ψ_i are normalized with respect to this inner product,

$$\int \mathcal{D}\phi\, \psi_i[\phi]\psi_j[\phi] = \delta_{ij} , \tag{2.3}$$

where δ_{ij} denotes the "Kronecker delta". Note that δ_{ij} can be a functional in general. E.g., if we are dealing with a "free" Hamiltonian, so

$$\mathcal{H} \propto \frac{\delta^2}{\delta\phi\delta\phi} = \int_{x,y} \Delta(x,y)\frac{\delta^2}{\delta\phi_x\delta\phi_y} ,$$

where we have introduced a "metric" $\Delta(x, y)$ on the space of dummy indices $\{x, y\}$. Then the eigenfunctionals are of the form

$$\psi_J[\phi] \propto e^{i\int_x \phi_x J_x} \, .$$

It follows that

$$(\psi_J, \psi_{J'}) \propto \delta(J - J') \, ,$$

i.e. the functional delta-function. Note that in this sense, the currents J behave as conjugates to the fields ϕ, just like momenta p conjugate to positions x in usual quantum mechanics. We assume that $\mathcal{H}$ is self-adjoint with respect to the inner product. By completeness of the set $\{|\psi_n\rangle\}$, we can derive the usual decomposition of unity

$$1 = \sum_n |\psi_n\rangle\langle\psi_n| \, .$$

Using this, it is further easy to derive the spectral theorem for $\mathcal{H}$,

$$\mathcal{H} = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \, . \tag{2.4}$$

One could in principle compute different weights α_i,

$$(\psi_i, \psi) = \alpha_i = \int \mathcal{D}\phi \, \psi_i \overline{\psi} \, . \tag{2.5}$$

This would then imply that an RG flow of any theory can be represented by a flow of coefficients multiplying different weights of theories. Plugging (2.2) into (1.3) gives

$$\partial_t \alpha_i(t) = \lambda_i \alpha_i(t) \, ,$$

or

$$\alpha_i(t) = \alpha_{i0} e^{\lambda_i t} \, .$$

The action is then formally given by

$$S = -\log\left[\sum_i \alpha_{i0} e^{\lambda_i t} \psi_i\right] \, . \tag{2.6}$$

Solving exact RG theory in this way of course depends crucially on whether we can diagonalise the "Hamiltonian" $\mathcal{H}$. It also depends on whether we can perform the inner products (2.5), which in general are rather tricky path integrals. Moreover, having found the spectrum of $\mathcal{H}$ and computed the integrals (2.5), it is hard to see how (2.6) can be put in the usual form as an integralal over space-time. Indeed, we would expect generically the final action to have non-local interaction terms.

Often, however, we are not interested in the full solution, but rather the IR behavior of the theory, i.e. when $t \to \infty$. In this case, the dominant eigenfunctional(s) in (2.6) will be the ones with highest weight. In the deep IR, the largest λ_i will dominate completely.[1]

[1] We assume $\mathcal{H}$ has a maximal eigenvalue. This is different from quantum mechanics, where a minimal eigenvalue, i.e. the vacuum, is assumed.

Assuming the original theory overlaps with eigenfunctionals of this eigenvalue, which is true generically, the action takes the form

$$S = -\log\left[\sum_{max} \alpha_{max0} e^{\lambda_{max} t} \psi_i\right] ,$$

where the sum now is over eigenfunctionals of maximal eigenvalue λ_{max}.

2.1 One-dimensional case

To get a very crude idea of what is going on, we now restrict ourselves to a toy-model with one dimension, where the problem reduces to the equation

$$\partial_t \psi = \partial_x^2 \psi + x \partial_x \psi \, . \tag{2.7}$$

This equation can be obtained by a very crude form of mean-field approximation, where we assume that the field is a constant over space-time, i.e. higher derivative modes play a small effect. In this case, we have the hamiltonian

$$\mathcal{H} = \partial_x^2 + x\partial_x = \partial_x^2 + \frac{1}{2}(x\partial_x + \partial_x x) - \frac{1}{2}[\partial_x, x] = (\partial_x + \frac{1}{2}x)^2 - \frac{1}{4}x^2 - \frac{1}{2} \, .$$

Note that

$$\partial_x + \frac{1}{2}x = e^{-\frac{1}{4}x^2} \partial_x e^{\frac{1}{4}x^2} \, .$$

Let us also perform the rescaling

$$\tilde{\psi} = e^{\frac{1}{4}x^2} \psi \, .$$

With this, the equation (2.7) can be rewritten as

$$\partial_t \tilde{\psi} = -\left[\hat{p}^2 + \frac{1}{4}\hat{x}^2 + \frac{1}{2}\right] = \hat{H}\tilde{\psi} \, ,$$

where we have introduced the usual momentum operator

$$\hat{p} = -i\partial_x \, .$$

In particular, note that

$$\hat{H} = -\left[\hat{p}^2 + \frac{1}{4}\hat{x}^2 + \frac{1}{2}\right] ,$$

is vary close to the hamiltonian of a harmonic oscillator. Indeed, if we define the ladder operators

$$\hat{a} = \frac{1}{2}\hat{x} + i\hat{p}$$
$$\hat{a}^\dagger = \frac{1}{2}\hat{x} - i\hat{p} \, ,$$

then the hamiltonian takes the form

$$\hat{H} = -(\hat{a}^\dagger \hat{a} + 1) \, .$$

Note that this differs from the usual harmonic oscillator by the minus in front, and a one instead of one half within the bracket.

Note that $\hat{H}$ is negative definite, and the highest eigenvalue is $E_0 = -1$, which is obtained by the vacuum ψ_0,

$$\hat{a}\tilde{\psi}_0 = 0 \tag{2.8}$$

$$\hat{H}\tilde{\psi}_0 = -\psi_0 \,. \tag{2.9}$$

Assuming that the initial state $\tilde{\psi}_I$ overlaps with this vacuum, this will become the dominant state in the deep IR, where $t \to \infty$. We therefore want to see what this vacuum looks like. Note that (2.9) reads

$$\partial_x^2 \tilde{\psi}_0 - \frac{1}{4} x^2 \tilde{\psi}_0 + \frac{1}{2} \tilde{\psi}_0 = 0 \,,$$

which has the general solution

$$\psi_0(x) = e^{-\frac{1}{2}x^2} \Big(C_1 + C_2\, Ei(x/\sqrt{2}) \Big)$$

where $\{C_1, C_2\}$ are constants, and we have reintroduced the original field $\psi = e^{-\frac{1}{4}x^2}\tilde{\psi}$. Here Ei denotes the imaginary error function. It should be noted that the function

$$f(x) = e^{-\frac{1}{2}x^2} Ei(x/\sqrt{2})$$

is not a normalizable function, as would be required in usual quantum mechanics. It does however tend to zero as $|x| \to \infty$.

The corresponding potential then reads

$$V(x) = -\log \left[e^{-\frac{1}{2}x^2} \Big(C_1 + C_2\, Ei(x/\sqrt{2}) \Big) \right] = \frac{1}{2} x^2 - \log \Big(C_1 + C_2\, Ei(x/\sqrt{2}) \Big)$$

This has the usual Gaussian quadratic part, corresponding to a free fixed point, plus a correction. We now consider three cases,

$$\begin{aligned} &\text{Case I:} && C_2 = 0 \\ &\text{Case II:} && C_1 = 0 \\ &\text{Case III:} && C_2 = \epsilon \,, \end{aligned}$$

where ϵ is a small number.

Case I, $C_2 = 0$

This is the trivial case, where the fixed point is just a usual Gaussian, with the potential

$$V(x) = \frac{1}{2} x^2 + C \,,$$

where C is some cosmological constant depending on the pre-factor C_1, $C = -\log(C_1)$.

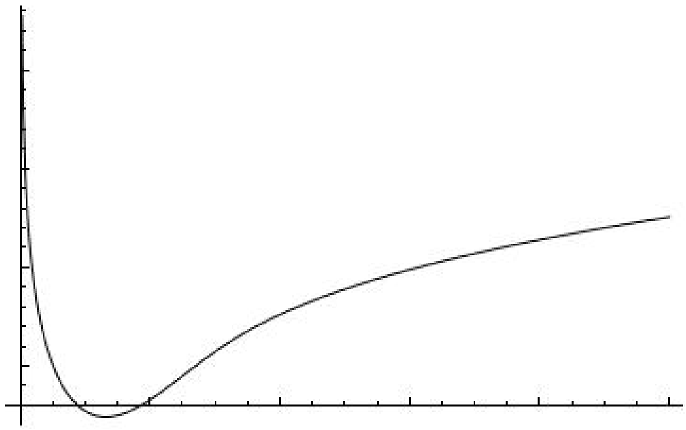

Figure 1. *Generic plot of the error function potential* (2.10).

Case II, $C_1 = 0$

This case is more interesting. The fixed point potential now takes the following form

$$V(x) = \frac{1}{2}x^2 - \log\left(Ei(x/\sqrt{2})\right) + C \,, \tag{2.10}$$

where now $C = -\log(C_2)$. We plot the generic form of this potential in figure 1. Note that although the potential has a gaussian term, the correction from the error function makes the curve distinctly non-gaussian.

Case III, $C_2 = \epsilon$

We now consider the most interesting case, where C_2 is non-zero, but small compared with C_1. For concreteness, we set $C_1 = 1$, and $C_2 = \epsilon$, where we assume ϵ is small. The potential then reads

$$V(x) = \frac{1}{2}x^2 - \log\left(1 + \epsilon\, Ei(x/\sqrt{2})\right) . \tag{2.11}$$

A generic potential of this form is plotted in figure 2.

The conclusion we would like to draw is the following. Assuming that the UV theory does have a non-zero overlap with both IR theories, then the IR will not only be a Gaussian fixed point but a fixed point composed of the two theories. The resulting potential looks rather intriguing - much like a slow-roll inflationary potential. Indeed, the smaller ϵ is, the flatter the plateau becomes. Moreover, the theory looks Gaussian up until the plateau, where it suddenly flattens out.

3 Multiple Dimensions and QFT

The next step is to generalize this to higher dimensions and ultimately to QFT, i.e. how far does the analogy between QM and RG stretch?

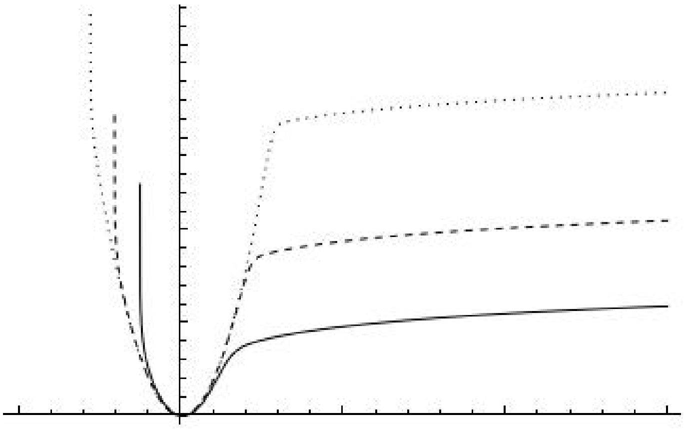

Figure 2. *Generic plots of the potential* (2.11) *for* $\epsilon = 0.1$ *(solid),* $\epsilon = 10^{-3}$ *(dahsed), and* $\epsilon = 10^{-6}$ *(dotted).*

4 Inflation

Let us assume that the scalar field is coupled to gravity,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + (\partial_\mu \phi)^2 + 2V(\phi)\right], \tag{4.1}$$

where $\kappa^2 = 8\pi G_N$ and

$$V(\phi) = \frac{1}{2}\phi^2 - \log \mathcal{C}_0 - \log\left[1 + \frac{\mathcal{C}}{\mathcal{C}_0}\sqrt{\frac{\pi}{2}}Ei\left(\phi/\sqrt{2}\right)\right]. \tag{4.2}$$

We can expand the potential around $\phi \approx 0$, finding

$$V(\phi) = -\log(\mathcal{C}_0) - \frac{\mathcal{C}}{\mathcal{C}_0}\phi + \frac{1}{2}\left(1 + \frac{\mathcal{C}^2}{\mathcal{C}_0^2}\right)\phi^2 + \dots. \tag{4.3}$$

We can identify $\log \mathcal{C}_0 = \Lambda$, where Λ is the cosmological constant. Furthermore, we notice that the mass of the scalar field is $m^2 = 1 + \mathcal{C}^2 e^{-2\Lambda}$, hence

$$\mathcal{C} = \pm e^{\Lambda}\sqrt{m^2 - 1}. \tag{4.4}$$

We can rewrite the full action as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - 2\Lambda + (\partial_\mu \phi)^2 + 2V(\phi)\right], \tag{4.5}$$

where

$$V(\phi) = \frac{1}{2}\phi^2 - \log\left[1 \pm \sqrt{\frac{\pi(m^2-1)}{2}}Ei\left(\phi/\sqrt{2}\right)\right]. \tag{4.6}$$

In this notation, for small $\phi \ll 1$ the potential is

$$V(\phi) \approx \mp\sqrt{m^2-1}\,\phi + \frac{m^2}{2}\phi^2 + \dots, \tag{4.7}$$

whereas for large $\phi \gg 1$, the potential behaves as

$$V(\phi) \to \log\left[\pm\frac{\phi}{\sqrt{m^2-1}}\right] + \dots . \tag{4.8}$$

5 Derivation

$$\phi'(x) = \phi(x) + \sigma\Psi[\phi] \tag{5.1}$$

$$S[\phi'] = S[\phi] + \sigma \int d^4x \Psi[\phi] \frac{\delta S}{\delta\phi(x)} \tag{5.2}$$

Measure:

$$\int \mathcal{D}\phi' = \int \mathcal{D}\phi \left(1 + \sigma \int d^4y \frac{\delta\Psi[\phi(y)]}{\delta\phi(y)}\right) \tag{5.3}$$

Partition function

$$Z = \int \mathcal{D}\phi' e^{-S[\phi']} = \int \mathcal{D}\phi \left(1 + \sigma \int d^4y \frac{\delta\Psi[\phi(y)]}{\delta\phi(y)}\right) \exp\left\{-S[\phi] - \sigma \int d^4x \Psi[\phi] \frac{\delta S}{\delta\phi(x)}\right\} \tag{5.4}$$

$$= \int \mathcal{D}\phi \exp\left\{-S[\phi] - \sigma \int d^4x \left[\Psi \frac{\delta S}{\delta\phi(x)} - \frac{\delta\Psi}{\delta\phi(x)}\right]\right\} \tag{5.5}$$

Now use

5.1 Polchinski's Equation

Next, we consider Polchinski's RG equation

$$\partial_t \psi = -\int_p K'(p^2) \left(\frac{\delta^2\psi}{\delta\phi_p \delta\phi_{-p}} + \frac{2p^2}{K(p^2)} \phi_p \frac{\delta\psi}{\delta\phi_p}\right) , \tag{5.6}$$

where $K'(p^2) = \mathrm{d}K(p^2)/\mathrm{d}p^2$. The corresponding "Hamiltonian"

$$\mathcal{H} = -\int_p K'(p^2) \left(\frac{\delta^2}{\delta\phi_p \delta\phi_{-p}} + \frac{2p^2}{K(p^2)} \phi_p \frac{\delta}{\delta\phi_p}\right)$$

also has a free theory as its only bounded eigenvector, but in this case we have

$$\psi = C \exp\left(-\int_p \frac{p^2}{K(p^2)} \phi_p^2\right) ,$$

with eigenvalue

$$\lambda = \int_p K'(p^2) \frac{2p^2}{K(p^2)} .$$