notes on probe-D-branes

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1 Wilsonian renormalisation group of probe-D-branes

The DBI action:

$$S_{\rm DBI} = \int dr L_{\rm DBI} = \int d^{p+1}x dr \mathcal{L}_{\rm DBI} = -\mathcal{N}_q \int d^{p+1}x dr r^p \sqrt{1 - A_0'^2} \tag{1.1}$$

The constant of motion

$$\frac{\delta S_{\rm DBI}}{\delta A_0} = \frac{\delta L_{\rm DBI}}{\delta A_0'} = \frac{\mathcal{N}_q r^p A_0'}{\sqrt{1 - A_0'^2}} = -d \tag{1.2}$$

and therefore

$$\frac{\delta S_{\rm B}}{\delta A_0} = -\frac{\delta S_{\rm DBI}}{\delta A_0} = d \tag{1.3}$$

so that when varied and evaluated on-shell $S = S_{\rm B} + S_{\rm DBI}$ vanishes. Then

$$\begin{aligned} S &= S_{\rm B}[\rho] + S_{\rm DBI}[\rho] \\ &= S_{\rm B}[\rho - \delta\rho] + S_{\rm DBI}[\rho - \delta\rho] + \int_{\rho-\delta\rho}^{\rho} d\rho' \partial_{\rho'} S_{\rm B} + \int_{\rho-\delta\rho}^{\rho} d\rho' \int d^{p+1}x \frac{\delta S_{\rm B}}{\delta A_0} A_0' + \int_{\rho-\delta\rho}^{\rho} dr L_{\rm DBI} \end{aligned} \tag{1.4}$$

The Wilsonian renormalisation group equation is

$$\begin{aligned} \partial_\rho S_{\rm B} &= -\int d^{p+1}x \left[\frac{\delta S_{\rm B}}{\delta A_0} \frac{\partial A_0}{\partial \rho} + \mathcal{L}_{\rm DBI} \right] \\ &= \int d^{p+1}x \frac{1}{\mathcal{N}_q \rho^p} \left[\mathcal{N}_q^2 \rho^{2p} + \left(\frac{\delta S_{\rm B}}{\delta A_0} \right)^2 \right] \sqrt{1 - A_0'^2} \end{aligned} \tag{1.5}$$

Using $\sqrt{1 - A_0'^2} = \frac{\mathcal{N}_q \rho^p}{\sqrt{\mathcal{N}_q^2 \rho^{2p} + \left(\frac{\delta S_{\rm B}}{\delta A_0}\right)^2}}$, the RG flow equation is

$$\partial_\rho S_{\rm B} = \mathcal{N}_q \rho^p \int d^{p+1}x \sqrt{1 + \frac{1}{\mathcal{N}_q^2 \rho^{2p}} \left(\frac{\delta S_{\rm B}}{\delta A_0} \right)^2} \tag{1.6}$$

Formally, we can expand this to get

$$\partial_\rho S_{\rm B} = \int d^{p+1}x\mathcal{N}_q\rho^p\left[1+\frac{1}{2}\rho^{-2p}\left(\frac{\delta S_{\rm B}}{\delta A_0}\right)^2-\frac{1}{8}\rho^{-4p}\left(\frac{\delta S_{\rm B}}{\delta A_0}\right)^4+\ldots\right]$$
$$= \int d^{p+1}x\mathcal{N}_q\rho^p\sum_{k=0}^{\infty}\binom{1/2}{k}\rho^{-2kp}\left(\frac{\delta S_{\rm B}}{\delta A_0}\right)^{2k} \tag{1.7}$$

Now we can write at ρ_0

$$S_{\rm B}^{\rm sub}[\rho_0] = S_{\rm B}[\rho_0]-S_{\rm c.t.}[\rho_0] = \frac{1}{2}\int d^{p+1}x\mathfrak{d}A_0 = \frac{1}{2}\int d^{p+1}xdA_0 - \frac{\mathcal{N}_q}{p+1}\int d^{p+1}x\sqrt{-g} \tag{1.8}$$

so that

$$\frac{\delta}{\delta A_0}(\mathfrak{d}A_0) = \frac{\delta}{\delta A_0}(dA_0) \tag{1.9}$$

and

$$A_0 = \frac{2\sqrt{-g(\rho_0)}}{(p+1)(d-\mathfrak{d}_0)} \tag{1.10}$$

We can now make all terms in $S_{\rm B}$ run and write out explicitly all generally possible counter-terms.

$$S_{\rm B} = \mathcal{N}_q\int d^{p+1}x\left[\frac{\sqrt{-g}\alpha}{p+1}+\frac{1}{2}\mathfrak{d}A_0-\sqrt{-g}\sum_{n=2}^{\infty}\frac{\lambda_n}{n}A_0^n\right] \tag{1.11}$$

with

$$\alpha(\rho_0)=1, \qquad \mathfrak{d}(\rho_0)=\mathfrak{d}_0, \qquad \sqrt{-g}\lambda_n(\rho_0)=0, \text{ at } \rho_0\to\infty, \tag{1.12}$$

set by the minimal-subtraction values of holographic renormalisation counter-terms. The zeroth term corresponds to the volume renormalisation and higher orders to multi-trace deformations. At orders of A_0^0, A_0^1 and A_0^2 we find from (1.6)

$$\partial_\rho\left(\sqrt{-g}\alpha\right) = (p+1)\sqrt{\rho^{2p}+\mathfrak{d}^2} \tag{1.13}$$
$$\partial_\rho\mathfrak{d} = -2\sqrt{-g}\frac{\mathfrak{d}\lambda_2}{\sqrt{\rho^{2p}+\mathfrak{d}^2}} \tag{1.14}$$
$$\partial_\rho\left(\sqrt{-g}\lambda_2\right) = c_1\lambda_2^2+c_2\lambda_3 \tag{1.15}$$

1.1 RG equation in the IR with zero temperature

Consider the $\rho\to 0$ regime of $\sqrt{-g}=\rho^{p+1}$, where

$$\partial_\rho S_{\rm B} = V_p\frac{\delta S_{\rm B}}{\delta A_0} = \mathcal{N}_qV_p\left[\mathfrak{d}-\sum_{n=1}^{\infty}\sqrt{-g}\lambda_{n+1}A_0^n\right] \tag{1.16}$$

We get

$$\partial_\rho\left(\sqrt{-g}\alpha\right) = (p+1)\mathfrak{d} \tag{1.17}$$
$$\partial_\rho\mathfrak{d} = -2\sqrt{-g}\lambda_2 \tag{1.18}$$
$$\partial_\rho\left(\sqrt{-g}\lambda_n\right) = n\sqrt{-g}\lambda_{n+1}, \text{ for } n\geq 2. \tag{1.19}$$

We find

$$S_{\rm B}[\rho \to 0] = \frac{1}{2(p+1)} V_p \left(1 + e^{A_0 \partial_\rho}\right) \sqrt{-g}\alpha \tag{1.20}$$

and using (1.16) we find

$$\partial_\rho \left(\sqrt{-g}\alpha\right) = 0 \Rightarrow \sqrt{-g}\alpha\Big|_{\rho\to 0} = Const. + \mathcal{O}\left(\rho\right) \tag{1.21}$$

and

$$\mathfrak{d}(\rho \to 0) = 0 \quad \Rightarrow \quad S_{\rm B}^{\rm ren} \to 0 \tag{1.22}$$

Writing $S_{\rm B} = \frac{1}{2} V_p d Z(\rho) A_0$ and using (1.16) we find

1.2 RG equation in the IR with non-zero temperature

Use $\rho = r_H + u$ and take $u \to 0$. If $\alpha(\rho)$ is analytic at $u = 0$ then

$$\mathfrak{d}(u) = \frac{r_H^{p+1/2}\alpha(r_H)}{2\sqrt{p+1}} \frac{1}{\sqrt{u}} + \mathcal{O}(\sqrt{u}) \tag{1.23}$$

2 Thermodynamics

$$\Omega_{\rm fun}(\rho) = -S_{\rm DBI}[\text{on shell}] = S_{\rm B}(\rho) \tag{2.1}$$